Universität Regensburg

Algebraic Topology II

Lecture by Pavel Sechin in Summer 2021

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CHAPTER 1

Higher Homotopy Groups

1.1 Homotopy Groups of Pairs

Higher Homotopy Groups

Definition 1.1.1.
Let (X, x) be a pointed topological space. We define the **homotopy groups** of X at the basepoint x by

$$\pi_n(X, x) := \{f : (I^n, \partial I^n) \to (X, x)\}/\sim$$

where the equivalence relation is given by homotopy equivalence, that is

$$f \sim g \quad \Leftrightarrow \quad \exists H : I^n \times I \to X \text{ s.t. } \begin{array}{l} \cdot H|_{I^n \times \{0\}} = f \\ \cdot H|_{I^n \times \{1\}} = g \\ \cdot H_t := H|_{I^n \times \{t\}} : (I^n, \partial I^n) \to (X, x) \end{array}$$

Lemma 1.1.2.
When $n \geq 2$ there is a group operation in $\pi_n(X, x)$ which is abelian.

Proof.
See exercise sheet 1 $\to$ works with Eckmann-Hilton argument. ■

Remark 1.1.3.
Instead of describing elements of higher homotopy groups as equivalence classes of maps

$$(I^n, \partial I^n) \to (X, x)$$

one can regard them as maps of the quotient

$$I^n/\partial I^n = S^n \to X$$

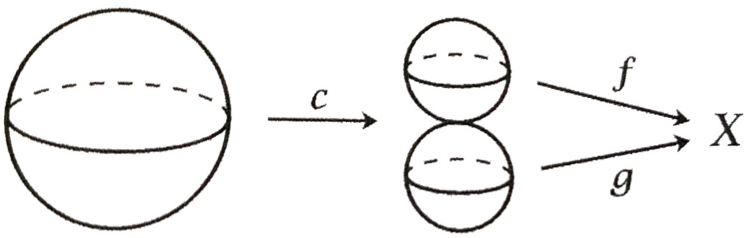
taking the basepoint $s_0 = \partial I^n/\partial I^n$ to x.

This means that we can also view $\pi_n(X, x)$ as homotopy classes of maps $(S^n, s_0) \to (X, x)$ where homotopies are through maps of the same form $(S^n, s_0) \to (X, x_0)$.

In this interpretation of $\pi_n(X, x)$, the group operation becomes more graphic: $f * g$ is the composition

$$S^n \xrightarrow{c} S^n \vee S^n \xrightarrow{f \vee g} X$$

where c collapses the equator S^{n-1} in S^n to a point (and we choose the basepoint s_0 to be an element of S^{n-1}).



Homotopy Groups of the Pair

Definition 1.1.4.
Let (X, A, x) be a pointed pair of topological spaces, $n \geq 1$.
We have inclusions:

$$\underset{\cong I^{n-1}\times I}{I^n} \supset \partial I^n \supset \underset{:=\partial I^{n-1}\times I \cup I^{n-1}\times\{0\}}{\mathcal{J}^{n-1}}$$

Here $\mathcal{J}^{n-1}$ is all of the boundary of I^n except for one face.

We now define the homotopy groups of the pair by

$$\pi_n(X, A, x) := \{f : (I^n, \partial I^n, \mathcal{J}^{n-1}) \to (X, A, x)\}/\sim$$

where the equivalence relation is given by homotopy equivalence, that is

$$f \sim g \quad \Leftrightarrow \quad \exists H : I^n \times I \to X \text{ s.t. } \begin{array}{l} \cdot H_0 = f \\ \cdot H_1 = g \\ \cdot \forall t : H_t : (I^n, \partial I^n, \mathcal{J}^{n-1}) \to (X, A, x) \end{array}$$

Lemma 1.1.5.
$\pi_n(X, A, x)$ has group structure for $n \geq 2$, which is abelian for $n \geq 3$.

Remark 1.1.6.
Just like before, we can interpret the elements of the relative homotopy group in terms of homotopy classes of maps $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$, since collapsing $\mathcal{J}^{n-1}$ to a point converts $(I^n, \partial I^n, \mathcal{J}^{n-1})$ into (D^n, S^{n-1}, s_0).
From this viewpoint the group operation is done via the map $c : D^n \to D^n \vee D^n$ collapsing $D^{n-1} \subset D^n$ to a point.

Interpretation 1.1.7.
An element of $\pi_n(X, A, x)$ is trivial if it fulfills the following *compression criterion*:

A map $f : (D^n, S^{n-1}, s_0) \to (X, A, x)$ represents zero in $\pi_n(X, A, x)$ iff it is homotopic rel S^{n-1} to a map with image contained in A.

Remark 1.1.8.

- $\pi_n(X, \{x\}, x) \cong \pi_n(X, x)$
- Any map $f : (X, A, x) \to (Y, B, y)$ where $x \in A$ induces a map $f_* : \pi_n(X, A, x) \to \pi_n(Y, B, b)$
  These induced maps fulfil the following properties:
  - $(f \circ g)_* = f_* \circ g_*$
  - $\mathbb{1}_* = \mathbb{1}$
  - homotopic maps $f \simeq g$ (via homotopy maps of the form $(X, A, x) \to (Y, B, y)$ induce the same map: $f_* = g_*$

- By the previous point the inclusions

$$\begin{aligned} i :&(A, \{x\}, x) \hookrightarrow (X, \{x\}, x) \\ j :&(X, \{x\}, x) \hookrightarrow (X, A, x) \end{aligned}$$

  induce maps on homology:

$$\pi_n(A, \{x\}, x) \xrightarrow{i_*} \pi_n(X, \{x\}, x) \xrightarrow{j_*} \pi_n(X, A, x)$$

  To eventually get a LES, we would like to define a map $\partial : \pi_n(X, A, x) \to \pi_{n-1}(A, \{x\}, x)$. Let $f : (I^n, \partial I^n, \mathcal{J}^{n-1}) \to (X, A, x)$ be an element of $\pi_n(X, A, x)$. We define ∂ by assigning to f:

$$\partial f := f|_{I^{n-1}\times\{1\}} : (I^{n-1}, x) \to (A, x)$$

  (which is the same as restricting f as a map $(D^n, S^{n-1}, s_0) \to (X, A, x)$ to S^{n-1}).

Theorem 1.1.9.
The following sequence of pointed sets and groups is exact:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_n(A,x) \arrow[r, "i_*"] & \pi_n(X,x) \arrow[r, "j_*"] & \pi_n(X,A,x) \arrow[dll, "\partial"] \\
 & \pi_{n-1}(A,x) \arrow[r, "i_*"] & \cdots \arrow[r, "j_*"] & \pi_1(X,A,x) \arrow[dll, "\partial"] \\
 & \pi_0(A,x) \arrow[r, "i_*"] & \pi_0(X,x) \arrow[r, "j_*"] & \pi_0(X,A,x)
\end{tikzcd}$$

Proof.

- Exactness at $\pi_n(X, A, x)$:

  1. $\partial \circ j_* = 0$:
     Let $f : (I^n, \partial I^n) \to (X, x)$ be a representative of an element of $\pi_n(X, x)$. Keep in mind, that this can equivalently be put as

$$f : (I^n, \partial I^n, \mathcal{J}^{n-1}) \to (X, \{x\}, x)$$

     where all of ∂I^n and $\mathcal{J}^{n-1}$ is being sent onto x.
     We then get (because j is just the inclusion):

$$j \circ f : (I^n, \partial I^n, \mathcal{J}^{n-1}) \to (X, A, x)$$

     Now applying ∂ the map $j \circ f$ is being sent to

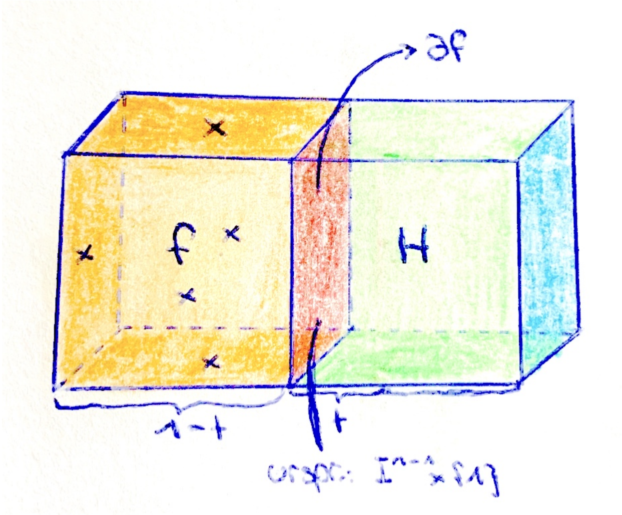
$$\partial f := f|_{I^{n-1}\times\{1\}} : (I^{n-1}, x) \to (A, x)$$

     However, since $I^{n-1} \times \{1\}$ is subset of ∂I^n, which in turn is sent onto x as described above, we get that ∂f is the constant map, sending everything onto x.
     The equivalence class of this map is by definition the neutral element.

  2. $\ker(\partial) \subset \operatorname{im}(j_*)$:
     We choose a representative $f : (I^n, \partial I^n, \mathcal{J}^{n-1}) \to (X, A, x)$ of an element of the kernel. By definition this means that $\partial f = f|_{I^{n-1}\times\{1\}}$ is homotopy equivalent to the constant map.

     Let $H : \overbrace{I^{n-1} \times I}^{=I^n} \to A \subset X$ be such a homotopy. This means that

$$\begin{aligned} H_0 &= \partial f \\ H_1 &= e_x, \\ H(\partial I^{n-1} \times I) &= \{x\} \end{aligned}$$



Next we tack H onto f as indicated in the picture above. We know all of the border of I^n except for the face indicated in red, to be sent onto x by f. Also $H(\partial I^{n-1} \times I) = \{x\}$ which means that H sends the faces of I^n indicated in green onto x. By H being a homotopy from ∂f to e_x, we also know that H_1 (the face indicated in blue) is mapped to x.
We can now define functions $F_t : I^n \to X$ (I^n is now the whole cuboid above but scaled such that it is a cube again). By the argumentation above all faces of this cuboid are mapped onto the constant function, thus $\partial I^n \to \{x\}$. Therefore, e.g. $F_{1/2} \in \pi_n(X, x)$. Those maps F_t shall be defined such that for increasing t longer initial segments of H are attached to f.

We can now see, that $j_*[F_{1/2}] = [j \circ F_{1/2}] = [f]$ because $F_{t/2}$ is a homotopy in t between $F_{1/2}$ and f.

- <u>Exactness at $\pi_n(A, x)$</u>:

1. $i_* \circ \partial = 0$:
Take $f : (I^{n+1}, \partial I^{n+1}, \mathcal{J}^n) \to (X, A, x)$ as a representative of an element of $\pi_{n+1}(X, A, x)$. Then

$$i \circ \partial f = i \circ f|_{I^n \times \{1\}} : \begin{array}{ll} I^n & \to A \xrightarrow{i} X \\ \partial I^n & \to \{x\} \mapsto x \end{array}$$

This map, ∂f included in $\pi_n(X, x)$ via i, however, is homotopy equivalent to the constant map in the most obvious way: Take f to be the homotopy

$$H = f : I^n \times I \to X$$
$$H_1 = f|_{I^n \times \{1\}} = \partial f, \quad H_0 = f|_{\underbrace{I^n \times \{0\}}_{\subset \mathcal{J}^n}} = e_x, \quad H_t : (I^n, \partial I^n) \to (X, x)$$

The last part is well-defined because we have seen above that ∂I^n is mapped onto x.

2. $\ker i_* \subset \operatorname{im} \partial$:
Let $f : (I^n, \partial I^n) \to (A, x)$ be a representative of an element of the kernel of i_*. Thus $i \circ f \overset{H}{\simeq} x$ via a homotopy $H : I^n \times I \to X$

$$H_0 = i \circ f, \quad H_1 = e_x, \quad H(\partial I^n \times I) = e_x$$

Then the map where $\overline{H}(x_1, ..., x_{n+1}) := H(x_1, ..., 1 - x_{n+1})$

$$\begin{tikzcd}
\overline{H} : I^{n+1} \arrow[r] & X \\
\cup & \cup \\
\partial I^{n+1} \arrow[r] & A \\
\cup & \cup \\
\mathcal{J}^n \arrow[r] & \{x\}
\end{tikzcd}$$

is an element in $\pi_{n+1}(X, A, x)$ that is mapped onto f by ∂.

- Exactness at $\pi_n(X, x)$:
  1. $j_* \circ i_* = 0$:
     Let $f : (I^n, \partial I^n, \mathcal{J}^{n-1}) \to (A, \{x\}, x)$ be a representative of an element in $\pi_n(A, x)$. Then:
     $$j \circ i \circ f : (I^n, \partial I^n, \mathcal{J}^{n-1}) \to (A, \{x\}, x) \to (X, \{x\}, x) \to (X, A, x)$$
     where $\partial I^n \to \{x\} \to \{x\} \mapsto x \in A$ and thus $\partial I^n \supset \mathcal{J}^{n-1} \mapsto x$.
     Since j and i are just inclusions and f maps I^n into A, we know $j \circ i \circ f$ to send I^n into A. Thus, by the proposition in lecture 3, $j \circ i \circ f$ is nullhomotopic.
  2. $\ker(j_*) \subset \mathrm{im}(i_*)$:
     Let $f : (I^n, \partial I^n, \mathcal{J}^{n-1}) \to (X, \{x\}, x)$ be a representative of an element in $\ker(j_*)$. By definition this means that $j \circ f : (I^n, \partial I^n, \mathcal{J}^{n-1}) \to (X, A, x)$ is homotopy equivalent to the constant map.
     By the proposition in lecture 3, this implies that $j \circ f$ is homotopic relative the boundary to a map g sending I^n into A. Then we can regard g as an element of $\pi_n(A, x)$.
     We see that $i \circ g \simeq f$.

■

1.2 Serre Fibrations

Definition 1.2.1.
A map $p : E \to B$ satisfies the **Homotopy Lifting Property** with respect to X (HLP) if for every commutative diagram

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, hook, "{(id,0)}"'] & E \arrow[d, "p"] \\
X \times I \arrow[r, "h"'] \arrow[ur, dashed, "\tilde{h}"] & B
\end{tikzcd}$$

exists a map $\tilde{h}$ as indicated such that

$$\tilde{h} \circ (id, 0) = g$$
$$p \circ \tilde{h} = h$$

Definition 1.2.2.
A map $p : E \to B$ is a **Serre fibration** if it has HLP with respect to I^n, $n \geq 0$.

Remark 1.2.3.
In Algebraic topology I we saw that if p is a covering, then p is a Serre fibration. (see homotopy lifting property).

Remark 1.2.4. Another property we can easily see for coverings is the following (which is a special case of the theorem below). Already in AT I we have seen the following exact sequence:

$$1 \to \pi_1(E,e) \to \pi_1(X,x) \to \underbrace{F}_{=\pi_0 F} \to \pi_0 E \to \pi_0 X$$

Since F is discrete $\pi_n(F) = 1$ for all $n \geq 1$. Thus, using the long exact sequence of homology groups, we get that

$$\pi_n(E,e) \cong \pi_n(X,x)$$

for all $n \geq 2$.

Theorem 1.2.5.
If $p : E \to B$ is a Serre fibration, then for $n \geq 1$, $e \in E$, $b = p(e)$, $B_0 \subset B$ and $E_0 := p^{-1}(B_0)$:

$$p_* : \pi_n(E, E_0, e) \xrightarrow{\cong} \pi_n(B, B_0, b)$$

Corollary 1.2.6. *(LES of a Serre fibration)*
For $p : E \to B$, $F := E_0 = p^{-1}(b)$ there is the following LES:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_n(F,e) \arrow[r] & \pi_n(E,e) \arrow[r] & \overset{\overset{p_*}{\cong}\pi_n(E,F,e)}{\pi_n(B,b)} \arrow[dll, "\partial"] \\
& \pi_{n-1}(F,e) \arrow[r] & \cdots \arrow[r] & \pi_1(B,b) \arrow[dll, "\partial"] \\
& \pi_0(F,e) \arrow[r] & \pi_0(E,e) \arrow[r] & \pi_0(B,b) \arrow[r] & 0
\end{tikzcd}$$

Proof.
Follows direct from the LES theorem by choosing $B_0 = \{b\}$ in the theorem above.

■

Lemma 1.2.7.
If $p : E \to B$ is a Serre fibration then every commutative diagram

$$\begin{tikzcd}
I^{n-1} \times \{e\} \arrow[d, hook] \arrow[r, phantom, "\cong"] & \mathcal{J}^{n-1} \arrow[r] \arrow[d, phantom, "\cap"] & E \arrow[d, "p"] \\
I^{n-1} \times I \arrow[r, phantom, "\overset{\phi}{\cong}"] & I^n \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

has a lift as indicated.

Proof.

■

Proof. (of theorem)
We have to show that $p_* : \pi_n(E, E_0, e) \xrightarrow{\cong} \pi_n(B, B_0, b)$ is an isomorphism.

- p_* is surjective
  Take an element $f : (I^n, \partial I^n, \mathcal{J}^{n-1}) \to (B, B_0, b)$. We would like to lift this map to a map $(I^n, \partial I^n, \mathcal{J}^{n-1}) \to (E, E_0, e)$ which would then be an element of $\pi_n(E, E_0, e)$ that is sent onto f by p.

By the lemma, we get a map $g : I^n \to E$ such that

$$\begin{array}{ccc} g : I^n & \longrightarrow & E \\ \cup & & \cup \\ \partial I^n & \longrightarrow & p^{-1}(B_0) = E_0 \\ \cup & & \cup \\ \mathcal{J}^{n-1} & \longrightarrow & \{e\} \end{array}$$

Thus a map $g : (I^n, \partial I^n, \mathcal{J}^{n-1}) \to (E, E_0, e)$ such that $p \circ g = f$.

- p_* is injective
  Let $f_0, f_1 : (I^n, \partial I^n, \mathcal{J}^{n-1}) \to (E, E_0, e)$ be elements of $\pi_n(E, E_0, e)$ such that $p \circ f_0 \overset{H}{\simeq} p \circ f_1$ ($[p \circ f_0] = [p \circ f_1]$).
  Choose such a homotopy $H : I^n \times I \to B$

$$H_0 = p \circ f_0, \quad H_1 = p \circ f_1, \quad H_t : (I^n, \partial I^n, \mathcal{J}^{n-1}) \to (B, B_0, b)$$

Define $T := I^n \times \partial I \cup \mathcal{J}^{n-1} \times I$. Then the following square commutes:

$$\begin{tikzcd} T \arrow[r, "G"] \arrow[d, phantom, "\cap"] & E \arrow[d] \\ I^n \times I \arrow[r, "H"'] & B \end{tikzcd}$$

where G is defined by

$$G(u,t) = \begin{cases} f_0(u) & t = 0 \\ f_1(u) & t = 1 \\ e & u \in \mathcal{J}^{n-1} \end{cases}$$

To apply the lemma, we now check that T is isomorphic to $\mathcal{J}^n$. When we regard $T = I^n \times \partial I \cup \mathcal{J}^{n-1} \times I$ we see that it indeed consists of all faces of I^{n+1} except for one face missing. This is exactly what we want for $\mathcal{J}^n$. However, this missing face is given by $\mathcal{J}^{n-1} \times I$ and thus not in the last coordinate. But we can fix that by using an isomorphism switching the last two coordinates (which we define on the cube):

$$\begin{tikzcd} \mathcal{J}^n \arrow[d, phantom, "\cap"] & T \arrow[l, "\cong"'] \arrow[d, phantom, "\cap"] \\ I^{n+1} & I^n \times I \arrow[l, "\cong", "\psi"'] \end{tikzcd}$$

where $\psi : I^n \times I \to I^{n+1}$ does exactly what was specified above:

$$\psi(t_1, ..., t_n, t_{n+1}) = (t_1, ..., t_{n+1}, t_n)$$

Thus, by the lemma, we get a lifted map $\tilde{H}$:

$$\begin{tikzcd} \mathcal{J}^n \cong T \arrow[r, "G"] \arrow[d, phantom, "\cap"] & E \arrow[d] \\ I^n \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"'] & B \end{tikzcd}$$

Because of commutativity $\tilde{H}$ fulfils: $\tilde{H}_0 = G_0 = f_0$ and $\tilde{H}_1 = G_1 = f_1$.
What we have to check though, is that for each $t \in I$, H_t is a map of triples.
To do so, we see what G does on $T = I^n \times \partial I \cup \mathcal{J}^{n-1} \times I$. For $\mathcal{J}^{n-1} \times I$, we know G to send everything onto the constant map, so we are fine here. Left to check is therefore only the same property for the boundary of $I^n \times \partial I$:

$$\begin{tikzcd} & E_0 \arrow[dd, "p"] \\ \partial I^n \times \{t\} \arrow[ur, "\tilde{H}_t"] \arrow[dr, "H_t"'] & \\ & B_0 \end{tikzcd}$$

Here H_t goes to B_0 because H is a homotopy of triples. By commutativity $\tilde{H}_t$ thus goes to $E_0 = p^{-1}(B_0)$.
Thus $\tilde{H}_t$ is a map of triples and therefore $\tilde{H}$ a homotopy from f_0 to f_1 just as we wanted.

■

1.3 Hopf Fibration

Definition 1.3.1.
Let $\mathcal{F}$ be a collection of topological spaces.
A continuous map $p : E \to B$ is called a **locally trivial bundle** (or **fiber bundle**) with fibers in $\mathcal{F}$ if:
For every $b \in B$, there exists an open neighbourhood U of b and an isomorphism over U

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, phantom, "\cong"] \arrow[dr, "p"'] & & F \times U \arrow[dl, "pr_2"] \\ & U & \end{tikzcd}$$

for some $F \in \mathcal{F}$.

- If this holds for $U = B$, the bundle is called **trivial**.
- B is called the **base** and F is called the **total space** of the bundle
- If $\mathcal{F} = \{F\}$, we say "with fiber F".

Proposition 1.3.2.
Every fiber bundle is a Serre fibration

Remark 1.3.3.
By this we get we get a very important result: Contrary to homology groups, the homotopy groups of a sphere of dimension greater than that of the sphere, do not have to be zero.
For this, we need that $S^3 \to S^2$ is a fiber bundle with fibers in S^1. This map is constructed as follows:

$$\begin{tikzcd} S^3 \arrow[r] \arrow[d, phantom, "\cap"] & S^2 \arrow[d, equal] \\ \mathbb{R}^4 \setminus \{0\} \arrow[d, equal] & \mathbb{C}P^1 \\ \mathbb{C}^2 \setminus \{0\} \arrow[ur, "pr"'] & \end{tikzcd}$$

By the proposition this map is also a Serre fibration, thus we now have a LES of homotopy groups:

$$\cdots \to \underset{}{\overset{=0}{\pi_n(S^1)}} \to \pi_n(S^3) \xrightarrow{\cong} \pi_n(S^2) \to \overset{=0}{\pi_{n-1}(S^1)} \to \cdots$$
$$\cdots \to \underset{=0}{\pi_2(S^1)} \to \underset{=0}{\pi_2(S^3)} \to \pi_2(S^2) \to \underset{\cong\mathbb{Z}}{\pi_1(S^1)} \to \underset{=0}{\pi_1(S^3)}$$

For example in $n = 3$ we have:

$$\pi_3(S^3) \xrightarrow{\cong} \pi_3(S^2)$$

$\pi_3(S^3)$, however, contains the identity element. Since S^3 is not contractible, the identity element is not homotopic to the point map and thus non-trivial.
Due to the isomorphism, we get the existence of a non-trivial element in $\pi_3(S^2)$.

Proof. (of proposition)
By definition of fiber bundles there is for each element $\alpha \in B$ an open neighbourhood U_α such that

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[dr, "p"'] & \cong & F \times U_\alpha \arrow[dl, "pr_2"] \\ & U_\alpha & \end{tikzcd}$$

Obviously $\{U_\alpha\}$ is an open cover of B.
By adding the inclusions in each of the bigger spaces we get the following diagram:

$$\begin{tikzcd} U_\alpha \times F \cong p^{-1}(U_\alpha) \arrow[d, "pr"'] & \subset & E \arrow[d, "p"] \\ U_\alpha \arrow[rr, hook] & & B \end{tikzcd}$$

1. The trivial fiber bundle is a Serre fibration:
   A fiber bundle is called trivial if the property that $p^{-1}(U) \cong F \times U$ via p holds for $U = B$ which in turn implies that $p^{-1}(B) = E = F \times B$ for some fiber F.
   What we now have to show is that for every commutative square

$$\begin{tikzcd} I^n \times \{0\} \arrow[r, "g"] \arrow[d, hook] & U \times F \cong E \arrow[d, "pr"] \\ I^n \times I \arrow[r, "h"'] & U = B \end{tikzcd}$$

   exists a lift $\tilde{h} : I^n \times I \to U \times F$ (that preserves commutativity):

$$\begin{tikzcd} I^n \times \{0\} \arrow[r, "g"] \arrow[d, hook] & U \times F \arrow[d, "pr"] \\ I^n \times I \arrow[r, "h"'] \arrow[ur, dashed, "\tilde{h}"] & U \end{tikzcd}$$

   But we can easily find such a map by setting

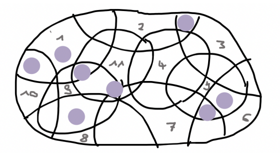
$$\tilde{h}(u,t) = (\underbrace{h(u,t)}_{\in U}, g(u,0)_2)$$

2. In the general case we have to find a lift for every commutative square

$$\begin{tikzcd} I^n \times \{0\} \arrow[r, "g"] \arrow[d, hook] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "h"'] & B \end{tikzcd}$$

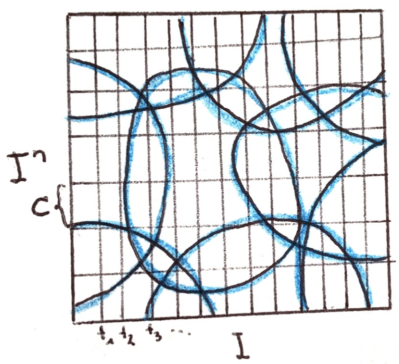
Let $\{U_\alpha\}$ be the trivialising open covering for p that we described above. Then $\{h^{-1}U_\alpha\}$ is an open covering of $I^n \times I$ (which is a compact metric space.
We will now apply the Lebesgue lemma (see AT I)

Lemma 1.3.4. *(Lebesgue lemma)*
Let (X, d) be a compact metric space and $(U_i)_{i\in I}$ an open cover of X.
Then there exists $\varepsilon > 0$ such that for every $x \in X$ the ball $B(x, \varepsilon)$ is contained in U_i for some i.



In the picture the space X is covered by eleven open sets. The lilac balls are supposed to symbolise that to each $x \in X$ exists at least one $i \in \{1, ..., 11\}$ such that the ball fits into the open set U_i.

By the Lebesgue lemma we can subdivide $I^n \times I$ into cubes $C \times [t_j, t_{j+1}]$ which are contained completely in at least one $h^{-1}(U_\alpha)$.



The open sets $h^{-1}(U_\alpha)$ are indicated in blue, the grid symbolizes the subdivision of $I^n \times I$ we get from the Lebesgue lemma.

Thus there are maps (restrictions of h)

$$C \times [t_j, t_{j+1}] \xrightarrow{h} U_\alpha \subset B$$

The procedure of the proof will now be the following: We start off by defining $\tilde{h}$ on lower dimensional faces of the smaller cubes C, increase the dimensions and then generalize the definition on the bigger cube $I^n \times I$.

Let $V^k \subset I^n$ be the union of the k-dimensional faces of these cubes C. We will now find a solution $H(k)$ to the following lifting problem by induction on k:

$$\begin{tikzcd}
I^n \times \{0\} \cup V^{k-1} \times [0, t_1] \arrow[r, "H(k-1)"] \arrow[d, hook] & E \arrow[d, "p"] \\
I^n \times \{0\} \cup V^k \times [0, t_1] \arrow[r, "h"'] \arrow[ur, dashed, "H(k)"] & B
\end{tikzcd}$$

By induction we assume that we already have a lift on $I^n \times \{0\} \cup V^{k-1} \times [0, t_1]$, that is where $H(k-1)$ comes from.

So how do we define $H(k)$?
On $I^n \times \{0\}$ this is already defined by $H(k-1)$ since we want the upper triangle (with the inclusion!) to be commutative.

Left for us to set is what $H(k)$ is supposed to do on $V^k \times [0, t_1]$.
Let W be a cube of V^k, so W is a k-dimensional cube. Its boundary consists therefore of $k-1$-dimensional cubes which are $k-1$-dimensional faces of C. Thus $\partial W \subset V^{k-1}$.

On $W \times \{0\}$ we already know the map which is given by g. So the lifting property for this specific cube looks as follows:

$$\begin{tikzcd}
\mathcal{J}^k \cong & W \times \{0\} \cup \partial W \times [0, t_1] \arrow[rr, "H(k-1)"] \arrow[d, hook] & & U_\alpha \times F \arrow[r, hook] \arrow[d, "pr"] & E \arrow[d, "p"] \\
\cap & & & & \\
I^{k+1} & \cong \quad W \times [0, t_1] \arrow[rr, "h"'] & & U_\alpha \arrow[r, hook] & B
\end{tikzcd}$$

(Remark to $\mathcal{J}^k \cong W \times \{0\} \cup \partial W \times [0, t_1]$: W is a k-dimensional cube and as such isomorphic to I^k, so $\partial W \times [0, t_1]$ makes up all of the boundary of I^{k+1} except for two faces. One of them is given by $W \times \{0\}$.)

By lemma 2.7 we would get our lift, if the map $U_\alpha \times F \to U_\alpha$ is a Serre fibration. We have seen this to be true in 1.

Thus we get a lift $W \times [0, t_1] \times I^{k+1} \to U_\alpha \times F$.
If we now glue all of those H_W together, which we can because they are all given on the boundary by the induction condition, we get a map $H(k)$.
So in the end we get for $k = n$ a map $I^n \times [0, t_1] \to E$:

$$\begin{tikzcd}
& E \arrow[d, "p"] \\
I^n \times [0, t_1] \arrow[ur] \arrow[r] & B
\end{tikzcd}$$

Left to do is now only to prove the very same thing on the succeeding intervals of $[0, t_1]$. But for this, we can use exactly the same arguments because now we have a different starting argument, we start with the square:

$$\begin{tikzcd}
I^n \times \{t_i\} \arrow[r, "g"] \arrow[d, hook] & E \arrow[d, "p"] \\
I^n \times I \arrow[r, "h"'] & B
\end{tikzcd}$$

Thus we get for all intervals lifting maps which we eventually can out together to a map H:

$$\begin{tikzcd}
I^n \times \{0\} \arrow[r, "g"] \arrow[d, hook] & E \arrow[d, "p"] \\
I^n \times I \arrow[r, "h"'] \arrow[ur, "H"] & B
\end{tikzcd}$$

■

Our next goal will be to show that $\pi_m(S^n) = 0$ for $m < n$. We have seen this to be a very useful piece of information last time in the LES of the Serre fibration. But before we can deal with that , we have to introduce one more notion: Cofibration

1.4 Cofibrations

Definition 1.4.1. (Homotopy Extension Property)
A map $i : A \to X$ of topological spaces satisfies the Homotopy Extension Property (HEP) for a space T if for a homotopy $h : A \times I \to T$ and a map $f : X \to T$ such that $f \circ i = h|_{A\times\{0\}}$ there exists $H : X \times I \to T$ such that $H|_{X\times\{0\}} = f$ and $H \circ (i \times id) = h$.

Remark 1.4.2.
We want to see the duality of the construction above to the Homotopy Lifting Property. This is done using the exponential object we know from Algebraic Topology I.
So a map from $A \times I \to T$ is the same as a map $A \to T^I$ (this exists because I is a nice object -> see AT I)

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "i"'] & T^I \arrow[d, "ev_0"] & = \{g : I \to T\} \arrow[d, mapsto, "ev_0"] \\ X \arrow[r, "f"'] \arrow[ur, dashed, "H"] & T & \ni \quad g(0) \end{tikzcd}$$

You have to check that $T^I \to T$ is a continuous map which is not hard using the universal property of the exponential object.
Finding H is now the same as finding a lift as indicated. The condition that $H|_{X\times\{0\}} = f$ is precisely the commutativity of the lower triangle ($ev_0 \circ H = f$) and the condition $H \circ (i \times id) = h$ the commutativity of the upper one ($H \circ i = h$).
Thus the Homotopy Extension Property is also a lifting property for some commutative squares.

Definition 1.4.3.
A map $i : A \to X$ is a cofibration if it satisfies HEP for all spaces.

Interpretation 1.4.4.
Instead of directly jumping to using the exponential object, we could stay directly by the definition and get the following commutative square:

$$\begin{tikzcd} A \arrow[r, hook, "\iota_0"] \arrow[d, "i"'] & A \times I \arrow[d, "i\times id"] \arrow[ddr, bend left, "h"] & \\ X \arrow[r, "\iota_0"'] \arrow[drr, bend right, "f"'] & X \times I \arrow[dr, dashed, "H"] & \\ & & Y \end{tikzcd}$$

If the solid part of the diagram commutes than the property of i being a cofibration states that the dotted line can be filled in.

So why introduce cofibrations? ([**cutler**]) It is a classical problem in the case of $i : A \hookrightarrow X$ being a subspace inclusion to ask when a given map $f' : A \to X$ can be extended to all of Y:

$$\begin{tikzcd} A \arrow[d, hook, "i"'] \arrow[dr, "f'"] & \\ X \arrow[r, dashed] & Y \end{tikzcd}$$

If i is a cofibration this question becomes one of homotopy classes: If f' is homotopy equivalent to a map which indeed extends to all of X (f in the definition) then f' can also be extended by the homotopy extension property.
So cofibrations are a means of translating a topological problem into one approachable by homotopy-theoretic methods.

When we have another look on the first diagram above we see that it looks very similar to the universal property of the pushout. However, cofibrations require only a weaker version of the pushout in that they do not need uniqueness of the induced map.

Example 1.4.5.

- For any space X the inclusion $\emptyset \hookrightarrow X$ is a cofibration.
- Every homeomorphism $i : A \to X$ is a cofibration.

Properties 1.4.6.
If $i : A \to X$ is a cofibration, then

1. *it is injective and a homeomorphism onto its image.*
2. *if X is Hausdorff, then $A \subset X$ is a closed subset.*

Proof.

1. To show that i is injective, we want to make suitable choices of the map h and the map f in the following diagram, which exists, since i is a cofibration.

$$\begin{tikzcd}
A \arrow[r, "\iota_0"] \arrow[d, "i"] & A \times I \arrow[d, "i \times id"] \arrow[ddr, bend left, "h"] & \\
X \arrow[r] \arrow[drr, bend right, "f"] & X \times I \arrow[dr, dashed, "\exists H"] & \\
& & Cyl(i)
\end{tikzcd}$$

   Let $h : A \times I \to Cyl(i)$ be the map defined as the projection $h = \pi_{A \times X}$ to the mapping cylinder, and let $f := \pi_X$.
   By construction of the mapping cylinder, the outer diagram commutes. Hence, we get a map H (in particular) satisfying $h = H \circ (i \times id)$. Since h is injective, so is $(i \times id)$ and in particular, i.

   To show that i is also a homeomorphism onto its image, it suffices to show that i^{-1} is continuous. But the equation $h = H \circ (i \times id)$ implies that $(i^{-1} \times id) = h^{-1} \circ H$. H is continuous and h is a homeomorphism onto its image. Therefore, i^{-1} is continuous.

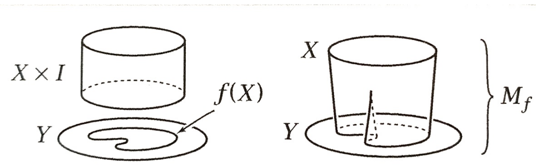
2.

■

In order to check that a map is a cofibration, by definition you have to check that it has HEP for all spaces, but it turns out you only have to check for one space which is the mapping cylinder.

Definition 1.4.7.
The mapping cylinder is the pushout of the following diagram:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, hook, "(id{,}0)"'] & X \arrow[d] \\
A \times I \arrow[r] & Cyl(i)
\end{tikzcd}$$



($Cyl(i)$ = mapping cylinder)

So how is it useful for cofibrations?

