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ALGEBRAIC NUMBER THEORY II

UNIVERSITÄT DUISBURG-ESSEN

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1 Introduction and Overview

There are some typical topics for a second course on Algebraic Number Theory, including *Tate's thesis* and *class field theory*. Both of these topics are closely related to the theory of Hecke L-functions; while *Tate's thesis* gives a purely Fourier-theoretic approach to the analytic continuation and the functional equation of such functions, *class field theory* allows one to relate Hecke characters to one-dimensional Galois representations. Nevertheless, both topics cover just a very particular aspect of L-functions.

Before we continue with mathematics, let me briefly recall the content of the Indian parable *blind men and an elephant*.

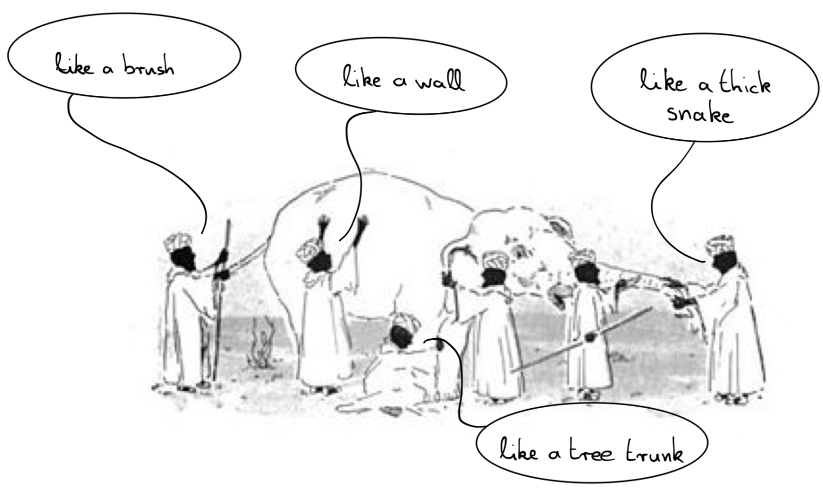


Figure 1.1: The blind men and the elephant; source: Wikipedia

> A group of blind men heard that a strange animal, called an elephant, had been brought to the town, but none of them were aware of its shape and form. Out of curiosity, they said: "We must inspect and know it by touch, of which we are capable". So, they sought it out, and when they found it they groped about it. The first person, whose hand landed on the trunk, said, "This being is like a thick snake". For another one whose hand reached its ear, it seemed like a kind of fan. As for another person, whose hand was upon its leg, said, the elephant is a pillar like a tree-trunk. The blind man who placed his hand upon its side said the

elephant, "is a wall". Another who felt its tail, described it as a rope. The last felt its tusk, stating the elephant is that which is hard, smooth and like a spear.[1]

[1] source: "Blind men and the elephant." Wikipedia: The Free Encyclopedia. Wikimedia Foundation, Inc. 31 March 2021.,

So, based on their experience of touching just a very particular part of the elephant's body they form their minds on what an elephant looks like. While everyone is right in a sense, no one will get a complete picture of the elephant.

Something similar happens, if you ask number theorists about L-functions. All of them will agree that they are of central importance for number theory. But if you ask ten number theorists about the most important aspect of L-functions, you will probably get ten different answers.

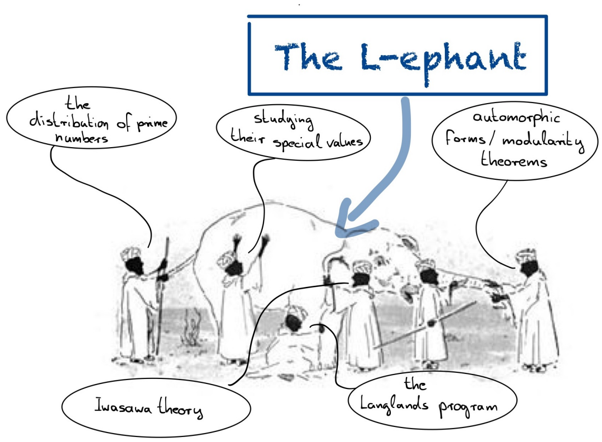


Figure 1.2: The L-ephant.

In this lecture, I will try to give you an idea about many different aspects of L-functions. Of course, I won't be able to go into as much depth as if I had focused on one single aspect. On the other hand, I think it makes more sense to get a vague picture of the entire L-ephant, than to understand its right leg in detail. Now, you might argue that certain aspects of the theory of L-functions are rather analytic, e.g., functional equations, distribution of primes, etc. But if there is one thing we can learn from the Indian parable, it is to be *open-minded* and try to understand different aspects of something we are interested in. That fits perfectly with the spirit of the University of Duisburg-Essen and its slogan ***Offen*** *im Denken*.

1.1 Overview

We will start with the Riemann zeta function and its basic properties. Afterwards, we will briefly say something about the importance of the Riemann zeta function for the distribution of primes. We will prove a weak form of the prime number theorem and briefly indicate the deeper relationship between the Riemann zeta function and prime numbers. Afterwards, we will turn our attention to the special values of the Riemann zeta function. First, we will prove Euler's formula which gives an explicit formula for the values of the Riemann zeta function at the positive even integers. Afterwards, we will briefly discuss odd zeta values, which are much more mysterious. Here, we will prove the irrationality of $\zeta(3)$, which is due to Apéry.

