Exercise sheet 1

Let M be a finitely generated module over a noetherian ring A.
Define *the length* $l_A(M) \in \mathbb{N} \cup \{0\}$ as the supremum of the length of the chains of submodules of M. The length of the ring A as a module over itself is often denoted as $l(A)$.

1. If A is a field, what is $l_A(M)$?

2. Let R be one of the rings below, and denote by $A := R_{nil(R)}$.
   Compute $l_A(A)$ for

   (a) $R = k[x]$;

   (b) $R = k[x, \varepsilon]/(\varepsilon^2)$;

   (c) $R = k[x, \varepsilon]/(\varepsilon^2, \epsilon x)$.

For a closed subscheme Z of a variety X with integral components $Z_1, \ldots, Z_k$ denote by *the fundamental class* of Z

$$[Z] := \sum_{i=1}^{k} m_i [Z_i]$$

where $m_i = l(\mathcal{O}_{Z,Z_i})$ is called the multiplicity of the component Z_i.

3. Let C_1, C_2 be two curves in $\mathbb{A}^2_k$ given by equations $f(x, y) = 0$, $g(x, y) = 0$.
   Compute the multiplicities of the components of $C_1 \cap C_2$ for

   (a) $f = y^2 - x^3$, $g = x$;

   (b) $f = y^2 - x^3$, $g = y$;

4. Let V be a smooth irreducible variety over k, $f : V \to \mathbb{P}^1_k$ is a dominant morphism. Show that

   $$[div(f)] = [f^{-1}(0)] - [f^{-1}(\infty)]$$

   where $[div(f)] := \sum_D \nu_D(f)[D]$ and the sum runs over all codimension 1 subvarieties of V, $f^{-1}(x)$ denotes the scheme-theoretic preimage of x.

5. Compute $\mathrm{Pic}(\mathbb{A}^n_k)$, $\mathrm{Pic}(\mathbb{P}^n_k)$.

   Hint. Identify Pic with the class group and use that $k[x_1, \ldots, x_n]$ is a UFD.

6. Compute $\mathrm{CH}_0(\mathbb{P}^1_k)$ for algebraically closed field k, for arbitrary field k.

7. Compute $\mathrm{CH}_0(\mathbb{A}^1_k)$ for algebraically closed field k, for arbitrary field k.

8. Which class in $\mathrm{Pic}(\mathbb{P}^1_k)$ has $T_{\mathbb{P}^1_k}$?

9. Given two curves C_1, C_2 in P^2_k, what is the product $[C_1], [C_2]$ in the Chow ring of $\mathbb{P}^2_k$?

Exercise sheet 2

Let X be a variety, $Vect_X \subset Coh_X$ a full subcategory of locally free sheaves on X as a subcategory of coherent sheaves.

1. Let $F \in Coh_X$ and assume that F has two finite resolutions $W_\bullet, V_\bullet$ in $Vect_X$:

   $$0 \to W_n \to W_{n-1} \to \ldots \to W_0 \to F \to 0, \quad 0 \to V_n \to V_{n-1} \to \ldots \to V_0 \to F \to 0,$$

   and an (objectwise) surjective morphism from $W_\bullet \to V_\bullet$.

   Show that in $K_0(Vect_X)$ there is an equality $\sum(-1)^i[W_i] = \sum(-1)^i V_i$.

2. Given two resolutions $V_\bullet, V'_\bullet$ of F construct a third one together with surjective morphisms to $V_\bullet, V'_\bullet$.

3. Assume that every coherent sheaf F on X has a finite resolution by locally free sheaves. Show that the morphism $K_0(X) \to G_0(X)$ is an isomorphism.

Recall that a local Noetherian ring $(R, \mathfrak{m})$ is called regular, if its maximal ideal $\mathfrak{m}$ is generated by $\dim R$ elements.

4. Show that a regular local ring is an integral domain.

   Hint. Recall that the associated graded quotient with respect to the $\mathfrak{m}$-adic filtration grR is a polynomial algebra.

5. If $\mathfrak{m} = (x_1, \ldots, x_d)$ where $d = \dim R$, show that $x_1, \ldots, x_d$ is a regular sequence.

6. Let M be a finitely generated module over a Noetherian local ring $(R, \mathfrak{m})$. Show that if the projective dimension of M is $r < \infty$, multiplication by $x \in \mathfrak{m}$ is injective on M, then the projective dimension of $M/(x)M$ is $r+1$.

   Hint. Relate projective dimension with the Tor-dimension.