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, hook, "{(id,0)}"'] & X \arrow[d, "j"] \arrow[ddr, bend left, "f"] & \\
A \times I \arrow[r, "\overline{h}"] \arrow[drr, bend right, "h"'] & Cyl(i) \arrow[ul, phantom, "\square"] \arrow[dr, dashed] & \\
& & T
\end{tikzcd}$$

The fact that the solid part of the diagram commutes means precisely that we can fill in the dotted map (pushout). So the maps of the mapping cylinder into some space T are in one-to-one correspondence with the data we need for the homotopy extension property.
How does this correspondence work? We have to extend the map $h : A \times I \to T$ to a map $X \times I \to T$. We do so by extending the diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr, bend left, "(id\times 0)"] & \\
A \times I \arrow[r] \arrow[drr, bend right, "i\times id"'] & Cyl(i) \arrow[dr, "s"] & \\
& & X \times I
\end{tikzcd}$$

Proposition 1.4.8.
For $i : A \to X$ the following are equivalent:

1. *i is a cofibration*
2. *i satisfies HEP for $Cyl(i)$*
3. *$s : Cyl(i) \to X \times I$ has a retraction.*

Proof.
(1) $\implies$ (2): by Def
(2) $\implies$ (3):
The data $(\overline{h}, j)$ can be extended to $H : X \times I \to Cyl(i)$:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, hook, "{(id,0)}"'] & X \arrow[d, "id\times\iota_0"] \arrow[ddr, bend left, "j"] & \\
A \times I \arrow[r] \arrow[drr, bend right, "\overline{h}"'] & X \times I \arrow[dr, dashed, "H"] & \\
& & Cyl(i)
\end{tikzcd}$$

We now have a candidate for the retraction. We have to check that the composition

$$Cyl(i) \xrightarrow{s} X \times I \xrightarrow{H} Cyl(i)$$

is the identity. But the map s is the pushout map and as such uniquely defined by the maps j and $\overline{h}$, so we have to check that the composition is the identity for either one of them:

$$\begin{tikzcd}
X \arrow[d] \arrow[dr, "{(id,\iota_0)}"] \arrow[drr, bend left, "j"] & & \\
Cyl(i) \arrow[r, "s"] & X\times I \arrow[r, "H"] & Cyl(i) \\
A\times I \arrow[u, "\overline{h}"] \arrow[ur, "i\times id"'] \arrow[urr, bend right, "\overline{h}"'] & &
\end{tikzcd}$$

Since H is the map given by HEP it extends when restricted to $i \times id$ the map $\overline{h}$. By the other condition of the HEP, when restricted to (id, ι_0) H extends j.
So we get the map from the mapping cylinder to itself to be determined by j and $\overline{h}$ and thus $H \circ s = id_{Cyl(i)}$.
<u>(3) $\implies$ (1):</u>
We now want to show that if we have such a retraction r than i fulfils the HEP.
Thus we need to show that the indicated map H in the following diagram exists:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, hook, "{(id,0)}"'] & X \arrow[d, "{(id\times\iota_0)}"] \arrow[ddr, bend left, "f"] & \\
A\times I \arrow[r, "i\times id"] \arrow[drr, bend right, "h"'] & X\times I \arrow[dr, dashed, "H"] & \\
& & T
\end{tikzcd}$$

To do so, we put in $Cyl(i)$ as an intermediate step:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, hook, "{(id,0)}"'] & X \arrow[d, "j"] \arrow[dr, bend left, "f"] & \\
A\times I \arrow[r, "\overline{h}"] \arrow[rr, bend right, "h"] \arrow[dr, "i\times id"'] & Cyl(i) \arrow[r, dashed, "{(h,f)}"] \arrow[d, "s"] \arrow[dr, "id"] & T \\
& X\times I \arrow[r, "r"] \arrow[ur, bend right, "H"'] & Cyl(i) \arrow[u, "{(h,f)}"']
\end{tikzcd}$$

Since all triangles in the above diagram commute, we see that H fulfils the necessary condition when restricted to $i \times id$. Using analogous arguments one can see the restriction condition on $id \times \iota_0$. ∎

Corollary 1.4.9.
If A is a closed subset in X and there exists a retraction $r : X \times I \to X \times \{0\} \cup A \times I$, then the inclusion $A \hookrightarrow X$ is a cofibration.

Proof.
The mapping cylinder $Cyl(A \to C)$, defined as the pushout of topological spaces is computed as the disjoint union on X and $A \times I$ glued along A. Therefore it is in bijection with $X \times \{0\} \cup A \times I$. Whenever A is closed in X, this bijection is a homeomorphism:

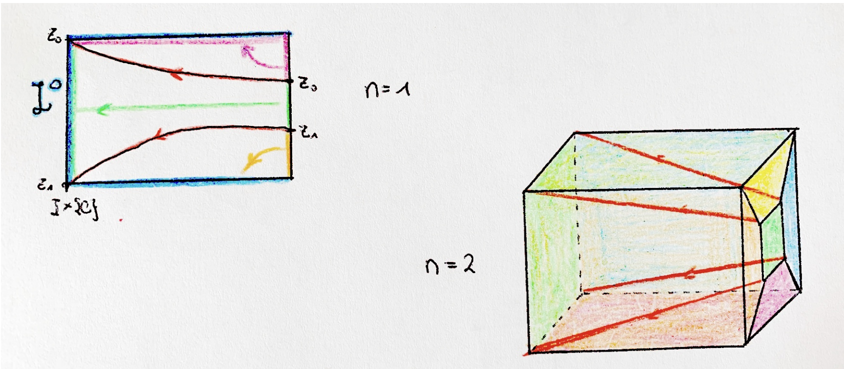
$$\begin{tikzcd}
Cyl(A\to X) \arrow[rr, "\varphi", "\cong"'] \arrow[dr, "s"'] & & X\times\{0\}\cup A\times I \arrow[dl, hook] \\
& X\times I &
\end{tikzcd}$$

To check that this is in fact a homeomorphism, take an open in $Cyl(A \to X)$ comes from two opens (U, W) where $U \subset X$, $W \subset A \times I$.
For φ to be open, the image of those opens has to be open. Instead we will show that the complement is closed in $X \times I$: $\underbrace{(X \setminus U) \times \{0\}}_{\text{closed in } X} \cup \underbrace{(A \times I \setminus W)}_{\text{closed in the cylinder}}$ ∎

Example 1.4.10. (Main example of cofibrations)
The inclusions $\partial I^n \subset I^n$ and $S^{n-1} \subset D^n$ are cofibrations.
Since those spaces are homeomorphic to each other, we only have to check this for one of them.
We need to construct a retraction

$$I^n \times I \to I^n \times \{0\} \cup \partial I^n \times I = \mathcal{J}^n$$

the construction method is called "push through a cardboard box".
We will illustrate how this is done in a lower dimensional case:
We start off by dividing the face not contained in $\mathcal{J}^0$ in three pieces. By pushing the points dividing the edge to the corners of the opposite face, each of the tree pieces are pushed onto one face included in $\mathcal{J}^0$.



Cofibrations and Pushouts

Proposition 1.4.11.
Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & B \arrow[d, "j'"] \\ X \arrow[r, "f'"'] & Y \end{tikzcd}$$

be a pushout diagram in `Top`.
If j has HEP for some space T, then so does j'. In particular, if j is a cofibration, then j' is also.

Proof.
We will use the remark that HEP is some sort of lifting property:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & B \arrow[r] \arrow[d] & T^I \arrow[d, "ev_0"] \\ X \arrow[r, "x"'] \arrow[urr, dashed, "H"] \arrow[ur, phantom, "\square"] & Y \arrow[r, "y"'] \arrow[ur, "\tilde{H}"'] & T \end{tikzcd}$$

So what happens is that

1. By HEP for j we get H

2. By pushout we get $\tilde{H}$ (so the upper triangle commutes by definition, but we have to check commutativity for the lower one)

3. By universal property $ev_0 \circ \tilde{H} = (Y \to T)$ (see below)

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] & B \arrow[d] \arrow[ddr, bend left, "ev_0 \circ (B \to T^I)"] & \\
X \arrow[r] \arrow[drr, bend right, "ev_0 \circ \tilde{H} \circ x"'] & Y \arrow[ul, phantom, "\square"] \arrow[dr, dashed] & \\
 & & T
\end{tikzcd}$$

We have to check that given two maps $Y \to T$ that they coincide, in particular we want to show that any given map y has to coincide with $ev_0 \circ \tilde{H}$ because of the pushout property.
For this we have to check that these maps restricted to B and X coincide (bc pushout).
So if we restrict $ev_0 \circ \tilde{H}$ to B, we get, because the square on the right commutes, the same as those maps $Y \to T$ restricted to B. So we are fine.
If we restrict those maps $Y \to T$ to X, we get again by commutativity that this is equal to $ev_0 \circ \tilde{H}$. ■

Proposition 1.4.12.
Let (X, A) be a relative CW complex. Then $A \to X$ is a cofibration.

Proof.
We will start by showing (via induction) that each inclusion $X^{(i)} \hookrightarrow X^{(i+1)}$ is a cofibration. Just like in AT I we will hereby use the notation of $A = X^{(-1)}$.

Take $n = 0$. When attaching 0-cells to A in order to get $X^{(0)}$, we have by definition the following pushout square:

$$\begin{tikzcd}
\emptyset = \coprod_{\alpha \in I_0} S^0 \arrow[r] \arrow[d, hook] & X^{(-1)} = A \arrow[d, hook] \\
\coprod_{\alpha \in I_0} D^0 \arrow[r] & X^{(0)} \arrow[ul, phantom, "\square"]
\end{tikzcd}$$

By this we know that $X^{(0)} = A \amalg \{v_\alpha\}_{\alpha \in I_0}$. To check that the inclusion $A = X^{(-1)} \hookrightarrow X^{(0)}$ is a cofibration, we have to find a map H, as indicated by the dotted map in the diagram below:

$$\begin{tikzcd}
A \arrow[r, "\iota_0"] \arrow[d] & A \times I \arrow[d, "i \times id"] \arrow[ddr, bend left, "h"] & \\
X^{(0)} \arrow[r] \arrow[drr, bend right, "f"'] & X^{(0)} \times I \arrow[dr, dashed, "\exists H"] & \\
 & & T
\end{tikzcd}$$

for any space T. By what we have seen before, that $X^{(0)} = A \amalg \{v_\alpha\}_{\alpha \in I_0}$, we can now easily define the map H as being h on $A \times I$ and f on the vertices $\{v_\alpha\}_{\alpha \in I_0}$. This is obviously continuous.

Next we will show that $X^{(i-1)} \hookrightarrow X^{(i)}$ is a cofibration. For this we first note that $S^{n-1} = \partial D^n$. We have seen in the lecture that the inclusion $\partial D^n \subset D^n$ is a cofibration. Therefore the inclusion

of the disjoint union is as well.
By definition of CW-structure, we again have the following pushout diagram:

$$\begin{tikzcd}
\coprod_{\alpha \in I_n} S^{n-1} \arrow[r] \arrow[d, hook, "\text{cofib.}"'] & X^{(n-1)} \arrow[d, hook] \\
\coprod_{\alpha \in I_n} D^n \arrow[r] \arrow[ur, phantom, "\square", very near end] & X^{(n)}
\end{tikzcd}$$

By another result of the lecture, we already know that in a pushout square, if the downturned map on the left is a cofibration, then so is the one on the right. Therefore the inclusion $X^{(n-1)} \hookrightarrow X^{(n)}$ is a cofibration, just as we wanted.

For our final step, we use that $X = \operatorname{colim}_n X^{(n)}$. This allows us to glue all inclusions $X^{(i-1)} \hookrightarrow X^{(i)}$, $i \geq 0$, together to get a map $A = X^{(-1)} \to X$. This map again is a cofibration. ∎

So up to now we have used the mapping cylinder as a means to detect cofibrations, but it actually has more to offer. Maybe even its main use is a different one: replacing a map by an associated cofibration.

Proposition 1.4.13.
Suppose we have a map of topological spaces $f : X \to Y$. Then we can factor it as an inclusion in the mapping cyclinder and a map from the mapping cyclinder to Y:

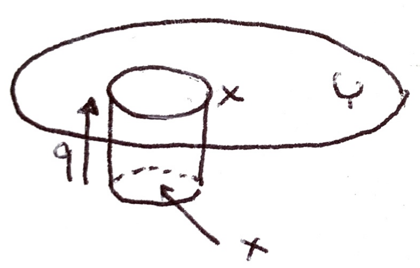
$$X \xrightarrow{j} Cyl(f) \xrightarrow{q} Y$$

such that

1. *j is a cofibration*
2. *q is a homotopy equivalence $\pi \circ q \simeq id_{Cyl(f)}$ rel Y*
3. *$q \circ j = f$*

Remark 1.4.14. (Construction of the maps q and j)

$$\begin{tikzcd}
 & Y \arrow[d] \arrow[dr, bend left, "id_Y"] & \\
X \arrow[r, "j"] \arrow[dr, "{(id,1)}"'] & Cyl(f) \arrow[r, "q"] & Y \\
 & X \times I \arrow[u, "\overline{h}"'] \arrow[r, "pr_X"'] & X \arrow[u, "f"']
\end{tikzcd}$$



Proof.

3. Obvious because by commutativity instead of going the direct path $q \circ j$ one can go the lower one and see directly that this is f.

1. Consider:

$$\begin{tikzcd}
X \times \partial I = X \amalg X \arrow[r, "f \amalg id_X"] \arrow[d, hook, "id \times incl"'] & Y \amalg X \arrow[d, "(\pi, j)"] & X \arrow[l, "in_X"'] \arrow[dl, bend left, "j"] \\
X \times I \arrow[r] & Cyl(f) \arrow[ul, phantom, "\square", very near start] &
\end{tikzcd}$$

we use here a somewhat different definition of the mapping cylinder: instead of gluing X to Y on one side, we do so on both sides.
The map $id \times incl$ is a cofibration. To check this to be true what you can do is use that $\partial I \hookrightarrow I$ is a cofibration but not just any cofibration but such that there exists a closed cofibration such that there exists a retraction of the cylinder and then you can extend this retraction to X times everything by identity on X and then you get a retraction from the cylinder on $X \times I$ to this union of a cylinder on $X \times \partial I$ and the face of $X \times I$.
In general it is not true, that the product of cofibrations is a cofibration but for such a nice cofibration like in this case it works.

Since this diagram is a pushout diagram, (π, j) too is a cofibration.

in_X is a cofibration for very obvious reasons because for this map you can easily see that it fulfils the HEP for any space, because Y lives separately from X and thus you can always just choose the constant homotopy on Y.

So in_X and (π, j) are cofibrations and since the composition of cofibrations is again a cofibration, so is j.

2. We need to construct the homotopy which is a map $H : Cyl(f) \times I \to Cyl(f)$ and the cylinder of f is a pushout because a direct product with the unit interval commutes with pushouts. Constructing such a map is the same as to construct a map on $(X \times I) \times I$ and $Y \times I$

$$\begin{tikzcd}
Y \times I \arrow[d] \arrow[dr] & \\
Cyl(f) \times I \arrow[r, "H"] & Cyl(f) \\
(X \times I) \times I \arrow[u] \arrow[ur] &
\end{tikzcd}$$

where the map $Y \times I \to Cyl(f)$ is given by $(y, i) \mapsto y = \pi(y)$, so nothing happens on y just like we want it to be the case, so on y it is always the identity map.
On the map $(X \times I) \times I \to Cyl(f) \times I$ something needs to happen, so we have $((x, t), \tau) \mapsto \overline{h}(x, t\tau)$ where $\overline{h} : X \times I \to Cyl(f)$.
What we have to check is that this maps defines a map H which gives us the right homotopy.
So we have to check what happens here: $((x, t), \tau) \mapsto \overline{h}(x, t\tau)$
For $\tau = 1$ we get here the canonical map $\overline{h}$. For $\tau = 0$:

$$\begin{tikzcd}
Cyl(f) \arrow[r, "q"] & Y \arrow[r, "\pi"] & Cyl(f) \\
X \times I \arrow[u] \arrow[r] & X \arrow[u, "f"'] \arrow[r] & X \times I \arrow[u, "\overline{h}"']
\end{tikzcd}$$

$$(x, \tau) \longmapsto x \longmapsto (x, 0)$$

■

Example 1.4.15.
We can use this proposition to define:

Definition 1.4.16.
If we are given a map $f : X \to Y$, we can define relative homotopy groups of this map as

$$\pi_n(X \xrightarrow{f} Y) := \pi_n(Cyl(f), X)$$

For this definition we have a long exact sequence because $\pi_n(Cyl(f) \cong \pi_n(Y)$

1.5 Higher Connectivity

This is a notion that generalizes the notions of simply path-connectedness and connectedness in the direction of higher homotopy groups.

Definition 1.5.1.
A map of topological spaces $i : A \to X$ is n-connected $(-1 \leq n \leq \infty)$ if it satisfies:

0. $X \neq \emptyset$
1. $\pi_0 A \twoheadrightarrow \pi_0 X \quad$ if $n \geq 0$
2. $\pi_i(X, A, a) = 0$ for all $a \in A$, $1 \leq i \leq n$

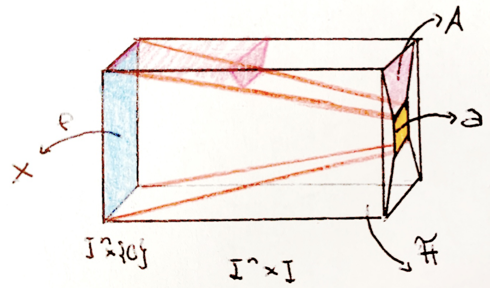
Remark 1.5.2.
If one chooses A to be just a point a, one gets a definition of higher connectivity of a space X. If $X \neq \emptyset$, then condition 1. is always fulfilled. Thus $X \neq \emptyset$ is n-connected, whenever $\pi_i(X, a) = 0$ for all $1 \leq i \leq n$.

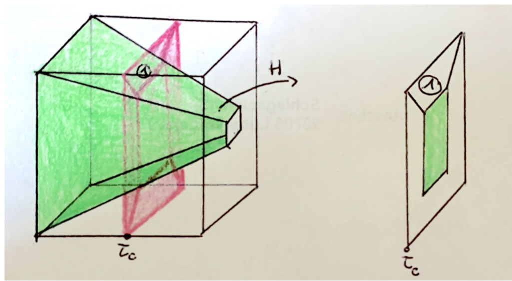
Proposition 1.5.3.
Let (X, A) be a pair in `Top`*, $n \geq 0$. Then the following conditions are equivalent:*

1. *Each map $(I^q, \partial I^q) \to (X, A)$ is homotopic to a constant map, $1 \leq q \leq n$ and $\pi_0 A \twoheadrightarrow \pi_0 X$*
2. *Each map of pairs $(I^q, \partial I^q) \to (X, A)$ is homotopic relative ∂I^q to a map into A, $0 \leq q \leq n$*
3. *$\pi_0 A \twoheadrightarrow \pi_0 X$, $\pi_q(A, a) \xrightarrow{\cong} \pi_q(X, a)$ for all $a \in A$, $1 \leq q < n - 1$, $\pi_n(A, a) \twoheadrightarrow \pi_n(X, a)$*
4. *$A \to X$ is n-connected.*

Proof.

(1)⇒ (2)
Let $q \geq 1$, $f : I^q \to X$ a map that is nullhomotopic via $H : I^n \times I \to X$, $H|_{I^n \times \{0\}} = a$, $a \in A$. We want to construct a homotopy rel ∂I^n to a map into A.





To perhaps explain in more detail what is happening in what is drawn above, here are some comments: Painted in green is the homotopy H we are already given. We can now define $\tilde{H}$ section-wise, so for any $\tau_0 \in I$ we need to define the pink part indicated, so all that is happening around the homotopy H. The new, pink part, can be divided into four parts, one is indicated by a circled 1 in the picture. This part is now defined to be the upper face of H up to this point τ_0. The line of part circled 1 that is subset of the boundary of the newly defined cube thus coincides with the boundary of the left face. The left face, however, is given by the map f which is a map pf pairs sending its boundary into A.
Also H is a homotopy of pairs, so all of its boundary is sent into A.

(2)⇒(4)
Let $q > 0$. We are given a map of triples $f : (I^n, \partial I^n, \mathcal{J}^{n-1}) \to (X, A, a)$ and we want to show that this is homotopic to the constant map as a map of triples.
We know that f is homotopic rel ∂I^n to a map into A:

$$f \simeq g : (I^n, \partial I^n, \mathcal{J}^{n-1}) \to (A, A, a)$$

So $[f]$ comes from $\pi_q(A, A, a) \to \pi_q(X, A, a)$

Lemma 1.5.4. $\pi_q(A, A, a) = 0$

Proof.
$id : A \to A$ is a Serre fibration so we get a LES where $\pi_q(A, A, a) = 0$. ∎

(3)⇔(4) are connected by LES of groups.

(4)⇒(1)
We are given a map of pairs $f : (I^q, \partial I^q) \to (X, A)$ and want to show that it is homotopy equivalent to the constant map.
We could have used that the relative homotopy groups are zero, but this is not a map of triples because it does not necessarily send $\mathcal{J}^{q-1}$ to a point but we can fix that:

$$\underbrace{\mathcal{J}^{q-1}}_{\simeq *} \hookrightarrow \partial I^q \hookrightarrow I^q$$

are cofibrations. Thus $f|_{\mathcal{J}^{q-1}} \simeq a$ and because these are cofibrations we can extend the homotopy to $H : I^q \times I \to X$, $H|_{I^q \times \{0\}} = f$, $H|_{I^q \times \{1\}} = g : (I^q, \partial I^q, \mathcal{J}^{q-1}) \to (X, A, a)$ which is a map of triples. As such it gives a class in the relative homotopy groups which we know to be trivial. In particular, g is homotopic to a constant map via a homotopy of triples. So f is homotopic to g via a homotopy of pairs, g is homotopic to the constant map via a homotopy of triples, so f is homotopic to the constant map via a homotopy of pairs.

∎

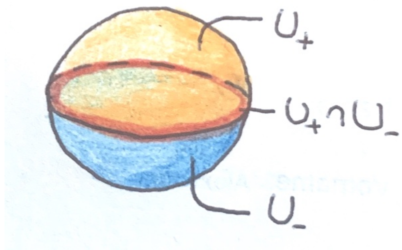
Theorem 1.5.5.
Suppose we have a map of topological spaces $f : X \to Y$, $X_0, X_1 \subset X$ such that $X_0^\circ \cup X_1^\circ = X$, $f(X_i) = Y_i$, $i = 0, 1$
$Y_0, Y_1 \subset Y$, $Y_0^\circ \cup Y_1^\circ = Y$.
Assume that

- *$f|_{X_i}$ is n-connected*
- *$f|_{X_0 \cap X_1}$ is $(n-1)$-connected*

for some $n \geq 1$.
Then f is n-connected.

Corollary 1.5.6.
S^n is $(n-1)$-connected.

Proof.
Proof by induction on n. Let $S^n = U_+ \cup U_-$.
So $U_+ \cap U_- \simeq S^{n-1}$ and $U_+ \simeq U_- \simeq *$.



$$
\begin{array}{ccc}
X_0 & & U_+ = Y_0 \\
\| & & \cap \\
X = * & \xrightarrow{f} & S^n = Y \\
\| & & \cup \\
X_1 & & U_- = Y_1
\end{array}
$$

So on the intersection $U_+ \cap U_-$ we get that f as a map from a point to S^{n-1} is $(n-2)$-connected by induction assumption.
The maps from $X_i \to Y_i$ are infinitely connected, since they are in fact homotopy equivalences ($Y_i \simeq *$, $X_i = *$).
Putting all of this together, S^n is $(n-1)$-connected. ■

How to reduce the question of n-connectedness of an arbitrary map, to the case of an inclusion?:
Suppose we have the following commutative diagram:

$$
\begin{tikzcd}
A \arrow[r, "\varphi|_A"] \arrow[d, hook, "i"'] & X \arrow[d, "f"] \\
B \arrow[r, "\varphi"'] \arrow[ur, dashed, "\psi"] & Y
\end{tikzcd}
$$

where i is a cofibration (in particular an inclusion).
We want to find a map ψ such that

$$
\psi \circ i = \varphi|_A
$$
$$
f \circ \psi \simeq \varphi \text{ rel } A
$$

One of the examples of cofibrations we introduced is the inclusion of the boundary in the cube:

$$
\begin{tikzcd}
\partial I^n \arrow[r, "\varphi|_{\partial I^n}"] \arrow[d, hook] & X \arrow[d, hook, "f"] \\
I^n \arrow[r, "\varphi"'] \arrow[ur, dashed, "\psi"] & Y
\end{tikzcd}
$$

where $f : X \to Y$ also is an inclusion and q-connected. Then we saw that ψ exists as we want it to (one of the equivalent descriptions of q-connectedness).

In the general case, we will not assume that f is either an inclusion or q-connected.
The claim is that if we can solve this problem for the commutative square:

$$\begin{tikzcd} A \arrow[r, "\tilde{\varphi}|_A"] \arrow[d, hook, "i"'] & X \arrow[d, "j"] \\ B \arrow[r, "\tilde{\varphi}"'] \arrow[ur, dashed, "\tilde{\psi}"] & Cyl(f) \end{tikzcd}$$

Find $\tilde{\psi}$ in that and then we can solve the original problem.

We have of course the mapping cylinder construction:

$$\begin{tikzcd} A \arrow[r, "\varphi|_A"] \arrow[d, "i"'] & X \arrow[d, "j"] \\ B \arrow[r, dashed, "\pi\circ\varphi"] \arrow[dr, "\varphi"'] & Cyl(f) \arrow[d, shift right, "q"'] \\ & Y \arrow[u, shift right, bend right, "r"'] \end{tikzcd}$$

The problem is that this diagram does not commute!

$$\pi \circ \varphi \circ i = \pi \circ f \circ \varphi|_A = \underbrace{\pi \circ q}_{\simeq id \text{ rel } Y} \circ j \circ \varphi|_A \simeq j \circ \varphi_A \text{ rel } A$$

So it is commutative up to homotopy.
But because i is a cofibration, we can extend this homotopy to B:

$$\begin{aligned} \exists H : B \times I \to Cyl(f) \\ H|_{B\times\{0\}} = \pi \circ \varphi \\ H|_{B\times\{1\}} = \tilde{\varphi} \\ \tilde{\varphi} \simeq \pi \circ \varphi \end{aligned}$$

$\tilde{\varphi}$ has the property that $\tilde{\varphi} \circ i = j \circ \varphi|_A$.
Suppose we have found $\tilde{\psi}$ such that $\tilde{\psi} \circ i = \varphi|_A = \tilde{\varphi}|_A$ and $j \circ \tilde{\psi} \simeq \tilde{\varphi}$ rel A

Now we go back to the space Y so we have to apply the map q:

$$f \circ \tilde{\psi} = q \circ j \circ \tilde{\psi} \simeq \tilde{\varphi} \text{ rel } A \simeq \underbrace{q \circ \pi}_{id_Y} \circ \varphi = \varphi$$

The next step towards the proof of the theorem is to generalise the lifting problem we had for the cube (proposition) to cubical complexes.

Definition 1.5.7.
A <u>subdivision</u> of I^n of width $\frac{1}{N}$ is the representation

$$I^n = \bigcup_{k_j} \underbrace{\prod_{j=1}^{n} \left[\frac{k_j}{N}, \frac{k_{j+1}}{N}\right]}_{n\text{-dim cube}}$$

A <u>cubical complex</u> $B \subset I^n$ is the union of the form

$$\prod_{j=1}^{n} \left[\frac{k_j}{N}, \frac{k'_j}{N+1}\right]$$

where $0 \leq k_j < N$ with $k'_j = \begin{cases} k_j \\ k_{j+1} \end{cases}$
We call cubes of this form elementary cubes.
Those are no n-dimensional cubes but rather of some dimension that might even be zero and thus a point.
The k-th skeleton

$$B(k) := \bigcup \leq k\text{-dim elementary cubes in } B$$

A subcomplex $A \subset B$ is a subset of the union of cubes of B.

Lemma 1.5.8.
Suppose that $f : X \to Y$ is n-connected. Let (B, A) be a cubical pair of dimension $\leq n$. Then every commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varphi|_A"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "\varphi"'] \arrow[ur, dashed, "\psi"] & Y \end{tikzcd}$$

has a lift ψ as indicated such that $\psi \circ i = \varphi|_A$ and $f \circ \psi \simeq \varphi$ rel A.

Proof.
Step 1 is to reduce to the case when f is an inclusion (just like we done before). This can be done because an inclusion of a cubical subcomplex is clearly a relative CW complex. The bigger cubical complex is obtained by attaching cubes to the subcomplex.
A relative CW complexes is always a cofibration and we can apply the trick of the mapping cylinder to reduce to the case where f is an inclusion. Eventually we will be able to use the proposition.

Step 2: Go by induction on the relative dimension of (B, A), $\dim(B, A) = m$ (dimension of the biggest elementary cube in B that is not in A).
So how do we go by induction? First we attach to A the $n-1$-skeleton of B and then all of B:

$$\begin{tikzcd} A \arrow[r, "\varphi|_A"] \arrow[d, hook] & X \\ A \cup B(m-1) \arrow[r, "\varphi|_{A \cup B(m-1)}"] \arrow[ur, dashed, "\psi(m-1)"] \arrow[d, hook] & Y \\ B \arrow[ur, "\varphi"'] & \end{tikzcd}$$

The relative dimension of the inclusion $A \hookrightarrow A \cup B(m-1)$ is at most $m-1$. So the map $X \to Y$ is m-connected. Since $m - 1 < m$ we can find by induction assumption the lift indicated, which makes the square commute up to homotopy relative to A.

we should now look at the following square:

$$\begin{tikzcd} A \cup B(m-1) \arrow[r, "\psi_{m-1}"] \arrow[d] & X \arrow[d, "f"] \\ B \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

Using the fact that the inclusion $A \cup B(m-1) \hookrightarrow B$ is a cofibration we can replace φ by $\tilde{\varphi}$ ($\tilde{\varphi} \simeq \varphi$ rel A), such that the square does now commute.

$$\begin{tikzcd}
A \cup B(m-1) \arrow[r, "\psi_{m-1}"] \arrow[d] & X \arrow[d, "f"] \\
B \arrow[r, "\tilde{\varphi}"'] & Y
\end{tikzcd}$$

B is obtained from $A \cup B(m-1)$ by attaching m-dimensional cubes which is the only thing we attach. So if we regard any m-dimensional cube, its boundary lies in $B(m-1)$, so

$$\begin{tikzcd}
\coprod \partial I^m \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & A \cup B(m-1) \arrow[r, "\psi_{m-1}"] \arrow[d] & X \arrow[d, "f"] \\
\coprod I^m \arrow[r] & B \arrow[r, "\tilde{\varphi}"'] & Y
\end{tikzcd}$$

If we look at the outer square, f is as we know an inclusion and an m-connected map by the assumption, so we can find a map $\tilde{\psi}$ by the proposition:

$$\begin{tikzcd}
\coprod \partial I^m \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & A \cup B(m-1) \arrow[r, "\psi_{m-1}"] \arrow[d] & X \arrow[d, "f"] \\
\coprod I^m \arrow[r] \arrow[urr, dashed, "\tilde{\psi}"] & B \arrow[r, "\tilde{\varphi}"'] \arrow[ur, dashed, "\psi"] & Y
\end{tikzcd}$$

such that the upper triangle commutes and the lower one up top homotopy.
ψ makes the upper part commutative and satisfies

$$f \circ \psi \simeq \tilde{\varphi} \text{rel } A \cup B(m-1)$$

■

Proof. (of the theorem)
Step 1: Reduction to the case of a pair (Y, X) (that means f is an inclusion)
To do so, we again need the mapping cylinder construction.
We factor our map as follows:

$$\begin{tikzcd}
X \arrow[r] & Cyl(f) \arrow[r] & Y \\
\cup & & \cup \\
X_i \arrow[r] & Cyl(f|_{X_i}) \arrow[r] \arrow[uu] & Y_i
\end{tikzcd}$$

(The map $Cyl(f|_{X_i}) \to Cyl(f)$ exists by the universal property, also the commutativity)
The important property though, is that this map is an inclusion of a subspace and

$$Cyl(f) = Cyl(f|_{X_1})^\circ \cup Cal(f|_{X_2}^\circ)$$
$$Cyl(f|_{X_0 \circ X_1}) = Cyl(f|_{X_0}) \cap Cyl(f|_{X_1})$$

(Easy exercise)

Step 2: Finding a subdivision of I^n Let $f : (I^n, \partial I^n) \to (Y, X)$ be a map of pairs. We would to show that this map is homotopic relative the boundary to a map inside X.
Find a subdivision of I^n:

$$I^n = K_0 \subset K_1$$

where K_0 and K_1 are cubical complexes such that $f(K_i) = Y_i$ and $f(K_i \cap \partial I^n) \subseteq X_i$.

We want to apply the Lebesgue lemma. Define $A_i := f^{-1}(Y \setminus Y_i^\circ) \cup f^{-1}(X \setminus X_i^\circ)$ closed in I^n, $A_0 \cap A_1 = \emptyset$.
Now we get an open cover of the cube: $I^n = (I^n \setminus A_0) \cup) I^n \setminus A_1)$. By the Lebesgue lemma we get a subdivision of I^n s.t. $\begin{cases} W \subset I^n \setminus A_0 \\ W \subset I^n \setminus A_1 \end{cases}$
Now we define

$$K_i := \bigcup_{\substack{f(W) \subset Y_i^\circ \\ f(W \cap \partial I^n) \subset X_i^\circ}} W$$

Step 3
Define $K_{01} = K_0 \cup K_1$; $K_i^\bullet$ is the (n-1)-skeleton of K_i, $i = 0, 1, 01$

$$\begin{aligned} g_{01} &: K_{01} \to X_{01} \\ g_0 &: (K_0 \cap \partial I^n) \cup K_{01}^\bullet \to X_0 \\ h_0 &: K_0 \to X_0 \end{aligned}$$

Also the same definition for g_1 and h_1.
How are these maps related to f?: They are all homotopic (relative at least to the boundary of the cube, maybe more) to f restricted to the respective subset.
We can glue h_0 and h_1 to

$$h : K_0^\bullet \cup K_1^\bullet \to X$$

To get the final map to which f will be homotopic relative the boundary, we want to extend h to the insides of all the cubes. But for the insides it will be easy, because the boundary will already go to the right place and the insides of the cubes don't intersect each other.

We have that the boundary of our cube goes inside X. If we intersect it with K_{01} everything on the boundary that is in K_0 is sent to X_0 and all that is in K_1 is sent to X_1. Also, all of the boundary is contained in the $(n-1)$-skeleton of the cube, so:

$$\begin{tikzcd} \partial I^n \cap K_{01} \arrow[r] \arrow[d, hook] & X_{01} \arrow[d, hook, "j_{01}"] \\ K_{01}^\bullet \arrow[r] \arrow[ur, dashed, "g_{01}", "h"'] & Y_{01} \end{tikzcd}$$

where $K_{01}^\bullet$ is a $(n-1)$-dimensional cubical complex and the inclusion $X_{01} \hookrightarrow Y_{01}$ is $(n-1)$-connected. Thus, by the previous lemma, we get a lift as indicated.

$$j_{01} \circ g_{01} \overset{h_{01}}{\simeq} f \text{ rel } \partial I^n \cap K_{01}$$

So we extend this map g_{01} to g_0

$$g_0 : \underbrace{K_0 \cap (\partial I^n \cup K_1^\bullet)}_{=(K_0 \cap \partial I^n) \cup K_{01}^\bullet} \to X_0$$

$K_0 \cap (\partial I^n \cup K_1^\bullet = (K_0 \cap \partial I^n) \cup$ because when intersecting K_0 with $K_1^\bullet$, which is the $(n-1)$-skeleton of K_1, no n-dimensional cubes will be in there, so it's the same as the intersection of $K_0^\bullet$ with $K_{01}^\bullet$ which is the same as $K_{01}^\bullet$.

Thus we can define g_0 by g_{01} on $K_{01}^\bullet$ and f on $K_0 \cap \partial I^n$.
We have to check that this is well-defined on the intersection (which is closed):

- $K_0 \cap K_{01}^\bullet$ is fine
- $\partial I^n \cap K_{01}^\bullet$ is exactly the case for which we found g_{01} to be the lift. So by commutativity of the triangle this is f.

Thus we can glue those two maps together to get g_0.

Moreover, g_{01} was homotopic to f (after inclusion) and we can glue this homotopy h_{01} to get the homotopy $g_0 \overset{H_0}{\simeq} f$ relative $K_0 \cap \partial I^n$.
So the homotopy is "active" only on $K_{01}^\bullet$ because on $K_0 \cap \partial I^n$ it's already f so it's a simple gluing procedure and g_0 is homotopic to f.

Now $K_0 \cap (\partial I^n \cup K_1^\bullet) \hookrightarrow K_0$ which is a cofibration since this is an inclusion of a cubicle complex and thus a relative CW complex.
We can extend the homotopy H_0 to $\psi : K_0 \times I \to Y_0$, $\psi|_{K_0 \times \{0\}} = f$, $\psi|_{K_0 \times \{1\}} = F_0$.

$$\begin{tikzcd}
K_0 \cap (\partial I^n \cup K_1^\bullet) \arrow[r, "g_0"] \arrow[d, hook] & X_0 \arrow[d] \\
K_0 \arrow[r, "F_0"'] \arrow[ur, dashed, "h_0"] & Y_0
\end{tikzcd}$$

This is a commutative diagram, that is what extending the homotopy gives us: instead of the map f we get some map that makes this diagram commute where we have the map g_0 instead of f.
The map $X_0 \to Y_0$ is n-connected and on the left hand side there is an inclusion of a cubicle complexes of dimension at most n and thus a cofibration.
Therefore we can find a lift h_0 as indicated.

By this we now have constructed, g_{01}, g_0 and h_0 and by the same construction, replacing all zeros with ones we can construct g_1 and h_1 starting with the same map g_{01}. So the claim is that those maps h_0 and h_1 glue together to give us the map h.