Afterwards, we will turn our attention to cyclotomic fields. In a first step, we will prove the Theorem of Kronecker-Weber which classifies all abelian extensions of $\mathbb{Q}$ and hence can be seen as a very explicit instance of class field theory for the base field $\mathbb{Q}$. We will introduce Dirichlet L-functions in this context. Then, we will discuss the analytic continuation and the functional equation of Dirichlet L-functions in a rather Fourier-theoretic way. This will make you familiar with the main ideas of *Tate's thesis* in a particular case. Then, we will express the Dedekind zeta function of abelian extensions of $\mathbb{Q}$ in terms of Dirichlet L-functions. If time permits, we will prove Kummer's criterion which gives a beautiful relation between special values of L-functions and class groups of cyclotomic fields.

At the end of the term, we will introduce Hecke characters for general number fields. We will give a small overview of how they relate to class field theory and Tate's thesis. Since we have already treated both topics in the more elementary case of the base field $\mathbb{Q}$, we will not go into details here.

2 The Riemann zeta function

In this chapter, we will define the Riemann zeta function. First, we will discuss its basic properties, afterwards we will discuss the relevance of the Riemann zeta function for the distribution of prime numbers. Finally, we will prove Euler's Theorem about the values of the Riemann zeta function at the even positive integers and discuss the irrationality of $\zeta(3)$.

2.1 Basic properties of the Riemann zeta function

The *Riemann zeta function* is defined for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ by the formula

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Here, we define $n^s := \exp(s \log n)$.

Lemma 2.1.1. *The series defining the Riemann zeta function converges absolutely and defines a holomorphic function in the half-plane* $\mathrm{Re}(s) > 1$.

Proof. For a real number $\delta > 0$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq 1 + \delta$, the series

$$\sum_{n=1}^{\infty} \frac{1}{|n^s|}$$

admits the convergent majorant $\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}}$, i.e., the series defining the Riemann zeta function converges absolutely and uniformly [1] on the domain $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 1 + \delta\}$. Now, recall from complex analysis that a uniform limit of holomorphic functions is again holomorphic, see for example Theorem III.1.3 in Freitag–Busam[2]. Since $\delta > 0$ was arbitrary, the claim follows. □

Next, we will prove that the Riemann zeta function admits an *Euler product*. We will prove this in slightly greater generality. A function $f\colon \mathbb{N} \to \mathbb{C}$ is called *completely multiplicative* if it satisfies $f(1) = 1$ and $f(n \cdot m) = f(n) \cdot f(m)$ for all $n, m \in \mathbb{N}$.[3]

[1] Recall that a sequence of functions $(f_n(s))_n$ with $f_n\colon U \to \mathbb{C}$ converges uniformly to f if and only if for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|f(x) - f_n(x)| < \epsilon$ for all $x \in U$ and $n \geq N$. A series of complex functions converges uniformly if and only if its partial sums converge uniformly.

[2] Eberhard Freitag and Rolf Busam. *Funktionentheorie*. Springer-Verlag, Berlin, 1993. ISBN 3-540-50618-7

[3] In other words, it is a homomorphism of monoids $f\colon (\mathbb{N}, \cdot) \to (\mathbb{C}, \cdot)$

Lemma 2.1.2. *Let $f\colon \mathbb{N} \to \mathbb{C}$ be a completely multiplicative function for which the series $\sum_{n=1}^{\infty} f(n)$ converges absolutely. Then*

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} (1 - f(p))^{-1},$$

where p runs through the set of all primes.[4]

[4] Recall that an infinite product $\prod_{i=1}^{\infty} a_i$ of complex numbers is said to converge if the sequence of partial products $P_n = \prod_{i=1}^{n} a_i$ has a non-zero limit.

Proof. The assumptions imply that $|f(n)| < 1$ for $n \geq 2$. Indeed, if we had $|f(n)| \geq 1$ for some $n \geq 2$ then $|f(n^k)| \geq 1$ for every $k \geq 1$ contradicting the absolute convergence of the sum $\sum_{n=1}^{\infty} f(n)$. In particular, we have $|f(p)| < 1$ for every prime p and obtain the geometric series

$$(1 - f(p))^{-1} = \sum_{k=0}^{\infty} f(p)^k.$$

Using the complete multiplicativity of f and the unique factorization in $\mathbb{Z}$, we obtain for every positive integer N the identity

$$\prod_{p \leq N} (1 - f(p))^{-1} = \sum_{\substack{n = p_1^{\alpha_1} \dots p_m^{\alpha_m} \\ p_i \leq N}} f(n) = \sum_{n=1}^{\infty} f(n) - \sum_{\substack{n \\ p \mid n \text{ for some } p > N}} f(n).$$

Now it follows

$$\left| \sum_{n=1}^{\infty} f(n) - \prod_{p \leq N} (1 - f(p))^{-1} \right| \leq \sum_{\substack{n \\ p \mid n \text{ for some } p > N}} |f(n)| \leq \sum_{n > N} |f(n)|.$$

The latter sum tends to zero as $N \to \infty$ by the absolute convergence of $\sum_{n=1}^{\infty} f(n)$ and the result follows. $\square$

Corollary 2.1.3. *The Riemann zeta function admits the following product formula for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$*

$$\zeta(s) = \prod_{p} \frac{1}{1 - p^{-s}},$$

where p runs through the set of all prime numbers. This formula is called the Euler product *of the Riemann zeta function. For a prime p, the term $\frac{1}{1-p^{-s}}$ is called the* Euler factor *at p.*

Proof. We apply the previous lemma to the completely multiplicative function $n \mapsto \frac{1}{n^s}$. $\square$

2.1.1 *The Gamma function*

As we will see, the Gamma function will play an important role for proving the functional equation of the Riemann zeta function. The Gamma function is defined for $z \in \mathbb{C}$ with $\operatorname{Re}(z) > 0$ by

$$\Gamma(z) := \int_0^{\infty} e^{-t} t^z \frac{dt}{t}.$$

To prove that the Gamma function is holomorphic, we recall the following result from complex analysis:

Lemma 2.1.4 (Leibniz rule). *Let $U \subseteq \mathbb{C}$ open and $a, b \in \mathbb{R}$ with $a < b$. Suppose that $f\colon [a,b] \times U \to \mathbb{C}$ is a continuous function, which is holomorphic for every $t \in [a,b]$. Then the function*

$$z \mapsto \int_a^b f(z,t)dt$$

is holomorphic on U.

