Grothendieck ring of varieties and birational geometry

SoSe 2022/2023, University of Regensburg

Fridays, 15-17 in M 102

Overview of the seminar

In a letter to Jean-Pierre Serre from 1964 Alexader Grothendieck has introduced the following ring $K_0(\mathrm{Var}_k)$, currently known as the Grothendieck ring of varieties:

$$K_0(\mathrm{Var}_k) := \mathbb{Z} < [X] | X \in \mathrm{Var}_k > / ([X \setminus Y] + [Y] - [X] \text{ if Y is a closed subvariety of X})$$

i.e. the abelian generators $[X]$ are classes of isomorphisms of reduced varieties X over k, and every closed subvariety Y of a variety X yields a relation $[X] = [X \setminus Y] + [Y]$. The multiplicative structure is induced by the direct product of varieties: $[X] \cdot [Y] = [X \times Y]$.

Despite simple definition of $K_0(\mathrm{Var}_k)$ the Grothendieck ring of varieties on the one hand remains a quite mysterious object, and on the other hand has shown to be useful in different areas of algebraic geometry, e.g. in studying zeta functions of 'motivic origin' and so-called motivic integration[1] (see e.g. [2]). In this seminar, however, we will be interested in the connections of $K_0(\mathrm{Var}_k)$ and similar invariants to birational geometry. All the following results are planned to be discussed in the seminar.

Note that if two varieties $X, Y \in \mathrm{Var}_k$ are piecewise isomorphic (i.e. there exists filtrations X_i, Y_i into loc. closed subvarieties s.t. $X_i \cong Y_i$; in particular, X and Y are birationally isomoprhic), then $[X] = [Y]$. It turns out that the converse fails, e.g. there exist varieties X, Y s.t. X and Y are not stably birational[2], but $[X \times \mathbb{A}^1] = [Y \times \mathbb{A}^1]$ (i.e. $[X] - [Y]$ is a non-trivial $[\mathbb{A}^1]$-divisor.

On the other hand Larsen-Lunts [6] have proved that the quotient of $K_0(\mathrm{Var}_k)$ by the ideal generated by $[\mathbb{A}^1]$ is isomorphic to a free abelian group generated by stable birational types. However, the $[\mathbb{A}^1]$-divisors in the Grothendieck ring of varieties remain quite mysterious, although many examples are known, and we will even see the machinery that allows to construct such examples from certain geometrical data.

At first we will study specialization of varieties. Given a sufficiently nice family $\mathcal{X} \to C$ over a curve C with a rational point x, one can ask what is the relation between the properties of the generic fibre of $\mathcal{X} \times_k k(C)$ and its special fibre $\mathcal{X} \times_C \{x\}$. We will show following the work of Nicaise-Shinder [9] that the class of the special fiber in $K_0(\mathrm{Var}_k)$ does not depend on the choice of the rational point and is determined just by the class of the generic fiber in $K_0(\mathrm{Var}_{k(C)})$. Together with the results of Larsen-Lunts this allows to show that stable rationality also specializes in nice enough families (this is was not known prior to this construction in $K_0(\mathrm{Var}_k)$). In fact, one can work with birational types similarly as $K_0(\mathrm{Var}_k)$ to obtain also specialization of rationality, as done by Konstevich-Tschinkel [5], and we will also study this approach.

The structural connection between $K_0(\mathrm{Var}_k)$ and birational geometry was explained by Zakharevich [10] as follows: there exists a spectrum[3] $K(\mathrm{Var}_k)$ whose 0-th homotopy group is $K_0(\mathrm{Var}_k)$ and which has a filtration coming from filtration of Var_k by dimension of varieties and whose n-th graded piece is the suspension spectrum of the classifying space of the groupoid of birational types of dimension n. In particular, in the corresponding spectral sequence one gets a homomorphisms from the group of birational automorphism of an irreducible variety X of dimension n to $K_0(\mathrm{Var}_k - len - 1)$ (which is defined similarly as above, but only by varieties of dimensions not greater than n). Moreover, a birational isomorphism of X can be extended to a piecewise automorphism if and only if all the differentials of the spectral sequence vanish on it.