To check that, we don't look inside n-dimensional cubes, we only care about the $(n-1)$-skeleton. We have to look what happens on the intersection of $K_0^\bullet$ and $K_1^\bullet$.
So we have to restrict h_0 to this intersection. If we restrict h_0 to $K_0 \cap (\partial I^n \cup K_1)$ we get g_0 and when we regard g_0, we see that this leaves us with g_{01}, just like we wished for. Same goes for h_1.
We can now glue both maps together and get that

$$h \simeq f \text{ rel } \partial I^n$$

Finally, we extend h to I^n:

$$W \subset I^n, \quad W \subset K_0, \quad \partial W \subset K_0^\bullet \cup H_1^\bullet$$

$$\begin{tikzcd}
\partial I^n \arrow[d, hook] & \cong & \partial W \arrow[rr, bend left, "h_0"] & \subset \; K_0^\bullet \cup K_1^\bullet \arrow[r, "h"] & X_0 \\
 & & \cap & & \cap \\
I^n & \cong & W \arrow[rr] \arrow[urr, dashed, "\tilde{h}_W"] & & Y_0
\end{tikzcd}$$

$$\begin{aligned}
\tilde{h} \simeq h_0|_W \text{ rel } \partial W &\simeq f \text{ rel } \partial I^n \\
\implies \tilde{h} &\simeq g \text{ rel } \partial I^n \\
\tilde{h} &: I^n \to X
\end{aligned}$$

■

Corollary 1.5.9. *("easy excision")*
Suppose $Y = U \cup V$ is an open covering, $W := U \cap V \neq \emptyset$.
If (V, W) is n-connected, then so is (Y, U):

$$\begin{tikzcd} W \arrow[r] \arrow[d, "n\text{-}conn"'] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ V \arrow[r] & Y \end{tikzcd}$$

Proof.
Consider

$$\begin{aligned} X = U &\hookrightarrow Y \\ X_0 = W &\xrightarrow[n\text{-conn}]{} Y_0 = V \\ X_1 = U &\xrightarrow[\infty\text{-conn}]{} Y_1 = U \\ X_{01} = W &\xrightarrow[\infty\text{-conn}]{} Y_{01} = W \end{aligned}$$

We can now apply the theorem. ■

Lemma 1.5.10.
Let $i : A \to X$ be a cofibration, $f : A \to A'$ a homotopy equivalence.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] \arrow[dr, phantom, "\square"] & A' \arrow[d] \\ X \arrow[r, "f'"'] & X' \end{tikzcd}$$

Them f' is a homotopy equivalence.

Proposition 1.5.11.
Consider the following pushout square:

$$\begin{tikzcd} \coprod S^n \arrow[r, "\varphi"] \arrow[d, "i"'] \arrow[dr, phantom, "\square"] & A \arrow[d, "f"] \\ \coprod D^{n+1} \arrow[r, "\Phi"'] & X \end{tikzcd}$$

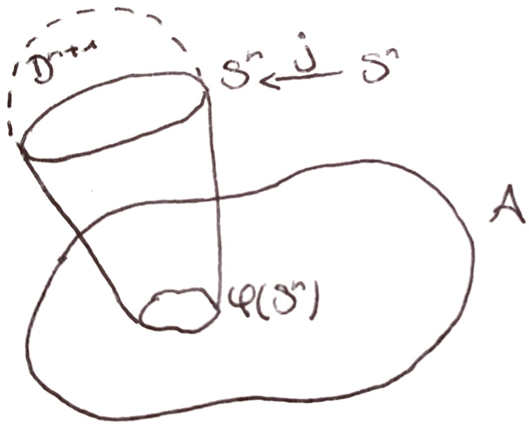
Then f is n-connected.

Example 1.5.12.
One can attach an $n+1$-dimensional disk to a point and you will get an $n+1$-dimensional sphere and this is then n-connected.

Proof.
First, reduce to the case where the map φ is not some abstract map but rather comes from the mapping cylinder construction.

$$\begin{tikzcd} \coprod S^n \arrow[r] \arrow[d, "i"'] \arrow[dr, phantom, "\square"] & Cyl(\varphi) \arrow[r] \arrow[d, "\tilde{i}"] \arrow[dr, phantom, "\square"] & A \arrow[d, "f"] \\ \coprod D^{n+1} \arrow[r] \arrow[rr, bend right, "\Phi"'] & Z \arrow[r] & X \end{tikzcd}$$

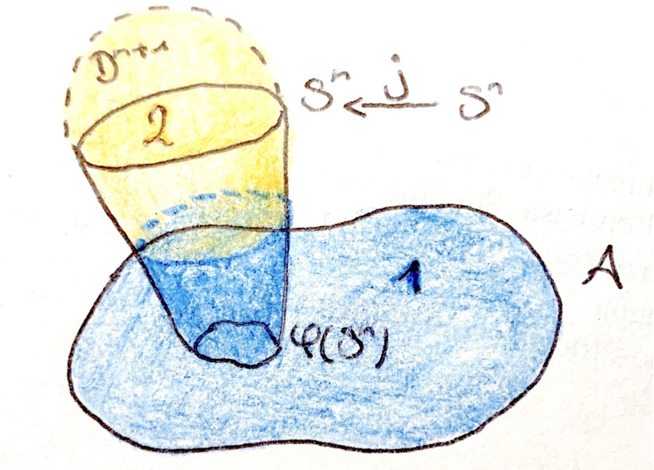
where Z is the pushout of the square on the left.



It is an easy exercise in category theory to show that if the left square is a pushout square and the outer square is one, then the square on the right is too.
Since taking the pushout preserves cofibrations, i being a cofibration implies that $\tilde{i}$ is a cofibration. By the lemma $Z \to X$ is therefore a homotopy equivalence.

Proving now that f is n-connected is equivalent to proving that $\tilde{i}$ is n-connected.

So now we find some opens in Z:



$$\begin{aligned} \text{1st open of } Z\text{:} \quad & A \cup_{Cyl(\varphi)} \coprod (S^n \times [0, \tfrac{1}{2}]) \simeq A \simeq Cyl(\varphi) \\ \text{2nd open of } Z\text{:} \quad & \coprod D^{n+1} \cup (S^n \times [\tfrac{1}{4}, 1]) \simeq \coprod D^{n+1} \simeq \coprod * \\ \text{intersection:} \quad & \simeq \coprod S^n \end{aligned}$$

(D^{n+1}, S^n) is a n-connected pair (one can see that for example by regarding the homotopy groups: those of D^{n+1} are zero, those of S^n are zero up dimension n. Therefore we get isomorphisms up to dimension n and in n a surjection.)
Also we can regard the disjoint union $(\coprod D^{n+1}, \coprod S^n)$ and it will not change anything.
By easy excision, $\tilde{i}$ is n-connected. ■

Remark 1.5.13.
By this, we have shown that attaching $n+1$-dimensional cells does not change the homotopy groups at least up to dimension n.

Corollary 1.5.14.
If (X, A) is a relative CW-complex. Then $(X, X^{(n)})$ is n-connected.

Proof.
By compactness argument

$$\pi_m X \cong \operatorname*{colim}_n X^{(n)}$$

■

Definition 1.5.15.
A map $f : X \to Y$ of CW-complexes is cellular, if $f(X^{(n)}) \subseteq Y^{(n)}$ for all n.

Theorem 1.5.16. *(Cellular approximation)*
Let X, Y be CW complexes, $B \subseteq X$ a subcomplex.
Any map $f : X \to Y$ such that $f|_B$ is cellular is homotopic to a cellular map g relative to B:

$$f \simeq g \text{ rel } B, \quad \text{where } g \text{ is cellular.}$$

Proof.
By induction on the skeletal filtration

$$\exists H^n : X \times I \to Y, \quad n \geq 0$$

1. $H^0|_{X\times\{0\}} = f$, $H^{n-1}|_{X\times\{1\}} = H^n|_{X\times\{0\}}$, $n \geq 1$
2. $H^n|_{X\times\{1\}}(X^{(i)} \subset Y^{(i)}$, $i \leq n$
3. H^n is constant on $X^{n-1} \cup B$

After having constructed those H^n, we can proceed as follows:

$$H : I \times I \to Y$$
$$H(x,t) := \begin{cases} H^i(x, 2^{i+1}(t - 1 + 2^{-1})), & \text{if } 1 - 2^{-i} \leq t \leq 1 - 2^{-i+1} \\ H^i(x,1) & \text{if } x \in X^{(i)}, t = 1 \end{cases}$$

So we have to attach infinitely many homotopies together on our CW-complex. We can do that because this whole CW-complex is a colimit of these finite skeletons, but we have to be clever about it.
We divide the interval into parts such that the next part is always two times smaller than the previous one. So we can go on up to infinity and always have an interval on which we can define the homotopy.

One has to check that H is continuous on $X^{(i)} \times I$, so it is continuous on $X \times I \cong \operatorname{colim} X^{(i)} \times I$.

Now we will define the maps H^i by induction:
We denote the end map that we will get $f_{n-1} := H^{n-1}|_{X\times\{1\}}$.
By induction assumption we know that this map is cellular, i.e. $f_{n-1}(X^{(i)}) \subset Y^{(i)}$, $i \leq n-1$.
We want to extend this homotopy to the next skeleton

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & X^{(n-1)} \arrow[r, "f_{n-1}"] \arrow[d] & Y^{(n-1)} \\ D^n \arrow[r] \arrow[ur, phantom, "\square"] & X^{(n)} \arrow[r, "f_{n-1}"'] & Y \end{tikzcd}$$

So we have a map of pairs $f_{n-1} : (D^n, S^{n-1}) \to (Y, Y^{(n-1)}$ which is homotopic rel S^{n-1} to a map $g : (D^n, S^{n-1}) \to (Y^{(n)}, Y^{(n-1)})$, since $(Y, Y^{(n)})$ is n-connected.
We glue g and the homotopy to get $H^n|_{X^{(n)}}$ which will start at f_{n-1} and end at some map that sends $X^{(n)}$ inside $Y^{(n)}$, so this will be cellular.
Since $X^{(n)} \hookrightarrow X$ is a cofibration, we extend H^n to $X \times I$ ■

1.6 Whitehead theorem

Proposition 1.6.1.
Let (Y, B) be an n-connected pair, (X, A) a relative CW-complex, rel $\dim(X, A) \leq n \leq \infty$. Then any map of pairs $f : (X, A) \to (Y, B)$ is homotopic to a map into B relative to A:

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, hook] \\ X \arrow[r, "f"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

If rel $\dim(X, A) < n$*, then the homotopy class of $(X \to B)$ is unique relative to A.*

Proof. We construct the needed homotopy by induction on skeletal filtration. Assume that X is obtained from A by attaching q-cells ($q \leq n$) which means we have the following pushout square:

$$\begin{tikzcd} \coprod \partial I^q \arrow[r] \arrow[d] & A \arrow[r, "f|_A"] \arrow[d] & B \arrow[d, hook, "i"] \\ \coprod I^q \arrow[r, "(\Phi_q)"'] \arrow[urr, dashed, "g"] \arrow[ur, phantom, "\square"] & X \arrow[r, "f"'] \arrow[ur, dashed, "G"'] & Y \end{tikzcd}$$

We look at the outer square. (Y, B) is n-connected by assumption. By one of the equivalences to n-connectedness, we can find a lift g as indicated for each cube I^q separately and then glue it together to the disjoint union.

- $\exists g : \coprod I^q \to B$ such that $i \circ g \simeq f \circ \Phi$ rel $\coprod \partial I^q$ via $H : (\coprod I^q) \times I \to Y$
- $G : X \to B$ is defined as $f|_A$ on A and g on $\coprod I^q$ (by the PO property)
- Now the upper triangle in the square on the right has to commute and the lower one as well at least up to homotopy. So we have to find a homotopy $h : X \times I \to Y$ between $i \circ G$ and f. Since the product with the unit interval commutes with pushouts the map h is again a pushout. So to define this map, it suffices to define $A \times I \to Y$ as the constant map:

$$\begin{tikzcd} A \times I \arrow[rr] \arrow[dr] & & Y \\ & A \arrow[ur, "f|_A"'] & \end{tikzcd}$$

  and $(\coprod I^q) \times I \xrightarrow{H} Y$.
  Those maps agree on $(\coprod I^q) \times I$.

<u>Induction step</u>: Suppose that we have constructed a lifting G_q up to homotopy on the q-th skeleton:

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & B \arrow[d, "i"] \\ X^{(q)} \arrow[r, "f"'] \arrow[ur, dashed, "G_q"'] & Y \end{tikzcd}$$

Thus $G_q|_A = f|_A$ and $i \circ G \overset{h_q}{\simeq} X^{(q)}$ rel A.
Since $X^{(q)} \hookrightarrow X^{(q+1)}$ is a cofibration, we can extend h_q to

$$H_q : X^{(q+1)} \times I \to Y$$

where $H_q|_{X^{(q+1)} \times \{0\}} = f|_{X^{q+1)}}$.

$$\begin{tikzcd}
X^{(q)} \arrow[r, "G_q"] \arrow[d, hook] & B \arrow[d] \\
X^{(q+1)} \arrow[r, "H_q|_{X^{(q+1)} \times \{1\}}"'] \arrow[ur, dashed, "G_{q+1}"'] & Y
\end{tikzcd}$$

By explanations before, $\exists G_{q+1} : X^{(q+1)} \to B$ such that $i \circ G_{q+1} \simeq_{\text{rel } X^{(q)}} H_q|_{X^{q+1}} \times \{1\} \simeq_{\text{rel } A} f|_{X^{(q+1)}}$.

For $n = \infty$ glue all homotopies using $X \cong \operatorname{colim}_q X^{(q)}$.

If rel $\dim(X, A) < n$, let $F_0, F_1 : X \to B$ be maps such that $F_j|_A = f|_A$, $i \circ F_j \overset{H_j}{\simeq} f$ rel A for $j = 0, 1$.
We already know that these maps F_0 and F_1 are homotopic as maps to Y. However, we want to prove here that they are homotopic as maps to B as well.
We do that by extending the homotopies that we have. Consider the following commutative square:

$$\begin{tikzcd}
X \times \partial I \bigcup_{A \times \partial I} A \times I \arrow[r, "(F_0, F_1)"] \arrow[r, dashed, shift right, "f"'] \arrow[d, hook] & B \arrow[d, "i"] \\
X \times I \arrow[r, "``H_0 * H_1''"'] \arrow[ur, dashed, "\exists h"'] & Y
\end{tikzcd}$$

where $H_0 * H_1$ is on $X \times [0, \frac{1}{2}]$ H_0 and $\overline{H}_1$ on $X \times [\frac{1}{2}, 1]$.
$(X \times I, X \times \partial I \bigcup_{A \times \partial I} A \times I)$ is a relative CW complex of relative dimension $= rel \dim(X, A) + 1 \leq n$.
Using the first part of the proposition $\exists h : X \times \to B$ such that $h|_{X \times \{0\}} = F_0$ and $h|_{X \times \{1\}} = F_1$ ■

Corollary 1.6.2.
Suppose that X is a CW-complex. Let $B \xrightarrow{g} Y$ be n-connected, $n \geq 0$.
If $\dim X < n$, then $[X, B] \xrightarrow{\cong} [X, Y]$ and if $\dim X = n$, then $[X, B] \twoheadrightarrow [X, Y]$.

Proof.
Let $(X, \emptyset)$ be a relative CW-complex. Then this has relative dimension equal to the dimension of X. If $\dim X \leq n$, then any map $X \to Y$ is homotopic to a map into B, thus we get a surjection $[X, B] \twoheadrightarrow [X, Y]$.
If $\dim X = rel \dim(X, \emptyset) < n$, then the second part of the proposition gives us uniqueness and thus injectivity. ■

Definition 1.6.3.
A map $g : Y \to Z$ is a <u>weak equivalence</u> if $\pi_0 Y \xrightarrow{\cong} \pi_0 Z$ and $\pi_i(Y, y) \underset{f_*}{\xrightarrow{\cong}} (Z, f(y))$ for all $i \geq 1$.

(The name we have so far used for this is ∞-connected)

Theorem 1.6.4. *(Whitehead Theorem)*
Let $f : Y \to Z$ be a map of CW-complexes.

1. *f is a homotopy equivalence $\Leftrightarrow$ f is a weak equivalence.*
2. *If $\dim Y \leq k$, $\dim Z \leq k$ and $f_* : \pi_q(Y, y) \xrightarrow{\cong} \pi_q(Z, f(y))$ for $q \leq k$, then f is a homotopy equivalence.*

Remark 1.6.5.
This theorem tells you, how much information is contained in these homotopy groups. In some time we will see that this information is in some sense already contained in the homology groups. Using this theorem, we see that for CW-complexes and thus for spaces that are homotopy equivalent to CW-complexes, the notion of homotopy equivalence is the same as the notion of isomorphisms on homotopy groups. By this we reduce the very hard question of homotopy theory, whether two spaces are homotopy equivalent to an algebraic question.

There is, however, in this reduction a minor drawback: it is not true that if we take two spaces and have, somehow, computed their homotopy groups and they turn out to be isomorphic as groups (for the first one, all other as abelian groups), that they are homotopy equivalent.
This is because the statement of the theorem is that, once you have a map f which induces isomorphisms on the homotopy groups <u>then</u> they are homotopy equivalent. It is not that this isomorphism of groups can be taken somewhat abstractly from somewhere.

So of course this is a reduction to an algebraic question, but perhaps in the end it just tells you how complicated homotopy groups in fact are and not how easy it is to do homotopy theory.

Proof.

1. $\Rightarrow$ Auge (this eye is the symbol for obvious because you can see with one eye that this is true)
   $\Leftarrow$ In the previous corollary we have seen that since f is ∞-connected $[X,Y] \xrightarrow{\cong} [X,Z]$ for all CW complexes X. By Yoneda lemma on hCW

   > (homotopy category of CW-complexes, which is the subcategory of $h\mathtt{Top}$ generated by CW-complexes and is also the same by the cellular approximation theorem and an exercise of Ex Sheet 2, as considering the category of CW complexes with cellular maps and cellular homotopies in an appropriate sense and considering the corresponding homotopy category)

   we see that Y and Z define the same Hom-functors on this category, so we see that the map f induces an isomorphism of these two functors and therefore f is an isomorphism. But an isomorphism in this category ($hCW \subset h\mathtt{Top}$) is a homotopy equivalence.

2. If f is k-connected, then $[Z,Y] \to [Z,Z]$ is surjective, so if we look at the identity in $[Z,Z]$, we can find a $g \in [Z,Y]$ such that $f \circ g = id$ in $[Z,Z]$.
   So we found a right inverse to f in hCW which (because it is an inverse to f) also induces an isomorphism on all the homotopy groups in the same range.
   By applying the same arguments we see that it also has a right inverse. So f has a right inverse and its right inverse has a right inverse and this implies that all these maps are isomorphisms and that the right inverse is just inverse to f and its inverse so its just f.
   Therefore f is an isomorphism in hCW.

■

Corollary 1.6.6.
A CW-complex X is contractible iff $\pi_0 X \simeq \{*\}$, $\pi_i(X,x) = 0$ for all i, x.

Example 1.6.7.
One can argue that S^∞ is contractible (without finding the homotopy we found in AT I).
It has a CW-complex structure: $S^n \hookrightarrow S^{n+1}$ should be $(S^\infty)^{(n)} \hookrightarrow (S^\infty)^{(n+1)}$.
Then $\pi_n(S^\infty) \cong \pi_n((S^\infty)^{(n+1)}) = \pi_n(S^{n+1}) = 0$

1.7 CW-Approximation

We have seen that for CW-complex the notion of a weak equivalence and the notion of a homotopy equivalence coincide but this is not true for arbitrary spaces.

So for an arbitrary space its homotopy groups do not give you as much information about its homotopy type as for CW-complexes.

However, if you are only interested in the homotopy groups of an abstract topological space, you can always reduce to the situation of a CW-complex.

Theorem 1.7.1.
Let $f : A \to Y$ be a k-connected map, $k \geq -1$.
Then for each $n > k$, $n \leq \infty$ (for $k = \infty$, $n = \infty$ because $\infty > \infty$), there exists a relative CW complex $A \hookrightarrow X$ which is obtained by only attaching cells in dimension $k+1, k+2, ..., n$ and there exists a map $X \to Y$ that makes the following diagram commute

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & Y \\ X \arrow[ur, "F"'] & \end{tikzcd}$$

and such that F is n-connected.

(If A is a CW-complex, then $A \hookrightarrow X$ is a subcomplex)

Proof.
Special cases: $n = 0$, $n = 1$ is an exercise (path-connected components and fundamental group)
Assume $n \geq 2$. Induction step: it suffices to treat $n = k+1$
Why can reduce to that? If $n \neq k+1$, then $n > k+1$. So first we solve the problem for $k+1$:

$$\begin{tikzcd} A \arrow[r, "k\text{-conn}", "f"'] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & Y \\ X^{(k+1)} \arrow[ur, "F", "(k+1)-\text{conn}"'] & \end{tikzcd}$$

now, since $n > k+1$, we can start

$$\begin{tikzcd} X^{(k+1)} \arrow[r, "F"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & Y \\ X^{(k+2)} \arrow[ur, "F'", "(k+1)-\text{conn}"'] & \end{tikzcd}$$

and if n is even bigger than $k+2$, you just continue to do this step. If $n = \infty$ one again has to glue these maps together (which we will not do here)

Thus it suffices to treat $n = k+1$. We reduce once again to the case of an inclusion:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, hook, "k\text{-}conn"'] & Cyl(f) \arrow[r, "k\text{-}conn"] & Y \\ X \arrow[ur, "\circlearrowleft" description] & & \end{tikzcd}$$

Thus we can assume that (Y, A) is a pair.

Now the proof works as follows: we fix a set of generators of $\pi_n(Y, A, a) = \{(D^n, S^{n-1}, *) \xrightarrow{\Phi_j} (X, A, a)\}_{j \in J}$
We are going to attach the cells of A along these maps. Since this is a pushout diagram by definition of CW-complexes (and X is actually filled in as the pushout to this diagram), we can define the map F indicated as a dashed line below as the canonical pushout map by fixing maps $A \to Y$ which

is f and $\coprod D^n \to Y$ which is given by these maps Φ_j:

$$\begin{tikzcd}
\coprod S^{n-1} \arrow[r, "\Phi_j|_{S^{n-1}}"] \arrow[d, hook] & A \arrow[r, hook, "f"] \arrow[d] & Y \\
\coprod D^n \arrow[r, "X_j"] \arrow[ur, phantom, "\square"] \arrow[rr, bend right, "\Phi_j"'] & X \arrow[ur, dashed, "F"] &
\end{tikzcd}$$

Here f is $(n-1)$-connected, so we want F to be n-connected.
We have a map of pairs:

$$(F, id_A) : (X, A) \to (Y, A)$$

which induces a map of LES:

$$\begin{tikzcd}
\pi_n(A) \arrow[r] \arrow[d, "\cong"] & \pi_n(X) \arrow[r] \arrow[d, "1."] & \pi_n(X,A) \arrow[r] \arrow[d, "2."] & \pi_{n-1}(A) \arrow[r] \arrow[d, "\cong"] \arrow[dr] & \pi_{n-1}X \arrow[r] \arrow[d, "3."] & 0 \\
\pi_n(A) \arrow[r] & \pi_n(Y) \arrow[r] & \pi_n(Y,A) \arrow[r] & \pi_{n-1}(A) \arrow[r] & \pi_{n-1}Y \arrow[r] & 0
\end{tikzcd}$$

So F sends $\{(D^n, S^{n-1}, *) \xrightarrow{X_j} (X, A, a)\} \twoheadrightarrow \{(D^n, S^{n-1}, *) \xrightarrow{\Phi_j} (X, A, a)\}_{j \in J}$.
But these maps Φ_j are the generators of the relative homotopy groups of the pair (Y, A), so $\pi_n(X, A) \twoheadrightarrow \pi_n(Y, A)$.

Because 2. is surjective by our construction $\implies$ 3. is injective (diagram chase)
Since we assume that $f : A \to Y$ is $n-1$-connected, this implies that the map $\pi_{n-1}A \twoheadrightarrow \pi_{n-1}Y$ is surjective. But because this map factors through 3. $\circ\ (\pi_{n-1}A \to \pi_{n-1}X)$, 3. is also surjective.
Thus, now 3. is an isomorphism.

By yet another diagram chase, 1. is surjective.

So F is n-connected. ■

Corollary 1.7.2. *(CW approximation)*
If Y is any topological space, then there exists a CW-complex X and a weak equivalence $X \to Y$.

Proof.
Take $A = \emptyset$ in the theorem. ■

Corollary 1.7.3.
Let Y be a CW-complex such that $\pi_i Y = 0$ for $0 \leq i \leq k$.
Then Y is homotopy equivalent to a CW-complex such that $X^{(k)} = \{*\}$

Proof.
We have the inclusion of a point into Y, $f : * \to Y$. Since we don't have any homotopy groups up to k, f is k-connected. By the theorem, we can find

$$\begin{tikzcd}
* \arrow[rr, "f"] \arrow[dr] & & X \\
& x \arrow[ur, "F"'] &
\end{tikzcd}$$

X is obtained from $*$ by attaching cells of dimension at least $k+1$ and therefore $X^{(k)} = \{*\}$ for X.
By the theorem, F is ∞-connected and thus a homotopy equivalence by the Whitehead theorem. ■

Remark 1.7.4.
This is sort of a reverse statement of what we have seen for spheres. An n-dimensional sphere has a CW-structure with the $(n-1)$-skeleton being a point and by e.g. the cellular approximation theorem it therefore has no homotopy groups in dimensions up to n because all the spheres would have to go cellularly to a point.
This corollary is an inverse statement because we have a CW-complex where there are up to homotopy no maps in degrees up to k in our CW-complex. Then we can change the CW-structure such that up to this degree there are no cells at all which is often of use for practical computations.

1.8 Excision for homotopy groups & applications

Reminder:
For homology groups, if we have an inclusion $A \overset{i}{\hookrightarrow} X$, then

$$H_*(X,A) \xrightarrow{\cong} \tilde{H}_*(X/A)$$

if e.g. i is a NDR.

This is however <u>not true</u> for π_*:

$$(D^2,S^1) \to (\underbrace{D^2/S^1}_{S^2},*)$$

On the right, we have the homotopy groups of the 2-dimensional sphere, what about the left hand side though? We can compute those homotopy groups using the LES of the pair:

$$\cdots \to \underbrace{\pi_1 D^2}_{=0} \to \underbrace{\pi_1 S^1}_{\cong\mathbb{Z}} \to \underbrace{\pi_1(D^2,S^1)}_{\cong\mathbb{Z}} \to \pi_0 S^1 \xrightarrow{\cong} \pi_0 D^2$$

And $\pi_i(D^2,S^1) = 0$ for $i \geq 2$. So we get that

$$\pi_1(D^2,S^1) \to \pi_1(S^2) \text{ is not injective.}$$
$$\pi_2(D^2,S^1) \to \pi_2(S^2) \text{ is not surjective.}$$

Excision for homotopy says that

$$\pi_i(X,A) \xrightarrow{\cong} \pi_i(X/A)$$

in a certain range of i.

Theorem 1.8.1. *(Excision for arbitrary topological spaces)*
Suppose we have a space $Y = Y_0 \cup Y_1$ covered by two open subsets which intersect non-trivially: $Y_{01} := Y_0 \cap Y_1 \neq \emptyset$.
Suppose (Y_1, Y_{01}) is p-connected, (Y_0, Y_{01}) is q-connected, where $p, q \geq 0$.
Then

$$\pi_i(Y_0, Y_{01}) \to \pi_i(Y, Y_1)$$

is an isomorphism for $i \leq p+q-1$ and is surjective for $i = p+q$.

$$\begin{tikzcd} Y_{01} \arrow[r, hook, "p-conn"] \arrow[d, hook, "q-conn"'] & Y_1 \arrow[d, hook] \\ Y_0 \arrow[r, hook] & Y \end{tikzcd}$$

Theorem 1.8.2. *(Excision for CW-complexes)*
Let $X = A \cup B$ be a CW-complexes covered by the two subcomplexes A and B (inclusions are cellular) and $C := A \cap B \neq \emptyset$. Suppose (A, C) is p-connected, (B, C) is q-connected, where $p, q \geq 0$. Then

$$\pi_i(A, C) \to \pi_i(X, B)$$

is an isomorphism for $i \leq p + q - 1$ and is surjective for $i = p + q$.

Lemma 1.8.3. *Excision for CW-complexes $\implies$ Excision for arbitrary topological spaces*

Proof. We will reduce the case of arbitrary topological spaces to the one of CW complexes. This is possible because

$$\begin{tikzcd}
C \arrow[rr, "p-conn"] \arrow[dr, "weq"] \arrow[dd, hook] & & A \arrow[dr, "weq"] & \\
& Y_{01} \arrow[rr, hook, "p-conn"] \arrow[dd, hook, "q-conn"] & & Y_1 \arrow[dd, hook] \\
B \arrow[dr, "weq"'] & & & \\
& Y_0 \arrow[rr, hook] & & Y
\end{tikzcd}$$

Since the map $C \to Y_{01}$ is a weak equivalence and $Y_{01} \hookrightarrow Y_1$ is p-connected, the composition is p-connected. Therefore we can find a CW-complex A which is obtained from C by attaching cells of dimension at least $p+1$, such that we have a weak equivalence $A \to Y_1$. The same can be done for B.

We define the CW-complex X as the pushout:

$$X := A \cup_C B$$

Since Y also is the pushout of its square, we get a map

$$A \cup_C B = X \to Y = Y_0 \cup_{Y_{01}} Y_1$$

So to finish the claim of being able to reduce to the situation of CW-complexes, we have to check that this map is a weak equivalence. For this we can use the higher-connectivity theorem. Since on the subspaces A and B we have weak equivalences and also on the intersection, this theorem tells us that the map above indeed is a weak equivalence. ■

Remark 1.8.4. One can also show that excision for arbitrary topological spaces implies the theorem on CW-complexes. For this one has to go from the CW-complexes to open subsets, so one has to explain that you can take some open neighbourhoods of A and B inside X and you can control their intersection.

Corollary 1.8.5. *("Quotient theorem")*
Let (X, A) be a p-connected CW-pair, let A be q-connected, where $p, q \geq 0$.
Then $\pi_i(X, A) \to \pi_i(X/A)$ is an isomorphism for $i \leq p + q$ and surjective for $i = p + q + 1$

Proof.

The cone of A is

$$CA := A \times I/(A \times \{1\})$$

CA is contractible.

$$\begin{tikzcd}
A \arrow[r, hook, "p-conn"] \arrow[d, hook, "q+1-conn"'] \arrow[dr, phantom, "\square" near end] & X \arrow[d] \\
CA \arrow[r] & X \cup CA
\end{tikzcd}$$

$CA \to X \cup CA$ is a CW-pair. $X \cap CA = A$.
Since CA is contractible, we get by the LES, that

$$\pi_i(CA, A) \xrightarrow{\cong} \pi_{i-1}A$$

for $i \geq 1$.

$$\pi_i(X, A) \to \pi_i(X \cup CA, CA)$$

is an isomorphism for $i \leq p+q$ and surjective for $i = p+q+1$ (by excision).
So now we have to relate $\pi_i(X \cup CA, CA)$ to the homotopy groups of the quotient. For this we can use (since CA is contractible, using LES) that

$$\pi_i(X \cup CA, CA) \cong \pi_i(X \cup CA)$$

Therefore instead we can relate $\pi_i(X \cup CA)$ to the quotient.

$$\begin{tikzcd}
A \arrow[r, hook, "p-conn"] \arrow[d, hook, "q+1-conn"'] \arrow[dr, phantom, "\square" near end] & X \arrow[d] \\
CA \arrow[r, "cofib"] \arrow[d, "h.e.q."'] \arrow[dr, phantom, "\square" near end] & X \cup CA \arrow[d] \\
* \arrow[r] & X/A
\end{tikzcd}$$

to argue why it has to be X/A in the corner down on the right, one can either use the categorical statement, that iff both squares are pushouts, then the outer square is as well. The pushout of the outer square, however, is exactly X/A.
Since $(X \cup CA, CA)$ is a relative CW-pair, thus the inclusion a cofibration and $CA \to *$ is a homotopy equivalence, by an exercise $X \cup CA \to X/A$ is a homotopy equivalence.
Thus we get:

$$\pi_i(X \cup CA) \xrightarrow{\cong} \pi_i(X/A)$$

One hast to be careful though and check that the composition

$$\pi_i(X, A) \to \pi_i(X \cup CA, CA) \xleftarrow{\cong} \pi_i(X \cup CA) \xrightarrow{\cong} \pi_i(X/A)$$

is the same composition given by the map, sending $\pi_i(X, A)$ to the homotopy groups of the quotient. ■

Corollary 1.8.6. *(Freudenthal's suspension theorem)*
Suppose X is a n-connected, pointed CW-complex. Then there are natural morphisms

$$\pi_i(X) \to \pi_{i+1}(\Sigma X)$$

which are isomorphisms for $i < 2n$ and surjective for $i = 2n$.
In particular,

$$\pi_i(S^{n+1}) \to \pi_{i+1}(S^{n+2})$$

is an isomorphism for $i \leq 2n$, so

$$\underbrace{\pi_2(S^2)}_{\cong\mathbb{Z}} \xrightarrow{\cong} \pi_3(S^3) \xrightarrow{\cong} \pi_4(S^4) \xrightarrow{\cong} \dots \xrightarrow{\cong} \pi_n(S^n) \xrightarrow{\cong} \dots$$

Proof.

$$\Sigma X = C_+X \cup C_-X, \quad C_+X \cap C_-X = X$$

Using the same arguments (using the LES and that the cones are contractible), we get that $(C_\pm X, X)$ is $(n+1)$-connected. So

$$\begin{tikzcd} \pi_{i+1}(C_+X, X) \arrow[r] \arrow[d, "\cong"'] & \pi_{i+1}(\Sigma X, C_-X) \\ \pi_i(X) \arrow[r, dashed] & \pi_{i+1}(\Sigma X) \arrow[u, "\cong"'] \end{tikzcd}$$

By excision we are done for $i + 1 \leq 2n$ ∎

Corollary 1.8.7.

$$\deg : \pi_n(S^n) \to H_n(S^n) \cong \mathbb{Z}$$

is an isomorphism for all n.

Proof.
By AT I : $\deg(id) = 1$.
We have seen before, that $\pi_n(S^n) \cong \mathbb{Z}$, generated by id_{S^n}. ∎

Definition 1.8.8.
We can define stable homotopy groups from X to Y by

$$[X, Y]^{st} := \operatorname*{colim}_n [\Sigma^n X, \Sigma^n Y)] \cong [\Sigma^k X, \Sigma^k Y]$$

where the latter isomorphism is given by the Freudenthal suspension theorem for $k >> 0$.
In particular, one can define the stable homotopy groups of any CW-complex as

$$\pi_i^{st}(X) := \operatorname*{colim}_k \pi_{i+1}(\Sigma^k X)$$

Corollary 1.8.9. *(Hurewicz theorem)*
If a topological space X is $(n-1)$-connected, $n \geq 2$, then $\tilde{H}_i(X) = 0$ for $i \leq n-1$ and $\pi_n(X) \xrightarrow[\cong]{h} H_n(X)$.

For a pair, if (X, A) is $(n-1)$-connected and A is simply path-connected, then $H_i(X, A) = 0$ for $i \leq n-1$. There is a relative Hurewicz-morphism

$$h : \pi_n(X, A) \xrightarrow{\cong} H_n(X, A)$$

which is an isomorphism.

Remark 1.8.10.
$\pi_1(X) \twoheadrightarrow H_1(X) \cong \pi_1(X)/_{[\pi_1(X), \pi_1(X)]}$
This is not just any surjection but rather the one, that identifies $H_1(X)$ with the abelianization of π_1.

Proof.
First of all, let's reduce the relative statement to the absolute one for CW-pairs.
Let (X, A) be a $(n+1)$-connected CW-pair, A is 1-connected. We have a map

$$\pi_i(X, A) \xrightarrow{\cong} \pi_i(X/A)$$

which is an isomorphism for $i \leq n$ by the quotient theorem. Also for homology

$$H_i(X, A) \xrightarrow{\cong} \tilde{H}_i(X/A)$$

which we can always do, because A is a NDR inside X.
So instead of considering this pair, we can consider the quotient $Y := X/A$. Y is an $(n-1)$-connected CW-complex. So if we can show that for the quotient the homology and homotopy groups up to degree n are isomorphic, then we are done.

Since (X, A) is $(n+1)$-connected, we know that the homotopy groups up to degree $n-1$ are 0. Left to show is only that the relative homology group in degree n of the quotient is isomorphic to the homotopy group of Y in degree n and that the relative homology groups up to degree $n-1$ are zero.
This reduces the relative statement for CW complexes to the absolute statement of CW-complexes.

Let Y be an $(n-1)$-connected CW-complex. By CW-approximation, we can assume that $Y^{(n-1)} = \{*\}$. It follows that $Y^{(n)} \cong \bigvee_{\alpha \in \mathcal{A}} S^n$. $Y^{(n+1)}$ is obtained by attaching $(n+1)$-cells e_β, $\beta \in \mathcal{B}$.

