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SoSe 2021
Sheet 2

Exercises
Algebraic Number Theory II

(due Sunday 25th of April, 20:00, to manuel.hoff@uni-due.de)

On each exercise sheet, one solution will be corrected in more detail. Please, indicate your choice.

Exercise 1. 5 P.

Show that the function $\omega(t) := \sum_{n=1}^{\infty} e^{-\pi n^2 t}$ is rapidly decreasing on $[1, \infty)$.

Exercise 2. 5 P.

Recall, that we have defined the Fourier transform of a Schwartz function f as

$$\widehat{f}(x) := \int_{\mathbb{R}} f(y) e^{-2\pi i x y} dy.$$

a) Prove that the function $f\colon \mathbb{R} \to \mathbb{R}$ with $f(x) := e^{-\pi x^2}$ is a Schwartz function.

b) Show that the function $f(x) := e^{-\pi x^2}$ is its own Fourier transform, i.e., $\widehat{f} = f$.

c) For $t \in \mathbb{R}_{>0}$ define $f_t\colon \mathbb{R} \to \mathbb{R}$ by $f_t(x) := e^{-\pi t x^2}$. Prove

$$\widehat{f_t} = \frac{1}{\sqrt{t}} f_{1/t}.$$

Hint: You are allowed to use the well-known formula $\int_{\mathbb{R}} \exp(-\pi x^2) dx = 1$.

Exercise 3. 5 P.

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, prove the formula

$$\Gamma(s)\zeta(s) = \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt.$$

Exercise 4. 5 P.

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ show the formula

$$\zeta(s)^2 = \sum_{n=1}^{\infty} \frac{d(n)}{n^s},$$

where $d(n)$ is the sum of divisors function

$$d(n) := \sum_{d \mid n} 1.$$