Proof. We refer to Lemma II.3.3 in Freitag–Busam[5]. □

[5] Eberhard Freitag and Rolf Busam. *Funktionentheorie*. Springer-Verlag, Berlin, 1993. ISBN 3-540-50618-7

Using the Leibniz rule it is not difficult to prove that the Gamma function is holomorphic.

Lemma 2.1.5. *The integral defining the Gamma function converges absolutely for* $\mathrm{Re}(z) > 0$*, where it represents a holomorphic function.*

Proof. We split the integral into two parts

$$\int_0^\infty e^{-t}t^z \frac{dt}{t} = \int_0^1 e^{-t}t^z \frac{dt}{t} + \int_1^\infty e^{-t}t^z \frac{dt}{t}$$

and discuss both integrals separately. Note that we have the equality

$$|t^{z-1}e^{-t}| = t^{x-1}e^{-t}$$

for $x = \mathrm{Re}(z)$. For any real number $x_0 > 0$, we find a constant $C > 0$ such that $t^{x-1} \leq Ce^{t/2}$ for all $0 < x \leq x_0$ and all $t \geq 1$. This estimate together with the existence of

$$\int_1^\infty e^{-t/2}dt$$

shows the absolute convergence of the second integral. For the absolute convergence of the first integral, we use the estimation $|t^{z-1}e^{-t}| < t^{x-1}$ for $t > 0$ and the existence of

$$\int_0^1 t^{x-1}dt \quad \text{for } x > 0.$$

The above estimates show that the functions

$$f_n(z) := \int_{1/n}^n e^{-t}t^z \frac{dt}{t}$$

converge uniformly to the Gamma function. Each of the functions f_n is holomorphic by the Leibniz rule. Therefore, the Gamma function is holomorphic as a uniform limit of holomorphic functions. □

Using integration by parts, it is not difficult to prove the following result.

Lemma 2.1.6. *The Gamma function satisfies for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ the functional equation*

$$\Gamma(z+1) = z\Gamma(z).$$

In particular, we have for a positive integer n the formula $\Gamma(n) = (n-1)!$.

Proof. We will prove this in the exercises. □

We can use the functional equation to extend the Gamma function to $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.

Lemma 2.1.7. *The Gamma function extends to a meromorphic function on all of $\mathbb{C}$ with simple poles at all non-positive integers and residue*[6]

$$\mathrm{res}_{z=-n}\Gamma(z) = \frac{(-1)^n}{n!} \quad \textit{for } n \in \mathbb{Z}_{\geq 0}.$$

Proof. We will prove this result in the exercises. □

[6] Recall: The *residue* of a meromorphic function f at $z_0 \in \mathbb{C}$ is given by the term a_{-1} in its Laurent expansion

$$f(z) = \sum_{k \in \mathbb{Z}} a_k (z - z_0)^k.$$

If f has at most a simple pole in z_0, we can compute the residue as follows:

$$\mathrm{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z).$$

For later reference, we will prove the completion formula for the Gamma function.

Proposition 2.1.8 (Completion formula). *For all $z \in \mathbb{C} \setminus \mathbb{Z}$ we have*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

Proof. Both of the functions $\Gamma(z)\Gamma(1-z)$ and $\frac{\pi}{\sin \pi z}$ have only simple poles at the integers. Let us first compute their residues. For $n \in \mathbb{N}_0$, we have

$$\mathrm{Res}_{z=-n}\, \Gamma(z)\Gamma(1-z) = \Gamma(1+n)\, \mathrm{Res}_{z=-n}\, \Gamma(z) = (-1)^n$$

and similarly one proves

$$\mathrm{Res}_{z=n}\, \Gamma(z)\Gamma(1-z) = (-1)^n.$$

Thus, the formula $\mathrm{Res}_{z=n}\, \Gamma(z)\Gamma(1-z) = (-1)^n$ holds for all integers. The leading term of the Taylor expansion of $\sin \pi z$ at $z = n$ is $(-1)^n \pi$. We deduce the formula

$$\mathrm{Res}_{z=n} \frac{\pi}{\sin \pi z} = (-1)^n.$$

Since both functions $\Gamma(z)\Gamma(1-z)$ and $\frac{\pi}{\sin \pi z}$ have only simple poles with the same residues, we deduce that

$$h(z) := \Gamma(z)\Gamma(1-z) - \frac{\pi}{\sin \pi z}$$

extends to an entire function[7] on $\mathbb{C}$. Now, the strategy is to apply Liouville's Theorem[8] to deduce that h is constant. Therefore, we need to prove the boundedness of h. Let us first prove that the function $h(z)$ is 'periodic up to sign', i.e.,

$$h(z+1) = -h(z).$$

[7] An *entire* function is a function which is holomorphic the whole complex plane.