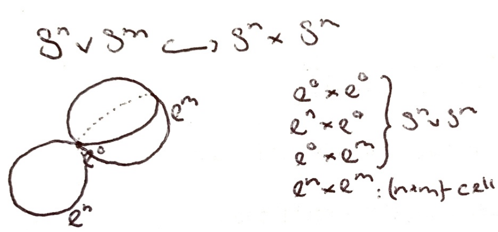
The claim is that

$$\begin{array}{ccc} \pi_n\left(\bigvee_{\alpha \in \mathcal{A}} S^n\right) & \xrightarrow{\qquad\qquad\cong\qquad\qquad} & \bigoplus \mathbb{Z}[S^n \overset{i_\alpha}{\hookrightarrow} \bigvee_{a \in \mathcal{A}} S^n] \\ \cong & \text{by compactness argument} & \cong \\ \operatorname{colim}_{A' \subset A \text{ finite}} \pi_n\left(\bigvee_{\alpha \in \mathcal{A}'} S^n\right) & \xrightarrow{\cong} & \operatorname{colim}_{A' \subset A \text{ finite}} \bigoplus \mathbb{Z}[i_\alpha] \end{array}$$

Thus it suffices to consider A' finite. By an exercise in AT I the inclusion

$$\bigvee_{\alpha \in \mathcal{A}'} S^n \hookrightarrow \prod_{\alpha \in \mathcal{A}'} S^n$$

is a closed subspace. In our case it is a closed CW-subcomplex, so this is a pair that is $(2n-1)$-connected.
The last part is due to the CW-structure of $\prod_{\alpha \in \mathcal{A}'} S^n$, since this is obtained by attaching $2n, 3n, 4n, \ldots$-cells:



Thus

$$\pi_n\left(\bigvee_{\alpha \in \mathcal{A}'} S^n\right) \xrightarrow{\cong} \pi_n\left(\prod_{\alpha \in \mathcal{A}'} S^n\right) \cong \bigoplus_{\alpha \in \mathcal{A}'} \pi_n(S^n) \cong \bigoplus_{\alpha \in \mathcal{A}'} \mathbb{Z}$$

Now we can look at the LES of the homotopy groups of the pair. $(Y, Y^{(n+1)})$ is $(n+1)$-connected and therefore

$$\pi_n(Y) \xleftarrow{\cong} \pi_n(Y^{(n+1)})$$
$$H_n(Y) \leftarrow H_n(Y^{(n+1)})$$

Then

$$\begin{tikzcd}
\pi_{n+1}(Y^{(n+1)}, Y^{(n)}) \arrow[r] \arrow[d, "\cong"] & \pi_n Y^{(n)} \arrow[r] \arrow[d, "\cong"] & \pi_n Y^{(n+1)} \arrow[r] \arrow[ddd, bend left, "h"', "\text{Hurewicz morphism}"] & 0 \\
\pi_{n+1}(\underbrace{Y^{(n+1)}/Y^{(n)}}_{=\bigvee_\beta S^{n+1}}) \arrow[d, "\cong"] & \bigoplus_{\alpha \in J'} \mathbb{Z} i_\alpha & & \\
\bigoplus_\beta \mathbb{Z} e_\beta & & & \widehat{H}_n(Y^{(n+1)}/Y^{(n)}) \arrow[d, phantom, "\cong"] \\
H_{n+1}(Y^{(n+1)}, Y^{(n)}) \arrow[r] \arrow[u, "\cong"] & H_n Y^{(n)} \arrow[r] \arrow[uu, "\cong"] & H_n Y^{(n+1)} \arrow[r] & 0 = H_n(Y^{(n+1)}, Y^{(n)})
\end{tikzcd}$$

Finally to reduce the case of an arbitrary topological space (or a pair) we should use the CW-approximation. CW-approximation - by definition- gives us some CW-complexes that have the same homotopy groups as the space we have at hand. In the Hurewicz theorem, homotopy groups are also homology groups, but we need:

Proposition 1.8.11.
If $f : X \to Y$ is a weak equivalence, then

$$H_*(X, A) \underset{\cong}{\xrightarrow{f_*}} H_*(Y, A)$$
$$H^*(Y, A) \underset{\cong}{\xrightarrow{f^*}} H^*(X, A)$$

for all $A \in$ Ab

Proof.
First, reduce to the inclusion using the mapping cylinder construction.

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[rr, bend right, "weq"'] & Cyl(f) \arrow[r, "heq"] & Y
\end{tikzcd}$$

$\implies$ $X \hookrightarrow Cyl(f)$ is a weak-equivalence, thus

$$\begin{tikzcd}
\pi_i(X) \arrow[r, hook] \arrow[rr, bend right, "\cong"'] & \pi_i(Cyl(f)) \arrow[r, "\cong"] & \pi_i(Y)
\end{tikzcd}$$

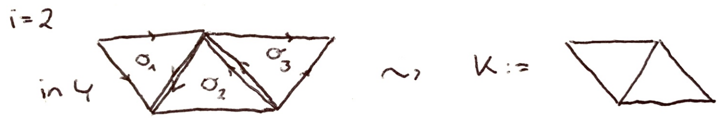
And the homotopy equivalence induces an isomorphism on the homology groups.

Secondly,

$$\begin{tikzcd}
\pi_i X \arrow[r] \arrow[d] & \pi_i Y \arrow[r] \arrow[d] & \pi_i(Y, X) \arrow[r] \arrow[d] & \pi_{i+1} X \arrow[r] \arrow[d] & \cdots \\
H_i X \arrow[r] & H_i Y \arrow[r] & H_i(Y, X) \arrow[r] & H_{i+1} X \arrow[r] & \cdots
\end{tikzcd}$$

We need that $H_i(Y, X) = 0$. It suffices to show that if (Y, X) is n-connected, then $H_i(Y, X) = 0$ for $i \leq n$.

So let (Y, X) be an n-connected pair and we want $H_i(X,Y) = 0$ for $i \leq n$.
We start with a class $\overline{\alpha} \in H_i(X,Y)$ and we can lift it to $\alpha = \sum_i n_i \sigma_o$, where $\sigma_{i \in I} : \Delta^j \to Y$.
$\partial\alpha$ is a chain in X (since $\overline{\alpha}$ defines a non-zero class in the homology)
We define a CW-complex $K := \coprod_{i \in I} \Delta^j / \sim$, where $\sim$ defines a relation which glues $(i-1)$-dimensional faces of Δ^j's that are sent to Y identically under different σ_i:



So we get a map σ from K to Y:

$$\begin{tikzcd} K \arrow[r, "\sigma"] & Y \\ \Delta^j \arrow[u] \arrow[ur, "\sigma_i"'] & \end{tikzcd}$$

We define a subcomplex $W \subset K$ consisting of $(i-1)$-faces of Δ^i that appear non-trivially in $\partial\alpha$. By assumption

$$\begin{array}{ccc} K & \xrightarrow{\sigma} & Y \\ \cup & & \cup \\ L & \longrightarrow & X \end{array}$$

The inclusion $X \subset Y$ is n-connected and $l \subset K$ is a pair of CW-complexes and the relative dimension is i which is $\leq n$.
Therefore $\exists g : K \to X$, $g \simeq \sigma$ rel L. An important property of the map σ is that we have α' in $C_i^{Sing}(K) \xrightarrow{\sigma} C_i(Y)$ that is sent onto α via σ (and $\overline{\alpha} \in H_i(K, L)$), so $\overline{\sigma}_*(\overline{\alpha}') = \overline{\alpha} = 0$ because $\overline{\sigma}_*(\overline{\alpha}') = Im(g_*(\overline{\alpha}')$ along $H_*(\underset{=0}{X}, X \to H_*(Y, X)$.

Finally if $f_* : H_*(X) \xrightarrow{\simeq} H_*(Y)$, then

$$H_*(X, A) \underset{\cong}{\xrightarrow{f_*}} H_*(Y, A)$$

$$H^*(Y, A) \underset{\cong}{\xrightarrow{f^*}} H^*(X, A)$$

for any $A \in \mathsf{Ab}$ by the universal coefficient theorems. ■

■

CHAPTER 2

Homotopy pullback & Homotopy pushouts

`Top` and h`Top` do not fulfil nice categorical properties (`Top` in regard to homotopy). If we regard a pushout or pullback diagram and start to substitute some of the spaces by something homotopic to them, the resulting pushout or pullback of the new diagram does not have to coincide with the previous one. Not even up to homotopy. In the very first section of this chapter, we will see examples of where such a substitution fails to coincide with pushouts and pullbacks.

In the following part, we aim to find a way around this problem. The method which we intend to apply is enriching the category `Top` itself by an additional structure: homotopy. The result will be a 2-category, a term which we also make precise.

2.1 Problems with categorical constructions in `Top` and h`Top`

- `Top` as a category does not "know" about homotopies

  Example 2.1.1.
  We start by regarding the following pushout diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \arrow[ddr, bend left, "t_2"] & \\ * \arrow[r] \arrow[drr, bend right, "t_1"'] & * \arrow[ul, phantom, "\square"] \arrow[dr, dashed, "t_1 t_2"] & \\ & & T \end{tikzcd}$$

  Now suppose we substitute the point $*$ with something homotopic to it: CX is contractible and the pushout will now be the suspension ΣX.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & CX := X \times I / X \times \{1\} \arrow[d] \\ * \arrow[r] & \Sigma X \arrow[ul, phantom, "\square"] \end{tikzcd}$$

  From the homotopy perspective those two diagrams should be the same. Therefore in the following diagram all horizontal maps are homotopy equivalences:

$$\begin{tikzcd} X \arrow[dd] \arrow[dr] & & X \arrow[ll, "id"'] \arrow[dd] \arrow[dr] & \\ & * & & \Sigma X \arrow[ll] \\ * & & * \arrow[ll, "id"] & \end{tikzcd}$$

  This cannot be true, however, the suspension is in general not contractible!

We get the same problem when regarding pullbacks, e.g.

$$\begin{tikzcd} * \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square" near start] & * \arrow[d, "x"] \\ * \arrow[r, "x"'] & X \end{tikzcd}$$

For again replacing the point with something contractible to it, we choose the pullback $\Pi_x X$ of the diagram where instead of a point in the upper right corner we put the exponential object (which exists because I is a very nice object)

$$\begin{tikzcd} \Pi_x X \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square" near start] & \Pi X \arrow[d, "ev_0"] \\ * \arrow[r, "x"'] & X \end{tikzcd} \qquad \Pi X := X^I = \{j : [0,1] \to X\}$$

The new pullback $\Pi_x X = \{j \mid j(0) = x\}$ is the set of all paths starting at x.
This space is contractible (even though ΠX in general is not) because all paths start at the same point x. Thus you can contract them simultaneously by adjusting the speed in which you go through them:

$$\begin{aligned} \Pi_x X \times I &\to \Pi_x X \\ (j, \tau) &\mapsto j_\tau(t) = j(t \cdot \tau) \end{aligned}$$

For $\tau = 0$ this gives $\overline{x}$, for τ_1 it is $id_{\Pi_x X}$. So $\Pi_x X$ is contractible.

However, if we now replace in the first diagram the point in the upper right corner by $\Pi_x X$, which as we now have checked is in fact homotopic to it and now construct the pullback, we get:

$$\begin{tikzcd} \Omega_x X \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square" near start] & \Pi_x X \arrow[d, "ev_1"] \\ * \arrow[r, "x"'] & X \end{tikzcd}$$

where $\Omega_x X$ is the set of loops at x in X.
$\Omega_x X$ is not always contractible, $\pi_0(\Omega_x X) \cong \pi_1(X, x)$.

The upshot of these examples is: pullbacks/pushouts in `Top` do not preserve homotopy equivalences.

- h`Top`$_*$ does not have some simple pushouts

  Example 2.1.2.
  For this example we will use some properties that we have not yet proven.
  First we claim that there is a fiber bundle

$$S^1 \hookrightarrow \mathbb{R}P^3 \to S^2$$

This is not hard to construct because $\mathbb{R}P^3 \cong SO(3)$:

$$\begin{tikzcd} & S^1 \arrow[r, hook] \arrow[dd, dash, "\cong"] & \mathbb{R}P^3 \arrow[r, "p"] \arrow[dd, dash, "\cong"] & S^2 \arrow[dd, dash, "\cong"] \\ U(1) \arrow[ur, dash, "\cong"] \arrow[dr, dash, "\cong"'] & & & \\ & SO(2) \arrow[r] & SO(3) \arrow[r] & SO(3)/SO(2) \end{tikzcd}$$

p is a fibration, $T \in \mathtt{Top}_*$.
Thus we get (we haven't yet defined what a fibration is nor that we get a LES)

$$[T, S^1] \to [T, \mathbb{R}P^3] \to [T, S^2]$$

Now a pushout in $h\mathtt{Top}_*$ will not exist. Assume that it does:

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, "z^2"'] & * \arrow[d] \arrow[ddr, bend left, "x"] & \\ S^1 \arrow[r] \arrow[drr, bend right, "\gamma"'] & T \arrow[ul, phantom, "\square" near start] & \\ & & X \end{tikzcd}$$

where $\gamma \in \pi_1(X, x)$. The fundamental group of S^1 is $\mathbb{Z}$, as we know. So the map z^2 induces a map on $\mathbb{Z}$ and corresponds as such to the map doubling the generating element.
The commutativity of the outer square says:

$$\gamma \circ z^2 = x \circ * \Leftrightarrow \gamma^{\cdot 2} = e \text{ in } \pi_1(X, x)$$

Therefore T should classify 2-torsion elements in π_1

$$[T, X] \cong \{j \in \pi_1(X, x) \mid j^2 = e\}$$

Unfortunately such a space T does not exist as an object in $h\mathtt{Top}$. To see that we now use the SES from above:

$$\begin{tikzcd} {[T, S^1]} \arrow[r] \arrow[d, equal] & {[T, \mathbb{R}P^3]} \arrow[r] \arrow[d, equal] & {[T, S^2]} \arrow[d, equal] \\ 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \end{tikzcd}$$

The upper row is exact, but the lower one is not. Thus T does not exist.

Idea as to how to fix these problems: include homotopies as piece of data in $\mathtt{Top}$ - develop some language to deal with this.

"Example"

Suppose we have a commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[ddr, bend left, "y_2"] & \\ * \arrow[drr, bend right, "y_1"'] & & \\ & & Y \arrow[from=2-3, to=3-2, phantom, "H"] \end{tikzcd}$$

and its a data of the homotopy $H : X \times I \to Y$ between the two compositions:

$$H|_{X \times \{0\}} = \overline{y_1}$$
$$H|_{X \times \{1\}} = \overline{y_2}$$

So here we see what kind of object classifies such an H:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & * \arrow[d, "t"] \arrow[ddr, bend left] & \\
* \arrow[r, "m"'] \arrow[drr, bend right] & \Sigma X \arrow[dr, dashed, "\exists"] & \\
& & Y
\end{tikzcd}
\qquad\qquad H : X \times I \to Y \leftrightsquigarrow \Sigma X \to Y$$

This is not a commutative square but there exists a precise homotopy that makes it commute. Given the homotopy H in the background we get a reasonably unique map as indicated by the dotted map.
So having this additional data really changes the outcome of the computation of the pushout.

We will show that this is, in fact, a homotopy pushout square. So this is really the right object to put there.

As a dual example:

$$\begin{tikzcd}
Y \arrow[drr, bend left] \arrow[ddr, bend right] \arrow[dr, phantom, "H"] & & \\
& & * \arrow[d, "x"] \\
& * \arrow[r, "x"'] & X
\end{tikzcd}$$

So

$$H : Y \times I \to X \leftrightsquigarrow Y \to X^I = \Pi X$$
$$H|_{Y \times \{0\}} = x = H|_{Y \times \{1\}} \leftrightsquigarrow Y \to \Omega_x X$$

and the following diagram will be a homotopy pullback square:

$$\begin{tikzcd}
\Omega_x X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square", very near start] & * \arrow[d] \\
* \arrow[r] & X
\end{tikzcd}$$

<u>Problem:</u> Concatenation of homotopies is not associative

2.2 Elements of 2-category theory

The next step will now be to make this notion of "a homotopy in the background" and "homotopy pullback square" precise. So in particular, we start with adding the new data to any category and receive a 2-category.

Definition 2.2.1.
A <u>2-category</u> C consists of

- objects (set/class) $Ob(X)$
- for every $x, y \in Ob(C)$: $HOM_C(x, y)$ a category with
  - objects: 1-morphisms $x \to y$

  - morphisms: 2-morphisms

- law of composition: for every $x, y, z \in Ob(C)$:

$$\underbrace{HOM(y,z) \times HOM(x,y)}_{\text{product of categories}} \xrightarrow{\circ_{x,y,z}} HOM(x,z)$$

  (a product of categories has pairs as objects as well as for morphisms)

- for every object $x \in Ob(C)$ an identity morphism $id_X \in HOM_C(x,x)$

that satisfies the following conditions:

- associativity of composition:

$$\begin{tikzcd}
HOM(z,t) \times HOM(y,z) \times HOM(x,y) \arrow[r, "id \times \circ_{x,y,z}"] \arrow[d, "\circ_{y,z,t} \times id"'] \arrow[dr, phantom, "\circlearrowleft"] & HOM(z,t) \times HOM(x,z) \arrow[d, "\circ_{x,z,t}"] \\
HOM(y,t) \times HOM(x,y) \arrow[r, "\circ_{x,y,t}"'] & HOM(x,t)
\end{tikzcd}$$

- neutral element $HOM(x,x)$:

$$\begin{tikzcd}
HOM(x,y) \times HOM(x,x) \arrow[r, "\circ_{x,x,y}"] \arrow[dr, "pr_1"'] & HOM(x,y) \\
 & HOM(x,y) \arrow[u, "id \times id_X"'] \arrow[ul, phantom, "\circlearrowleft"]
\end{tikzcd}$$

  and dually for neutrality on the left.

A 2-category C is a <u>(2,1)-category</u> if $HOM_C(x,y)$ is a groupoid.

Example 2.2.2.

1. `Cat` is a 2-category where $HOM_{\mathtt{Cat}}(C,D)$ has functors as objects and as morphisms natural transformations.

2. `Gpd` $\subset$ `Cat` is a (2,1)-category.

   Every natural tranformation $\eta F \implies G$, for $G, F : C \to D$
   is an isomorphism if C, D are groupoids.

In every 2-category C we have different types of compositions:
For $x, y, z \in Ob(C)$:

- Horizontal composition:

$$\begin{tikzcd}
x \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "g"'] \arrow[r, phantom, "\Downarrow \eta"] & y \arrow[r, "k"] & z
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
x \arrow[r, bend left=60, "k \circ f"] \arrow[r, bend right=60, "k \circ g"'] \arrow[r, phantom, "\Downarrow k \circ \eta"] & z
\end{tikzcd}$$

  Sometimes, because $k \circ \eta$ is a composition of a 1-morphism with a 2-morphism it is denoted differently. However, here we will not do that.

- Vertical composition:

$$\begin{tikzcd}
x \arrow[rr, bend left=60, "f"] \arrow[rr, "g"'] \arrow[rr, bend right=60, "s"'] & \arrow[phantom, u, "\Downarrow q"] \arrow[phantom, d, "\Downarrow \xi"] & y
\end{tikzcd}$$

  where $\xi \circ q : f \implies s$.

Example 2.2.3.

3. Let A be an abelian category (e.g. $\mathtt{Ab}$, $\mathtt{Mod}_R$), then $Ch(A)$ is a (2,1)-category:
   Define for $C_*, D_* \in Ch(A)$ the morphism $HOM(C_*, D_*)$ by

   - objects: 1-morphisms (chain morphisms):

   $$\{(f_n : C_n \to D_n)_{n\in\mathbb{Z}} \mid d \circ f_n = f_{n-1} \circ d\}$$

   - morphisms = 2-morphisms (chain-homotopies):

   $$Mor((f_n),(g_n)) := \{h_n : C_n \to D_{n+1} \mid h_{n-1} \circ d + d \circ h_n = f_n - g_n\}$$

   $$\begin{tikzcd}
   & C_n \arrow[r] \arrow[dl, "h_n"'] & C_{n-1} \arrow[dl, "h_{n-1}"] \\
   D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1}
   \end{tikzcd}$$

   We have to check that this is a category, so we have to understand how to compose two homotopies.

   $$(f_n) \overset{h}{\Rightarrow} (g_n) \overset{H}{\Rightarrow} (r_n)$$
   $$(H \circ h)_n = H_n + h_n$$

   The result of the so defined composition is again a chain homotopy:

   $$\begin{aligned}(H \circ h)_{n-1} \circ d + d \circ (H \circ h)_n &= (H_{n-1} + h_{n-1}) \circ d + d \circ (H_n + h_n) \\ &= H_{n-1} \circ d + h_{n-1} \circ d + d \circ H_n + d \circ h_n \\ &= r_n - g_n + g_n - f_n = r_n - f_n\end{aligned}$$

4. (Main example): $\mathtt{Top}$ and $\mathtt{Top}_*$:

   - Objects: Topological spaces
   - Morphisms: For $X, Y \in \mathtt{Top}$: $HOM(X,Y)$ consists of
     - 1-morphisms: $f : X \to Y$ continuous maps
     - 2-morphisms:

     $$Mor(f,g) = \{H : X \times I \to Y \mid H_{X\times\{0\}} = f,\ H_{X\times\{1\}} = g\}/\sim$$

     where the equivalence relation is given by $H_0 \sim H_1$ if:

     $$\begin{aligned}&\exists F : (X \times I) \times I \to Y \\ &F|_{X\times I\times\{0\}} = H_0 \\ &F|_{X\times I\times\{1\}} = H_1 \\ &F(x,0,\tau) = f(x), \quad F(x,1,\tau) = g(x)\end{aligned}$$

Lemma 2.2.4.
Concatenation of homotopies gives $HOM(X, Y)$ a structure of a groupoid.

Proof.

$$
\begin{tikzcd}
x \arrow[r, bend left=60, "f"] \arrow[r, "g"] \arrow[r, bend right=60, "k"'] & y
\end{tikzcd}
\qquad \text{with } H_1 : f \Rightarrow g, \; H_2 : g \Rightarrow k
$$

Define the composition as their concatenation:

$$(H_2 \circ H_1)(t) = \begin{cases} H_1(2t), & t \in [0, 1/2] \\ H_2(2t-1) & t \in [1/2, 1)] \end{cases}$$

We need to check that $\circ$ respects equivalence relation:

$$H_1 \sim H_1' \implies H_2 \circ H_1 \sim H_2 \circ H_1'$$

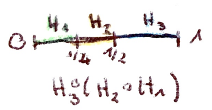
and that it is associative:
Let $a \in (0, 1)$. Define a new kind of concatenation

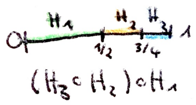
$$H_2 \circ_a H_1 := \begin{cases} H_1(\frac{1}{a}t), & t \in [0, a] \\ H_2(\frac{1}{1-a}(t-a)), & t \in [a, 1] \end{cases}$$

We want to show that these concatenations coincide for different a's. Let therefore $a, b \in (0, 1)$. To show that $H_2 \circ_a H_1 \overset{F}{\sim} H_2 \circ_b H_1$, define F:

$$F(x, t, \tau) := \begin{cases} H_1(\frac{1}{a+(b-a)\tau}t), & t \in [0, a + (b-a)\tau] \\ \dots \end{cases}$$
$$= H_2 \circ_{a+(b-a)\tau} H_1$$

Now we can see that the concatenation is associative:





Finally $H \in Mor(f, g) \rightsquigarrow \overline{H} : g \implies f$, $H \circ \overline{H} \sim const_g$,

$$F(x, t, \tau) := \begin{cases} H(x, (1-\tau)2t) & t \in [0, 1/2] \\ \overline{H}(x, (1-\tau)(2t-1), & t \in [1/2, 1] \end{cases}$$

■

Remark 2.2.5.
What we have just done is very similar to what we did in AT I. This is due to the fact that if Y^X exists, then $HOM(X, Y) \cong \Pi_1(Y^X)$, the fundamental groupoid of the exponential object.
The point of what we just did, is that we can define $HOM(X, Y)$ even if the exponential object does not exist.

Definition 2.2.6.
A <u>2-commutative diagram</u> is a (2,1)-category C with

- objects X_0, X_1, X_2
- 1-morphisms: f_{01}, f_{02}, f_{12}
- 2-morphism: $H_{012} : f_{01} \circ f_{12} \implies f_{02}$

$$\begin{tikzcd}
 & X_1 \arrow[dd, "f_{12}"] \\
X_0 \arrow[ur, "f_{01}"] \arrow[dr, "f_{02}"'] \arrow[r, phantom, "H_{012}"] & {} \\
 & X_2
\end{tikzcd}$$

A 2-commutative square consists of

- four 1-morphisms u, v, f, g
- and a 2-morphism $H : g \circ v \implies f \circ u$.

$$\begin{tikzcd}
A \arrow[r, "v"] \arrow[d, "u"'] & B \arrow[d, "g"] \arrow[dl, phantom, "\Swarrow H"] \\
C \arrow[r, "f"'] & D
\end{tikzcd}$$

Remark 2.2.7.
One can think of a 2-commutative square as of two triangles:

$$\begin{tikzcd}
A \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, "g \circ v"] & B \arrow[d, "g"] \arrow[dl, phantom, "H"] \\
C \arrow[r, "f"'] \arrow[ur, phantom, "\circlearrowleft"] & D
\end{tikzcd}$$

where the lower triangle commutes in the sense that the homotopy that makes it commute can be chosen to be trivial and the upper triangle commutes up to homotopy.

If we want to expand our diagrams we therefore have more conditions.

Definition 2.2.8.
A <u>2-commutative 3-simplex</u> has

- objects X_0, X_1, X_2, X_3
- 1-morphisms $f_{ij} : X_i \to X_j$, $i < j$
- 2-morphisms H_{ijk}, $i < j < k$

$$\begin{tikzcd}
 & X_3 & \\
X_0 \arrow[ur, "f_{03}"] \arrow[rr, "f_{02}"] \arrow[dr, "f_{01}"'] & & X_2 \arrow[ul, "f_{23}"'] \\
 & X_1 \arrow[uu, "f_{13}"] \arrow[ur, "f_{12}"'] &
\end{tikzcd}$$

in the groupoid $HOM(X_0, X_3)$ there are four objects: $f_{23} \circ f_{12} \circ f_{01}$, $f_{23} \circ f_{02}$, $f_{13} \circ f_{01}$, f_{03}.
Between those objects there are homotopies:

$$\begin{tikzcd}
f_{23} \circ f_{12} \circ f_{01} \arrow[r, Rightarrow, "f_{23} \circ H_{012}"] \arrow[d, Rightarrow, "H_{123} \circ f_{01}"'] & f_{23} \circ f_{02} \arrow[d, Rightarrow, "H_{023}"] \\
f_{13} \circ f_{01} \arrow[r, Rightarrow, "H_{013}"'] & f_{03}
\end{tikzcd}$$

For a 2-commutative 3-simplex, this square commutes.

Remark 2.2.9.
We will not formally define a 2-commutative diagram, but a hint on what it should be:
A 2-commutative diagram in C has objects, 1-morphisms and 2-morphisms and whenever there is a 3-simplex this should be commutative.

Definition 2.2.10.
Let G be a (2,1)-category.
The homotopy category of G is hG:

- Objects: = Objects of G
- Morphisms:

$$Mor_{hG}(x, y) := \pi_0(HOM_G(x, y)) =: [x, y]$$

$\pi_0(HOM_G(x, y))$ is the set of objects of a groupoid up to isomorphism.

Example 2.2.11. h`Top`

Definition 2.2.12.
A 1-morphism $f : X \to Y$ in G is an equivalence if $[f]$ in hG is an isomorphism.
(Alternatively, $\exists g : y \to x$, $\exists g \circ f \implies id_x$, $\exists f \circ g \implies id_y$)

Example 2.2.13.

- in `Top`: equivalences = homotopy equivalences
- in `Gpd`: equivalences = equivalences of categories

Definition 2.2.14.
Let G be a (2,1)-category.
A weak (respectively strict) 2-pullback of

$$\begin{tikzcd} & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

in G consists of (X, u, v, H)

$$\begin{tikzcd} X \arrow[r, "v"] \arrow[d, "u"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] \arrow[ur, phantom, "\Swarrow H"] & B \end{tikzcd}$$

that satisfies the following conditions:

1. Existence:

$\exists \alpha : T \to X$, $h_u : u \circ \alpha \implies u'$, $h_v : v \circ \alpha \implies v'$:

$$\begin{tikzcd}
T \arrow[dr, "\alpha"] \arrow[drr, bend left, "v'"] \arrow[ddr, bend right, "u'"'] & & \\
& X \arrow[r, "v"] \arrow[d, "u"'] & C \arrow[d, "g"] \arrow[dl, phantom, "\Swarrow H"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

$H' : g \circ v' \implies f \circ u'$.
This square has to commute in the following way:

$$\begin{tikzcd}
T \arrow[dr, "\alpha"] \arrow[drr, bend left, "v'"] \arrow[ddr, bend right, "u'"'] & & \\
& X \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, "g \circ v"] \arrow[r, phantom, shift right=4ex, "\circlearrowleft"] & C \arrow[d, "g"] \arrow[dl, phantom, "\Swarrow H"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

as well as - for the outer square including T:

$$\begin{tikzcd}
T \arrow[d] \arrow[drr] \arrow[ddr] \arrow[ddrr] & & \\
X \arrow[rr] \arrow[dr] \arrow[drr] & & C \arrow[d] \\
& A \arrow[r] & B
\end{tikzcd}$$

In the latter diagram one can clearly see several simplices. Since we have chosen $X \to B$ to be $g \circ v$ and $g \circ v'$, we have trivial homotopies in the faces XBC and TBC, so there is no condition on the homotopies h_u, h_v, H, H' coming from the simplex $TXBC$. but there is a homotopy coming from the simplex $TXAB$.
So recall how we write this compatibility for 2-morphisms in a 3-simplex. We start with the longest path from the initial vertex to the last vertex, here this is $T \to X \to A \to B$, which is $f \circ u \circ \alpha$. Now, $f \circ u$ is homotopic to $g \circ v$ using thes 2-morphism H:

$$\begin{tikzcd}
f \circ u \circ \alpha \arrow[r, Rightarrow, "f \circ h_u"] & f \circ u' \\
g \circ v \circ \alpha \arrow[u, Rightarrow, "H \circ \alpha"] \arrow[r, Rightarrow, "g \circ h_v"'] & g \circ v' \arrow[u, Rightarrow, "H'"']
\end{tikzcd}$$

commutes.

2. Uniqueness:
Assume for (T, u', v', H') exist two such triples (α, h_u, h_v) and $(\tilde{\alpha}, \tilde{h}_u, \tilde{h}_v)$ satisfying these conditions.
Then $\exists \alpha \overset{\overline{h}}{\implies} \tilde{\alpha}$ such that the diagram we obtain again is 2-commutative.

So we again have to unravel what it means: One only has to look at two simplices in this diagram:

$$\begin{tikzcd}
T \arrow[rr, "\tilde{\alpha}"] \arrow[dr, "\alpha"] \arrow[ddr, bend right, "u'"'] & & X \arrow[dl, "id"'] \arrow[ddl, bend left, "u"] \\
& X \arrow[d, "u"] & \\
& A &
\end{tikzcd}$$

and when you write down the compatibility of this diagram, what you obtain is the following:

$$\begin{tikzcd}
u \circ \alpha \arrow[r, Rightarrow, "h_u"] \arrow[d, Rightarrow, "u\circ\overline{h}"'] \arrow[dr, phantom, "\circlearrowleft"] & u' \\
u \circ \tilde{\alpha} \arrow[ur, Rightarrow, "\tilde{h}_u"'] &
\end{tikzcd}$$

This gives us $\tilde{h}_u \circ (u \circ \overline{h}) = h_u$ and for the other one we need $\tilde{h}_v \circ (v \circ \overline{h}) = h_v$.

What is the difference between strict and weak 2-pullbacks? If the homotopy α is unique then one calls the pullback strict.

Remark 2.2.15.
We will mostly work with weak 2-pullbacks, for the reason that they exist in `Top`. Strict 2-pullbacks do not always exist.
In some (2,1)-categories strict 2-pullbacks always exist.

Definition 2.2.16.
In `Top` weak 2-pullbacks are called homotopy pullback.
Dually weak 2-pushouts are called homotopy pushout.

Lemma 2.2.17 (Uniqueness of weak 2-pullbacks)**.**
Assume that we are given two weak 2-pullback squares

$$\begin{tikzcd}
X \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\Swarrow H"] & C \arrow[d, "g"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
X' \arrow[r, "v'"] \arrow[d, "u'"'] \arrow[dr, phantom, "\Swarrow H'"] & C \arrow[d, "g"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

Then $\exists \alpha : X \to X'$, $\beta : X' \to X$ equivalences such that the diagram we obtain is 2-commutative.

Proof.
By the uniqueness part of the definition of a weak 2-pullback: we get $\alpha : X' \to X$, $\beta : X \to X'$, $h_u : u \circ \alpha \implies u'$, $h_{u'} : u' \circ \beta \implies u$, $h_v : v \circ \alpha \implies v$, $h_{v'} : v' \circ \beta \implies v$

Take $\beta \circ \alpha : X' \to X'$. If we find homotopies that make the following diagram commutative there is a 2-morphism $\beta \circ \alpha \implies id_{X'}$, since the identity as well makes this diagram commute.

$$
\begin{tikzcd}
X' \arrow[dr, dashed, "\beta\circ\alpha"] \arrow[drr, bend left, "v'"] \arrow[ddr, bend right, "u'"'] \arrow[drr, phantom, "?"] \arrow[ddr, phantom, "?"] & & \\
& X' \arrow[r, "v'"] \arrow[d, "u'"'] & C \arrow[d, "g"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
X' \arrow[dr, bend left, "id_{X'}"] \arrow[dr, bend right, "\beta\circ\alpha"'] \arrow[dr, phantom, "\Uparrow"] \arrow[drr, bend left, "v'"] \arrow[ddr, bend right, "u'"'] & & \\
& X' \arrow[r, "v'"] \arrow[d, "u'"'] & C \arrow[d, "g"] \\
& A \arrow[r, "f"] & B
\end{tikzcd}
$$

So from all the data we got, we have to somehow construct those homotopies:

$$
\begin{tikzcd}
u'\circ\beta\circ\alpha \arrow[rr, Rightarrow] \arrow[dr, Rightarrow, "h_{u'}\circ\alpha"'] & & u' \\
& u\circ\alpha \arrow[ur, Rightarrow, "h_u"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
v'\circ\beta\circ\alpha \arrow[rr, Rightarrow] \arrow[dr, Rightarrow, "h_{v'}\circ\alpha"'] & & v' \\
& v\circ\alpha \arrow[ur, Rightarrow, "h_v"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
f\circ u'\circ\beta\circ\alpha \arrow[r, Rightarrow, "h_{u'}\circ\alpha"] \arrow[dr, phantom, "\beta\text{-diagram}\circ\alpha"] & f\circ u\circ\alpha \arrow[r, Rightarrow, "h_u"] \arrow[dr, phantom, "\alpha\text{-diagram}"] & f\circ u' \\
g\circ v'\circ\beta\circ\alpha \arrow[r, Rightarrow] \arrow[u, Rightarrow, "H'\circ\beta\circ\alpha"] & g\circ v\circ\alpha \arrow[r, Rightarrow] \arrow[u, Rightarrow, dotted, "H\circ\alpha"] & g\circ v' \arrow[u, Rightarrow]
\end{tikzcd}
$$

Now we claim that this diagram is commutative. Recall that we are working now in a 1-category so commutativity is just the one we are used to.

Doing this dually will give you that β and α are equivalences. ■

Proposition 2.2.18.

1. *in* `Top` *show that*

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & \Sigma X
\end{tikzcd}
$$

is a homotopy pushout.

2. *Assume that in G there exists always the homotopy pushout*

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & * \arrow[d] \\
* \arrow[r] & \Sigma X
\end{tikzcd}
$$

(this means that for all objects X, exists an object ΣX that makes this square a weak 2-pushout square)
Then $\Sigma : hG \to hG$ is a functor.

Proposition 2.2.19.
Let $*$ be a final object in G (i.e. $HOM(T, *) = \{*\}$).
Let the following diagram be a weak 2-pullback square.

$$
\begin{tikzcd}
F \arrow[r, "i"] \arrow[d, "\pi_F"'] & X \arrow[d, "f"] \arrow[dl, phantom, "\Swarrow"] \\
* \arrow[r, "b"'] & B
\end{tikzcd}
$$

Then

$$[T,F] \xrightarrow{i_*} [T,X] \xrightarrow{f\circ} [T,B]$$

is exact, i.e. the preimage of $T \to * \xrightarrow{b} B \in [T,B]$ *is in the image of* i_*.

Proof.
Given $T \xrightarrow{g} X,\ f\circ g \xRightarrow{h} b \circ \pi_T$.
This means that we have a 2-commutative square

$$\begin{tikzcd}
T \arrow[dr, "\alpha"'] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "\pi_T"'] & & \\
& F \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& * \arrow[r, "b"'] & B
\end{tikzcd}$$

which means precisely that $\exists \alpha : T \to F$ s.t. $i \circ \alpha \implies g$. ∎

Remark 2.2.20.
Dually: Given a 2-pushout square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
* \arrow[r] \arrow[ur, phantom, "\square"] & Q
\end{tikzcd}$$

$$[A,T] \to [B,T] \to [Q,T]$$

is exact. Note that $*$ is assumed to be an initial object.

Proposition 2.2.21.
Let

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[r, "\beta"] \arrow[d, "g"] \arrow[dl, phantom, "\swarrow h_1"] & E \arrow[d, "k"] \arrow[dl, phantom, "\swarrow h_2"] \\
C \arrow[r, "\gamma"'] & D \arrow[r, "\delta"'] & F
\end{tikzcd}$$

be a 2-commutative diagram.
Assume that

- *both small squares are weak 2-pullbacks then the big square*

$$\begin{tikzcd}
A \arrow[r, "\beta\circ\alpha"] \arrow[d, "f"'] & E \arrow[d, "k"] \arrow[dl, phantom, "\swarrow"] \\
C \arrow[r, "\delta\circ\gamma"'] & F
\end{tikzcd}$$

  is a weak 2-pullback square.

