Dr Johannes Sprang **SoSe 2021**
Manuel Hoff **Sheet 1**

Exercises
Algebraic Number Theory II

(due Monday 19th of April, 20:00, to manuel.hoff@uni-due.de)

On each exercise sheet, one solution will be corrected in more detail. Please, indicate your choice.

Exercise 1. 5 P.

Use integration by parts to prove that the Gamma function satisfies the functional equation

$$\Gamma(z+1) = z\Gamma(z) \quad \text{for all } z \in \mathbb{C} \text{ with } \mathrm{Re}(z) > 0.$$

Deduce the formula $\Gamma(n+1) = n!$ for a non-negative integer n.

Exercise 2. 5 P.

In this exercise, we will use the functional equation of the Gamma function to prove the following result. The Gamma function extends to a holomorphic function on $\mathbb{C} \setminus \{0, -1, -2, \dots\}$ satisfying the functional equation $\Gamma(z+1) = z\Gamma(z)$. For a non-negative integer n, the Gamma function has a simple pole at $z = -n$ with residue

$$\mathrm{Res}_{z=-n}\, \Gamma(z) = \frac{(-1)^n}{n!}.$$

Exercise 3. 5 P.

We have already seen that the Gamma function extends to a meromorphic function on $\mathbb{C}$ with simple poles at the non-positive integers. One might ask about its zeros. Prove that $\Gamma(z)$ is a non-vanishing function on $\mathbb{C} \setminus \{0, -1, -2, \dots\}$.

Exercise 4. 5 P.

Use the Euler product of the Riemann zeta function to prove for all real $s > 1$ the estimate

$$0 < \left| \log \zeta(s) - \sum_{p \text{ prime}} \frac{1}{p^s} \right| < \frac{1}{2}.$$

Conclude that the series

$$\sum_{p \text{ prime}} \frac{1}{p}$$

diverges and that there are infinitely many primes.