[8] Recall, that Liouville's Theorem says that a bounded entire function is constant.

Of course, the function $\frac{\pi}{\sin \pi z}$ is periodic up to sign and the same property for $\Gamma(z)\Gamma(1-z)$ follows from the following computation:

$$\begin{aligned}\Gamma(z+1)\Gamma(1-(z+1)) &= z\Gamma(z)\Gamma(-z)\\ &= -\Gamma(z)(-z)\Gamma(-z) = -\Gamma(z)\Gamma(1-z).\end{aligned}$$

We conclude that also h is 'periodic up to sign'. In particular, the function $|h(z)|$ is a periodic function. Thus, it suffices to prove that $h(z)$ is bounded on the vertical strip

$$V_0 = \{z \in \mathbb{C} \mid 0 \leq \mathrm{Re}(z) < 1\}.$$

Indeed, let us first remark that $h(z)$ is bounded on the compact set[9]

[9] Recall that every continuous function on a compact set is bounded.

$$V_0 \cap \{|\mathrm{Im}(z)| \leq 1\}.$$

For the boundedness of h on $V_0 \cap \{|\mathrm{Im}(z)| > 1\}$ it suffices to prove that both functions Γ and $\frac{\pi}{\sin \pi z}$ are bounded on this set. It is not difficult to see that $\frac{\pi}{\sin \pi z}$ is bounded on $V_0 \cap \{|\mathrm{Im}(z)| > 1\}$, so let us turn our attention to $\Gamma(z)$. For $z \in V_0$ with $\mathrm{Im}\, z > 1$, we have

$$|\Gamma(z)| = \frac{|\Gamma(z+1)|}{|z|} \leq |\Gamma(z+1)| \leq \int_0^\infty e^{-t} t^{\mathrm{Re}(z+1)} \frac{dt}{t} = \Gamma(\mathrm{Re}(z+1)).$$

Now, observe that the function $\Gamma(\mathrm{Re}(z+1))$ is bounded since Γ is bounded on the compact interval $[1,2]$.

Thus, we have shown that the function

$$h(z) := \Gamma(z)\Gamma(1-z) - \frac{\pi}{\sin \pi z}$$

is an entire bounded function. By Liouville's Theorem, h has to be constant. To conclude that $h = 0$, let us observe

$$h(-z) = -h(z).$$

This equation implies $h(0) = 0$ and hence $h = 0$. □

Corollary 2.1.9. *We have $\Gamma(1/2) = \sqrt{\pi}$, and for $n \in \mathbb{N}$*

$$\Gamma\left(\frac{1}{2} + n\right) = \sqrt{\pi} \prod_{k=0}^{n-1} \left(k + \frac{1}{2}\right).$$

Proof. The first formula follows immediately from Proposition 2.1.8:

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(1 - \frac{1}{2}\right) = \frac{\pi}{\sin \pi/2} = \pi.$$

The second formula follows from the first formula using the functional equation

$$\Gamma(z+1) = z\Gamma(z).$$

□

Outlook

[10] In the upcoming lectures, will see many interesting aspects of the Riemann zeta function. But also the Gamma function is an interesting function. We have already seen the formula

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

This formula together with the functional equation gives us the values of the Gamma function at all rational numbers with denominator 2. In particular, $\Gamma(1/2)$ is a transcendental number. One might ask about the nature of other values $\Gamma(\frac{\bullet}{d})$ at rational numbers with denominator $d > 2$. Surprisingly, the nature of these values is closely related to periods of elliptic curves and abelian varieties. Let us give a simple example. It is not so difficult to compute the following integral:

$$\Omega := \int_1^\infty \frac{1}{\sqrt{x^3 - x}} = \frac{\Gamma(\frac{1}{4})^2}{2^{3/2}\pi^{1/2}}.$$

Of course, this formulas doesn't look very interesting at first glance. But, it has the following interesting arithmetic interpretation. The equation

$$E\colon y^2 = x^3 - x,$$

is an example of an (affine) elliptic curves with complex multiplication. Such elliptic curves play an important role in arithmetic geometry. Now, observe that the right hand side of the defining equation of the elliptic curve E appears in the above integral formula for Ω. More precisely, it can be shown that the differential form $\omega := dx/y$ is an example of a *global differential form* on the above elliptic curve E. This gives the following re-interpretation of the above integral formula:

$$\int_1^\infty \frac{dx}{y} = \int_1^\infty \frac{1}{\sqrt{x^3 - x}} = \frac{\Gamma(\frac{1}{4})^2}{2^{3/2}\pi^{1/2}}.$$

In algebraic geometry, such integrals are called *period integrals* and their values are called *periods*[11]. Thus, the innocent looking integral formula turns out to give an interesting relation between the Gamma value $\Gamma(1/4)$, π and the period of an elliptic curve with complex multiplication. This is only the tip of the iceberg; there are much more general relations between Gamma values and periods (e.g. the Chowla-Selberg formula). This arithmetic interpretation of Gamma values can finally be used to prove deep transcendence results for Gamma values, for example:

Theorem (Chudnovsky)**.** *The values $\Gamma(\frac{1}{4})$ and π are algebraically independent. In particular, $\Gamma(\frac{1}{4})$ is transcendental.*

[10] At the end of the section, we will often give an outlook on interesting topics. Reading these parts of the lecture notes is voluntary. They are neither relevant for understanding the upcoming lectures nor for the final exam.