- *If the right and the big square are weak 2-pullbacks then so is the left one.*

Proof. (sketch)
We have to check existence and uniqueness. Let's start with existence:

$$\begin{tikzcd}
T \arrow[rrd, dashed, "\omega"] \arrow[rrrd, bend left, "v"] \arrow[rrdd, "\gamma\circ u"] \arrow[rdd, bend right, "u"'] & & & \\
& A & B \arrow[r, "\beta"] \arrow[d, "g"] & E \arrow[d, "k"] \\
& C \arrow[r, "\gamma"'] & D \arrow[r, "\delta"'] & F
\end{tikzcd}$$

The homotopy we are given is the homotopy $k \circ v \implies \delta \circ \gamma \circ u$.
Therefore we get a map $\omega : T \to B$ and also homotopies $g \circ \omega \overset{h_g}{\implies} \gamma \circ u$ and $\beta \circ \omega \overset{h_\omega}{\implies} v$

By the universal property of $ABCD$, we obtain a map $\xi : T \to A$ and homotopies h_f, h_α

$$\begin{tikzcd}
T \arrow[rd, dashed, "\xi"] \arrow[rrd, bend left, "\gamma"] \arrow[rdd, bend right, "u"'] & & \\
& A \arrow[r, "\alpha"'] \arrow[d, "f"] \arrow[r, phantom, shift left=3, "h_\alpha"] \arrow[d, phantom, shift right=3, "h_f"] & B \arrow[d, "g"] \\
& C \arrow[r, "\gamma"'] & D
\end{tikzcd}$$

We need to check that

$$\begin{tikzcd}
k\circ\beta\circ\alpha\circ\xi \arrow[r, Rightarrow] \arrow[d, Rightarrow] & k\circ v \arrow[d, Rightarrow] \\
\delta\circ\gamma\circ f\circ\xi \arrow[r, Rightarrow] & \delta\circ j\circ u
\end{tikzcd}$$

commutes. This equates to the following diagram

$$\begin{tikzcd}
& k\circ\beta\circ\alpha\circ\xi \arrow[rr, Rightarrow] \arrow[dl, Rightarrow] & & k\circ\beta\circ\omega \arrow[r, Rightarrow] & k\circ v \arrow[ddl, Rightarrow] \\
\delta\circ g\circ\alpha\circ\xi \arrow[rr, Rightarrow] \arrow[dr, Rightarrow] & & \delta\circ g\circ\omega \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & TBEDF & \\
& \delta\circ TABCD & & & \\
& \delta\circ\gamma\circ f\circ\xi \arrow[rr, Rightarrow] & & \delta\circ\gamma\circ u &
\end{tikzcd}$$

The reason why we discuss this, is that one can draw a diagram with just one commutative triangle and one commutative square and that square above, that is left, in fact comes just from those two homotopies:

$$\begin{tikzcd}
T \arrow[d, "\xi"'] \arrow[dr, "\omega"] & & \\
A \arrow[r, "\alpha"'] & B \arrow[r, "\beta"] \arrow[d, "g"] & E \arrow[d, "k"] \\
& D \arrow[r, "\delta"'] & F
\end{tikzcd}$$

From this you obtain a commutative square of homotopies.

$$\begin{tikzcd} HOM(T,B) \arrow[r, bend left, "k\circ\beta"] \arrow[r, bend right, "\delta\circ j"'] \arrow[r, phantom, "\Downarrow"] & HOM(T,F) \end{tikzcd}$$

So the following square commutes, because those two are functors and there is a natural transformation between them or in other words we can translate the conditions of the composition in (2,1)-category into the fact that this homotopy in the square makes it a natural transformation of functors.

$$\begin{tikzcd} k\circ\beta\circ\omega \arrow[r, Rightarrow] & \delta\circ g\circ\omega \\ k\circ\beta\circ\alpha\circ\xi \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \delta\circ g\circ\alpha\circ\xi \arrow[u, Rightarrow] \end{tikzcd}$$

This finishes the existence part.

Uniqueness:

$$\begin{tikzcd} T \arrow[dr, bend right, "\xi"'] \arrow[dr, "\xi'"] \arrow[drr, bend left, "\alpha\circ\xi'"] \arrow[drr, "\alpha\circ\xi"] & & \\ & A \arrow[r] & B \end{tikzcd}$$

By second pullback square there exists $\alpha\circ\xi \overset{?}{\Longrightarrow} \alpha\circ\xi'$.
By the first pullback square... we can again use the uniqueness part but we have to be careful, because it tells you that given fixed morphisms $T\to B$ and $T\to C$, then the morphism to A is unique up to homotopy. But here we have two different morphisms $T\to B$: $\alpha\circ\xi$ and $\alpha\circ\xi'$ and so for $\alpha\circ\xi$ the candidate for the map to A is ξ.
So what you have to change here are not the map to B but the homotopy inside.

$$\begin{tikzcd} T \arrow[dr, "?", "\xi'"'] \arrow[drr, bend left, "\alpha\circ\xi"] \arrow[ddr, bend right, "u"'] & & \\ & A \arrow[r, "\alpha"'] \arrow[d] & {} \arrow[d] \\ & C \arrow[r] & D \end{tikzcd}$$

we can modify the homotopy ? such that ξ' also is a map that makes this diagram commute. Therefore we obtain a map $\xi \implies \xi'$ and this is what we needed in the end. ■

Corollary 2.2.22. *(Puppe long exact sequences)*
Assume $*$ is both an initial and final object in C and that all (necessary) weak 2-pushouts exist. In particular there is a functor

$$\Sigma : hC \to hC$$

Given a weak 2-pushout square

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\square"] & Q \end{tikzcd}$$

we get for all $T \in Ob(C)$ a sequence

$$[T, A] \to [T, B] \to [T, Q] \to [T, \Sigma A] \to [T, \Sigma B] \to \cdots$$

*Dually, let $*$ be an initial object (e.g. $*$ in **Top**). Define a loop functor $\Omega : hC \to hC$ which is a weak 2-pullback of the following diagram:*

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square", very near start] & * \arrow[d] \\ * \arrow[r] & X \end{tikzcd}$$

Given a weak 2-pullback square

$$\begin{tikzcd} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\square", very near start] & X \arrow[d] \\ * \arrow[r] & B \end{tikzcd}$$

yields a LES we get for all $T \in Ob(C)$ a sequence

$$\cdots \to [T, \Omega X] \to [T, \Omega B] \to [T, F] \to [T, X] \to [T, B]$$

Those two LES are exact in pointed sets.

Proof.
Let the following diagram consist of weak 2-pushout squares

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\square", very near end] & Q \arrow[r] \arrow[d] \arrow[ur, phantom, "\square", very near end] & \Sigma A \arrow[d, "\Sigma i"] \\ & * \arrow[r] \arrow[ur, phantom, "\square", very near end] & \Sigma B \end{tikzcd}$$

■

Definition 2.2.23.
$f : X \to Y$ is a <u>fibration</u> if it has HEP for all spaces T.

Theorem 2.2.24.
*In **Top** and **Top**$_*$, we can look at the diagram*

$$\begin{tikzcd} & C \arrow[d, "f"] \\ A \arrow[r] & B \end{tikzcd}$$

where f is a fibration.
*Then we can compute the pullback as usual in **Top***

$$\begin{tikzcd} A \times_B C \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C \arrow[d, "f"] \\ A \arrow[r] & B \end{tikzcd}$$

which is a homotopy pullback if f is a fibration.

Dually

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d] & Y \arrow[d] \\
Z \arrow[r] & Z \amalg_X Y
\end{tikzcd}$$

is a homotopy pushout if i is a cofibration.

Proof.

- Existence:
  If we are given a diagram

$$\begin{tikzcd}
T \arrow[rrd, bend left, "v"] \arrow[rdd, bend right, "u"'] & & \\
& & C \arrow[d, "f"] \arrow[dl, phantom, "\Leftarrow H"] \\
& A \arrow[r, "g"'] & B
\end{tikzcd}$$

  we should find a map $T \to A \times_B C$

$$\begin{tikzcd}
T \arrow[rd, "{(u,v')}"] \arrow[rrd, bend left, "v"] \arrow[rdd, bend right, "u"'] & & \\
& A \times_B C \arrow[r] \arrow[d, "\pi_1"] \arrow[r, phantom, bend left, "h_v"] & C \arrow[d, "f"] \\
& A \arrow[r, "g"'] \arrow[ur, phantom, "\circlearrowleft"] & B
\end{tikzcd}$$

  By HLP, $\exists h_v : v \implies v'$ where $v' : f \circ v' = g \circ u$ and the compatibility is that $f \circ h_v = H$ which follows directly from the HLP.

- Uniqueness:
  Assume we are given two maps $T \to A \times_B C$:

$$\begin{tikzcd}
T \arrow[rd, bend left, "{(u_2,v_2)}"] \arrow[rd, bend right, "{(u_1,v_1)}"'] \arrow[rrd, bend left, "v"] \arrow[rdd, bend right, "u"'] & & \\
& A \times_B C \arrow[r] \arrow[d] & C \arrow[d, "f"] \\
& A \arrow[r, "g"'] & B
\end{tikzcd}$$

  We need to find $H = (H_1, H_2) : (u_1, v_1) \implies (u_2, v_2)$ such that $H_1 \circ u_1 \implies u_2$ and $H_2 \circ v_1 \implies v_2$.
  We have $\pi_1 \circ (u_1, v_1) = u_1 \overset{h_{u_1}}{\implies} u$ and $\pi_1 \circ (u_2, v_2) = u_2 \overset{h_{u_2}}{\implies} u$.

At the moment we have three different maps $T \to A$: u, u_1 and u_2. We now just define the homotopy $u_1 \overset{H_1}{\Longrightarrow} u_2$ using the homotopies we have:

$$\begin{tikzcd}
u_1 \arrow[rr, Rightarrow, "H_1"] \arrow[dr, Rightarrow, "h_{u_1}"'] & & u_2 \arrow[dl, Rightarrow, "h_{u_2}"] \\
& u &
\end{tikzcd}$$

Similarly, for H_2. H_1 and H_2 give us H.

∎

2.3 2-pushouts and 2-pullbacks (continued)

Let G be a (2,1)-category.

Definition 2.3.1.
A <u>2-commutative cube</u>

$$\begin{tikzcd}
A \arrow[r] \arrow[drr] \arrow[dd] & B \arrow[drr] \arrow[dd] & & \\
& & C \arrow[r] \arrow[dd] & D \arrow[dd] \\
A' \arrow[r] \arrow[drr] & B' \arrow[drr] & & \\
& & C' \arrow[r] & D'
\end{tikzcd}$$

Remark 2.3.2.

There is some logic in the choice of notation here, setting primes behind all objects on the bottom, whilst ordering alphabetically within the top and bottom face.
What you do is, you choose an order of those four side-faces. Stating at the face at the back which we thus will give the number one, the one on the left will be number two, the one on the right three and lastly the face on the front will be number four.
In accordance with this numbering, we denote the maps on the upper face by Greek letters and of course on the bottom face analogously with primes

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[drr, "\beta"'] \arrow[dd] & B \arrow[drr, "\gamma"] \arrow[dd] & & \\
& & C \arrow[r, "\delta"'] \arrow[dd] & D \arrow[dd] \\
A' \arrow[r, "\alpha'"] \arrow[drr, "\beta'"'] & B' \arrow[drr, "\gamma'"] & & \\
& & C' \arrow[r, "\delta'"'] & D'
\end{tikzcd}$$

Now the homotopies on the side-faces will be called F_i with the corresponding number of the side-face as index i

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "f_A"'] \arrow[drr, "\beta"] & B \arrow[d, "f_B"] \arrow[drr, "\gamma"] \arrow[dl, phantom, "F_1"] \arrow[dr, phantom, "H"] & & \\
A' \arrow[r, "\alpha'"] \arrow[drr, "\beta'"'] \arrow[dr, phantom, "F_2"] & B' \arrow[drr, "\gamma'"] & C \arrow[r, "\delta"] \arrow[d, "f_C"] \arrow[dr, phantom, "H'"] & D \arrow[d, "f_D"] \arrow[dl, phantom, "F_3 \quad F_4"] \\
& & C' \arrow[r, "\delta'"'] & D'
\end{tikzcd}$$

So a 2-commutative cube consists of

- objects: $A, B, C, D, A', B', C', D'$
- 1-morphisms: $\alpha, \beta, ..., \alpha', ...f_A, ...$
- 2-morphisms: $F_1, ..., F_4, H, H'$

such that in the groupoid $HOM(A, D')$ (between the object with only arrows going out of it to the one with none leaving but only arriving) we have some morphisms between those paths which we can put in the following diagram:

$$\begin{tikzcd}
& \delta' \circ f_C \circ \beta \arrow[dl, Rightarrow, "\delta' \circ F_2"] & f_D \circ \delta \circ \beta \arrow[l, Rightarrow, "F_4 \circ \beta"'] & \\
\delta' \circ \beta' \circ f_A & & & f_D \circ \gamma \circ \alpha \arrow[ul, Rightarrow, "f_D \circ H"'] \arrow[dl, Rightarrow, "F_3 \circ \alpha"] \\
& \gamma' \circ \alpha' \circ f_A \arrow[ul, Rightarrow, "H' \circ f_A"] & \gamma' \circ f_B \circ \alpha \arrow[l, Rightarrow, "\gamma' \circ F_1"'] &
\end{tikzcd}$$

This diagram should commute.

Remark 2.3.3.
1 relation for 2-morphisms of the cube corresponds to the cube having one interior.
If you delete one of the two-dimensional faces of the cube - and by delete a face, we understand forgetting about the homotopy on this face, than you cannot find the closed circuit which gives you some conditions on the 2-morphisms.
This is because, if you delete a face of a cube, you cannot fill it in, it is not closed anymore.

Remark 2.3.4.
One could use the definition of 2-commutative 3-simplices to define a cube by splitting the cube up in 3-simplices. Then one could check that the conditions one gets from those simplices coincide with the condition in the definition above. This, however, is far to cumbersome, since we have no intention to define all kinds of 2-commutative diagrams.

Definition 2.3.5.
A 2-commutative square

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \arrow[d, "\gamma"] \arrow[dl, phantom, "H"] \\
C \arrow[r, "\delta"'] & D
\end{tikzcd}$$

is equivalent to a 2-commutative square

$$\begin{tikzcd} A' \arrow[r, "\alpha'"] \arrow[d, "\beta'"'] & B' \arrow[d, "\gamma'"] \arrow[dl, phantom, "\Swarrow H'"] \\ C' \arrow[r, "\delta'"'] & D' \end{tikzcd}$$

if there is a 2-commutative cube as before with vertical arrows f_A, f_B, f_C, f_D equivalences.

Lemma 2.3.6.
This is indeed an equivalence relation on 2-commutative squares.

Proof.

- Reflexivity is obvious because we take f_A, f_B, f_C, f_D to be the identities.
- Transitivity: “compose” the cubes
- Symmetry is non-obvious!

  Regard two squares in an 1-category $\mathcal{A}$:

$$\begin{tikzcd} \underset{=[1]\times[1]}{I} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{A} \end{tikzcd}$$

  Any such is given by a functor F_1 respectively F_2 as indicated.
  If $\eta(i) : F_1(i) \to F_2(i)$ is an isomorphism for all $i \in I$, then there exists an inverse to the natural transformation

$$\eta^{-1} : \eta^{-1}(i) \implies (\eta(i))^{-1}$$
$$\eta^{-1} : F_2 \implies F_1$$

  This is the same claim that was already stated, that a natural transformation between groupoids is always an isomorphism.
  So in the 1-category $\mathcal{A}$ once we have an morphism from one diagram to the other, it is just a matter of inverting all those arrows to get the inverse back to the other one (because the inverse is unique and to check the commutativity of all the squares, one has to say that η^{-1} indeed is a natural transformation. This is straight forward.)

  Thus in 1-category theory the claim is almost obvious.
  But in $(2,1)$-category theory it is not clear that such things happen. Given a 2-commutative cube, we need to construct a sort of “inverse” 2-commutative cube. This is not trivial anymore because an inverse morphism to an equivalence does not have to be unique.

  To construct the inverse 2-commutative cube, we need to construct the inverse on objects.
  We had a map $f_A : A \to A'$, so let $g_A : A' \to A$ be sort of quasi-inverse to f_A (both are equivalences). Fix a homotopy $h_A : g_A \circ f_A \implies id_A$ and similarly for B, C, D.
  We need to construct homotopies G_1, G_2, G_3, G_4 that will make this new inverse cube a 2-commutative cube. So this new cube will have the same top face as the bottom face of the previous one, so there is already a homotopy H' and the same bottom face as the top face before, thus equipped with the homotopy H.
  Those two together with the homotopies G_i have to satisfy the 2-commutative cube condition.

  Step 1: Define 2-morphisms G_1, G_2, G_3, G_4.
  One of the side-faces of the previous cube is

$$\begin{tikzcd}
A \arrow[r, "\beta"] \arrow[d, "f_A"'] & C \arrow[d, "f_C"] \arrow[dl, phantom, "\swarrow F_2"] \\
A' \arrow[r, "\beta'"'] & C'
\end{tikzcd}$$

Adding the inverse cube we get

$$\begin{tikzcd}
A \arrow[r, "\beta"] \arrow[d, "f_A"'] & C \arrow[d, "f_C"] \arrow[dl, phantom, "\swarrow F_2"] \\
A' \arrow[r, "\beta'"'] \arrow[d, "g_A"'] & C' \arrow[d, "g_C"] \arrow[dl, phantom, "\swarrow G"] \\
A \arrow[r, "\beta"'] & C'
\end{tikzcd}$$

So we need to construct a homotopy

$$g_C \circ \beta' \implies \beta \circ g_A$$

This is not defined yet, but what we can use is those homotopies h_A and h_C. We can precompose with f_A and g_A:

$$\begin{tikzcd}
g_C \circ \beta' \arrow[d, Rightarrow, dotted, "G"] & g_C \circ \beta' \circ f_A \circ g_A \arrow[l, Rightarrow, "g_C \circ \beta' \circ h_A"'] \\
\beta \circ g_A & g_C \circ f_C \circ \beta \circ g_A \arrow[l, Rightarrow, "h_C \circ \beta \circ g_A"] \arrow[u, Rightarrow, "g_C \circ F_2 \circ g_A"']
\end{tikzcd}$$

So we define G_2 such that this diagram commutes. We can do that because all of this happens within some groupoid, thus taking inverses is possible.
The main claim here is that having defined G we can define the composition of the two squares:

$$\begin{tikzcd}
A \arrow[r, "\beta"] \arrow[d, "g_A \circ f_A"'] & C \arrow[d, "g_C \circ f_C"] \arrow[dl, phantom, "\swarrow K_2"] \\
A \arrow[r, "\beta"'] & C
\end{tikzcd}$$

<u>claim:</u>

$$\begin{tikzcd}
g_C \circ f_C \circ \beta \arrow[r, Rightarrow, "g_C \circ F_2"] \arrow[dr, Rightarrow, "K_2"'] & g_C \circ \beta' \circ f_A \arrow[d, Rightarrow, "G_2 \circ f_A"] \\
& \beta \circ g_A \circ f_A
\end{tikzcd}$$

This is the definition of K_2. The claim is now that even though we defined it in terms of G_2 and F_2 it has an easy description:

$$K_2 = (\beta \circ h_A)^{-1} \circ (h_C \circ \beta)$$

To do that we precompose with f_A:

$$\begin{tikzcd}
g_C \circ \beta' \circ f_A \arrow[d, Rightarrow, "G_2 \circ f_A"] & g_C \circ \beta' \circ f_A \circ g_A \circ f_A \arrow[l, Rightarrow, "g_C \circ \beta' \circ h_A \circ f_A"'] \\
\beta \circ g_A \circ f_A & g_C \circ f_C \circ \beta \circ g_A \circ f_A \arrow[l, Rightarrow, "h_C \circ \beta \circ g_A \circ f_A"] \arrow[u, Rightarrow, "g_C \circ F_2 \circ g_A \circ f_A"']
\end{tikzcd}$$

Also we can add on the left:

$$\begin{tikzcd}
g_C \circ f_C \circ \beta \arrow[r, Rightarrow, "g_C \circ \underline{F_2}"'] \arrow[d, Rightarrow, "\underline{h_C} \circ \beta"'] & g_C \circ \beta' \circ f_A \arrow[d, Rightarrow, "G_2 \circ f_A"] & g_C \circ \beta' \circ f_A \circ g_A \circ f_A \arrow[l, Rightarrow, "g_C \circ \beta' \circ \underline{h_A} \circ f_A"'] \\
\beta & \beta \circ g_A \circ f_A \arrow[l, Rightarrow, "\beta \circ \underline{h_A}"] & g_C \circ f_C \circ \beta \circ g_A \circ f_A \arrow[l, Rightarrow, "\underline{h_C} \circ \beta \circ g_A \circ f_A"] \arrow[u, "g_C \circ \underline{F_2} \circ g_A \circ f_A"']
\end{tikzcd}$$

The square on the right commutes by definition of G_2 and the big square also commutes because what we do here is, we apply only three homotopies (underlined in the diagram). Those three homotopies they are written here in different order and with different morphisms applied to them but using the axioms of homotopies it is not hard to show that you can change the order of those compositions and make the outer square commute.

Since the outer square and the square on the right both are commutative, so is the square on the left. But the commutativity of this square on the left is just the claim.

Step 2:

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[dr, "\beta"] \arrow[dd] & B \arrow[dr, "\gamma"] \arrow[dd] & & \\
& C \arrow[r, "\delta"'] \arrow[dd] & D \arrow[dd] & \\
A' \arrow[r, "\alpha'"] \arrow[dr, "\beta'"] \arrow[dd] & B' \arrow[dr, "\gamma'"] \arrow[dd] & & \\
& C' \arrow[r, "\delta'"'] \arrow[dd, "g_C"] & D' \arrow[dd] & \\
A \arrow[r, "\alpha'"] \arrow[dr, "\beta'"'] & B \arrow[dr, "\gamma'"] & & \\
& C \arrow[r, "\delta'"'] & D &
\end{tikzcd}$$

Since f_A is an equivalence, 2-commutativity of the lower ("the inverse") cube is equivalent to the 2-commutativity of the composed cube.
We need to check that the homotopies defined in the lower cube

$$\begin{tikzcd}
& \delta \circ g_C \circ \beta' \arrow[dl, Rightarrow] & & g_D \circ \delta' \circ \beta' \arrow[ll, Rightarrow] & \\
\delta \circ \beta \circ g_A & & \text{lower cube} & & g_D \circ \gamma' \circ \alpha' \arrow[ul, Rightarrow] \arrow[dl, Rightarrow] \\
& \gamma \circ \alpha \circ g_A \arrow[ul, Rightarrow] & & \gamma \circ g_B \circ \alpha' \arrow[ll, Rightarrow] &
\end{tikzcd}$$

satisfy this commutativity condition for the upper cube. The claim is that it suffices to check that the corresponding diagram for the upper cube.
Because f_A is an equivalence the composition with f_A, $\circ f_A$, is an equivalence of categories (/ corresponding groupoids) and in particular it is a fully faithful functor.
This will work with any groupoid, but here we will work with $HOM(A', D)$ composition with f_A will give a groupoid $HOM(A, D)$ and because f_A is an equivalence and thus has an inverse it is easy to check that these conditions of the existence of an inverse transform into those HOM-groupoids as an equivalence of categories, in very much the same way that in 1-category the condition that a morphism is an isomorphism translates to the fact that it induces an isomorphism on the Hom-sets.

So because this is fully faithful, to check that two morphisms agree, it suffices to see that after we precompose with f_A.

$$
\begin{tikzcd}
 & \delta\circ g_C\circ\beta'\circ f_A \arrow[dl, Rightarrow] & g_D\circ\delta'\circ\beta'\circ f_A \arrow[l, Rightarrow] & \\
\delta\circ\beta\circ g_A\circ f_A & \text{lower cube} & & g_D\circ\gamma'\circ\alpha'\circ f_A \arrow[ul, Rightarrow, "\ldots\circ H'\circ\ldots"'] \arrow[dl, Rightarrow] \\
 & \gamma\circ\alpha\circ g_A\circ f_A \arrow[ul, Rightarrow] & \gamma\circ g_B\circ\alpha'\circ f_A \arrow[l, Rightarrow] &
\end{tikzcd}
$$

To this there is the diagram of the top cube attached:

$$
\begin{tikzcd}
 & & & g_D\circ\delta'\circ f_D\circ\beta \arrow[dl, Rightarrow] & g_D\circ f_D\circ\delta\circ\beta \arrow[l, Rightarrow] & \\
 & \delta\circ g_C\circ\beta'\circ f_A \arrow[dl, Rightarrow] & g_D\circ\delta'\circ\beta'\circ f_A \arrow[l, Rightarrow] & & & g_D\circ f_D\circ\gamma\circ\alpha \arrow[ul, Rightarrow] \arrow[dl, Rightarrow] \\
\delta\circ\beta\circ g_A\circ f_A & \text{lower cube} & & g_D\circ\gamma'\circ\alpha'\circ f_A \arrow[ul, Rightarrow, "\ldots\circ H'\circ\ldots"'] \arrow[dl, Rightarrow] & g_D\circ\gamma'\circ\beta\circ\alpha \arrow[l, Rightarrow] & \\
 & \gamma\circ\alpha\circ g_A\circ f_A \arrow[ul, Rightarrow] & \gamma\circ g_B\circ\alpha'\circ f_A \arrow[l, Rightarrow] & & &
\end{tikzcd}
$$

Ignoring the morphisms that have to be composed and precomposed with the homotopies, the 2-morphisms are given by

$$
\begin{tikzcd}
 & & & g_D\circ\delta'\circ f_D\circ\beta \arrow[dl, Rightarrow, "F_2"'] & g_D\circ f_D\circ\delta\circ\beta \arrow[l, Rightarrow, "F_4"'] & \\
 & \delta\circ g_C\circ\beta'\circ f_A \arrow[dl, Rightarrow] & g_D\circ\delta'\circ\beta'\circ f_A \arrow[l, Rightarrow] & & \text{top cube} & g_D\circ f_D\circ\gamma\circ\alpha \arrow[ul, Rightarrow, "H"'] \arrow[dl, Rightarrow, "F_2"] \\
\delta\circ\beta\circ g_A\circ f_A & \text{lower cube} & & g_D\circ\gamma'\circ\alpha'\circ f_A \arrow[ul, Rightarrow, "\ldots\circ H'\circ\ldots"'] \arrow[dl, Rightarrow] & g_D\circ\gamma'\circ\beta\circ\alpha \arrow[l, Rightarrow, "F_1"'] & \\
 & \gamma\circ\alpha\circ g_A\circ f_A \arrow[ul, Rightarrow] & \gamma\circ g_B\circ\alpha'\circ f_A \arrow[l, Rightarrow] & & &
\end{tikzcd}
$$

After composing those two, we get the commutativity of the bigger cube. For this, we only have to add one morphism:

$$\begin{tikzcd}[column sep=small]
 & & & g_D\circ\delta'\circ f_D\circ\beta \arrow[dl, Rightarrow, "F_2"'] & g_D\circ f_D\circ\delta\circ\beta \arrow[l, Rightarrow, "F_4"'] & \\
 & \delta\circ g_C\circ\beta'\circ f_A \arrow[dl, Rightarrow] & g_D\circ\delta'\circ\beta'\circ f_A \arrow[l, Rightarrow] & & \text{top cube } (\circ f_A) & g_D\circ f_D\circ\gamma\circ\alpha \arrow[ul, Rightarrow, "H"'] \arrow[dl, Rightarrow, "F_3"] \arrow[ddl, Rightarrow, "K_3"] \\
\delta\circ\beta\circ g_A\circ f_A & \text{lower cube} & & g_D\circ\gamma'\circ\alpha'\circ f_A \arrow[ul, Rightarrow, "\ldots\circ H'\circ\ldots"] \arrow[dl, Rightarrow, "G_3"] & g_D\circ\gamma'\circ\beta\circ\alpha \arrow[l, Rightarrow, "F_1"] & \\
 & \gamma\circ\alpha\circ g_A\circ f_A \arrow[ul, Rightarrow, "H"] & \gamma\circ g_B\circ\alpha'\circ f_A \arrow[l, Rightarrow, "G_1"'] & \text{always} & \gamma\circ g_B\circ f_B\circ\alpha \arrow[lll, Rightarrow, bend left] &
\end{tikzcd}$$

Taking a look back at the two cubes atop of each other, we see that the new downward 2-morphism has to be K_3 and the leftward arrow K_1.
The commutativity of the boundary of "this" diagram is the 2-commutativity of the composed cube.

Step 3: 2-commutativity of the composed cube
We now have the following cube:

$$\begin{tikzcd}
A \arrow[rr, "\alpha"] \arrow[dr, "\beta"] \arrow[dd, "g_A\circ f_A"'] & & B \arrow[dr] \arrow[dd] \arrow[d, phantom, "\swarrow H"] & \\
 & C \arrow[rr] \arrow[dd, "g_C\circ f_C"] & {} & D \arrow[dd] \\
A \arrow[rr] \arrow[dr, "\beta"'] \arrow[r, phantom, "\swarrow H"] & {} & B \arrow[dr] \arrow[d, phantom, "\swarrow H"] & \\
 & C \arrow[rr] & {} & D
\end{tikzcd}$$

The commutativity that we have to check is the following:

$$\begin{tikzcd}
 & \delta\circ g_C\circ f_C\circ\beta \arrow[dl, Rightarrow, "\delta\circ K_2"'] & g_D\circ f_D\circ\delta\circ\beta \arrow[l, Rightarrow] & \\
\delta\circ\beta\circ g_A\circ f_A & & & g_D\circ f_D\circ\gamma\alpha \arrow[ul, Rightarrow] \arrow[dl, Rightarrow] \\
 & \gamma\circ\alpha\circ g_A\circ f_A \arrow[ul, Rightarrow] & \gamma\circ g_B\circ f_B\circ\alpha \arrow[l, Rightarrow] &
\end{tikzcd}$$

To do so we can use the properties of the homotopies K_2 and K_4 that were described in step

1. So we can insert

$$
\begin{tikzcd}
 & \delta\circ g_C\circ f_C\circ\beta \arrow[dl, Rightarrow, "\delta\circ K_2"'] \arrow[d, Rightarrow, "\delta\circ h_C\circ\beta"] \arrow[dl, phantom, shift right=6, "\circlearrowleft"] & g_D\circ f_D\circ\delta\circ\beta \arrow[l, Rightarrow, "\delta\circ\beta\circ h_A"'] \arrow[dl, Rightarrow, "h_D\circ\delta\circ\beta"] \arrow[l, phantom, shift right=4, "\circlearrowleft"] & \\
\delta\circ\beta\circ g_A\circ f_A \arrow[r, Rightarrow] & \delta\circ\beta & \gamma\circ\alpha & g_D\circ f_D\circ\gamma\alpha \arrow[l, Rightarrow] \arrow[ul, Rightarrow] \arrow[dl, Rightarrow] \arrow[dl, phantom, shift left=6, "\circlearrowleft"] \\
 & \gamma\circ\alpha\circ g_A\circ f_A \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] & \gamma\circ g_B\circ f_B\circ\alpha \arrow[l, Rightarrow] \arrow[u, Rightarrow] \arrow[ul, phantom, shift left=4, "\circlearrowleft"] &
\end{tikzcd}
$$

We can also insert H and the claim is that the squares now subdivided by it commute.

$$
\begin{tikzcd}
 & \delta\circ g_C\circ f_C\circ\beta \arrow[dl, Rightarrow, "\delta\circ K_2"'] \arrow[d, Rightarrow, "\delta\circ h_C\circ\beta"] \arrow[dl, phantom, shift right=6, "\circlearrowleft"] & g_D\circ f_D\circ\delta\circ\beta \arrow[l, Rightarrow, "\delta\circ\beta\circ h_A"'] \arrow[dl, Rightarrow, "h_D\circ\delta\circ\beta"] \arrow[l, phantom, shift right=4, "\circlearrowleft"] & \\
\delta\circ\beta\circ g_A\circ f_A \arrow[r, Rightarrow] & \delta\circ\beta & \gamma\circ\alpha \arrow[l, Rightarrow, "H"'] & g_D\circ f_D\circ\gamma\alpha \arrow[l, Rightarrow] \arrow[ul, Rightarrow] \arrow[dl, Rightarrow] \arrow[dl, phantom, shift left=6, "\circlearrowleft"] \\
 & \gamma\circ\alpha\circ g_A\circ f_A \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] & \gamma\circ g_B\circ f_B\circ\alpha \arrow[l, Rightarrow] \arrow[u, Rightarrow] \arrow[ul, phantom, shift left=4, "\circlearrowleft"] &
\end{tikzcd}
$$

For the one of the left we only have h_A and H composed with some morphisms, check its commutativity:
We have a two functors and a natural transformation induced by H

$$
\begin{tikzcd}[column sep=huge]
HOM(A,A) \arrow[r, bend left=50, "\delta\circ\beta"{name=U}] \arrow[r, bend right=50, "\gamma\circ\alpha"'{name=D}] & HOM(A,D) \arrow[from=D, to=U, Rightarrow, "H"]
\end{tikzcd}
$$

In the groupoid $HOM(A,A)$ we have the identity object id_A and $g_A \circ f_A$ and the homotopy $h_A : g_A \circ f_A \to id_A$.
Now when we apply the functor $\delta \circ \beta$ we get $\delta \circ \beta$ for the identity and $\delta \circ \beta \circ g_A \circ f_A$ for the composition.
On the other hand, for $\gamma \circ \alpha$ we get $\gamma \circ \alpha \circ g_A \circ f_A$ and $\gamma \circ \alpha$.
Now we have a natural transformation between those two functors which gives us the commutativity.

Analogously done for the other square, we have its commutativity and thus the commutativity of the whole diagram.

■

Proposition 2.3.7.
If a 2-commutative square is equivalent to a weak 2-pullback square, then it is a weak 2-pullback.

Proof. (sketch)
Let the following diagram be a 2-pullback diagram to which ABCD is equivalent

$$
\begin{tikzcd}
A' \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\square"] & B' \arrow[d] \\
C' \arrow[r] & D'
\end{tikzcd}
$$

So we have a cube and can also construct the inverse cube

$$
\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[dr, "\beta"'] \arrow[dd] & B \arrow[drr, "\gamma"] \arrow[dd] & & \\
& & C \arrow[r, "\delta"'] \arrow[dd] & D \arrow[dd] \\
A' \arrow[r, "\alpha'"] \arrow[dr, "\beta'"'] \arrow[dd] & B' \arrow[drr, "\gamma'"] \arrow[dd] & & \\
& & C' \arrow[r, "\delta'"'] \arrow[dd, "g_C"] & D' \arrow[dd] \\
A \arrow[r, "\alpha'"] \arrow[dr, "\beta'"'] & B \arrow[drr, "\gamma'"] & & \\
& & C \arrow[r, "\delta'"'] & D
\end{tikzcd}
$$

We want to show that the top face is a 2-pullback, so we start with an object X

$$
\begin{tikzcd}
& X \arrow[d, bend left, "v"] \arrow[ddr, bend right, "u"'] & & \\
A \arrow[r, "\alpha"] \arrow[dr, "\beta"'] \arrow[dd] & B \arrow[drr, "\gamma"] \arrow[dd] & & \\
& & C \arrow[r, "\delta"'] \arrow[dd] & D \arrow[dd] \\
A' \arrow[r, "\alpha'"] \arrow[dr, "\beta'"'] \arrow[dd] & B' \arrow[drr, "\gamma'"] \arrow[dd] & & \\
& & C' \arrow[r, "\delta'"'] \arrow[dd, "g_C"] & D' \arrow[dd] \\
A \arrow[r, "\alpha'"] \arrow[dr, "\beta'"'] & B \arrow[drr, "\gamma'"] & & \\
& & C \arrow[r, "\delta'"'] & D
\end{tikzcd}
$$

- compose the data (u, v, G) with f_B, f_C and f_D to get

$$
\begin{tikzcd}
X \arrow[rr, bend left] \arrow[dr, bend right] & & B' \arrow[d] \arrow[dl, phantom, "\Leftarrow"] \\
& C' \arrow[r] & D'
\end{tikzcd}
$$

by the weak 2-pullback property, we get from this a morphism $X \to A'$ and homotopies.

- compose with $g_A : A' \to A$ to get $\omega : X \to A$

$$\begin{tikzcd}
 & X \arrow[dl, dashed, "\omega"'] \arrow[d, bend left, "v"] \arrow[ddr, bend left, "u"] \arrow[dddl, dashed] & & \\
A \arrow[r, "\alpha"] \arrow[drr, "\beta"'] \arrow[dd] & B \arrow[drr, "\gamma"] \arrow[dd] & & \\
 & & C \arrow[r, "\delta"'] \arrow[dd] & D \arrow[dd] \\
A' \arrow[r, "\alpha'"] \arrow[drr, "\beta'"'] \arrow[dd] & B' \arrow[drr, "\gamma'"] \arrow[dd] & & \\
 & & C' \arrow[r, "\delta'"'] \arrow[dd, "g_C"] & D' \arrow[dd] \\
A \arrow[r, "\alpha'"] \arrow[drr, "\beta'"'] & B \arrow[drr, "\gamma'"] & & \\
 & & C \arrow[r, "\delta'"'] & D
\end{tikzcd}$$

■

Remark 2.3.8.
This proposition works dually for the pushout.