7. Show that if $(R, \mathfrak{m})$ is a regular local ring of dimension d, then it has global dimension equal to d.

   Hint. It suffices to compute projective/Tor-dimension of $R/\mathfrak{m}$.

Recall that a scheme X is called regular, if every local ring $\mathcal{O}_{X,p}$ is a regular local ring.

8. Let X be a quasi-projective regular variety over a field k. Show that every coherent sheaf on X has a finite resolution by locally free sheaves.

9. Construct an exact sequence of coherent sheaves on $\mathbb{P}^1_k \times \mathbb{P}^1_k$:

   $$0 \to p_1^*\mathcal{O}(-1) \otimes p_2^*\mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}^1_k \times \mathbb{P}^1_k} \to \mathcal{O}_\Delta \to 0,$$

   where Δ is the diagonal closed subvariety $\mathbb{P}^1$.

10. Show that $K_0(\mathbb{P}^1_k)$ is generated by $\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)$ and compute $K_0(\mathbb{P}^1_k)$.

    Hint. Use the Fourier-Moukai transform: given a vector bundle on $\mathbb{P}^1$, pull it back to the $\mathbb{P}^1 \times \mathbb{P}^1$, multiply by the class of the diagonal and push it forward using the other projection.

Exercise sheet 3

1. Let X be a smooth variety, D_1, D_2 are two Cartier divisors such that $D_1 \cap D_2$ is empty. Show that in $K_0(X)$ one has the relation
$$(\mathcal{O}_X - \mathcal{O}_X(-D_1)) \cdot (\mathcal{O}_X - \mathcal{O}_X(-D_2)) = 0.$$

2. Let X be an irreducible scheme.
   (a) Given an exact sequence of vector bundles over X:
   $$0 \to U \to V \to W \to 0$$
   of ranks $m, m+n, n$, respectively, construct an isomorphism of line bundles $\Lambda^m U \otimes \Lambda^n W \cong \Lambda^{n+m} V$.
   (b) Construct a functorial morphism $det : K_0(X) \to Pic(X)$ that sends V of rank r to $\Lambda^r V$.

3. Show that if C is a smooth curve, then $K_0(C) = \mathbb{Z} \oplus Pic(C)$.

   Hint. Use the localization sequence.

4. Let X be a smooth variety over k. Show that there are natural pullback maps $K_0(X) \to K_0(X_F)$, $\mathrm{CH}^*(X) \to \mathrm{CH}^*(X_F)$ for any field extension F/k, and that they are isomorphisms if F is purely transcendental over k.

   Find examples of X and F for which these maps are not surjective.

   Hint. Use the generic constancy and homotopy invariance properties.

5. Compute K_0 and CH^* of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ as well as the class of the diagonal in these groups.

   Hint. Use the localization sequence.

6. Let Q be a smooth hypersurface of degree 2 (quadric) in $\mathbb{P}^3_k$. Assuming that k is algebraically closed, compute $K_0(Q)$ and $\mathrm{CH}^*(Q)$ as rings.

7. Let X be the blow-up of $\mathbb{P}^2_k$ at a rational point.
   Compute $\mathrm{CH}^*(X)$ as a ring.

   Hint. Use the adjunction formula to determine self-intersection of the exceptional divisor.

8. Let C be a smooth conic (a curve of degree 2 in $\mathbb{P}^2$).
   (a) Show that if C has a 0-cycle of degree 1, then C is isomorphic to $\mathbb{P}^1$.
   Hint. Define a morphism from C to $\mathbb{P}^1$ that becomes an isomorphism over $\overline{k}$.
   (b) Assume that C has no rational points. Compute $\mathrm{CH}^*(C)$.
   Hint. Show that $\mathrm{Pic}(X) \to \mathrm{Pic}(X_K)$ is injective for any projective X and K/k.
   (c) Show that any morphism between two conics without rational points must have an odd degree.
   Hint. Look at the pushforward of the unit in K_0 and use the filtration on G_0 by dimension of the support.

Exercise sheet 4

Let A^* be an oriented cohomology theory of smooth varieties over a field k, the ring $A^*(\mathrm{Spec}\ k)$ is denoted A.

1. Check that the pullbacks and pushforwards for the theory A^* are morphisms of A-modules

2. Let $X = X_1 \coprod X_2$ in Sm_k. Show that $A^*(X) \cong A^*(X_1) \times A^*(X_2)$ as rings.

3. Let $F_{1,1} \in A[x,y]/(x^2,y^2)$ be the expression of $c_1(\mathcal{O}(1,1))$ in $A^*(\mathbb{P}^1 \times \mathbb{P}^1)$. Show that
$$F_{1,1} = x + y - [\mathbb{P}^1]_A xy$$
where $[\mathbb{P}^1]_A := p_*(1_{\mathbb{P}^1})$, $p : \mathbb{P}^1_k \to \mathrm{Spec}\ k$.