[1] The morphisms from $K_0(\mathrm{Var}_k)$ to abelian groups are usually called motivic measures.
[2] Recall that X and Y are stably birational if there exists $n, m \geq 0$ s.t. $X \times \mathbb{A}^n$ is birational to $Y \times \mathbb{A}^m$.
[3] i.e. an element of the stable homotopy category.

Lin-Shinder-Zimmermann [8] have given a geometric construction of an additive map

$$c : \mathrm{Bir}_2/k \to \mathbb{Z}[\mathrm{Var}_k^0],$$

proved its vanishing and deduced that $K_0(\mathrm{Var}_k^{\leq n})$ for $n \leq 2$ can be explicitly described in terms of groups of birational classes of varieties of dimensions up to 2.

Lin-Shinder [7] extended this construction to arbitrary dimensions $c : \mathrm{Bir}_n/k \to K_0[\mathrm{Var}_k^{\leq n-1}]$ so that one allows to compute the differential in the above mentioned spectral sequence. More precisely, they show exactness of the following sequence:

$$\mathrm{Bir}_n/k \xrightarrow{c} K_0[\mathrm{Var}_k^{\leq n-1}] \to K_0[\mathrm{Var}_k^{\leq n}] \to \mathbb{Z}[\mathrm{Bir}_n/k] \to 0.$$

One of the non-trivial birational relation between two varieties is a so-called link, roughly it looks as the following commutative diagram:

$$\begin{tikzcd}
& & Bl_X\mathcal{X} \cong Bl_Y\mathcal{Y} \arrow[dl] \arrow[dr] & & \\
X \arrow[r] & \mathcal{X} \arrow[rr, dashed, "\phi"] & & \mathcal{Y} & Y \arrow[l]
\end{tikzcd}$$

Lin-Shinder show that if X and Y are not stably birational, then the invariant c does not vanish on the birational isomorphism ϕ between $\mathcal{X}$ and $\mathcal{Y}$. Thus, if one constructs such 'motivically non-trivial' L-links e.g. when $\mathcal{X} = \mathcal{Y}$ one obtains examples of non-trivial birational automorphisms and can distinguish them via their invariants c.

For example, they use such a link between certain elliptic curves and their second Jacobian (with $\mathcal{X}$ being a 3-dimensional quadric and $\mathcal{Y} = \mathbb{P}^3$) to show that the Cremona groups[4] Cr_n, $n \geq 3$, for some fields are 'very big', e.g. over an algebraically closed field, it contains a direct sum of the subgroup of pseudo-regularizable elements and a free abelian group of order equal to the cardinality of k.

They also apply this method to obtain similar result for Cr_n, $n \geq 4$, for all subfields of $\mathbb{C}$, and in order to do that they study the construction of Hassett-Lai [4] of K3 surfaces that are linked through $\mathbb{P}^4$.

Plan of the seminar

The dates and division into talks below are slightly preliminary, in the direction that some talks perhaps will require more time. There are also two additional talks (A1, A2) that could be in principle given at any time and are supposed to be more elementary than the other material in this seminar.

Talk 0 (Pavel, 26.04). Introduction to $K_0(\mathrm{Var}_k)$ and the Burnside ring. Overview of the seminar.

Talk 1 (–, 03.05). Generators and relations of $K_0(\mathrm{Var}_k)$ and $K_0(\mathrm{Var}_k)$.

The goal of this talk is to prove some structural results about the Grothendieck ring of varieties. First, explain that over any field $K_0(\mathrm{Var}_k)$ can be defined using as generators only smooth varieties, and the relations of the form $[X] = [Z] + [X \setminus Z]$ where Z is a smooth closed subvariety of X ([1, Th. 3.1]).

Second, show that one can define $K_0(\mathrm{Var}_k)$ only using smooth projective generators and the relatioins of the form $[Bl_Z X] - [E] = [X] - [Z]$ where Z is a smooth closed subvariety of X, E is the exceptional divisor of the blow-up $Bl_Z X$ ([Th. 3.2, loc.cit.]). For this proof one assumes that the base field has characteristic 0 and uses Hironaka's resolution of singularities and the weak factorization theorem that should be quickly reminded.