2.4 Homotopy pushouts and pullbacks

To construct homotopy pullbacks / pushouts it suffices to replace one of the maps with a fibration / cofibration. This is because this process of replacing gives you an equivalent square of which you can compute the usual pullback.
For cofibrations we have already seen that one can factor any map as a cofibration composed with a homotopy equivalence. However, we have not seen yet the same to be true about fibrations: We need to factor any map $f : X \to Y$ as $X \xrightarrow{s} W(f) \xrightarrow{p} Y$ with $p \circ s = f$, where s is a homotopy equivalence, p is a fibration.

1. $(ev_0, ev_1) : Y^I \to Y \times Y$ is a fibration for all Y.
   In order to see that we need to solve the lifting problem for any space T:

$$\begin{tikzcd}
T \arrow[r, "H"] \arrow[d, "id\times\{0\}"'] & Y^I \arrow[d, "(ev_0,ev_1)"] \\
T \times I \arrow[r, "(H_1,H_2)"'] \arrow[ur, dashed, "\tilde{H}"] & Y \times Y
\end{tikzcd}$$

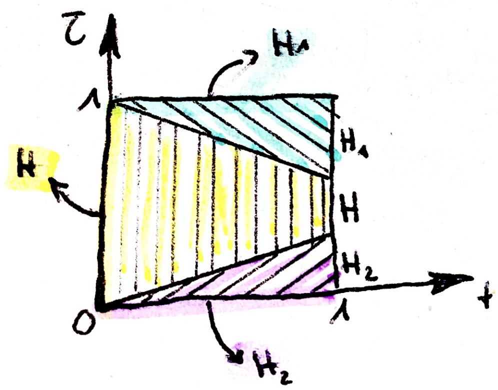
   The map $H : T \to Y^I$ corresponds to a map $H : T \times I \to Y$. The commutativity of this square implies that

$$H : T \times I \to Y$$
$$H|_{T\times\{0\}} = H_1|_{T\times\{0\}} \quad H|_{T\times\{1\}} = H_2|_{T\times\{0\}}$$

   we need

$$\tilde{H} : (T \times I) \times I \to Y$$

Given three faces of a square we have to fill it with the homotopy $\tilde{H}$.
The idea on how to construct such an $\tilde{H}$ is actually the same as for pushing through a cardboard box:



2. fibrations are closed under pullbacks and compositions

3. For any space Z, the map $Z \to *$ is a fibration

Using those three properties we can define $W(f)$ as the pullback:

$$\begin{tikzcd}
 & W(f) \arrow[r] \arrow[d, "{(q,p)}"'] \arrow[dr, phantom, "\square" very near start] & Y^I \arrow[d, "{(ev_0,ev_1)}"] \\
 & X \times Y \arrow[r, "f\times id"'] \arrow[dl] \arrow[dr] & Y \times Y \\
X \arrow[dr] & & Y \arrow[dl] \\
 & * &
\end{tikzcd}$$

Because (q, p) is a fibration and the projections are also pullbacks of fibrations, both q and p are fibrations.
We still have to define s. So what is the space $W(f)$?

$$W(f) = \{(x, \gamma) \mid \gamma \in Y^I, \gamma(0) = f(x)\}$$

We define $s : X \to W(f)$ by $x \mapsto (x, \overline{f(x)})$, so p will send it for all $x \in X$ to $f(x)$, exactly as we want it to. What is left to check is that s is a homotopy equivalence. For this, note that we have a map $q : W(f) \to X$ and $q \circ s = id_X$. The other composition $s \circ q : W(f) \to W(f)$ is homotopic to the identity via a homotopy H, which we will define now:

$$\begin{aligned} H : W(f) \times I &\to W(f) \\ (x, \gamma, \tau) &\mapsto (x, \gamma_\tau) \end{aligned}$$

where $\gamma_\tau(t) := \gamma(t(1 - \tau))$.

Theorem 2.4.1.
Homotopy pullbacks and pushouts exist in Top *and* Top_*

Proof.
We are given a diagram

$$\begin{tikzcd} & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Consider a cube

$$\begin{tikzcd} E_{fg} \arrow[rr, "v"] \arrow[dr, "u"] \arrow[dd, "id"'] & & C \arrow[dr, "g"] \arrow[dd, "s"] & \\ & A \arrow[rr, "f"] \arrow[dd] & & B \arrow[dd, "id"] \\ E_{fg} \arrow[rr, "\pi"] \arrow[dr, "u"'] & & W(g) \arrow[dr, "p"] & \\ & A \arrow[rr, "f"'] & & B \end{tikzcd}$$

also we have to specify a homotopy H_1 on the top face, H_2 on the back face.
The space E_{fg} is given by

$$E_{fg} = \{(a, c, \gamma) \mid \gamma(0) = g(c), \gamma(1) = f(a)\} \subset A \times C \times B^I$$

So the map $v : E_{fg} \to C$ is just the projection on the second component, also u is the projection onto A.

$$\begin{aligned} H_1 : E_{f,g} \times I &\to B \\ (a, c, \gamma, \tau) &\mapsto \gamma(\tau) \end{aligned}$$

H_1 is a homotopy between $g \circ v$ and $f \circ u$. $H_2 : s \circ v \implies \pi$ defined by

$$\begin{aligned} H_2 : E_{f,g} \times I &\to W(g) \\ (a, c, \gamma, \tau) &\mapsto (c, \tilde{\gamma}_\tau) \end{aligned}$$

where $\tilde{\gamma}_\tau(t) := \gamma(t \cdot \tau)$.
The final step is to check 2-commutativity of the cube is the condition

$$p \circ H_2 = H_1$$

but by definition they are actually equal.
$\implies$ (E_{fg}, v, u, H_1) is a homotopy pullback of

$$\begin{tikzcd} & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

and is often called standard homotopy pullback.

So this is the case for homotopy pullbacks, for homotopy pushouts one can define a space $Q_{f,g}$ and a homotopy H making the following square commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "u"] \\ C \arrow[r, "v"'] & Q_{f,g} \end{tikzcd}$$

and this space (which is also often called standard homotopy pushout) will be defined as

$$Q_{f,g} := B \amalg C \amalg A \times I / \begin{array}{c} (a,1) \sim g(a) \\ (a,0) \sim f(a) \\ [* \times I] \end{array}$$

where the last condition is for the pointed case and $*$ is the distinguished point in Top_* ■

Remark 2.4.2.
It is possible to show that $Q_{f,g}$ and $E_{f,g}$ are homotopy pullbacks / pushout directly.

Example 2.4.3.
Homotopy pushouts:

- $$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\swarrow H"] & B \arrow[d] \\ * \arrow[r] & Cone(f) \end{tikzcd}$$

  is a homotopy pushout, ($Cone(f)$ is sometimes called the homotopy cofiber)

- If i is a cofibration then the pushout can be computed as the quotient space:

  $$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\square"] & B \arrow[d] \\ * \arrow[r] & B/A \end{tikzcd}$$

- $$\begin{tikzcd} * \amalg * = S^0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & * \arrow[d] \\ * \arrow[r] & S^1 \end{tikzcd}$$

  More generally for an n-sphere in the upper left corner:

  $$\begin{tikzcd} S^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & * \arrow[d] \\ * \arrow[r] & S^{n+1} \end{tikzcd}$$

For any space X:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \arrow[ul, phantom, "\square", very near start] \end{tikzcd}$$

Homotopy pullbacks:

- $$\begin{tikzcd} F_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\square", very near start] & E \arrow[d, "f"] \\ * \arrow[r, "b"'] & B \arrow[from=1-2, to=2-1, Rightarrow, shorten=20pt] \end{tikzcd}$$

  where

  $$F_b := \{(e, \gamma) \mid \gamma \in B^I, \gamma(0) = b, \gamma(1) = f(e)\}$$

  This is the homotopy fiber.

- $$\begin{tikzcd} \Omega_x X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square", very near start] & * \arrow[d, "x"] \\ * \arrow[r, "x"'] & B \arrow[from=1-2, to=2-1, Rightarrow, shorten=20pt] \end{tikzcd}$$

Fact (Milnor):

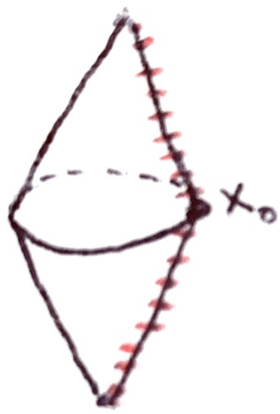
Homotopy pullbacks of CW-complexes exist as CW-complexes.
(have homotopy type of a CW-complex)

2.5 Pointed vs Unpointed

By exercise 4.1 there are functors $\Sigma : h\mathtt{Top} \to h\mathtt{Top}$ and $\Sigma' : h\mathtt{Top}_* \to h\mathtt{Top}_*$ where

$$\Sigma X := X \times I / \begin{array}{l} (x,0) \sim (x',0) \\ (x,1) \sim (x',1) \end{array}$$

$$\Sigma'(X, x_0) := X \times I / \begin{array}{l} (x,0) \sim (x',0) \sim (x_0, t_0) \\ (x,1) \sim (x',1) \sim (x_0, t_0) \end{array} = \Sigma X / \{x_0\} \times I$$



In $\mathtt{Top}_*$, there is the loop functor $\Omega : h\mathtt{Top}_* \to h\mathtt{Top}_*$, $\Omega(X, x_0) := \Omega_{x_0}X \cong \{\gamma \in X^I | \gamma(0) = \gamma(1) = x_0\}$ (pointed by $\overline{x_0}$).

This is but one way to define the loop functor on objects. One can also describe it as the following homotopy pullback:

$$\begin{tikzcd}
\Omega_{x_0}X \arrow[r] \arrow[d] & * \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\
* \arrow[r] & (X, x_0)
\end{tikzcd}$$

Lemma 2.5.1.
Those two functors are adjoint in the pointed case: $\Sigma' \dashv \Omega$
(In fact, this is not only true for $h\mathtt{Top}_*$ but in hC of any (2,1)-category with zero object and Σ', Ω)

Proof. (sketch)
Suppose we have our distinguished space (X, x_0).

$$\begin{tikzcd}
(X, x_0) \arrow[r] \arrow[d] & * \arrow[ddr, bend left] & \\
* \arrow[drr, bend right] \arrow[r, phantom, "\Swarrow H"] & & \\
 & & (T, t_0)
\end{tikzcd}$$

$[\Sigma'(X, x_0), (T, t_0)]$ is defined by H, perhaps up to some equivalence relation (the bent arrows are unique, there is no choice in them)
$[(X, x_0), \Omega(T, t_0)]$ is defined by H.
This shows that those two sets are isomorphic. Left to check would be naturality and so on... ∎

Corollary 2.5.2.

1. $\underset{[S^k, \Omega_{x_0}X]_{Top_*}}{\pi_k(\Omega_{x_0}X)} \cong \underset{[\Sigma' S^k, X]_{Top_*}}{\pi_{k+1}(X, x_0)}$ *for all* $k \geq 0$

2. $\forall (X, x_0)$ *exists because of the suspension a canonical map $X \to \Omega\Sigma' X$ from X to the loops of the suspension of X. This is just obtained from the adjunction of those two functors.*

   If you look at the homotopy groups, we have a map $\pi_k(X) \to \pi_k(\Omega\Sigma' X) = \pi_{k+1}(\Sigma' X)$. So this is the map that appears in Freudenthal's suspension theorem.

Lemma 2.5.3.
The forgetful functor $\mathtt{Top}_* \to \mathtt{Top}$ preserves homotopy pullbacks.

We have not really talked about functors between (2,1)-categories yet, but in this particular case, it is like the most stupid functor there could be:

- On objects, we just forget the point: $(X, x_0) \mapsto X$
- On groupoids $HOM((X, x), (Y, y)) \hookrightarrow HOM(X, Y)$ (in fact this is a subcategory).

Proof. (Sketch)
If we were to forget about homotopies, the functor is just one between 1-categories. In that case it has an adjoint: $\mathtt{Top} \to \mathtt{Top}_*$, $X \mapsto X_+ := (X \amalg *, *)$.
The claim is that this is the left adjoint:

$$HOM_{\mathtt{Top}_*}(X_+, (Y, y)) \cong HOM_{\mathtt{Top}}(X, Y)$$

The left hand side is defined
On objects: $X \amalg * \to Y, \quad x \to y$
On morphisms:

$$\begin{tikzcd} X \amalg * \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow \eta"] & Y \end{tikzcd}$$

where $\eta : (X \amalg *) \times I \to Y$ is pointed $* \times I \to y$

The right hand side
on objects by maps $X \to Y$ which is the same data as on the left hand side, because the additional point there has to be sent onto y and does not offer additional choices. Morphisms are homotopies

$$\begin{tikzcd} X \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow \eta"] & Y \end{tikzcd}$$

Those $\eta : X \times I \to Y$ are again the same as for the left hand side because

$$(X \amalg *) \times I \cong X \times I \amalg * \times I$$

From this isomorphism of groupoids follows that the claim holds: Say we are given a homotopy pullback of pointed spaces and we would like to see that if we forget all these points this is still a homotopy pullback square:

$$\begin{tikzcd} (A,a) \arrow[r] \arrow[d] & (B,b) \arrow[d] \arrow[dl, phantom, "H"] \\ (C,c) \arrow[r] & (D,d) \end{tikzcd} \rightsquigarrow \begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \arrow[dl, phantom, "H"] \\ C \arrow[r] & D \end{tikzcd}$$

So we have to check that for any space T and morphisms as indicated, there is a map $T \to A$ as well as homotopies

$$\begin{tikzcd} T \arrow[dr, dashed] \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "u"'] & & \\ & A \arrow[r] \arrow[d] & B \arrow[d] \arrow[dl, phantom, "H"] \\ & C \arrow[r] & D \end{tikzcd}$$

The isomorphism of groupoids that we have seen to exist beforehand, allows us to translate this problem into $\mathtt{Top}_*$:

$$\begin{tikzcd} T_+ \arrow[dr, dashed] \arrow[drr, bend left, "v_+"] \arrow[ddr, bend right, "u_+"'] & & \\ & (A,a) \arrow[r] \arrow[d] & (B,b) \arrow[d] \arrow[dl, phantom, "H"] \\ & (C,c) \arrow[r] & (D,d) \end{tikzcd}$$

and since there was a homotopy that also translated into $\mathtt{Top}_*$, we get by the universal property of the pullback square, a map that fills in the dashed one.

If we had two maps $T \to A$ in the right square, we would get two in the left one. There we would have a homotopy that again translates back into $\mathtt{Top}$. ■

Remark 2.5.4.
The forgetful functor $\mathtt{Top}_* \to \mathtt{Top}$ does not preserve homotopy pushouts, it never even preserved usual pushouts.

Example 2.5.5.
for well pointed spaces $(A, a) \amalg^h_{\mathtt{Top}_*} (B, b) = A \vee B$ but in $\mathtt{Top}$ $A \amalg^h B = A \amalg B$ ($\emptyset \to X$ is a cofibration, thus the pushout computed along that map is the standard one)

Suppose we are given a homotopy pullback in the pointed category
$$\begin{tikzcd} (F,f) \arrow[r] \arrow[d] & (E,e) \arrow[d] \arrow[dl, Rightarrow, "H"'] \\ * \arrow[r] & (X,x) \end{tikzcd}$$

Therefore we get two things:

1. Puppe long exact sequence ($\mathtt{Top}_*$ has a zero object)
$$\begin{tikzcd} \cdots \arrow[r] & \pi_k(\Omega_e E) \arrow[r] \arrow[d, equal] & \pi_k(\Omega_x X) \arrow[r] & \pi_k(F,f) \arrow[r] & \pi_k(E,e) \arrow[r] & \pi_k(X,x) \\ & \pi_{k+1}(E,e) \arrow[r] & \pi_{k+1}(X,x) \arrow[u, "\cong"] & & [(S^k,*),(X,x)]_{\mathtt{Top}_*} & \end{tikzcd}$$
similar to the long exact sequence of a Serre fibration.

2. $$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[d] \arrow[dl, Rightarrow, "h"'] \\ * \arrow[r, "x"] & X \end{tikzcd}$$ is a homotopy pullback in $\mathtt{Top}$. Starting from this diagram, do we get a long exact sequence of homotopy groups?
The problem is
   a) There might not exist $e \in E$ over $x \in X$
   b) Even if we find $f \in F$ such that
$$\begin{tikzcd} f \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & e \arrow[d] \\ * \arrow[r] & x \end{tikzcd}$$
   the homotopy h is in general not pointed!

This problem has sort of a simple solution which needs the following fact, that we will not proof in much detail:

Fact:

Given a 2-commutative diagram in $\mathtt{Top}$
$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "f"] \arrow[dl, Rightarrow, "h"'] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$
where $a \in A$, then
$$\begin{tikzcd} \pi_k(A,a) \arrow[r] \arrow[dd] & \pi_k(B,i(a)) \arrow[d] \\ & \pi_k(D,f(i(a))) \arrow[d, "\cong"', "h_a"] \\ \pi_k(C,j(a)) \arrow[r] & \pi_k(D,g(j(a))) \end{tikzcd}$$

commutes.
The homotopy h induces the isomorphism h_a that makes the diagram commute. $h_a = h|_{a\times I}$ is a path in D.

Dependence of π_k on the basepoint

Let (X, x_0) be a space, x_1 and x_2 shall be points in X, with a path $\gamma \in \pi_1(X)$ between them.

$$\{\gamma \in X^I \mid \gamma(0) = x_0, \gamma(1) = x_1\} =: \Pi_{x_0,x_1}X \xrightarrow[\simeq]{\gamma\circ} \Pi_{x_0,x_2}X$$

Similarly,

$$\Pi_{x_2,x_0}X \xrightarrow[\simeq]{\circ\gamma} \Pi_{x_1,x_0}X$$

Given this γ we can get an equivalence

$$\begin{tikzcd}
\Omega_{x_1}X = \Pi_{x_1x_1} \arrow[r, "\gamma\circ", "\simeq"'] \arrow[dr, "\simeq", "\gamma\circ-\circ\gamma"'] & \Pi_{x_1,x_2}X \\
 & \Pi_{x_2,x_2}X = \Omega_{x_2}X \arrow[u, "\circ\gamma^{-1}"']
\end{tikzcd}$$

This gives us an isomorphism

$$\begin{tikzcd}
\pi_k(\Omega_{x_1}X) \arrow[r, "\cong"'] \arrow[d, phantom, "\cong"] & \pi_k(\Omega_{x_2}X) \arrow[d, phantom, "\cong"] \\
\pi_{k+1}(X, x_1) \arrow[r, "\cong\gamma"] & \pi_{k+1}(X, x_2)
\end{tikzcd}$$

This isomorphism (the lower one) depends on $\gamma \in \Pi_1 X$

Exercise: Every homotopy pullback square is equivalent to the standard pullback square.

Lemma 2.5.6.
Given a homotopy pullback square $\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[d, "f"] \arrow[dl, phantom, "\Leftarrow h"] \\ * \arrow[r, "x"'] & X \end{tikzcd}$
there is a LES of homotopy groups if we choose $e \in E$ such that $f(e)$ and x lie in the same path connected component of X ($f(e) = x \in \pi_0 X$) and $f_0 \in F$ is in the fiber of X such that $f(f_0) = e$ in $\pi_0 E$.

$$\cdots \to \pi_k(F, f_0) \to \pi_k(E, e) \to \pi_k(X, x) \to \pi_{k-1}(F, f_0) \to \cdots$$

Proof.
By the exercise the diagram in the lemma is equivalent to the standard pullback square
$$\begin{tikzcd}
W(f)_* \arrow[r] \arrow[d] & W(f) \arrow[d, "f"] & := \{(e,\gamma) \in X^I | \gamma(0) = f(e)\} \arrow[d, mapsto] \\
* \arrow[r, "x"'] & X & \gamma(1)
\end{tikzcd}$$
By definition we therefore have the cube

$$\begin{tikzcd}
& W(f)_* \arrow[rr] \arrow[dd] & & W(f) \arrow[dd] \\
F \arrow[ur, "\simeq"] \arrow[rr, crossing over] \arrow[dd] & & E \arrow[ur, "\simeq"] & \\
& * \arrow[rr] & & X \\
* \arrow[ur] \arrow[rr] & & X \arrow[ur] \arrow[from=uu, crossing over] &
\end{tikzcd}$$

where all the horizontal are equivalences and therefore induce isomorphisms on the homotopy groups. So if we get a LES for the standard square, we get one for the other square as well, using the fact that homotopy commutative square allow us to identify homotopy groups between each other.

For the standard homotopy pullback square, it suffices to choose a point $e \in E$ and a path $\gamma : f(e) \to x$ to make $\begin{tikzcd} W(f)_x \arrow[r] \arrow[d] & (W(f),(e,\gamma)) \arrow[d] \\ * \arrow[r] & (X,x) \end{tikzcd}$ pointed. So once we get a point over the point x, this is also a homotopy pullback square in the pointed category because the right downward map is a fibration and the diagram is the usual pullback. The pullbacks in `Top` and $\texttt{Top}_*$ are the same by the same explanation we did for homotopy pullbacks.

Thus we get a LES of homotopy groups for this diagram. Using the homotopy equivalences in the cube, we also get a LES for the square we actually started with. ∎

2.6 On the importance of the homotopy fiber

Let (X, A, a) be a pointed pair.

Lemma 2.6.1.
Slogan: Relative homotopy groups can be redefined as the homotopy groups of the fiber.

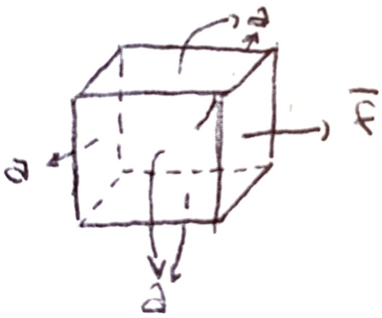
Let $\begin{tikzcd} (F,f) \arrow[r] \arrow[d] & (A,a) \arrow[d] \arrow[dl, Rightarrow] \\ * \arrow[r] & (X,x) \end{tikzcd}$ *be a homotopy pullback square in* $\texttt{Top}_*$.

Then $\pi_k(F, f) \cong \pi_{k+1}(X, A, a)$ *for all* $k \geq 0$.

Proof. (Sketch)
Let's start with an element in $[f] \in \pi_{k+1}(X, A, a)$:

$$\begin{array}{ccc}
I^{k+1} & \xrightarrow{f} & X \\
\cup & & \cup \\
\partial I^{k+1} & \xrightarrow{f} & A \\
\cup & & \cup \\
\mathcal{J}^k & \longrightarrow & \{a\}
\end{array}$$

Note that the middle map gives us a pointed map on the quotient: $\begin{tikzcd} S^k = \partial I^{k+1}/\mathcal{J}^k \arrow[r, "\overline{f}"] \arrow[d] & (A,a) \arrow[d] \arrow[dl, Rightarrow, "h"'] \\ * \arrow[r] & (X,a) \end{tikzcd}$



We can consider the whole cube as the homotopy from the right face to the left one: $I^{k+1} = I^k \times I \xrightarrow{f} X$.
f defines a homotopy $h : I^k \times I \to X$ from $\overline{f}$ to $\overline{a}$. This is a pointed homotopy.

Thus we have a map $\omega : S^k \to (F, f)$ (by existence property of the weak pullback). So we define

$$\pi_{k+1}(X, A, a) \xrightarrow{\cong} \pi_k(F, f)$$
$$[f] \mapsto [\omega]$$

We will use the 5-lemma to see that this map is an isomorphism:

$$\begin{tikzcd}
\pi_{k+1}(A,a) \arrow[r] \arrow[d, equal] & \pi_{k+1}(X,a) \arrow[r] \arrow[d, equal] & \pi_k(F,f) \arrow[r] & \pi_k(A,a) \arrow[r] \arrow[d, equal] & \pi_k(X,x) \arrow[d, equal] \\
\pi_{k+1}(A,a) \arrow[r] & \pi_{k+1}(X,a) \arrow[r] & \pi_{k+1}(X,A,a) \arrow[r] \arrow[u] & \pi_k(A,a) \arrow[r] & \pi_k(X,a)
\end{tikzcd}$$

By 5-lemma we are done.
(One actually would still have to check that the squares on the left commute) ∎

Definition 2.6.2. (Redefinition)
A map $p : E \to X$ is <u>n-connected</u> if all homotopy fibers of f are $(n-1)$-connected.

Remark 2.6.3.
What are the homotopy fibers? We choose an element x in the base space X and construct the homotopy pullback
$$\begin{tikzcd}
F_x \arrow[r] \arrow[d] & E \arrow[d] \\
* \arrow[r, "x"'] & X
\end{tikzcd}$$
F_x is the homotopy fiber over $x \in X$.

For $n = 0$ the old definition was that $\pi_0 E \twoheadrightarrow \pi_0 X$ is surjective. Why does it follow from the new definition? If it is not surjective, we take $x \notin Im(\pi_0 E \to \pi_0 X)$. Then the homotopy fiber is the emptyset:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E \arrow[d] \\
* \arrow[r, "x"'] & X
\end{tikzcd}$$

and $\emptyset$ is not (-1)-connected.

If $F_x \neq \emptyset$ for all fibers, then $x \in Im(\pi_0 E \to \pi_0 X)$.

For $n > 0$, let by the old definition $\pi_i(X, E, e) = 0$ for $0 < i \leq n-1$, then for some $f.$: $\pi_i(X, E, e) \cong \pi_{i-1}(F_2, f.)$ where $x = p(e)$.

Definition 2.6.4.
A 2-commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & A \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\ B \arrow[r] & C \end{tikzcd}$$

in Top is a n-weakly homotopy pullback (or cartesian) if $X \to A \times^h_C B$ is n-connected.

The following proposition is very important because it gives us an criterion to check whether a 2-commutative square is a n-weakly homotopy pullback. Also for CW-complexes whether it is a homotopy pullback.

Proposition 2.6.5.
A 2-commutative square $\begin{tikzcd} X \arrow[r] \arrow[d] & A \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\ B \arrow[r, "j"'] & C \end{tikzcd}$ *is a n-weakly homotopy pullback iff $\forall b \in B$*

$$\begin{tikzcd} X \times^h_B \{b\} \arrow[r] \arrow[d, equal] & A \times^h_C \{j(b)\} \arrow[d, equal] \\ X_b & A_{j(b)} \end{tikzcd}$$

is n-connected.

Remark 2.6.6.
This tells us that checking that the square is a homotopy pullback of CW complexes is equivalent to stating that homotopy fibers are the same for two different parallel arrows.

Proof.

By the actual homotopy pullback of this square: $\begin{tikzcd} X \arrow[r] \arrow[dr, bend right] & B \times^h_C A =: W \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ & B \arrow[r, "j"] & C \end{tikzcd}$

We want to relate the n-connectivity of $X \to B \times^h_C A = W$ to the n-connectivity of the fibers of the map $X \to B$ and of f. Obeservations:

1. The homotopy fibers of f and the map $W \to B$ are homotopy equivalent: $\begin{tikzcd} W_b \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ * \arrow[r, "b"] & B \arrow[r] & C \end{tikzcd}$ Since both smaller squares are homotopy pullbacks, the outer square is as well and therefore $W_b \simeq A_{j(b)}$

2. $\begin{tikzcd} X_b \arrow[rr] \arrow[dr] & & W_b = A_{j(b)} \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\ & X \arrow[r] \arrow[dr] & W \arrow[d] \arrow[l, phantom, near start, "\Leftarrow"] \\ & & B \end{tikzcd}$ So the question of the n-connectedness of the map $X \to W$ is a question of the n-connectedness of the map $X_b \to W_b$ and thus a consequence of 5-lemma.

From this we get two LES on which we apply the 5-lemma:

$$\begin{tikzcd}
\pi_{k+1}B \arrow[r] \arrow[d, equal] & \pi_k(X_b) \arrow[r] \arrow[d] & \pi_k X \arrow[r] \arrow[d] & \pi_k B \arrow[r] \arrow[d, equal] & \pi_{k-1}X_b \arrow[r] \arrow[d] & \pi_{k-1}X \\
\pi_{k+1}B \arrow[r] & \pi_k(W_b) \arrow[r] & \pi_k W \arrow[r] & \pi_k B \arrow[r] & \pi_{k-1}W_b \arrow[r] & \cdots
\end{tikzcd}$$

There is, however, something to consider regarding the 0-th homotopy groups:

$$\begin{tikzcd}
\pi_1 B \arrow[r] \arrow[d] & \pi_0 X_b \arrow[r] \arrow[d] & \pi_0 X \arrow[r] \arrow[d] & \pi_0 B \arrow[d, equal] \\
\pi_1 B \arrow[r] & \pi_0 W_b \arrow[r] & \pi_0 W \arrow[r] & \pi_0 B
\end{tikzcd}$$

Thus if $\pi_0 X \twoheadrightarrow \pi_0 W$ then by 5-lemma $\pi_0 X_b \twoheadrightarrow \pi_0 W_b$.
In the other direction, suppose $\pi_0 X_b \twoheadrightarrow \pi_0 W_b$ is surjective for all b.
Then $w \in \pi_0 W \mapsto \tilde{b} \in \pi_0 B \implies \exists \tilde{\omega} \in \pi_0 W_{\tilde{b}} \implies \tilde{x} \in \pi_0 C_{\tilde{b}} \implies x \in \pi_0 X$ that is mapped onto $w \in \pi_0 W$.

■

Corollary 2.6.7.
Let $\begin{tikzcd} X \arrow[r] \arrow[d] & A \arrow[d] \arrow[dl, phantom, "\swarrow"] \\ B \arrow[r, "j"'] & C \end{tikzcd}$ *be a 2-commutative square in* **CW**.

1. *it is homotopy pullback (=cartesian)* $\Leftrightarrow \forall b \in B : X_b \simeq A_{j(b)}$
2. $\forall b$*:* $X_b \simeq A_{j(b)} \Leftrightarrow \forall a \in A :$

$$\begin{array}{ccc} X_a & \simeq & B_{f(a)} \\ \| & & \| \\ X \times^h_A \{a\} & & B \times^h_C \{f(a)\} \end{array}$$

Remark 2.6.8.
The next theorem we are going to proof is the Blakers-Massey Theorem. This is, however, nothing more than the homotopic version of excision.
In future we will nevertheless refer by excision to the old version.

Theorem 2.6.9 (Blakers-Massey: homotopic version of excision)**.**
Let

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d] \arrow[dl, phantom, "\swarrow H"] \\
C \arrow[r] & X
\end{tikzcd}$$

be a 2-commutative square in **Top**.
If f is m-connected, g is n-connected ($m, n \geq 0$), then the square is $(m+n-1)$-weakly homotopy cartesian.

Remark 2.6.10. Blakers-Massey $\implies$ Excision for CW-complexes:
Let f, g be cofibrations (inclusions of CW-complexes). We have a pushout square, because when f, g are cofibrations then it automatically is a homotopy pushout (so H is constant).
The assumptions were the same and the conclusion was, so what we need to show is that

$$\pi_i(B, A) \to \pi_i(X, C)$$

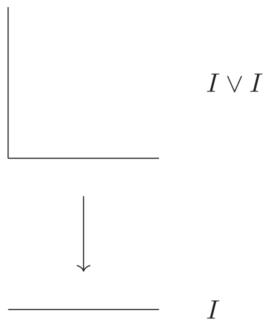
is surjective for $i = m + n$ and isomorphisms for $i \leq m + n - 1$.
But we have seen last time that $\pi_{i-1}(Fib(f))$ and $\pi_i(X, C) \cong \pi_{i-1}(Fib(u))$.
So we are done by previous statements.

2.7 Quasi-fibrations

Definition 2.7.1.
A map $f : E \to B$ is a quasi-fibration if for all $b \in B$ its fibers are the same as its homotopy fibers, i.e. $f^{-1}(b) \to E \times^h_B \{b\}$ is a weak equivalence.

Example 2.7.2.

- fibrations are quasi-fibrations
- Serre fibrations of CW-complexes are quasi-fibrations
  This is because if you have a Serre fibration of CW-complexes its homotopy fibers are also CW-complexes. You might compare this fiber of a Serre fibration to the homotopy fiber. The homotopy fibers of a Serre fibration can be computed using the LES of Serre fibrations. The homotopy groups of the homotopy groups can also be computed by a LES.

  So 5-lemma applied to LES and by Whitehead theorem, we are done.
- 

  Clearly all the fibers are homotopy equivalent to the point so the homotopy fibers of the map are also equivalent to the point because it is a homotopy equivalence.
  Therefore this is a quasi-fibration which is neither Serre-fibration nor fibration.

Remark 2.7.3.
If b is in the image of f and we have another point b' which lies in the same path-connected component as b ($b' \sim b$ in $\pi_0 B$), then $b' \in \mathrm{im}(f)$.
Because if b is in the image of f, it implies that the homotopy fiber over this point is non-empty and therefore if the fiber should be weakly equivalent to the homotopy fiber, the fiber should also be non-trivial.
So the homotopy fiber for b and b' are the same by one of the exercises and therefore if one of the fibers is non-empty, so is the other one.

In particular, if the base B is path-connected ($\pi_0 B = *$), then f is surjective.

Warning: Quasi-fibrations are not stable under pullback.

For the purpose of the next lemma, we reformulate the weak equivalence:

$$\begin{array}{c} (E, p^{-1}(b)) \longrightarrow (B, b) \text{ is a weak equivalence} \\ \Updownarrow \\ f^{-1}(b) \longrightarrow E \times^h_B \{b\} \text{ is a weak equivalence} \end{array}$$

To see that these two conditions are equivalent, we just have to look at the LES of homotopy groups. The LES for $(E, p^{-1}(b))$ involves E, the fiber and the relative homotopy group. So if the upper map is a weak equivalence then the relative homotopy group of the left hand side is the same as for

(B, b). So in this LES you get homotopy groups of E, B and the fiber and comparing this LES with the LES of the homotopy fiber you get that the homotopy groups of the upper left pair are the same as for the ones on the lower right side.

Definition 2.7.4.
$A \subset B$ is distinguished with respect to $f : E \to B$ if $f^{-1}(A) \to A$ is a quasi-fibration.

Lemma 2.7.5.
If $A \subset B$ is distinguished, then the following are equivalent:

1. $f : (E, f^{-1}(A)) \to (B, A)$ *is a weak equivalence*
2. $f : (E, f^{-1}(a)) \to (B, a)$ *is a weak equivalence for all* $a \in A$

Proof. (Sketch)

$$\begin{tikzcd}
f^{-1}A \arrow[r, hook, "i"] \arrow[d] & E \arrow[d] \\
A \arrow[r, hook, "j"] & B
\end{tikzcd}$$

(1) is equivalent to the statement that the homotopy fibers of i and j are the same (weakly equivalent)
(2) is the same as that the homotopy fiber of f at a is the fiber $f^{-1}(a)$ for all $a \in A$

But we have proven last time that the claim that the homotopy fibers of i and j are weakly equivalent is the same as saying that the square is infinitely weakly cartesian. Also the equivalent statement to (2) is the same statement. ∎

Proposition 2.7.6 (Locality of quasi-fibrations)**.**
Let $p : E \to B$ be any continuous map, $B = B_0 \cup B_1$, such that $B_0^\circ \cup B_1^\circ = B$; ($B_{01} := B_0 \cap B_1$)

If B_0, B_1, B_{01} are distinguished for p, then so is B (i.e. p is a quasi-fibration).

Theorem 2.7.7 (May, "Weak equivalences and quasi-fibrations")**.**
Let $f : (X, X_0, X_1) \to (Y, Y_0, Y_1)$ be a map of triples, assume that $X_0^\circ \cup X_1^\circ = X$ and $Y_0^\circ \cup Y_1^\circ = Y$. If $(X_i, X_{01}) \to (Y_i, Y_{01})$ is a weak equivalence for $i = 0, 1$, then $(X, X_i) \to (Y, Y_i)$ is a weak equivalence, for $i = 0, 1$.

Proof. (of proposition)
By lemma above $p : (\underbrace{p^{-1}B_i}_{E_i}, \underbrace{p^{-1}B_0 \cap p^{-1}B_1}_{E_{01}}) \to (B_i, B_{01})$ is a weak equivalence.
Then $(E, E_i) \to (B, B_i)$ is a weak equivalence for $i = 0, 1$. Again by lemma, this means that $(E, p^{-1}(b)) \to (B, b)$ is a weak equivalence, $b \in B_0 \cup B_1 = B$. ∎

2.8 Mather's Cube Theorems

Theorem 2.8.1 (Mather's First Cube Theorem)**.**
Let

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[drr] \arrow[dd, "f_A"] & B \arrow[drr, "\gamma"] \arrow[dd, "\beta"] \arrow[ddl, phantom, "F_1"] & & \\
& & C \arrow[r, "\delta"] \arrow[dd] \arrow[r, phantom, "H"] \arrow[ddll, phantom, "F_2"] & D \arrow[dd] \\
A' \arrow[r, "\alpha'"] \arrow[drr] & B' \arrow[drr, "\beta'"] & & \\
& & C' \arrow[r] & D'
\end{tikzcd}$$

be a 2-commutative cube in **Top** *or* $\mathbf{Top}_*$.
Assume that the the top (H) and the bottom (H') are homotopy cocartesian.