4. Let C be a smooth plane curve of degree d. Express the class $[C \to \mathbb{P}^2]_A$ in $A^*(\mathbb{P}^2)$ in terms of $1, z, z^2$ with coefficients in $(1, [\mathbb{P}^1]_A) \subset A$.

5. Compute the ring $A^*(X)$ where X is the blow-up of $\mathbb{P}^2$ at a rational point.

6. Let C be a smooth projective conic. Compute $A^*(C)$ assuming that $A*$ satisfies generic constancy property.

Exercise sheet 5

Let A^* be an oriented cohomology theory of smooth varieties over a field k.

Let X, Y be smooth projective varieties over k. An element in $A^{\dim X}(X \times Y)$ for an irreducible X, is called A-correspondence from X to Y.

1. Check that the following composition of correspondences is associative and contains a unit (the class in $A^{\dim X}(X \times X)$ for all smooth projective X): for $\alpha \in A^{\dim X}(X \times Y), \beta \in A^{\dim Y}(Y \times Z)$

$$\beta \circ \alpha := (p_{XZ})_*(p_{XY}^*(\alpha) \cdot p_{YZ}^*(\beta)) \in A^{\dim X}(X \times Z)$$

where p_{ij} are projections from $X \times Y \times Z$.

The composition of correspondences defines the category $Corr_A(k)$ with Ob – smooth projective varieties over k and $\mathrm{Hom}_{Corr_A}(X, Y) = \oplus_i A^{\dim X_i}(X_i \times Y)$ where the sum is taken over irreducible components of X.

2. Construct a functor $Sm_k \to Corr_A(k)$.

Recall that the unit of a (non-commutative) ring R decomposes as n orthogonal projectors if $1_R = \sum_{i=1}^n p_i$ and $p_i \circ p_j = p_j \circ p_i = 0$ if $i \neq j$, $p_i \circ p_i = p_i$ and $p_i \neq 0$ for all i.

3. Decompose $\mathrm{id}_X \in A^{\dim X}(X \times X)$ in 2 orthogonal projectors for $X = \mathbb{P}^1$ and arbitrary A^*.
4. Decompose $\mathrm{id}_X \in A^{\dim X}(X \times X)$ for $X = \mathbb{P}^n$ and $A^* = \mathrm{CH}^*$ into $n+1$ projectors p_i. For an arbitrary Y identify $\mathrm{Hom}_{Corr_{\mathrm{CH}}}(Y, X) \circ p_i$ with $\mathrm{CH}^i(Y)$.
5. Let $F_R \in \mathrm{FGL}(R)$. Show that there exist a unique series $[-1] \cdot_F t \in t \cdot R[[t]]$ such that $F_R(t, [-1] \cdot_F t) = 0$.
6. Let $F_R \in \mathrm{FGL}(R)$, and R is a $\mathbb{Q}$-algebra. Show that there exist a unique series $\eta(t) \in t + t^2\mathbb{Q}[[t]]$ such that $\eta(F_R(x, y)) = \eta(x) + \eta(y)$.

   Hint. Show that $\eta = \int (\partial_x F(x, y))|_{x=0,y=t}^{-1} dt$ works.

Exercise sheet 6

Let A be o.c.t. and let $\mathrm{PM}_A^{eff}(k)$ be the Karoubi envelope of $Corr_A$. and we have seen that the variety $\mathbb{P}^1$ decomposes into the direct sum $\mathbb{Z}_A(0) \oplus \mathbb{Z}_A(1)$ where $\mathbb{Z}_A(0) \cong M_A(\mathrm{Spec}\ k)$.

1. Check that PM_A^{eff} has a symmetric monoidal structure such that

$$M_A(X \times Y) \cong M_A(X) \otimes M_A(Y).$$

For an object M in PM_A^{eff}, $i \geq 0$, let $M(i)$ denote $M \otimes \mathbb{Z}_A(1)^{\otimes i}$.

2. Let X be a smooth projective variety, V a vector bundle on X of rank $r+1$. Show that there is isomorphism of motives $M_A(\mathbb{P}_X(V)) \cong \oplus_{i=0}^r M_A(X)(i)$.

   Hint. Use the projective bundle formula and the Yoneda lemma.