[11] Periods on a d-dimensional smooth and proper variety X over $\mathbb{Q}$ are defined by integrating an algebraic differential forms ω of degree i along a cycle $C \in H_i(X(\mathbb{C}), \mathbb{Z})$, i.e.,

$$\int_C \omega.$$

In our case, the path

$$\gamma := \{t \in [1, \infty) \mid (t, \sqrt{t^3 - t})\}$$

represents a non-trivial element $\gamma \in H_1(E(\mathbb{C}), \mathbb{Z})$ and so Ω is indeed a period in the above sense.

2.2 The functional equation

Our next goal is to extend the Riemann zeta function to a meromorphic function on $\mathbb{C} \setminus \{1\}$ and to prove its functional equation. Let us start with some facts about rapidly decreasing functions.

2.2.1 The classical theta function

As a preparation for the proof of the functional equation, we will introduce the classical theta function and prove that it is rapidly decreasing in the following sense:

Definition 2.2.1. Let $D \subseteq \mathbb{R}$ be an unbounded subset. A function $f\colon D \to \mathbb{C}$ is called *rapidly decreasing* if for every positive integer $N \in \mathbb{N}$ we have[12] $|t|^N |f(t)| \to 0$ as $|t| \to \infty$. For $D = \mathbb{N}$, we will call such a function a *rapidly decreasing sequence*.

[12] Alternatively, one can demand for any positive integer $N \in \mathbb{N}$ that $t \mapsto |t|^N |f(t)|$ is bounded on $D \cap [c, \infty)$ for all sufficiently large real numbers c.

Beispiel 2.2.2. The following functions are examples of rapidly decreasing functions:

(a) The function $f(t) := e^{-t}$ is rapidly decreasing on $D = \mathbb{R}_{>0}$.

(b) The function $n \mapsto e^{-n^2}$ is rapidly decreasing on $D = \mathbb{Z}$.

(c) The function $f(t) := t^{-2021}$ is not rapidly decreasing on $D = [1, \infty)$.

In the proof of the analytic continuation and the functional equation, the *classical theta function* $\theta\colon \mathbb{R}_{>0} \to \mathbb{R}$ given by

$$\theta(t) := \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} = 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 t}$$

will play an important role. Note that this sequence converges absolutely[13] for any real number $t \in \mathbb{R}_{>0}$. The values of the classical theta series are always ≥ 1. Of course, this implies that the theta function is not rapidly decreasing, but the following Lemma shows that the closely related function

[13] Indeed, we have $e^{-n^2 t} = (e^{-nt})^n$. For sufficiently large n, we have $e^{-nt} < 1$ and the series can be estimated by a convergent geometric series.

$$\omega(t) := \frac{1}{2}\left(\theta(t) - 1\right) = \sum_{n=1}^{\infty} e^{-\pi n^2 t}$$

is rapidly decreasing.

Lemma 2.2.3. *The function $\omega(t)$ is rapidly decreasing on $[1, \infty)$.*

Proof. We will prove this in the exercises. □

For later reference, let us record the following elementary property of rapidly decreasing functions:

Lemma 2.2.4. *Let $f\colon [1,\infty) \to \mathbb{R}$ be a continuous function which is rapidly decreasing. Then, for any complex number $s \in \mathbb{C}$, the integral*

$$g(s) := \int_1^\infty f(t)t^s dt$$

converges absolutely and defines a holomorphic function on $\mathbb{C}$.

Proof. Since f is rapidly decreasing, there exists a constant $C > 0$ such that

$$|f(t)|t^{\mathrm{Re}(s)+2} \leq C$$

for all $t \geq 1$, i.e.,

$$|f(t)||t^s| \leq \frac{C}{t^2}.$$

Now, the absolute convergence follows from the convergence of the integral $\int_1^\infty \frac{1}{t^2} dt$. The function g is holomorphic since it is the uniform limit of the holomorphic functions[14]

$$g_n(s) := \int_1^n f(t)t^s dt.$$

□

[14] Here, we use the Leibniz rule, i.e., Lemma 2.1.4.

2.2.2 *The functional equation*

The functional equation of the Riemann zeta function will follow from the following transformation behaviour of the theta series

$$\theta(t) = \frac{1}{\sqrt{t}}\theta(1/t). \tag{2.1}$$

For the moment, we postpone the proof of (2.1) and deduce the functional equation of the Riemann zeta function from the transformation behaviour of the theta series.

Theorem 2.2.5. *Let us define the* completed Riemann zeta function *as*

$$\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

The completed zeta function admits a holomorphic continuation to $\mathbb{C} \setminus \{0,1\}$ with simple poles at $s = 0$ and $s = 1$ and satisfies the functional equation

$$\xi(s) = \xi(1-s).$$

Proof. In a first step, let us relate the completed Riemann zeta function to the theta function. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we have

$$\begin{aligned}\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \int_0^\infty t^{s/2}\pi^{-s/2}e^{-t}\frac{dt}{t} \\ &= \sum_{n=1}^{\infty} \int_0^\infty \left(\frac{t}{n^2\pi}\right)^{s/2} e^{-t}\frac{dt}{t} \\ &\overset{[15]}{=} \sum_{n=1}^{\infty} \int_0^\infty t^{s/2}e^{-\pi t n^2}\frac{dt}{t} \\ &\overset{[16]}{=} \int_0^\infty t^{s/2}\omega(t)\frac{dt}{t} \end{aligned} \tag{2.2}$$