If the left (F_2) and the rear (F_1) are *(strict) homotopy cartesian,*
(weak) n-weakly homotopy cartesian,

then the right (F_3) and the front (F_4) are *(strict) homotopy cartesian.*
(weak) n-weakly homotopy cartesian.

Remark 2.8.2.
For CW-complexes, the weak version ($n = \infty$) implies the strict version.

Proof. (sketch, strict version for CW)
Preparation step: We can assume that f_A, f_B, f_C are fibrations, $A = A' \times_{B'} B$, so F_2 is trivial

By Exercise the rear face
$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \arrow[dl, phantom, "F_1"] \\ A' \arrow[r] & B' \end{tikzcd}$$
is equivalent to
$$\begin{tikzcd} \overline{A} \arrow[r] \arrow[d] & \overline{B} \arrow[d] \arrow[dl, phantom, "F_1"] \\ \overline{A}' \arrow[r] & \overline{B}' \end{tikzcd}$$
the standard homotopy pullback.
So by definition, this means that there is the following 2-commutative cube:

$$\begin{tikzcd} \overline{A} \arrow[rr] \arrow[dr, "\simeq"] \arrow[dd] & & \overline{B} \arrow[dr, "\simeq"] \arrow[dd] & \\ & A \arrow[rr] \arrow[dd] & & B \arrow[dd] \\ \overline{A}' \arrow[rr] \arrow[dr, "\simeq"] & & \overline{B}' \arrow[dr, "\simeq"] & \\ & \overline{A}' \arrow[rr] & & \overline{B}' \end{tikzcd}$$

where all the horizontal maps are homotopy equivalences. We now just add it to the previous cube.

claim: the composition with this cube does not break the assumptions of the theorem.
For example the top face:

$$\begin{tikzcd} \overline{A} \arrow[r, "\simeq"] \arrow[d] & A \arrow[r] \arrow[d] & C \arrow[d] \arrow[dl, phantom, "H"] \\ \overline{B} \arrow[r, "\simeq"] & B \arrow[r] & D \end{tikzcd}$$

The right square is a homotopy pushout. The left is as well because of the equivalences. We can see that by introducing the pushout Q

$$\begin{tikzcd} \overline{A} \arrow[rr, "\simeq"] \arrow[dd] & & A \arrow[r] \arrow[dl] \arrow[dd] & C \arrow[dd] \arrow[ddl, phantom, "H"] \\ & Q \arrow[dr] & & \\ \overline{B} \arrow[ur, "\simeq"] \arrow[rr, "\simeq"] & & B \arrow[r] & D \end{tikzcd}$$

Because the composition of two equivalences is an equivalence, the map $Q \to B$ is an equivalence. So the left and the right square are homotopy pushout squares and therefore so is the outer one.

Now the rear face is the standard homotopy pullback. The right map is a fibration, so since it is a pullback, so is the left one.

We can factor the map $C \to C'$ as
$$\begin{tikzcd} C \arrow[dd] \arrow[dr, "\omega"] & \\ & \overline{C} \arrow[ul, dashed, bend right, "\omega^{-1} heq"'] \arrow[dl, "\overline{f}_C"] \\ C' & \end{tikzcd}$$
where $\overline{A}f_C$ is a fibration and ω and its inverse commute to the identity up to h_C.

We now change the cube by replacing C by $\overline{C}$:

$$\begin{tikzcd} A \arrow[rr, "\alpha"] \arrow[dr, "\omega\circ\beta=\overline{\beta}"] \arrow[dd, "f_A"] & & B \arrow[dr, "\gamma"] \arrow[dd] & \\ & \overline{C} \arrow[rr, "\delta\circ\omega^{-1}"] \arrow[dd, "\overline{f}_C"] & & D \arrow[dd, "f_D"] \\ A' \arrow[rr] \arrow[dr, "\beta'"] & & B' \arrow[dr] & \\ & C' \arrow[rr] & & D' \end{tikzcd}$$

Regard the face

$$\begin{tikzcd} C \arrow[r, "\delta"] \arrow[d, "\omega"] & D \arrow[d, equal] \arrow[dl, phantom, "\Leftarrow h_C"] \\ \overline{C} \arrow[r, "\delta\circ\omega^{-1}"] \arrow[d] & D \arrow[d] \arrow[dl, phantom, "\Leftarrow F_4"] \\ C' \arrow[r] & D' \end{tikzcd}$$

$\rightsquigarrow$ $\overline{F}_4$ is the composition of $\delta \circ h_C$ and F_4. Similarly $\overline{H}, \overline{F_2}$. This gives a 2-commutative cube.

Finally we change $\overline{\beta}$ to a morphism such that F_2 is trivial.

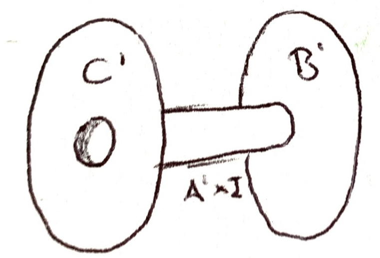
$$\begin{tikzcd} A \arrow[r] \arrow[d] & \overline{C} \arrow[d, "\overline{f}_C"] \arrow[dl, phantom, "\Leftarrow"] \\ A' \arrow[r] & C' \end{tikzcd}$$

$\overline{f}_C$ is a fibration. Therefore we can lift the homotopy from $\overline{\beta}$ to another map, then the new map between A and $\overline{C}$ that makes the square commutative without any homotopies.

<u>Construction step:</u>

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] \arrow[dd, "f_A"] & & B \arrow[dr] \arrow[dd, "f_B"] & \\ & C \arrow[rr] \arrow[dd, "f_C"] & & C \cup_A^h B \arrow[dd, "q"] \\ A' \arrow[rr] \arrow[dr, "\alpha'"] & & B' \arrow[dr] & \\ & C' \arrow[rr] & & C' \cup_{A'} B' \end{tikzcd}$$

where $C \cup_A^h B$ is the standard homotopy pushout and $C' \cup_{A'} B' := C' \amalg B' \amalg A'^I / \begin{matrix} (a,0) \sim \alpha'(a) \\ (a,1) \sim \beta'(a) \end{matrix}$



On this very concrete space you just define q by these maps, so on C it goes to C' by f_B on B it goes to B' by f_B and on the cylinder of A it goes by $f_A \times id$.
Because of the commutativity of those two squares (no homotopy) you get that q is a map of topological spaces.
So the cube is 2-commutative, but the only two non-trivial homotopies are h and h' in the upper and lower face.

Note that this new cube also satisfies the assumption of the first cube theorem.
We will now prove this theorem for this particular cube. For this we need to prove for example that the front face is homotopy cartesian.

<u>claim:</u> q is a quasi-fibration.
For this we need the locality property we have seen for quasi-fibrations. To do so, let $U_0 := C' \cup A' \times [0, 2/3)$ and $U_1 := B' \cup A' \times [1/3, 1]$ be open in $C' \cup^h_{A'} B'$. Their intersection is $U_{01} = A' \times [1/3, 2/3]$.

- $$\begin{tikzcd} q : q^{-1}(U_{01}) \arrow[r] \arrow[d, phantom, "\cong"] & U_{01} \arrow[d, equal] \\ A \times (1/3, 2/3) \arrow[r, "f_A \times id"] & A' \times (1/3, 2/3) \end{tikzcd}$$ is a quasi-fibration because neither the fibers nor the homotopy fibers are changed by the product, since they are homotopy equivalent as the interval is contractible.

- $$\begin{tikzcd} C \arrow[d, "f_C"] & q^{-1}(U_0) \cong Cyl(\alpha) \arrow[l] \arrow[d, "q"] & A \times (0,1] \arrow[l] \arrow[d, "f_A"] \\ C' & U_0 \simeq Cyl(\alpha') \arrow[l, "heq"'] & A' \arrow[l] \end{tikzcd}$$ The left square is a homotopy pullback as well as the outer square. Therefore the right square is homotopy cartesian.

  fiber of q at (a', t) $(t > 0)$ = fiber of f_A at a'. We would like to compare it to the homotopy fiber of q at this point. The fiber of q at (a', t) $(t > 0)$ = fiber of q at $(a', 1)$, whereas the fiber of f_A at a' = homotopy fiber of f_A at a' which in turn is isomorphic the homotopy fiber of f_A at $(a', 1)$.

  The fiber of q at c' is the same as the fiber of f_C at C' which is the homotopy fiber of f_C at C' and because we saw this square to be cartesian, this is the homotopy fiber of q at C'.

  Let's look at the square that we want to show to be a homotopy pullback (of CW complexes):

$$\begin{tikzcd} C \arrow[r] \arrow[d, "f_C"] & C \cup^h_A B \arrow[d, "q"] \\ C' \arrow[r] & C' \cup^h_{A'} B' \end{tikzcd}$$

It is equivalent that for all c' the homotopy fibers of f_C at c' are isomorphic to the homotopy fibers of q at c'.
The first are the fibers of f_C at c', the latter the fibers of q at c. Those fibers are the same by construction, so the homotopy fibers are also the same.

Therefore this is a homotopy pullback and we have proven the first Mather's Cube Theorem for this particular cube (we have done it for the front face, but there is a symmetry of the cube along the diagonal crossing $C \cup^h_A B$).

Final step:
Because of 2-commutativity of cubes, we obtain maps and homotopies

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd, "f_A"'] & & B \arrow[dr] \arrow[dd, "f_B" near start] \arrow[rrd, bend left] & & \\
& C \arrow[rr] \arrow[dd, "f_C" near start] \arrow[rrr, bend right] & & C \cup^h_A B \arrow[dd, "q" near start] & D \arrow[dd, "f_D"] \\
A' \arrow[rr] \arrow[dr, "\alpha'"] & & B' \arrow[dr] \arrow[rrd, bend left] & & \\
& C' \arrow[rr] \arrow[rrr, bend right] & & C' \cup_{A'} B' \arrow[r] & D'
\end{tikzcd}$$

Let's look at these two squares separately:

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "f_C"] & C \cup^h_A B \arrow[r, "heq"] \arrow[d, "q"] & D \arrow[d, "f_D"] \\
C' \arrow[r] & C' \cup^h_{A'} B' \arrow[r] & D'
\end{tikzcd}$$

The left square is homotopy cartesian by construction step, the right by assumption, thus

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & D \arrow[d] \arrow[dl, phantom, "\Swarrow F_3"] \\
C' \arrow[r] & D'
\end{tikzcd}$$

is homotopy cartesian.

■

Remark 2.8.3.
The weak version is proved the same way once one checks "pasting" of n-weakly homotopy cartesian squares.

Theorem 2.8.4. *(Second Mather's Cube Theorem)*
Let

$$
\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd] & & B \arrow[dr] \arrow[dd] & \\
& C \arrow[rr] \arrow[dd] & & D \arrow[dd] \\
A' \arrow[rr] \arrow[dr] & & B' \arrow[dr] & \\
& C' \arrow[rr] & & D'
\end{tikzcd}
$$

be a 2-commutative cube in $\mathtt{Top}$ *or* $\mathtt{Top}_*$.
If the vertical faces (F_1, F_2, F_3, F_4) *are homotopy cartesian and the bottom face* (H') *is homotopy cocartesian, then the top face is also homotopy cocartesian.*

Proof. (for CW)
We can construct sort of a new cube here by universal properties of homotopy pushouts:

$$
\begin{tikzcd}
A \arrow[rrrr] \arrow[ddr] \arrow[ddd] & & & & B \arrow[dll] \arrow[ddr] \arrow[ddd] & \\
& & C \cup_A B \arrow[ddd] \arrow[drrr, dashed] & & & \\
& C \arrow[ur] \arrow[rrrr] \arrow[ddd] & & & & D \arrow[ddd] \\
A' \arrow[rrrr] \arrow[ddr] & & & & B' \arrow[dll] \arrow[ddr] & \\
& & C' \cup_{A'} B' \arrow[drrr, dashed] & & & \\
& C' \arrow[ur] \arrow[rrrr] & & & & D'
\end{tikzcd}
$$

For this new cube (with $C \cup_A^h B$, $C' \cup_{A'} B'$), we have the assumptions of the first Mather's cube theorem:

- the bottom face is a homotopy pushout: by definition
- the top face is a homotopy pushout: by definition
- the left face is a homotopy pullback: by assumption
- the rear face is a homotopy pullback: by assumption
- all vertical arrows are cartesian

Hence we get that

$$
\begin{tikzcd}
C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C \cup_A^h B \arrow[r] \arrow[d] & D \arrow[d] \\
C' \arrow[r] & C' \cup_{A'} B' \arrow[r] & D'
\end{tikzcd}
$$

the left square is homotopy cartesian, the big square is homotopy cartesian by assumption.
It is not true for arbitrary topological spaces or arbitrary weak 2-pullbacks, but in the case of CW-complexes and with the further assumption that $\pi_0 C' \twoheadrightarrow \pi_0(C' \cup_{A'} B')$ is surjective the right square is homotopy cartesian. (we will prove this Lemma next time)

This implies that $C \cup_A^h B \xrightarrow{\cong} D$ ∎

Lemma 2.8.5.
Let

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] \arrow[dl, phantom, "\Swarrow u"] & C \arrow[d] \arrow[dl, phantom, "\Swarrow v"] \\ D \arrow[r] & E \arrow[r] & F \end{tikzcd}$$

be a 2-commutative diagram in **CW**.
If the left square and the big square are homotopy pullbacks and

$$\pi_0 D \twoheadrightarrow \pi_0 E$$

is surjective, then the right square is also a homotopy pullback.

Proof.

For CW-complexes it suffices that for all $e \in E$
$$\begin{tikzcd} B_e \arrow[r] \arrow[d, equal] & C_{v(e)} \arrow[d, equal] \\ B \times_E \{e\} & C \times_F \{v(e)\} \end{tikzcd}$$
are weak (= homotopy) equivalences.
Homotopy fibers as well as all homotopy pushouts and pullbacks we are constructing of CW-compelexes have the homotopy type of CW-complexes by theorem of Milnor.

By exercise 5.2 it suffices to do that for representatives of $\pi_0 E$. This is because all homotopy fibers of points in a path-connected component are homotopy equivalent (though not canonically, choice of path necessary). In this case, where we are looking at the fibers of points in the same path-connected component - and we know the isomorphism to not be canonical - we still can make our diagram with fibers commutative, by first setting a path connecting the points in E and then regard the image of the path in F. Afterwards we can choose the isomorphisms on the fibers such that they are induced by this path and its image, so the diagram remains commutative.

Therefore we can now only look at one point in each path-connected component. Now we can use the condition that $\pi_0 D \twoheadrightarrow \pi_0 E$ is surjective, so instead of choosing a point in E we can choose a point $d \in D$ which is sent to a point $u(d) \in E$.
So if we prove the homotopy equivalence for all points of this form we are done.

We take such a point d and use the pullback property of the left and outer square:

$$\begin{tikzcd} A_d \arrow[r, "heq", "\text{l-square}"'] \arrow[rr, bend right, "\text{o-square}", "heq"'] & B_{u(d)} \arrow[r] & C_{v(u(d))} \end{tikzcd}$$

It follows that the map $B_{u(d)} \to C_{v(u(d))}$ is also a homotopy equivalence. ■

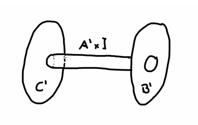
Remark 2.8.6.
In the proof of the second Mather's cube theorem, where we needed the lemma we just proved, we needed to assume that the map on π_0 was surjective.
The situation was that we had a cube with a base face

$$\begin{tikzcd} A' \arrow[rr] \arrow[dd] & & B' \arrow[dl] \arrow[dd] \\ & C' \cup_{A'} B' \arrow[dr, "\square"] & \\ C' \arrow[ur] \arrow[rr] & & D' \end{tikzcd}$$

So what we needed to assume was that $\pi_0 C' \twoheadrightarrow \pi_0(C' \cup_{A'} B')$ was surjective. But the map $C' \cup_{A'} B' \to D'$ is a homotopy equivalence by the assumption that the whole square is a homotopy pushout. So the assumption is equivalent to the one that $\pi_0 C' \twoheadrightarrow \pi_0 D'$ is surjective:

$$\begin{tikzcd}
\pi_0 C' \arrow[r, two heads] \arrow[dr, two heads] & \pi_0(X' \cup_{A'} B') \arrow[d, "\simeq"] \\
 & \pi_0 D'
\end{tikzcd}$$

This is in fact equivalent, in this square, to $\pi_0 A' \twoheadrightarrow \pi_0 B'$ being surjective. One can easily see these to be equivalent e.g. by looking at the standard homotopy pushout:



Because if $\pi_0 A' \twoheadrightarrow \pi_0 B'$ is surjective this means that we can connect every point in B' to a point in A', thus B' has the same connected components as the cylinder and the cylinder is glued to C'.

Assume that $\pi_0 A' \to \pi_0 B'$ is not surjective:
For CW connected and path-connected components are the same so it means that B' can be decomposed to:

$$\begin{tikzcd}
A' \arrow[r, "0-conn"] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{B}' \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{B}' \amalg T \simeq B' \arrow[d] \\
C' \arrow[r, "0-conn"] & \overline{D}' \arrow[r] & \overline{D}' \amalg T \simeq D'
\end{tikzcd}$$

The homotopy equivalence to D' follows because the left and right squares being homotopy pushouts implies that the outer one is too.

Let's look at one of the faces of the cube we had, now knowing that we can decompose this way:

$$\begin{tikzcd}
C & & D \simeq \overline{D} \amalg W \arrow[d, "f_D"] \\
C' \arrow[r] & \overline{D}' \arrow[r] & \overline{D}' \amalg T
\end{tikzcd}$$

where W is the preimage of T, $W := F_D^{-1}(T)$. Now lets take the homotopy pullback square here. We claim that this is just $\overline{D}$. One can check it by the universal property, maybe you can prove that $\overline{D}' \to \overline{D}' \amalg T$ is a fibration or you can look at the homotopy fibers...
Even without checking, it might be believable that since T is not in the image of this map, it has no influence on the homotopy pullback and without T this map is just the identity and the homotopy pullback is just the pullback of the preimage of this $\overline{D}'$ in D.

$$\begin{tikzcd}
C & \overline{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & D \simeq \overline{D} \amalg W \arrow[d, "f_D"] \\
C' \arrow[r] & \overline{D}' \arrow[r] & \overline{D}' \amalg T
\end{tikzcd}$$

So we have the original object C in the upper left corner and the total square is a pullback and thus the left square is a pullback as well.

$$\begin{tikzcd}
C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & D \simeq \overline{D} \amalg W \arrow[d, "f_D"] \\
C' \arrow[r] & \overline{D}' \arrow[r] & \overline{D}' \amalg T
\end{tikzcd}$$

The claim is that we can factor one face of the cube in this fashion.
The original cube was:

$$
\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd] & & \overline{B} \arrow[rr] \arrow[dr] \arrow[dd] & & B \arrow[dr] \arrow[dd] & \\
& C \arrow[rr] \arrow[dd] & & \overline{D} \arrow[rr, "f_D"] \arrow[dd] & & \overline{D}\amalg W \arrow[dd] \\
A' \arrow[rr] \arrow[dr] & & \overline{B}' \arrow[rr] \arrow[dr] & & \overline{B}'\amalg T \arrow[dr] & \\
& C' \arrow[rr] & & \overline{D}' \arrow[rr] & & \overline{D}'\amalg T
\end{tikzcd}
$$

In the bottom face we have started with this decomposition as a composition of two squares where we just add the T. Then we checked that we can also do the decomposition of the front facing face and now let me do that similarly for the right face.
We can take the pullback of the triangle

$$
\begin{tikzcd}
& D\amalg W \arrow[d] \\
\overline{B}'\amalg T \arrow[r] & \overline{D}'\amalg T
\end{tikzcd}
$$

and this is going to be B, but over T we have W and over $\overline{D}'$ we have $\overline{D}$, so we have a pullback face

$$
\begin{tikzcd}
\overline{B} \arrow[dr] \arrow[dd] \arrow[ddr, phantom, "\square", near start] & \\
& \overline{D} \arrow[dd] \\
\overline{B}' \arrow[dr] & \\
& \overline{D}'
\end{tikzcd}
$$

and B decomposes as $B \simeq \overline{B}\amalg W$ because the right face of the cube is a homotopy pullback square.

Recall that what we wanted to show was that if the bottom face is a homotopy pushout and all of the side faces are homotopy pullbacks then the top face is a homotopy pushout.

$$
\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd] \arrow[ddr, phantom, "\square", near start] \arrow[drr, phantom, "\square", near start] & & \overline{B}\simeq\overline{B}\amalg W \arrow[rr] \arrow[dr] \arrow[dd] \arrow[ddr, phantom, "\square", near start] \arrow[drr, phantom, "\square", near start] & & B \arrow[dr] \arrow[dd] \arrow[ddr, phantom, "\square", near start] & \\
& C \arrow[rr] \arrow[dd] \arrow[ddrr, phantom, "\square", near start] & & \overline{D} \arrow[rr, "f_D"] \arrow[dd] \arrow[ddrr, phantom, "\square", near start] & & \overline{D}\amalg W\simeq D \arrow[dd] \\
A' \arrow[rr] \arrow[dr] & & \overline{B}' \arrow[rr] \arrow[dr] & & \overline{B}'\amalg T \arrow[dr] & \\
& C' \arrow[rr] \arrow[urr, phantom, "\square", near end] & & \overline{D}' \arrow[rr] \arrow[urr, phantom, "\square", near end] & & \overline{D}'\amalg T
\end{tikzcd}
$$

The top right square is the usual pushout and thus in particular a homotopy pushout.
Therefore if we show that

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \overline{B} \arrow[d] \\ C \arrow[r] & \overline{D} \end{tikzcd}$$

is a homotopy pushout, then the whole upper face is also a homotopy pushout and this is actually the face we started with:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

So the upshot is: last time we used the lemma, which required an additional condition we did not actually have when we used it. We now got rid of this additional condition by decomposing the total cube into two cubes where in the left one the condition is satisfied.

2.9 James reduced product = James construction & Freudenthal's suspension theorem

Let (X, e) be a pointed connected CW-complex.

Definition 2.9.1.
The James construction $\mathcal{J}(X)$ of (X, e) is

$$\mathcal{J}(X) := \operatorname*{colim}_{k\geq 0} \mathcal{J}^k(X)$$
$$\text{where } \mathcal{J}^k(X) = \{(x_1, ..., x_r) \mid 1 \leq r \leq k, x_i \in X\} \amalg */ \sim$$

where the equivalence relation settles that whenever in this sequence of points of X there is the distinguished point e we can just throw it away:

$$(x_1, ..., x_i, e, x_{i+2}, ..., x_r) \sim (x_1, ...x_i, x_{i+2}, ..., x_r)$$
$$(e) \sim *$$

Remark 2.9.2.
We can easily give $\mathcal{J}^k(X)$ a CW-structure. We have a surjective map $X^{\times k} \twoheadrightarrow \mathcal{J}^k(X)$ because the elements in $\mathcal{J}^k(X)$ are k-tupels of elements of X (if we assume that whenever $r < k$ we just fill up with e's)
What is glued along this map are subspaces of the form $X \times ... \times \{e\} \times X \times ... \times ...\{e\} \times ... \times X$. $X^{\times k}$ has a CW-structure given by the product of CW-complexes and because (X, e) is a pointed CW-complex, of course e is the 0-cell in the CW-structure of X. So the subspaces are CW-subcomplexes and we can take the quotient space which will have the CW-structure such that the map $X^{\times k} \twoheadrightarrow \mathcal{J}^k(X)$ is cellular.

$$X \times ... \times \{e\} \times X \times ... \times ...\{e\} \times ... \times X \hookrightarrow X^{\times k} \twoheadrightarrow \mathcal{J}^k(X)$$

(clearly $\mathcal{J}^k(X) \subset \mathcal{J}^{k+1}(X)$ is a cellular inclusion, thus the colimit is again a CW-complex)

One can think about $\mathcal{J}(X)$ as the free associative monoid on (X, e) because we have just taken all the points in X and allowed them to multiply which means concatenate them and we do not have any new relations on them except for e being the unit element of this monoid.

Theorem 2.9.3.
$\mathcal{J}(X)$ is homotopy equivalent to the loops of the reduced suspension of X $\Omega\Sigma' X$.

Remark 2.9.4.
In fact, there is a map $\lambda : \mathcal{J}X \to \Omega\Sigma' X$ which is a map of H-spaces.

Here by the loops I do not mean the homotopy pullback, so a space defined up to homotopy equivalence, but just we can take some concrete model of the loops, usually just the loops in the space.
The loops of any space have a structure of H-spaces (proved in AT I on some exercise sheet). For an H-space the fundamental group is always commutative.
The structure of an H-space is the concatenation of loops.

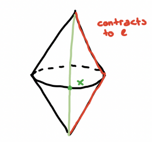
On $\mathcal{J}(X)$ the H-space structure is given by the concatenation of tupels.

Moreover we said that $\mathcal{J}(X)$ is sort of a free associative monoid, so if you want to define such a map of H-spaces it suffices to define it at just any one point x_i, so a sequence of length 1.

We are going to explain how this map is defined, but will not check anything about it.

$$\lambda(x) := \overline{(x, t)}$$

This is the interval in the normal suspension, the bar indicates that we take the quotient to get the reduced suspension. In the reduced suspension this is a loop.



One should show that those two spaces in the theorem are not just abstractly homotopic equivalent but that λ induces the homotopy equivalence.

Remark 2.9.5.
This is the geometric input that allows you to prove the Freudenthal's suspension theorem because it connects the homotopy groups of X with the homotopy groups of $\Omega\Sigma' X$.
So instead of looking at $\Omega\Sigma' X$ where we do not now how it is constructed, we can instead look at $\mathcal{J}(X)$ for which we have an explicit construction. We are going to see that from this claim the Freudenthal's suspension theorem easily follows.

Proof. for compact X ($\Leftrightarrow$ X has only finitely many cells)
There is a continuous map which is the concatenation

$$\begin{aligned} T : X \times \mathcal{J}X &\to \mathcal{J}X \\ (x, (x_1, ..., x_r)) &\mapsto (x, x_1, ..., x_r) \end{aligned}$$

This is a continuous map for all CW complexes X but for compact ones this is very easy to check because $X \times \mathcal{J}X \cong \operatorname{colim}_{n\geq 0} X \times \mathcal{J}^n X$ and on the finite level to prove that this map is continuous you can lift it to the products of X as covers and there the map clearly is continuous and then one just has to check that it factors.
But this uses that X is compact.
T is just the colimit of the maps defined at the finite level

$$X \times \mathcal{J}^n X \xrightarrow{T} \mathcal{J}^{n+1} X$$

Lemma 2.9.6.

$$\begin{tikzcd} X \times \mathcal{J}^n X \arrow[r, "T"] \arrow[d, "pr_2"] & \mathcal{J}^{n+1} X \arrow[d] \\ \mathcal{J}^n X \arrow[r] & * \end{tikzcd}$$

is a homotopy pushout for all n.

Proof.
By induction on n.
$n = 0$: $\mathcal{J}^0 X = e$ and $\mathcal{J}^1 X$ are sequences of length 1 were we identify the point e with the distinguished point so this is just $\mathcal{J}^1 X = X$, so the diagram is

$$\begin{tikzcd} X \cong X \times * \arrow[r, "T"] \arrow[d, "pr_2"] & X \arrow[d] \\ * \arrow[r] & * \end{tikzcd}$$

and obviously a pushout.

$n \to n + 1$:
We factor T through a space $X \triangleright \mathcal{J}^{n+1}(X) \subset X \times \mathcal{J}^{n+1}(X)$ defined by:

$$X \triangleright \mathcal{J}^{n+1}(X) := X \times \mathcal{J}^n(X) \cup_{\{e\}\times\mathcal{J}^n(X)} \{e\} \times \mathcal{J}^{n+1}(X)$$

Those are all CW-complexes and all the maps are cellular (inclusions of CW-complexes).

$$\begin{tikzcd}
X \triangleright \mathcal{J}^{n+1}(X) \arrow[d, "\overline{T}"] & := & X \times \mathcal{J}^n(X) \arrow[dll, "T"] & \cup_{\{e\}\times\mathcal{J}^n(X)} & \{e\} \times \mathcal{J}^{n+1}(X) \arrow[dllll, "id"] \\
\mathcal{J}^{n+1}(X) & & & &
\end{tikzcd}$$

The main claim is that we have a commutative diagram

$$\begin{tikzcd}
X \times \mathcal{J}^n X \arrow[rr, "pr_2"] \arrow[d] \arrow[dd, bend right=60, "T"'] & & \mathcal{J}^n X \arrow[d, hook] \\
X \triangleright \mathcal{J}^{n+1} X \arrow[r, hook] \arrow[d, "\overline{T}"] \arrow[ur, phantom, "2"] & X \times \mathcal{J}^{n+1} X \arrow[r, "pr_2"] \arrow[d, "T"] \arrow[dl, phantom, "1"] & \mathcal{J}^{n+1} X \arrow[d] \arrow[dl, phantom, "3"] \\
\mathcal{J}^{n+1} X \arrow[r, hook] & \mathcal{J}^{n+2} X \arrow[r] & *
\end{tikzcd}$$

In this diagram the outer square is the original one from the claim of the lemma mirrored. The map T is factored through the map $\overline{T}$.

1 is a pushout:
The maps that appear are inclusions of CW-complexes so they are closed cofibrations. Therefore the pushout is the homotopy pushout.
Check set-theoretically: all spaces are compact hausdorff as X is and the map from the pushout $\mathcal{J}^{n+1}(X) \to \mathcal{J}^{n+2}(X)$ is a continuous map of compact hausdorff spaces and to check that is a homeomorphism it suffices to see that it is a bijection.
So you can compute the pushout on the level of sets and see that it maps bijectively to $\mathcal{J}^{n+2}(X)$ and since $X \triangleright \mathcal{J}^{n+1} X \hookrightarrow X \times \mathcal{J}^{n+1}(X)$ is an inclusion you just have to look at what kind of points glue together along respectively $\overline{T}$ and T and see that it is the same.

2 is a pushout ($\implies$ a homotopy pushout)

$$\begin{tikzcd}
\{e\} \times \mathcal{J}^n X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \times \mathcal{J}^n X \arrow[r] \arrow[d, "j"] & \mathcal{J}^n X \arrow[d] \\
\{e\} \times \mathcal{J}^{n+1} X \arrow[r] & X \triangleright \mathcal{J}^{n+1} X \arrow[r] & \mathcal{J}^{n+1} X
\end{tikzcd}$$

The left square is the definition of the space in the down right corner. The map $X \triangleright \mathcal{J}^{n+1} X \to \mathcal{J}^{n+1} X$ comes from the inclusion followed by the projection, so the composition of the lower row is the identity, as well as of the top row.

$$\begin{tikzcd}
\{e\} \times \mathcal{J}^n X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] \arrow[rr, bend left, "id"] & X \times \mathcal{J}^n X \arrow[r] \arrow[d, "j"] & \mathcal{J}^n X \arrow[d] \\
\{e\} \times \mathcal{J}^{n+1} X \arrow[r] \arrow[rr, bend right, "id"'] & X \triangleright \mathcal{J}^{n+1} X \arrow[r] & \mathcal{J}^{n+1} X
\end{tikzcd}$$

So the outer square is a pushout by trivial reasons and since the left square is by definition a pushout, the right square is too.

$$\left.\begin{array}{c} 1 + 2 + 3 \\ 2 \end{array}\right\} \implies \left.\begin{array}{c} 1 + 3 \\ 1 \end{array}\right\} \implies 3$$

where "+" denotes that we regard the "union" of theses squares. We know that 1 + 2 + 3 is a pushout by induction assumption. ∎

Corollary 2.9.7.

$$\begin{tikzcd} X \times \mathcal{J}X \arrow[r, "T"] \arrow[d, "pr_2"] & \mathcal{J}X \arrow[d] \\ \mathcal{J}X \arrow[r] & * \end{tikzcd}$$

is a homotopy pushout.

Proof.
To compute the homotopy pushout, we replace one of the arrows with a cofibration:

$$\begin{tikzcd} X \times \mathcal{J}X \arrow[r, "T"] \arrow[d, hook] & \mathcal{J}X \arrow[d] \\ CX \times \mathcal{J}X \arrow[r] \arrow[ur, phantom, "\square"] & W \end{tikzcd}$$

This the usual 1-pushout. One can do the same for every finite level:

$$\begin{tikzcd} X \times \mathcal{J}^{(n)}X \arrow[r, "T"] \arrow[d, hook] & \mathcal{J}^{(n+1)}X \arrow[d] \\ CX \times \mathcal{J}^{(n)}X \arrow[r] \arrow[ur, phantom, "\square"] & W^{(n)} \end{tikzcd}$$

Thus we have defined $W^{(n)}$ and W. Again using compactness arguments all of that for instance (but also for the other spaces) $X \times \mathcal{J}X \cong \operatorname{colim}_{n\geq 0} X \times \mathcal{J}^n X$.
So using that colimits commute with pushouts we get that

$$W \cong \operatorname*{colim}_{n\geq 0} W^{(n)}$$

$W^{(n)}$ is contractible by the lemma, so W is a colimit of contractible spaces (you can even check of subspaces $W^{(n)} \hookrightarrow W^{(n+1)}$). We conclude that W is also contractible. For this one can use that Whitehead theorem tells us that

$$\pi_k W \cong \operatorname*{colim}_k \pi_k W^n \cong \operatorname{colim} 0 = 0$$

implies that W is contractible.

∎

Finally for compact X ($\Leftrightarrow$ X has finitely many cells):

$$\begin{tikzcd} X \times \mathcal{J}X \arrow[rr, "\pi_2"] \arrow[dr, "T"'] \arrow[dd] & & \mathcal{J}X \arrow[dr] \arrow[dd] & \\ & \mathcal{J}X \arrow[rr] \arrow[dd] & & * \arrow[dd] \\ X \arrow[rr] \arrow[dr] & & * \arrow[dr] & \\ & * \arrow[rr] & & \Sigma' X \end{tikzcd}$$

is a 1-commutative cube (in particular 2-commutative)

- The bottom face is a homotopy pushout
- The rear face a homotopy pullback (because for instance the projection $\mathcal{J}X \to *$ is a fibration and this is the usual pullback)
- The top face is a homotopy pushout by the corollary
- The left face is a homotopy pullback:
  The homotopy pullback is the product of the two spaces X and $\mathcal{J}X$. What we have here in the top left corner is the same space but the map to the pullback is not the identity map but rather

  $$X \times \mathcal{J}X \xrightarrow{\pi_1 \times T} X \times \mathcal{J}X$$

  To show that the left face is a homotopy pullback we have to check that this map is a homotopy equivalence. For this it suffices that the following square is a homotopy pullback square because then it follows from the map at the bottom being a homotopy equivalence (identity) that the top map is too.

  $$\begin{tikzcd} X \times \mathcal{J}X \arrow[r, "\pi_1 \times T"] \arrow[d, "pr_1"'] & X \times \mathcal{J}X \arrow[d, "pr_1"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

  It suffices to compare homotopy fibers of the vertical maps. Since projections are fibrations, homotopy fibers are the same as fibers. Moreover X is path-connected, thus it is enough to regard the homotopy fibers of just one point in X. Luckily we can just take the point e over which nothing happens:

  $$\begin{tikzcd} \{e\} \times \mathcal{J}X \arrow[r, "id", "\simeq"'] \arrow[d, equal] & \{e\} \times \mathcal{J}X \arrow[d, equal] \\ e \arrow[r, equal] & e \end{tikzcd}$$

By the first Mather's cube theorem we obtain that the front or the right face (which are the same, actually) are both homotopy pullbacks.

But the fact that

$$\begin{tikzcd} \mathcal{J}X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma' X \end{tikzcd}$$

is a homotopy pullback means that when we take the homotopy pullback of

$$\begin{tikzcd} & * \arrow[d] \\ * \arrow[r] & \Sigma' X \end{tikzcd}$$

which is the loop space of the reduced suspension $\Omega\Sigma' X$ this is homotopic to $\mathcal{J}X$ ■

Corollary 2.9.8 (Freudenthal's suspension theorem = special case of BM)**.**
X is n-connected, $X \to \Omega\Sigma' X$ is $(2n+1)$-connected

Remark 2.9.9.
Why is it a special case of the Blakers-Massey theorem?
If we draw the square

$$\begin{tikzcd} X \arrow[r, "n+1"] \arrow[d, "n+1"'] & * \arrow[d] \\ * \arrow[r] & \Sigma' X \end{tikzcd}$$

Because X is n-connected the map $X \to *$ is $n+1$-connected because the fiber of this map is just X and the connectedness of the map is related to the connectedness of the fiber by -1.

$$\begin{tikzcd} X \arrow[rr, "n+1"] \arrow[dd, "n+1"'] \arrow[dr, dashed] & & * \arrow[dl] \arrow[dd] \\ & \Omega\Sigma' X & \\ * \arrow[ur] \arrow[rr] & & \Sigma' X \end{tikzcd}$$

Proof.
Assume $X^{(i)} = \{*\}$ for all $i \leq n$. Therefore by construction if we take the CW-skeleton filtration of the James complex this is just $(\mathcal{J}X)^{(2n+1)} = X$.
e.g.

$$\begin{aligned} \mathcal{J}^2 X &\twoheadleftarrow X \times X \\ * &\leftarrow e \times e \\ * &\leftarrow e \times \geq n+1\text{-cells} \\ * &\leftarrow \geq n+1\text{-cells} \times e \\ (2n+2)\text{-cells} &\leftarrow \geq n+1\text{-cells} \times \geq n+1\text{-cells} \end{aligned}$$

Thus $\mathcal{J}^2 X$ is X with $\geq (2n+2)$-cells attached.

$$\begin{tikzcd} & \Omega\Sigma' X \\ X \arrow[ur] \arrow[dr, "(2n-1)-conn"'] & \circlearrowleft \\ & \mathcal{J}X \arrow[uu, "\lambda"', "heq"] \end{tikzcd}$$

■

Remark 2.9.10.
In particular, we proved unconditionally that the homotopy groups of spheres are

$$\underbrace{\pi_n(S^n)}_{\cong \mathbb{Z}} \xrightarrow[\cong]{h_n} H_n(S^n)$$

Remark 2.9.11.