Recall the *splitting principle*: for every vector bundle V over a smooth variety X there exist a morphism $f : Y \to X$ from a smooth variety Y such that f^*V decomposes as a direct sum of line bundles and $f^* : A^*(X) \to A^*(Y)$ is injective.

3. Let V, W be two rank 2 vector bundles. Compute $c_2^A(V \otimes W)$ as a polynomial in Chern classes of V and W.

4. Let V be vector bundle of rank r on X, show that $c_1^{\mathrm{CH}}(V) = c_1^{\mathrm{CH}}(\Lambda^r V)$.

5. Let $s : X \to V$ be the zero section of a rank r vector bundle. Show that $s^*s_*1_X = c_r^A(V)$ in $A^r(X)$.

   Hint. Reduce to the case where $V \cong \oplus L_i$ and prove by induction on rank of V.

6. Using Chern classes construct a multiplicative operation (i.e. a natural transformations of presheaves of rings)

$$ch : K_0 \to CH^* \otimes \mathbb{Q}.$$

For a smooth projective variety X define its Chern numbers:
given a partition $(1^{\times n_1}, 2^{\times n_2}, \ldots)$ of $d = \dim X$, i.e. $d = \sum_{i=1}^d i \cdot n_i$, let

$$\deg \prod_{i=1}^{d} (c_i^{\mathrm{CH}}(T_X))^{n_i} \in \mathbb{Z}$$

be the corresponding Chern number.

7. Let $\pi : W \to \mathbb{P}^1$ be a projective morphism from a smooth variety W such that the fibers over two rational points x, y are smooth divisors W_x, W_y.

   Show that Chern numbers of W_x and W_y are the same. In other words, naive cobordism relation preserves Chern numbers.

Exercise sheet 7

1. Let $i : Z \to X$ be a regular embedding of smooth varieties, of codimension d and with the normal bundle N.

   Show that $c_d([\mathcal{O}_Z]) = \pm(d-1)![Z]$ in Chow groups of X.

2. Let C be a smooth projective curve in $\mathbb{P}^2$ of degree d. Use the Riemann-Roch formula to express $ch(i_*([L]))$ where L is a line bundle on C. Deduce the classical Riemann-Roch formula from it, i.e. $\chi(L) = \deg L - g + 1$.

3. (Borel-Serre identity)

   Let X be a smooth projective variety of dimension d, let Td be the Todd class associated to the Chern character. Show that

$$ch(\sum_r (-1)^r [\Omega^r_{X/k}])\mathrm{Td}(T_{X/k}) = c_d(T_{X/k}).$$

4. Let Q be a smooth projective quadric over an algebraically closed field k. Compute $A^*(Q)$ for any oriented cohomology theory A^*.

   Hint. There exist a linear projective space inside Q whose complement is an affine bundle over a projective space.

Exercise sheet 8

1. Show that the class of the diagonal of the projective space $[\mathbb{P}^n \xrightarrow{\Delta} \mathbb{P}^n \times \mathbb{P}^n]_A$ in $A^n(\mathbb{P}^n \times \mathbb{P}^n)$ can be written as $z_1^n + z_2^n + \sum_{i,j\geq 1} c_{ij} z_1^i z_2^j$ where c_{ij} can be expressed as universal polynomials in the coefficients of the formal group law of A.

   Hint. Identify the diagonal with the zero locus of a section of a vector bundle on $\mathbb{P}^n \times \mathbb{P}^n$.

2. Show that $[\mathbb{P}^k]_A$ can be expressed as the universal polynomial in coefficients of the formal group law of A.

   Hint. Take the pushforward of the class of the diagonal from $\mathbb{P}^n \times \mathbb{P}^n$ to $\mathbb{P}^n$ using Exercise **??**.

3. For a smooth projective variety X let $h^{p,q}(X)$ denote the dimension of $\mathrm{H}^q(\Omega^p_X)$. Show that $\deg c_n(T_X) = \sum_{p,q} (-1)^{p+q} h^{p,q}$.

   Hint. Use the Riemann-Roch theorem for the Chern character and the Borel-Serre identity.

   Note that in characteristic 0 using the Hodge theory one can derive then that $\deg c_n(T_X) = \chi_{top}(X)$.

For the next exercise one should assume the existence of the total Steenrod operation

$$St^{tot} : \mathrm{CH}^*/p \to \mathrm{CH}^*/p$$

which is the stable multiplicative operation with $\gamma_{St}(x) = x + x^p$.

