Exercise sheet 7
Algebraic Geometry I
Winter term 2017/2018

Exercise 1
Let \( (X, \mathcal{O}_X) \) be an arbitrary ringed space and \( f \in \mathcal{O}_X(X) \) a global section. Show that the set
\[
Y := \{ x \in X \mid f_x \text{ is invertible} \}
\]
is an open subset \( Y \subseteq X \). Deduce the equivalence:
\[
f \text{ is invertible} \iff \forall x \in X : f_x \text{ is invertible}
\]

Exercise 2
Let \( X \) be a topological space and \( \mathcal{F} : \text{Ouv}(X)^{op} \to \text{Ab} \) a presheaf of abelian groups on \( X \). For a morphism \( f : \mathcal{F} \to \mathcal{G} \) of presheaves of abelian groups on \( X \), we define the Kernel presheaf \( \text{Ker}(f) \) by
\[
U \mapsto \text{Ker}(f(U))
\]
and the Image presheaf \( \text{Im}(f) \) by
\[
U \mapsto \text{Im}(f(U))
\]
both with the obvious restriction maps. Show that \( \text{Ker}(f) \) is a sheaf (also called the Kernel sheaf) if \( \mathcal{F} \) and \( \mathcal{G} \) are sheaves. Show that the same is not true for \( \text{Im}(f) \). The sheafification \( a(\text{Im}(f)) \) is called the Image sheaf and by abuse of notation denoted by the same symbol. We say that a sequence
\[
\mathcal{F}' \xrightarrow{f'} \mathcal{F} \xrightarrow{f} \mathcal{F}''
\]
of presheaves (respectively sheaves) of abelian groups on \( X \) is exact, if \( \text{Im}(f') = \text{Ker}(f) \) (where \( \text{Im}(f') \) denotes the image sheaf in the sheaf case). Suppose we have given an exact sequence
\[
0 \to \mathcal{F}' \xrightarrow{f'} \mathcal{F} \xrightarrow{f} \mathcal{F}'' \to 0
\]
of presheaves (respectively of sheaves) of abelian groups on \( X \). Decide whether the induced sequence
\[
0 \to \mathcal{F}'(X) \xrightarrow{f'(X)} \mathcal{F}(X) \xrightarrow{f(X)} \mathcal{F}''(X) \to 0
\]
of abelian groups is exact or just partly exact.

Exercise 3
Let \( f : A \hookrightarrow B \) be an an injective morphism of rings such that \( B \) is a
finitely generated $A$-module (i.e., $f$ is of finite type and integral). Show that the induced morphism $\text{Spec}(f): \text{Spec}(B) \rightarrow \text{Spec}(A)$ is surjective.

**Exercise 4**

Recall all the appearing terms in the following statement, called the *Noether Normalization* (cf. Theorem A.26 of the Skript), and write down a complete proof:

Let $k$ be a field, $A := k[X_1, \ldots, X_n]$ and $B := A/I$ for an ideal $I \neq A$. Then there exists algebraically independent elements $t_1, \ldots, t_d \in A$ such that $k[t_1, \ldots, t_d] \subseteq B$ is integral and injective.