Finally, one shows the following result of Larsen-Lunts that connects the Grothendieck ring of varieties to the stable birationality. Let SB_k be the set of stable birational classes of varieties over k. Then show [6, Th. 2.3] (cf. [1.7, loc.cit.]) using the Bittner's theorem (see [Rem. 2.4, loc.cit.]):

$$K_0(\mathrm{Var}_k)/\mathbb{L} \xrightarrow{\sim} \mathbb{Z}[SB_k].$$

[4] the group of birational automorphisms of a projective space.

Talk 2 (–, 10.05). Konstevich-Tschinkel theorem.

Prove the the main result of [5] about specialization of birational types (Theorems 1 and 4, loc.cit.). The contents of the talk do not involve $K_0(\mathrm{Var}_k)$, but the proof of the main theorem is similar to the proof of the Bittner's theorem above, and the statement is motivated by the specialization in $K_0(\mathrm{Var}_k)$ that we will study later.

Talks 3-4 (–, 17.05 and 24.05).

The goal of this talk is to prove existence of different specialization maps for $K_0(\mathrm{Var}_k)$ following Nicaise-Shinder [9]. Even though it is not so hard to write down a formula that gives the right answer [Th. 3.1.1], it takes some work that it is independent of the chosen snc-model. Nicaise-Shinder use for this purposes log-geometry and the task of the speaker is to explain the necessary facts and constructions thereof in order to show that specialization is well-defined [App. A.3, A.4, loc.cit.]

In the end please also explain how the above results show specialization for stable birationality types [Th. 4.2.11, loc.cit.] and provide some examples [Th. 4.3.1 and/or Th. 4.3.5, loc.cit.]).

Talk 5 (–, 31.05).

This talk should give an overview of the paper [10]. The results thereof will not be used in the later talks, so the actual content could be adapted to the speaker's taste.

Talk 6-7 (–, 07.06 and 14.06).

In these two talks we follow the paper [8] to understand the simplest examples of the invariant c and how it affects the structure of $K_0(\mathrm{Var}_k^{\leq 2})$.

Talk 8 (–, 21.06).

Explain the general results of [7] about the invariant c (Section 2), introduce L-links and geometric construction of non-trivial L-links between curves (Sections 3.1–3.3).

Talk 9 (–, 28.06).

Explain the geometric constructions of [4] and how they extended in [7, Sec. 3.4] and then used to obtain results about the Cremona group Cr_4.

Talk A1(–, ??)

Theorem of Noether-Castelnuovo on the description of the Cremona group $Cr_2(k)$. Preferably, this talk should happen before Talk 6.

Talk A2(–, ??).

Present a different strategy of the proof of Lin-Shinder's theorem on Cremona groups from [3]. The talk should introduce median graphs and its relation to Cremona groups.

References

[1] Bittner, Franziska. "The universal Euler characteristic for varieties of characteristic zero." Compositio Mathematica 140.4 (2004): 1011-1032.

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[3] Genevois, Anthony, Anne Lonjou, and Christian Urech. "On a Theorem by Lin and Shinder through the Lens of Median Geometry." arXiv preprint arXiv:2312.05197 (2023).

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[5] Kontsevich, Maxim, and Yuri Tschinkel. "Specialization of birational types." Inventiones mathematicae 217 (2019): 415-432.

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[8] Lin, Hsueh-Yung, Evgeny Shinder, and Susanna Zimmermann. "Factorization centers in dimension two and the Grothendieck ring of varieties." arXiv preprint arXiv:2012.04806 (2020).

[9] Nicaise, Johannes, and Evgeny Shinder. "The motivic nearby fiber and degeneration of stable rationality." Inventiones mathematicae 217 (2019): 377-413.

[10] Inna Zakharevich. "The annihilator of the Lefschetz motive." Duke Math. J. 166 (11) 1989 - 2022, (2017).