This will be helpful, since $\theta(t)$ satisfies a nice functional equation which will imply the corresponding functional equation for $\xi(s)$. Let us express the functional equation for $\theta(t)$ in terms of the function $\omega(t) = \frac{1}{2}(\theta(t) - 1)$. Using the functional equation (2.1), we get

$$\begin{aligned}\omega(1/t) = \frac{1}{2}(\theta(1/t) - 1) &= \frac{1}{2}(\sqrt{t}\theta(t) - 1) \\ &= \frac{1}{2}(\sqrt{t}(1 + 2\omega(t)) - 1) = \sqrt{t}\omega(t) + \frac{\sqrt{t}}{2} - \frac{1}{2}.\end{aligned}$$

The strategy is now to use the formula (2.2), i.e.,

$$\xi(s) = \int_0^\infty t^{s/2}\omega(t)\frac{dt}{t}$$

to prove both, the functional equation and the meromorphic continuation. Unfortunately, the integral on the right hand side does not converge for general $s \in \mathbb{C}$. By Lemma 2.2.4, for general $s \in \mathbb{C}$, the convergence at ∞ is not problematic, since $\omega(t)$ is a rapidly decreasing function. But, for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \leq 1$, the integral does not converge absolutely near 0. So let us split the integral into a problematic part and an unproblematic part:

$$\int_0^\infty t^{s/2}\omega(t)\frac{dt}{t} = \underbrace{\int_0^1 t^{s/2}\omega(t)\frac{dt}{t}}_{\text{converges only for } \mathrm{Re}(s)>1} + \underbrace{\int_1^\infty t^{s/2}\omega(t)\frac{dt}{t}}_{\text{converges for all } s\in\mathbb{C}}.$$

Luckily, we can use the transformation behaviour of ω and the substitution $t \mapsto 1/t$ to write the problematic part in a more convenient

[15] Here, we have substituted $\frac{t}{n^2\pi}$ by t. Maybe, you have already wondered why we use the logarithmic differential $\frac{dt}{t}$ instead of dt. One reason is that the logarithmic differential $\frac{dt}{t}$ is invariant under substitutions of the form $t \mapsto c \cdot t$ for a constant $c \in \mathbb{R}$.

[16] Here, we have used the absolute convergence to interchange integration and summation. More precisely, we have used the following fact from analysis: If f_n is a sequence of Lebesgue measurable functions and if $\sum \int |f_n| < \infty$ or $\int \sum |f_n| < \infty$, then

$$\sum \int f_n = \int \sum f_n.$$

way:

$$\begin{aligned}
\int_0^1 t^{s/2}\omega(t)\frac{dt}{t} &= \int_1^\infty \omega(1/t)t^{-s/2}\frac{dt}{t} \\
&= \int_1^\infty \left(\sqrt{t}\omega(t) + \frac{\sqrt{t}}{2} - \frac{1}{2}\right) t^{-s/2}\frac{dt}{t} \\
&= \int_1^\infty \omega(t)t^{\frac{1-s}{2}}\frac{dt}{t} + \frac{1}{2}\int_1^\infty t^{\frac{-1-s}{2}}dt - \frac{1}{2}\int_1^\infty t^{-1-\frac{s}{2}}dt \\
&= \underbrace{\int_1^\infty \omega(t)t^{\frac{1-s}{2}}\frac{dt}{t}}_{\text{converges for all } s\in\mathbb{C}} - \frac{1}{1-s} - \frac{1}{s}.
\end{aligned}$$

By combining what we have shown above, we obtain

$$\xi(s) = \int_1^\infty t^{s/2}\omega(t)\frac{dt}{t} + \underbrace{\int_1^\infty \omega(t)t^{\frac{1-s}{2}}\frac{dt}{t}}_{\text{converges for all } s\in\mathbb{C}} - \frac{1}{1-s} - \frac{1}{s}. \tag{2.3}$$

The right hand side of this equation is a meromorphic function on $\mathbb{C}\setminus\{0,1\}$ with simple poles at $s=0$ and $s=1$. Thus, we have succeeded to find a meromorphic continuation of $\xi(s)$. The functional equation follows from the fact that the right hand side of (2.3) is invariant under the substitution $s\mapsto 1-s$. □

In the above proof, we have written the completed Riemann zeta function as an integral

$$\xi(s) = \int_0^\infty t^{s/2}\omega(t)\frac{dt}{t}.$$

This is a special case of the following definition:

Definition 2.2.6. For a given function $f\colon \mathbb{R}_{>0} \to \mathbb{R}$, we define its Mellin transform by the formula

$$M_f(s) := \int_0^\infty f(t)t^s\frac{dt}{t},$$

whenever the integral exists.

Outlook

The classical theta function

$$\theta(t) := \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = 1 + 2\sum_{n=1}^\infty e^{-\pi n^2 t}$$

admits various generalizations which appear in different branches of mathematics, e.g., number theory, algebraic geometry, mathematical physics and analysis. Let us indicate the relation to algebraic geometry. One can define the following generalization of the classical theta function. For $z \in \mathbb{Z}$ and $\tau \in \mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\}$, we define the *classical theta function* as

$$\Theta(\tau, z) := \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z}.$$

This series converges absolutely for all $(\tau, z) \in \mathbb{H} \times \mathbb{C}$ and is holomorphic in τ and z. Indeed, this is a generalization of our classical theta function

$$\theta(t) := \Theta(it, 0).$$

Before we describe the relevance of the Jacobi theta functions for algebraic geometry, let us make a brief detour to (complex) elliptic curves. For fixed $\tau \in \mathbb{H}$, let us write Λ_τ for the subgroup $\langle 1, \tau \rangle$ of $(\mathbb{C}, +)$ generated by 1 and τ. Such subgroups are called *lattices* in $\mathbb{C}$. The quotient

$$\mathbb{C}/\Lambda_\tau$$

turns out to be a complex manifold of dimension 1. Even better, it can be shown that $\mathbb{C}/\Lambda_\tau$ are the $\mathbb{C}$-valued points of an elliptic curve [17] defined over $\mathbb{C}$. Thus, it is not only a complex manifold but a complex manifold which 'comes from an algebraic variety'. Conversely, it can be shown that the $\mathbb{C}$-valued points of any elliptic curve over $\mathbb{C}$ are isomorphic to $\mathbb{C}/\Lambda_\tau$ for some $\tau \in \mathbb{C}$. This correspondence is very important for studying isomorphism classes of elliptic curves[18].