4. Let X be a smooth projective variety of dimension n, Define the Segre classes $s_k(V)$ of a vector bundle V on X in Chow groups by the formula $\sum_{k\geq 0} s_k(V)t^k = (1 + \sum_{k>0} c_k(V)t^k)^{-1}$. The number $s_n(X) := \deg(s_n(T_X))$ is called the Segre number of X.

   Show that $s_n(X)$ is always divisible by 2 if $n > 0$.

   Hint. Note that for $p = 2$ one can show that $\mathrm{Td}_{St}(T_X) = \sum_{i\geq 0} s_i(T_X)t^i$.

Define the topological filtration $\tau^\bullet$ on G_0 by $\tau^i G_0(X)$ generated by coherent sheafs with codimension of support greater or equal to i.

5. Check that c_i^{CH} vanishes on $\tau^{i+1}K_0$ and that it becomes an additive operation on $\tau^i K_0$.

6. Show that $gr^i_\tau K_0 \otimes \mathbb{Q}$ is isomorphic to $\mathrm{CH}^i \otimes \mathbb{Q}$ as presheaves of abelian groups.

Exercise sheet 9

For a projective smooth variety X of dimension d let $S_d(X) \in \mathbb{Z}$ be the Chern number that is computed as $\deg(\sum_{i=1}^d \lambda_i^d)$ where λ_i are Chern roots of $-T_X$.

1. Show that if X, Y are smooth projective of positive (pure) dimensions d_X, d_Y, then $S_{d_X+d_Y}(X \times Y) = 0$.

2. Show that if $d = p^n - 1$, then $S_d(X)$ is always divisible by p.

   Hint. Identify $S_d(X)$ as a coefficient of some monomial of $h([X])$ where $h : \mathbb{L} \to \mathbb{Z}[b_1, b_2, \ldots]$. Check that this coefficient is zero by looking at h mod p.

3. Let H be a smooth hypersurface of degree p in $\mathbb{P}^{p^n}$. Show that p^2 does not divide $S_{p^n-1}(H)$.

4. Using the double point relation show that the class of a conic C in Ω equals the class of a projective line.

5. Show that $[\mathbb{P}^2]$ and $[\mathbb{P}^1 \times \mathbb{P}^1]$ freely generate Ω^{-2}.

Exercise sheet 10

If A^* is an o.c.t., denote by PM_A the category of A-motives of smooth projective varieties. Note that if $p : A^* \to B^*$ is a morphism of o.c.t., then it induces a functor $\mathrm{PM}_A \to \mathrm{PM}_B$.

1. Show that the kernel of the morphism of (non-commutative) rings

$$\mathrm{End}(M_\Omega(X)) \to \mathrm{End}(M_{\mathrm{CH}}(X))$$

consists of nilpotents. Conclude that the functor $\mathrm{PM}_\Omega \to \mathrm{PM}_{\mathrm{CH}}$ induces an isomorphism of the classes of isomorphisms of irreducible objects.

2. Show that $\Omega^*(X)$ is generated over $\mathbb{L}$ in degrees $0, 1, \ldots, \dim X$.

   Hint. Note that the generators of a graded $\mathbb{L}$-module M can be chosen to be $M/\mathbb{L}^{<0}M$.

3. Show that the stable multiplicative operation $H : \Omega^* \to \mathrm{CH}^*[b_1, b_2, \ldots]$ becomes an isomorphism after tensoring it with $\mathbb{Q}$.

For the next exercise let $F_{K(n)}(x, y)$ be a formal group law over $\mathbb{F}_p$ such that $p \cdot_{K(n)} x = x^{p^n}$. If $n = 1$ one can take $F_m = x + y + xy$ to be $F_{K(1)}$, for higher n we assume that $F_{K(n)}$ exists[1]. The corresponding free o.c.t. $K(n)^* := \Omega^* \otimes_{\mathbb{L}} \mathbb{F}_p$ is called an algebraic n-th Morava K-theory.

4. Let Q be an odd-dimensional quadric. Assuming that Q has no 0-cycles of odd degree[2] show that there are no Tate summands in the Chow motive of Q.

   Hint. Write explicitly all the projectors of Tate summands over $\overline{k}$ and note that none of these can be defined over the base field.

5. Let Q be a quadric of dimension $2^n - 1$. Show that $M_{K(n)}(Q)$ contains a Tate summand.

   Hint. It suffices to find elements $a, b \in K(n)(Q)$ such that $(\pi_Q)_*(a \cdot b) = 1$.

[1] One can explicitly write a logarithm of a formal group law F defined over $\mathbb{Z}$ such that F mod p is $F_{K(n)}$.

[2] which is equivalent to Q having no rational points by Springer's theorem