Let us now come back to the Jacobi theta function. As we have indicated above, we can associate to $\tau \in \mathbb{H}$ a complex elliptic curve $\mathbb{C}/\Lambda_\tau$. One might hope, that $\Theta(\tau, \cdot)$ is a Λ_τ-periodic function on $\mathbb{C}$. If this were the case, one would obtain a well-defined function on the elliptic curve $\mathbb{C}/\Lambda_\tau$. Unfortunately, it turns out that the Jacobi theta function is not Λ_τ-periodic. Nevertheless, it satisfies a nice transformation behaviour for maps of the form $z \mapsto z + \lambda$ for $\lambda \in \Lambda_\tau$. If one makes this transformation behaviour explicit, it turns out that the Jacobi theta function (and certain generalizations) give an explicit description of sections of line bundles on elliptic curves. This plays an important role in the study of elliptic curves, their line bundles and their cohomology. This can even be generalized to abelian varieties which can be seen as higher dimensional generalizations of elliptic curves. Finally, let us observe that elliptic curves and abelian varieties have many interesting applications to number theory. This closes the circle and we are back in the world of number theory where we belong to, at least in this lecture.

[17] If you have never seen an elliptic curve, you can think about an elliptic curve (over $\mathbb{C}$) as a curve given by the vanishing locus of the equation

$$y^2 = x^3 - Ax - B$$

for certain $A, B \in \mathbb{C}$ with the property that the polynomial $x^3 - Ax - B$ has only simple roots.

[18] Finally, this leads to an explicit description of the $\mathbb{C}$-valued points of certain moduli spaces of elliptic curves.

2.3 Fourier Theory

In this section, we deduce the functional equation of the classical theta series

$$\theta(t) := \sum_{n\in\mathbb{Z}} e^{-\pi i n^2 t} = 1 + 2\sum_{n=1}^{\infty} e^{-\pi i n^2 t}.$$

We will give a purely Fourier-theoretic proof. So let us start with recalling[19] some basic facts from Fourier analysis. Let us write $C^\infty(\mathbb{R})$ for the set of all infinite differentiable complex-valued functions on $\mathbb{R}$.

[19] By the way, don't be afraid if you are not familiar with Fourier analysis. We will recall all relevant statements and definitions.

Definition 2.3.1. The space of *Schwartz* functions consists of all $f \in C^\infty(\mathbb{R})$ such that all derivatives $f^{(n)}$ are rapidly decreasing on $\mathbb{R}$. We will write $\mathcal{S}(\mathbb{R})$ for the space of all Schwartz functions on $\mathbb{R}$. For a Schwartz function $f \in \mathcal{S}(\mathbb{R})$ let us define its *Fourier transform* as

$$\widehat{f}(x) := \int_{\mathbb{R}} f(y) e^{-2\pi i x y} dy.$$

An important example for a function in the Schwartz space is the function $f(x) = e^{-\pi x^2}$. In the exercises, we will verify that it is indeed rapidly decreasing. This function has the important property that it is its own Fourier transform. More generally, we have:

Lemma 2.3.2. *For $t \in \mathbb{R}_{>0}$ let us consider the function $f_t(x) = e^{-\pi t x^2}$. We have*

$$\widehat{f_t} = \frac{1}{\sqrt{t}} f_{1/t}.$$

Proof. This will be shown in the exercises. □

We will use the following Theorem without proof:

Theorem 2.3.3 (Fourier Inversion Theorem on $\mathbb{R}$)**.** *For $f \in \mathcal{S}(\mathbb{R})$, we have $\widehat{f} \in \mathcal{S}(\mathbb{R})$ and the Fourier inversion formula holds:*

$$\widehat{\widehat{f}}(x) = f(-x)$$

Proof. For a proof, we refer to Theorem 2.2.14. in Loukas Grafakos' book on 'Classical Fourier Analysis'[20]. □

[20] Loukas Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2008. ISBN 978-0-387-09431-1

The functional equation of the classical theta series will follow from the Poisson summation formula:

Theorem 2.3.4 (Poisson summation formula)**.** *For $f \in \mathcal{S}(\mathbb{R})$, we have*

$$\sum_{k\in\mathbb{Z}} f(k) = \sum_{k\in\mathbb{Z}} \widehat{f}(k).$$

Before we give the proof of the Poisson summation formula, let us explain how this implies the functional equation of θ.

Corollary 2.3.5. *The classical theta series satisfies the functional equation:*

$$\theta(t) = \frac{1}{\sqrt{t}}\theta(1/t).$$

Proof. Lemma 2.3.2 gives for the function $f_t(x) = e^{-\pi t x^2}$ the following explicit formula for the Fourier transform

$$\widehat{f_t} = \frac{1}{\sqrt{t}} f_{1/t}.$$

The functional equation of the theta function follows now immediately from the Poisson summation formula:

$$\theta(t) = \sum_{k\in\mathbb{Z}} f_t(k) = \sum_{k\in\mathbb{Z}} \widehat{f_t}(k) = \sum_{k\in\mathbb{Z}} \frac{1}{\sqrt{t}} f_{1/t}(k) = \frac{1}{\sqrt{t}}\theta(1/t).$$

□

For the proof of the Poisson summation formula, we will need the Fourier expansion of periodic functions, i.e., functions $f\colon \mathbb{R} \to \mathbb{C}$ with $f(x+1) = f(x)$. We will identify such functions with functions on $S^1 := \mathbb{R}/\mathbb{Z}$. We will also use the following Theorem form Fourier Analysis without proof.

Theorem 2.3.6 (Fourier Expansion on S^1)**.** *For $f \in C^\infty(\mathbb{R}/\mathbb{Z})$ and $x \in \mathbb{R}$, we have*

$$f(x) = \sum_{k\in\mathbb{Z}} c_k(f) e^{2\pi i k x}, \tag{2.4}$$

where $c_k(f) := \int_0^1 f(t)e^{-2\pi i k t}dt$. The sum (2.4) converges absolutely and uniformly in x. The elements of the sequence $(c_k(f))_{k\in\mathbb{Z}}$ are called Fourier coefficients *and form a rapidly decreasing sequence on $\mathbb{Z}$.*

Proof. For a proof, we refer to Theorem 2.2.14. in Loukas Grafakos' book on 'Classical Fourier Analysis'[21]. □

[21] Loukas Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2008. ISBN 978-0-387-09431-1

Let us now deduce the Poisson summation formula using Fourier theory:

Proof of Poisson summation. For a given Schwartz function $f \in \mathcal{S}(\mathbb{R})$ let us define

$$F(x) := \sum_{k\in\mathbb{Z}} f(x+k).$$

Since f and all its derivatives are rapidly decreasing, this sum converges absolutely and defines a smooth function on $\mathbb{R}$. Furthermore, the function F is periodic:

$$F(x+1) = \sum_{k\in\mathbb{Z}} f(x+k+1) = \sum_{k\in\mathbb{Z}} f(x+k) = F(x).$$

Thus, by Theorem 2.3.6, it admits a Fourier expansion

$$F(x) = \sum_{k\in\mathbb{Z}} c_k(F) e^{2\pi i k x},$$

with $c_k(F) := \int_0^1 F(t)e^{-2\pi i k t}dt$. The result follows from the following computation:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} f(k) = F(0) &= \sum_{k\in\mathbb{Z}} \int_0^1 F(t)e^{-2\pi i k t}dt \\
&= \sum_{k\in\mathbb{Z}} \int_0^1 \sum_{l\in\mathbb{Z}} f(t+l)e^{-2\pi i k t}dt \\
&= \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \int_l^{l+1} f(t)e^{-2\pi i k (t-l)}dt \\
&= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} f(t)e^{-2\pi i k t}dt = \sum_{k\in\mathbb{Z}} \widehat{f}(k).
\end{aligned}$$

Here, we have interchanged summation and integration by absolute convergence, compare footnote[16]. □

Outlook

Fourier analysis is everywhere. Perhaps, you might have already seen examples of Fourier expansions for periodic functions in terms of the sin and cos functions. In Theorem 2.3.6, we have seen the following formula for a periodic function f

$$f(x) = \sum_{k\in\mathbb{Z}} c_k(f) e^{2\pi i k x},$$

where $c_k(f) := \int_0^1 f(t)e^{-2\pi i k t}dt$. By writing $e^{2\pi i k x} = \cos(2\pi i k x) + i\sin(2\pi i k x)$, we obtain the following version of the Fourier expansion formula

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(2\pi i k x) + b_k \sin(2\pi i k x)), \tag{2.5}$$

with a_k and b_k defined for $k \geq 1$ by the formula

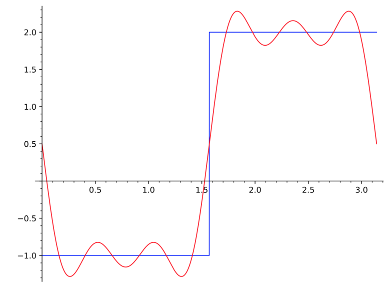
$$\begin{aligned}
a_k &= c_k + c_{-k} \\
b_k &= i(c_k - c_{-k}).
\end{aligned}$$

This allows one to write periodic functions as an infinite linear combination of simple trigonometric functions. The above Fourier expansion formula in (2.5) has the advantage that the coefficients b_k vanish if f is real-valued. The fact that the sequence of Fourier coefficients a_k and b_k (respectively c_k) is rapidly decreasing has important applications. One often obtains a quiet good approximation of f by considering only the truncated sequences[22]

$$\frac{a_0}{2} + \sum_{k=1}^{N} (a_k \cos(2\pi ikx) + b_k \sin(2\pi ikx)), \qquad \text{for some } N.$$

This has many applications, even outside of mathematics. Many important modern developments would not exist without Fourier Analysis. Just to mention a few of them: efficient compression of data (mp3, mp4, jpg), bandpass filters, image processing, face recognition, etc.. This is also a good place to recommend the following nice video[23] about a long forgotten machine – the Harmonic Analyzer.

[22] Approximation of a piecewise linear function:



[23] Here is a link to the YouTube video about the Harmonic Analyzer:

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