- unconditionally we can prove Hurewicz theorem:

$$\begin{cases} \pi_n(\bigvee_\alpha S^n) \cong \bigoplus \mathbb{Z}\alpha \\ \pi_{n+1}(Y^{(n+1)}, \underbrace{Y^{(n)}}_{=\bigvee S^n}) \twoheadrightarrow \pi_{n+1}(\underbrace{Y^{(n+1)}/Y^{(n)}}_{=\bigvee S^{n+1}}) \cong \bigoplus \mathbb{Z}\beta \end{cases} \qquad \| \bigvee S^n \to \prod S^n \text{ is } (2n-1)\text{-conn.}$$

That this map is surjective follows because $Y^{(n+1)}$ is obtained by attaching n-cells:

$$
\begin{tikzcd}
S^{n+1} \arrow[r, "\beta"] & Y^{(n+1)}/Y^{(n)} \\
D^{n+1} \arrow[u] \arrow[r] & Y^{(n+1)} \arrow[u] \\
S^{n} \arrow[u] \arrow[r] & Y^{(n)} \arrow[u]
\end{tikzcd}
$$

If you look in the proof of the Hurewicz theorem, you will see that one does not need this map to be an isomorphism but only surjective.

- we will prove relative Hurewicz using Serre spectral sequences.

2.10 Proof of Blakers-Massey Theorem

($n, m \geq 2$ or $n = 0$) for CW.

Theorem 2.10.1 (Blakers-Massey: homotopic version of excision)**.**
Let

$$
\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \arrow[d, "\gamma"] \arrow[dl, phantom, "\Swarrow H"] \\
C \arrow[r, "\delta"] & X
\end{tikzcd}
$$

be a 2-commutative homotopy pushout in $\mathtt{Top}$.
If α is m-connected, β is n-connected ($m, n \geq 0$), then the square is $(m+n-1)$-weakly homotopy cartesian.

Remark 2.10.2.
Note that δ is m-connected, γ is n-connected ("easy excision").
If α is m-connected one can assume A and B to be CW-complexes and replace α by an inclusion of CW-complexes which is attaching cells of dimension at least $m+1$.
Now the homotopy pushout can be taken to be the usual pushout, so D is obtained from C by attaching cells of dimension at least $m+1$. Therefore δ is m-connected.

This is in fact a particular case of Blakers-Massey which we are about to prove.

Proof.
<u>For $m = 0$:</u>

$$
\begin{tikzcd}
A \arrow[rr, "0-conn"] \arrow[dd, "n-conn"'] \arrow[dr] & & B \arrow[dd, "n-conn"] \\
& P \arrow[ur] \arrow[dl] & \\
C \arrow[rr] & & D
\end{tikzcd}
$$

Everything is commutative with some homotopies that I will not draw. We just explained that the arrow on the right is n-connected, if the one on the left is.
Now if you take the homotopy pullback, the fact that the map on the right is n-connected implies that the map $P \to C$ is n-connected because n-connectedness is the claim about the homotopy fibers of the map and in the pullback square the homotopy fibers of these two maps are homotopy equivalent.
We have a triangle

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, "n-conn"'] & & P \arrow[dl, "n-conn"] \\
& C &
\end{tikzcd}$$

From this follows that $A \to P$ is $(n-1)$-connected because

$$\begin{tikzcd}
\pi_i A \arrow[rr] \arrow[dr, "\cong"'] & & \pi_i P \arrow[dl, "\cong"] \\
& \pi_i C &
\end{tikzcd}$$

for $i \leq n-1$. So we get an isomorphism $\pi_i A \to \pi_i P$. We do not get, however, that it is n-connected, because if both downturned maps are surjective, it does not follow that the map $\pi_i A \to \pi_i P$ is surjective.

For $n, m \geq 2$:

Reduction step.
Choose $d \in D$ and construct a 2-commutative cube by taking the homotopy pullback $\tilde{C}$ of this pushout square along the inclusion of this point d and find a homotopy F_4:

$$\begin{tikzcd}
& \tilde{C} \arrow[rr] \arrow[dd] \arrow[drr, phantom, "\square"] & & * \arrow[dd, "d"] \\
A \arrow[rr] \arrow[dr] & & B \arrow[dr] \arrow[r, phantom, "\Swarrow F_4"] & \\
& C \arrow[rr] & & D
\end{tikzcd}$$

Next we take the homotopy pullback $\tilde{A}$ of the left face and find a homotopy, as well as for the right face

$$\begin{tikzcd}
\tilde{A} \arrow[dr, "\tilde{\beta}"] \arrow[dd, "f_A"] & & \tilde{B} \arrow[dr, "\tilde{\gamma}"] \arrow[dd, "f_B"'] & \\
& \tilde{C} \arrow[rr, "\tilde{\delta}"] \arrow[dd, "f_C"] \arrow[drr, phantom, "\square"] & & * \arrow[dd, "d"] \\
A \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] \arrow[r, phantom, "\Swarrow F_2"] & & B \arrow[dr, "\gamma"] \arrow[r, phantom, "\Swarrow F_4"] & \\
& C \arrow[rr, "\delta"'] & & D
\end{tikzcd}$$

To finish the cube, we need an arrow $\tilde{A}$ to $\tilde{B}$. $\tilde{B}$ is the pullback of the diagram

$$\begin{tikzcd}
\tilde{B} \arrow[r] \arrow[d] & * \arrow[d] \\
B \arrow[r] & D
\end{tikzcd}$$

So to find the map we wish for, we want to use the existence property of the homotopy pullback and find a diagram

$$\begin{tikzcd}
\tilde{A} \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \tilde{B} \arrow[r] \arrow[d] & * \arrow[d, "d"] \\
& B \arrow[r, "\gamma"'] & D
\end{tikzcd}$$

A map $\tilde{A} \to B$ is just $\alpha \circ f_A$ and $\tilde{A} \to *$ is $\tilde{\delta} \circ \tilde{\beta}$. Now we have to find a homotopy between the compositions:

$$\begin{tikzcd}
\tilde{A} \arrow[drr, bend left, "\tilde{\delta}\circ\tilde{\beta}"] \arrow[ddr, bend right, "\alpha\circ f_A"'] & & \\
& \tilde{B} \arrow[r] \arrow[d] & * \arrow[d, "d"] \\
& B \arrow[r, "\gamma"'] & D
\end{tikzcd}$$

$\gamma \circ \alpha \circ f_A \overset{\text{down face}}{\Longrightarrow} \delta \circ \beta \circ f_A \overset{\text{left face}}{\Longrightarrow} \delta \circ f_C \circ \tilde{\beta} \overset{\text{front face}}{\Longrightarrow} d \circ \tilde{\delta} \circ \tilde{\beta}$
This is precisely where we needed to end so the composition of these homotopies gives us the one we need. Therefore we obtain $\tilde{\alpha}$:

$$\begin{tikzcd}
\tilde{A} \arrow[drr, bend left, "\tilde{\gamma}\circ\tilde{\beta}"] \arrow[ddr, bend right, "\alpha\circ f_A"'] \arrow[dr, dashed, "\tilde{\alpha}"] & & \\
& \tilde{B} \arrow[r] \arrow[d] \arrow[ur, phantom, "F_1"] \arrow[dl, phantom, "F_1"] & * \arrow[d, "d"] \\
& B \arrow[r, "\gamma"'] & D
\end{tikzcd}$$

Finally we have a cube filled with homotopies:

$$\begin{tikzcd}
\tilde{A} \arrow[rr, dashed] \arrow[dr, "\tilde{\beta}"] \arrow[dd, "f_A"'] & & \tilde{B} \arrow[dr, "\tilde{\gamma}"] \arrow[dd, "f_B"] & \\
& \tilde{C} \arrow[rr, "\tilde{\delta}"] \arrow[dd, "f_C"] \arrow[dr, phantom, "\square"] & & * \arrow[dd, "d"] \\
A \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] \arrow[r, phantom, "F_2"] & & B \arrow[dr, "\gamma"] \arrow[r, phantom, "F_4"] & \\
& C \arrow[rr, "\delta"'] & & D
\end{tikzcd}$$

<u>Exercise:</u> This is a 2-commutative cube.

Because this is a 2-commutative cube, we can draw another cube inside it of which we use the pullback of the down and top face:

$$\begin{tikzcd}
\tilde{A} \arrow[rrr, dashed] \arrow[ddr, "\tilde{\beta}"] \arrow[ddd, "f_A"'] & & & \tilde{B} \arrow[ddr, "\tilde{\gamma}"] \arrow[ddd, "f_B"] & \\
& & \tilde{P} \arrow[ur] \arrow[dl] \arrow[ddd] \arrow[r, phantom, "\square"] & & \\
& \tilde{C} \arrow[rrr, "\tilde{\delta}"] \arrow[ddd, "f_C"] \arrow[d, phantom, "\square"] & & & * \arrow[ddd, "d"] \\
A \arrow[rrr, "\alpha"] \arrow[ddr, "\beta"'] \arrow[r, phantom, "F_2"] & & & B \arrow[ddr, "\gamma"] \arrow[r, phantom, "F_4"] & \\
& & P \arrow[ur] \arrow[dl] & & \\
& C \arrow[rrr, "\delta"] & & & D
\end{tikzcd}$$

Now the claim is that there is a map $\tilde{P} \to P$ such that we obtain a fully 2-commutative 3-dimensional diagram:

$$\begin{tikzcd}
\tilde{A} \arrow[rrr, dashed] \arrow[drr] \arrow[ddr, "\tilde{\beta}"] \arrow[ddd, "f_A"'] & & & \tilde{B} \arrow[ddr, "\tilde{\gamma}"] \arrow[ddd, "f_B"] & \\
& & \tilde{P} \arrow[ur] \arrow[dl] \arrow[ddd] \arrow[r, phantom, "\square"] & & \\
& \tilde{C} \arrow[rrr, "\tilde{\delta}"] \arrow[ddd, "f_C"] \arrow[d, phantom, "\square"] & & & * \arrow[ddd, "d"] \\
A \arrow[rrr, "\alpha"] \arrow[drr] \arrow[ddr, "\beta"'] \arrow[r, phantom, "F_2"] & & & B \arrow[ddr, "\gamma"] \arrow[r, phantom, "F_4"] & \\
& & P \arrow[ur] \arrow[dl] & & \\
& C \arrow[rrr, "\delta"] & & & D
\end{tikzcd}$$

What we will now be interested in is just one piece of the diagram, the top square (we can reduce to that since it is a pushout square by the 2nd Mather's cube theorem)

$$\begin{tikzcd}
\tilde{A} \arrow[r] \arrow[d] & \tilde{P} := \tilde{C} \times \tilde{B} \arrow[d] \\
A \arrow[r, "w"] & P := B \times^h_D C
\end{tikzcd}$$

Claim: This square is a homotopy pullback.
For this we will look at the fibers of the vertical arrows.

$$\begin{tikzcd}
& \tilde{P}_{w(a)} \\
\tilde{A}_a \arrow[ur] \arrow[dr, "\simeq"'] & \\
& \tilde{C}_{\beta(a)} \arrow[uu]
\end{tikzcd}$$

So to prove this claim it suffices to check that the map $\tilde{C}_{\beta(a)} \to \tilde{P}_{w(a)}$ is a homotopy equivalence or that the square
$$\begin{tikzcd} \tilde{P} \arrow[r] \arrow[d] & \tilde{C} \arrow[d] \\ P \arrow[r] & C \end{tikzcd}$$
is also a homotopy pullback.
Instead of looking at the vertical maps, let us look at the horizontal fibers. The fibers for the map $P \to C$ are the same as for $B \to D$ and those for $\tilde{P} \to \tilde{C}$ as those for $\tilde{B} \to *$ because both these squares are homotopy pullbacks.
But the homotopy fibers of $\tilde{B} \to *$ and $B \to D$ are also the same, because the right face is a homotopy pullback square and therefore we are done (it follows that $\tilde{C}_{\beta(a) \to \tilde{P}_{w(a)}}$ is a homotopy equivalence and thus the claim).

Now, if the claim is fulfilled, that the square is a homotopy pullback, what follows is that we do not have to prove Blakers-Massey theorem for the square below, we only need to prove it for the square above because the connectedness of w and $\tilde{w}$ are the same in

$$\begin{tikzcd} \tilde{A} \arrow[r, "\tilde{w}"] \arrow[d] & \tilde{P} := \tilde{C} \times \tilde{B} \arrow[d] \\ A \arrow[r, "w"] & P := B \times^h_D C \end{tikzcd}$$

$\implies$ w is $(n+m-1)$-connected, because the connectedness of some arrow is the connectedness of the homotopy fiber and since those are equivalent $\tilde{w}$ is also $(n+m-1)$-connected.

Main step:
We are reduced to the situation

$$\begin{tikzcd} \tilde{A} \arrow[r, "m-conn"] \arrow[d, "n-conn"'] & \tilde{B} \arrow[d] \\ \tilde{C} \arrow[r] \arrow[ur, phantom, "\square"] & * \end{tikzcd}$$

Let's just see what is implied by this being a homotopy pushout because the claim that the pushout is contractible, so just a point, is a very strong claim. For example look at the homotopy pushout

$$\begin{tikzcd} \tilde{A} \arrow[r, "m-conn", "\tilde{\alpha}"'] \arrow[d, "n-conn"', "\tilde{\beta}"] & \tilde{B} \arrow[r] \arrow[d] & * \arrow[d] \\ \tilde{C} \arrow[r] \arrow[ur, phantom, "\square"] & * \arrow[r] \arrow[ur, phantom, "\square"] & \Sigma' \tilde{B} \end{tikzcd}$$

On the other hand because both small squares are homotopy pushouts, the outer one is too and therefore $\Sigma' \tilde{B} \simeq \tilde{C}/\tilde{A}$.

Assume $\tilde{\alpha}$ and $\tilde{\beta}$ are inclusions of CW-complexes. Also we can add:

$$\begin{tikzcd} \tilde{A} \arrow[r, "m-conn", "\tilde{\alpha}"'] \arrow[d, "n-conn"', "\tilde{\beta}"] & \tilde{B} \arrow[r] \arrow[d] & * \arrow[d] \\ \tilde{C} \arrow[r] \arrow[d] \arrow[ur, phantom, "\square"] & * \arrow[r] \arrow[d] \arrow[ur, phantom, "\square"] & \Sigma' \tilde{B} \\ * \arrow[r] \arrow[ur, phantom, "\square"] & \Sigma' \tilde{C} & \end{tikzcd}$$

So by the same argument, $\Sigma' \tilde{C} \simeq \tilde{B}/\tilde{A}$. Moreover we can do the following addition:

$$\begin{tikzcd}
\tilde{A} \arrow[r, "m-conn", "\tilde{\alpha}"'] \arrow[d, "n-conn"', "\tilde{\beta}"] & \tilde{B} \arrow[r] \arrow[d] & * \arrow[d] \\
\tilde{C} \arrow[r] \arrow[d] \arrow[ur, phantom, "\square"] & * \arrow[r] \arrow[d] \arrow[ur, phantom, "\square"] & \Sigma'\tilde{B} \arrow[d] \\
* \arrow[r] \arrow[ur, phantom, "\square"] & \Sigma'\tilde{C} \arrow[r] \arrow[ur, phantom, "\square"] & \Sigma'\tilde{C} \vee \Sigma'\tilde{B}
\end{tikzcd}$$

On the other hand, because all small squares are homotopy pushouts, the complete outer square is too. Thus $\underline{\Sigma'\tilde{C} \vee \Sigma'\tilde{B} \simeq \Sigma\tilde{A}}$

What we need to look at is the map from $\tilde{A}$ to the homotopy pullback, which is in this case just the product

$$\tilde{A} \xrightarrow{\omega} \tilde{B} \times \tilde{C}$$

$$\begin{tikzcd}
\Sigma'\tilde{A} \arrow[r, "\omega"] \arrow[dr, "\simeq"', "\text{check!}"] & \Sigma'(\tilde{B} \times \tilde{C}) \arrow[r, "\text{ex. session}", "\simeq"'] \arrow[d, phantom, "\circlearrowleft"] & \Sigma'\tilde{B} \vee \Sigma'\tilde{C} \vee \Sigma'(\tilde{B} \wedge \tilde{C}) \\
 & \Sigma'\tilde{B} \vee \Sigma'\tilde{C} \arrow[ur, hook] &
\end{tikzcd}$$

Now what do we know? We know that for the following square the map $\tilde{B} \to *$ is n, $\tilde{C} \to *$ m-connected:

$$\begin{tikzcd}
\tilde{A} \arrow[rr, "m-conn", "\tilde{\alpha}"'] \arrow[dd, "n-conn"', "\tilde{\beta}"] \arrow[dr, dashed] & & \tilde{B} \arrow[dd, "n-conn"] \\
 & \tilde{B} \times \tilde{C} \arrow[ur] \arrow[dl] \arrow[dr, phantom, "\square"] & \\
\tilde{C} \arrow[rr, "m-conn"] & & *
\end{tikzcd}$$

Thus $\tilde{B}$ is $(n-1)$-connected $(n-1 \geq 1)$, $\tilde{C}$ is $(m-1)$-connected $(m-1 \geq 1)$.
Therefore $\tilde{B} \wedge \tilde{C}$ is $(n+m-1)$-connected, $\Sigma(\tilde{B} \wedge \tilde{C})$ is $(n+m)$-connected.

Now we look at the relative homology of the map $\Sigma'\tilde{B} \vee \Sigma'\tilde{C} \hookrightarrow \Sigma'\tilde{B} \vee \Sigma'\tilde{C} \vee \Sigma'(\tilde{B} \wedge \tilde{C})$ which is

$$\tilde{H}_i(\Sigma'(\tilde{B} \times \tilde{C}), \Sigma'\tilde{A}) \cong \tilde{H}_i(\Sigma'(\tilde{B} \wedge \tilde{C})) = 0$$

for $i \leq n+m-1$. For the homology groups we have the suspension isomorphism, so we have

$$\tilde{H}_i(\Sigma'(\tilde{B} \times \tilde{C}), \Sigma'\tilde{A}) \cong \tilde{H}_{i-1}(\tilde{B} \times \tilde{C}, \tilde{A})$$

which is 0 for $i - 1 \leq n + m$.

Now applying the relative Hurewicz we get:

$$\pi_i(\tilde{B} \vee \tilde{C}, \tilde{A}) = 0$$

for $i \leq n+m-1$.
For this last step we needed at least $\tilde{A}$ to be simply connected which is true because by assumption $n, m \geq 2$ and thus $\tilde{A} \to \tilde{B}$ as well as $\tilde{A} \to \tilde{B}$ are at least 2-connected and so induce isomorphisms on π_1. Since both $\tilde{B}$ and $\tilde{C}$ are at least 1-connected, those are both 0 and thus $\tilde{A}$ simply connected). ■

2.11 Some remarks on homotopy (co)limits

This section is somewhat complementary: there will be no proofs. Here some notions are expressed which we have studied in some special cases but not in their full generality.

Let I be a diagram, so a 1-category. There is the 1-category of such diagrams $\mathtt{Top}^I$ which is the 1-category of functors $I \to \mathtt{Top}$.
There is a functor

$$\begin{aligned} C : \mathtt{Top} &\to \mathtt{Top}^I \\ X &\mapsto (i \mapsto X) \end{aligned}$$

Inside the morphisms of this category there are homotopy (or weak) equivalences.

$$heq^I \subset Mor(\mathtt{Top}^I)$$

These are "levelwise" or "objectwise" homotopy equivalences.

An example. Assume I to be of the form $I = \bullet \to \bullet \leftarrow \bullet$.
A morphism in $Mor(\mathtt{Top}^I)$ would be of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f_X"'] & Y \arrow[d, "f_Y"'] & Z \arrow[l] \arrow[d, "f_Z"'] \\ X' \arrow[r] & Y' & Z' \arrow[l] \end{tikzcd}$$

such that both squares commute in the category $\mathtt{Top}$ (so no 2-commutativity needed yet).
If now all f_X, f_Y, f_Z were homotopy equivalences, then $(f_\bullet)$ would be in heq^I.

Thus

$$heq^I = \{(f_C) \mid f_C \text{ is a homotopy equivalence}\}$$

Now we define a category, where we take $\mathtt{Top}^I$ and invert globally these object- or levelwise homotopies: $\mathtt{Top}^I[heq^I]^{-1}$.
What does that mean? It is a 1-category that, if it exists, satisfies the universal property of inverting these morphisms:
Whenever there is a functor $F : \mathtt{Top}^I \to C$ where C is some chosen category and $F(heq^I) \subseteq Isom(C)$ then there exists exactly one $\overline{F} : \mathtt{Top}^I[heq^I]^{-1}$ that makes the following diagram commutative

$$\begin{tikzcd} \mathtt{Top}^I \arrow[r] \arrow[dr, "F"'] & \mathtt{Top}^I[heq^I]^{-1} \arrow[d, dashed, "\exists! \overline{F}"] \\ & C \end{tikzcd}$$

So $\mathtt{Top}^I[heq^I]^{-1}$ is a 1-category satisfying this universal property $(\forall F ... \exists! \overline{F} ...)$

Example: If $I = \bullet$, then $\mathtt{Top}^I = \mathtt{Top}$, so objectwise homotopy equivalences are just homotopy equivalences and inverting them $\mathtt{Top}[heq]^{-1} = h\mathtt{Top}$.
This was seen in AT I in a exercise, where we checked that giving a functor on the category $\mathtt{hTop}$ is the same as giving a functor on $\mathtt{Top}$ such that it sends all homotopy equivalences to isomorphisms. This is precisely checking this universal property.

Having seen this example we will denote $h(\mathtt{Top}^I) := \mathtt{Top}^I[heq^I]^{-1}$.
The constant functor C we defined above, factors as

$$\begin{tikzcd} \mathtt{Top} \arrow[r, "C"] \arrow[d] & \mathtt{Top} \arrow[d] \\ h\mathtt{Top} \arrow[r, dashed, "\exists! C"] & h(\mathtt{Top}^I) \end{tikzcd}$$

because the composition $\mathtt{Top} \xrightarrow{C} \mathtt{Top}^I \to h(\mathtt{Top}^I)$ sends all homotopy equivalences to isomorphisms.

Now that we introduced this category (even though we do not know whether it exists) we can define homotopy limits and colimits globally.
The following is one of many equivalent definitions:

Definition 2.11.1.
We have the functor $h\mathtt{Top} \xrightarrow{C} h(\mathtt{Top}^I)$. It it has a left or a right adjoint, they are called homotopy (co)limits.

$$\begin{tikzcd} h\mathtt{Top} \arrow[r, "C"] & h(\mathtt{Top}^I) \arrow[l, bend right, "h\,\mathrm{colim}"'] \arrow[l, bend left, "h\lim"] \end{tikzcd}$$

$$h\,\mathrm{colim} \dashv C \dashv h\lim$$

Remark 2.11.2.

- When we started the discussion of 2-categories, we said that there are problems with homotopy limits and colimits because they do not respect the homotopy equivalence between diagrams. Here in this new setting this is not a problem, because $h\,\mathrm{colim}$ and $h\lim$ are defined on the category where homotopic diagrams become isomorphic. So they respect homotopy equivalences between diagrams.
- This definition does not need any additional categorical structure e.g. for the definition of homotopy pushouts and pullbacks this definition does not need any 2-categorical structure. So this definition is purely in 1-categorical world (but in this setting it is not clear, that $h(\mathtt{Top}^I)$ exists).

So why did we need to introduce all the technical apparatus such as (2,1)-categories, fibrations, cofibrations and so on?
The answer lies in the following problem we have: even for simple diagrams I (e.g. $\to \bullet \leftarrow \bullet$ or $\bullet \to \bullet$)

$$h(\mathtt{Top}^I) \ncong (h\mathtt{Top})^I$$

and is not easy to describe.

Suppose we have a diagram like that:

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f_X"'] & Y \arrow[d, "f_Y"] \\ X' \arrow[r, "\alpha'"'] & Y' \end{tikzcd}$$

Suppose f_X and f_Y are homotopy equivalences. It does not follow that $\exists g_X, g_Y$ homotopy inverses such that the new diagram commutes:

$$\begin{tikzcd} X' \arrow[r, "\alpha'"] \arrow[d, "g_X"'] & Y' \arrow[d, "g_Y"] \\ X \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

For example

$$\begin{tikzcd}
* \arrow[r, "\alpha"] \arrow[d, "f_X"'] & S^1 \arrow[d, "id=f_Y"] \\
I \arrow[r, two heads, "\alpha'"'] \arrow[d, "g_X"] & S^1 \arrow[d, "g_Y"] \\
* \arrow[r, "\alpha"] & S^1
\end{tikzcd}$$

Since we assume $I \twoheadrightarrow S^1$ to be surjective and assuming the diagram were commutative, that is $g_Y \circ \alpha' = \alpha \circ g_X$, then $g_Y = \overline{\alpha}$. This is because if α' is surjective, then we can find to each point in S^1 a preimage in I. This is mapped by g_X at just some point and then by α again onto just one point in S^1. But we assumed commutativity, so g_Y also has to send every point in S^1 to just one point.
Then, however, $g_Y \circ id$ is just a map on a point and as such definitely no homotopy equivalence $S^1 \to S^1$.

We also do not know how to describe morphisms in $h(\mathtt{Top}^I)$.
Assume we are given three objects $\alpha : X \to Y$, $\alpha' : X' \to Y'$ and $\alpha'' : X'' \to Y''$

$$\begin{tikzcd}
 & X' \arrow[r, "\alpha'"] \arrow[dl, "heq"'] \arrow[drr] \arrow[d, phantom, "\circlearrowleft"] & Y' \arrow[dl, "heq"'] \arrow[drr] \arrow[d, phantom, "\circlearrowleft"] & & \\
X \arrow[r, "\alpha"'] & Y & & X'' \arrow[r, "\alpha''"'] & Y''
\end{tikzcd}$$

This diagram gives us a morphism $\alpha \to \alpha''$ in $h(\mathtt{Top}^I)$. Different choices within this diagram would maybe still give us the same morphism but possibly not. So we need to compare them. If we could invert the homotopy equivalences, as we tried before and failed making it commutative, then we would have a morphism in the original category $\mathtt{Top}^I$ and we would split it and get a morphism $\alpha \to \alpha''$ without any homotopies.
Unfortunately we cannot do so.

Remark 2.11.3.
For $I = [1] : \bullet \to \bullet$ then $(h\mathtt{Top}^I)$ is the homotopy category of pairs of topological spaces.
Also if you look at pairs where the map from the first to the second space is a cofibration, then it allows you what the maps between such pairs is.
This is why we talked about well-pointed spaces, because a well-pointed space is precisely a space where a map from the point to a chosen point is a cofibration. That's how we in the end defined the relative homotopy groups: we could always replace the map by a cofibration and then define everything.

Comparison with our approach

Let's look at the diagram $I = \bullet \to \bullet \leftarrow \bullet$.
Then $\mathtt{Top}^I$ is a (2,1)-category with

- Objects: $Ob(\mathtt{Top}^I)$: $X \xrightarrow{\alpha} Y \xleftarrow{\beta} Z$
- 1-Morphisms: diagrams

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "f_X"] & Y \arrow[d, "f_Y"] \arrow[dl, phantom, "\Swarrow h_1"] \arrow[dr, phantom, "\Searrow h_2"] & Z \arrow[l, "\beta"'] \arrow[d, "f_Z"] \\
X' \arrow[r, "\alpha'"'] & Y' & Z' \arrow[l, "\beta'"]
\end{tikzcd}$$

Here we allow the diagram to not be commutative, but the homotopy is part of the data:

$$(\alpha, \beta) \xrightarrow{(f_X, f_Y, f_Z, h_1, h_2)} (\alpha', \beta')$$

- 2-morphisms: 2-commutative cubes

$$\begin{tikzcd}
& X \arrow[rr] \arrow[d] \arrow[ddl] & & Y \arrow[d] \arrow[ddl] & & Z \arrow[ll] \arrow[d] \arrow[ddl] \\
& X' \arrow[rr] \arrow[ddl] & & Y' \arrow[ddl] & & Z' \arrow[ll] \arrow[ddl] \\
\overline{X} \arrow[rr] \arrow[d] & & \overline{Y} \arrow[d] & & \overline{Z} \arrow[ll] \arrow[d] & \\
\overline{X}' \arrow[rr] & & \overline{Y}' & & \overline{Z}' \arrow[ll] &
\end{tikzcd}$$

Thus the data consists of 6 maps of topological spaces and 7 homotopies.

Proposition 2.11.4.

1. *As a (2,1)-category*

$$h(\mathsf{Top}^I) \simeq h(\mathsf{Top}^I) := \mathsf{Top}^I[heq]^{-1}$$

2. $h\lim(X \to Y \leftarrow Z) \cong X \times_Y^h Z$.
   $h\operatorname{colim}(X \to Y \leftarrow Z) \cong X \cup^h Z$

Other Approaches

- Model categories: define abstractly "fibrations" and "cofibrations" in $\mathtt{Top}^I$
  so that one can "strictify" all homotopy commutative I-diagrams.
  But this strictification is the one we have seen for a square which is homotopy commutative:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "cofib"'] & \bullet \arrow[d, "fib"] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

For such a square we can change one of the arrows

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

and it will be just normal commutative.
Fibrations and cofibrations for more general diagrams will be something that allows you to change the diagram into a commutative one.

Then homotopy (co)limits can be computed as usual (co)limits of a replacement diagram with all arrows (co)fibrations. (This is more of a computational aspect)

- $(\infty, 1)$-categories
  There exist n-morphisms for all $n \geq 1$. All n-morphisms were $n \geq 1$ are invertible.
  These $(\infty, 1)$-categories are a generalisation of (2,1)-categories and they give us homotopy (co)limits but not via some computational aspect but rather universal property. This is what we have seen in (2,1)-category theory and it took us some time to say what the universal property is.

  We need those $(\infty, 1)$-categories because for more complicated diagrams than $I = (\bullet \to \bullet \leftarrow \bullet)$ weak 2-(co)limits do not exist in $\mathtt{Top}$!
  This is the same problem we started with and the reason why that happens is because in (2,1)-categories we have contracted lots of information: We said morphisms between morphisms are homotopies but only up to homotopy between homotopies. Now if you look at more complicated diagrams, if you forget this information you lose the universal property.
  The way out are $(\infty, 1)$-categories where you have morphisms, maps between topological spaces, 2-morphisms which are homotopies, 3-morphisms that represent homotopies between homotopies and so on.
  Once you have all this information together, in this gadget you can finally say what is the universal property of homotopy (co)limits.

Two examples of homotopy (co)limits

- Let X be a "nice" topological space and $\{U_i\}_{i \in I}$ an open cover of X.
  We can form a semi-simplicial diagram in topological spaces using this open cover, the Čech diagram:

$$\begin{array}{ccccccc}
 & \coprod_{i,j,k\in I} U_{ijk} & \Rrightarrow & \coprod_{i,j\in I} \overset{U_i\cap U_j=}{U_{ij}} & & \coprod_{i\in I} U_i & \longrightarrow X \\
 & \cong & & \cong & & := & \\
\cdots \Rrightarrow & \mathcal{U}\times_X \mathcal{U}\times_X \mathcal{U} & \Rrightarrow & \mathcal{U}\times_X \mathcal{U} & \underset{pr_2}{\overset{pr_1}{\rightrightarrows}} & \mathcal{U} &
\end{array}$$

  $\check{C}(\{U_i\}$ is a semi-simplicial diagram in $\mathtt{Top}$.
  The claim is that $h\operatorname{colim} \check{C}(\{U_i\}) \xrightarrow{\cong} X$.

  Corollary 2.11.5. *If we are given a map $f : X \to Y$ between two nice topological spaces and two coverings $\{U_i\}_{i\in I}$, $\{V_i\}_{i\in I}$ open covers such that $f(U_i) \subset V_i$.*
  If $U_{i_1} \cap ... \cap U_{i_k} \to V_{i_1} \cap ... \cap V_{i_k}$ are homotopy equivalences, then f is a homotopy equivalence.

  Actually you can prove that corollary using higher connectivity theorem (at least if I is finite, it had some corollary where we concluded from a map being n-connected on two covering subspaces and their intersection was $(n-1)$-connected, that the map was n-connected) by induction on $|I|$.

  It is a corollary of the claim $h\operatorname{colim} \check{C}(\{U_i\}) \xrightarrow{\cong} X$ because the homotopy colimit by definition depends only on the diagram up to homotopy equivalence, so there is a map $\check{C}(f) : \check{C}(\{U_i\}) \to \check{C}(\{V_i\}$ which is by assumption of the corollary objectwise a homotopy equivalence.
  Hence $h\underset{\cong X}{\operatorname{colim}}\check{C}(\{V_i\} \xrightarrow{\cong} h\underset{\cong X}{\operatorname{colim}}\check{C}\ (\{V_i\})$.

- Let G be a discrete group, $G \circlearrowright X$ G acts on X via a.
  This gives us a diagram of the type $BG \xrightarrow{a} \mathtt{Top}$.

  One can define the "homotopy quotient" of this action as

$$X//G := h\operatorname{colim} a$$

Two properties of this is that it is homotopy equivalent to the usual quotient $\simeq X/G$ if the action is free and "good"
If it is not free one can describe it as $(X \times EG)/G$ where we replace X by something homotopy equivalent to it such that the action is free $EG \simeq *$, G acts on EG freely and "good".
E.g. if $X = *$, its homotopy quotient $*//G \simeq EG/G \simeq K(G, 1)$.
This is because $EG \to EG/G$ is a covering with Galois group G. But since EG is contractible, $EG \simeq *$, the higher homotopy groups of EG/G are the same as those of EG so they are all zero and the fundamental group is G because again EG is contractible and the Galois group is G (see AT I).

2.12 Seifert-van Kampen theorem revisited

Idea:
$\Pi_1 : \mathtt{Top} \to \mathtt{Gpd}$ has a right adjoint $\overline{B} : \mathtt{Gpd} \to \mathtt{Top}$ on the level of (2,1)-categories.
Hence Π_1 preserves weak 2-pushouts (and $\overline{B}$ preserves weak 2-pullbacks)

- π_1 is a strict functor between (2,1)-categories
$$\Pi_1(X) = HOM_{\mathtt{Top}}(*, X) := \Pi_1(X^*)$$
- $\overline{B}(C)$ where C is a groupoid is defined by
$$\overline{B}(C) := |NC|$$
which is the geometric realisation of the nerve.
  - The nerve N sends natural transformations to simplicial homotopies and $|-|$ sends simplicial homotopies to homotopies.
  So those two combined give a functor between groupoids
  $$\overline{B} : HOM_{\mathtt{Gpd}}(C, D) \to HOM_{\mathtt{Top}}(\overline{B}C, \overline{B}D)$$
  functorial in C, D.
  - If $C = \coprod C_i$ then $\overline{B}(C) \cong \coprod \overline{B}(C_i)$.
  So you can always assume that C is connected.
  - If one applies $\overline{B}$ to a connected groupoid (and every connected groupoid is equivalent to BG for some G) then what we get is $\overline{B}(BG) \simeq K(G, 1)$.
  There are two parts to proving that. The first is to compute the fundamental group. In AT 1 Exercise 7.2(a) we constructed a functor $BG \to \Pi_1\overline{B}(BG)$ and in part (b) we checked that $G \to \pi_1(\overline{B}(BG), *)$ is an isomorphism.
  The second is that $\overline{B}(BG) = \overline{E}G/G$, where $\overline{E}G$ is contractible.

Proposition 2.12.1.
$X \in \mathtt{CW}$, $C \in Gpd$

$$\begin{array}{ccc} HOM_{Top}(X, \overline{B}C) & \xrightarrow{\Pi_1 -} & HOM_{Gpd}(\Pi_1 X, \Pi_1 \overline{B}C) \\ & & {\scriptstyle equivalence \atop of\ groupoids} \Big\uparrow \\ & & HOM_{Gpd}(\Pi_1 X, C) \end{array}$$

The claim of this proposition is that $HOM_{Top}(X, \overline{B}C) \to HOM_{Gpd}(\Pi_1 X, \Pi_1 \overline{B}C)$ is an equivalence.

Proof.
WLOG: $C = BG$. We can assume that because we can replace everything by equivalent things, so if C is not connected but rather the disjoint union of groupoids, then $\overline{B}C$ will be the dosjoint union of topological spaces and $/Pi_1\overline{B}C$ also the disjoint union of groupoids. So everything kind of

splits and it suffices to regard the case of one connected component.
So if C is connected, you can choose just one point of the groupoid, consider its automorphisms and by one of the first lemmata in AT I the inclusion of BG of this point with its automorphisms into C is an equivalence of categories.
To prove that the equivalence we would like to prove is stable under changing the HOM groupoids by equivalent groupoids and the objects defining the HOM groupoids by equivalent objects in (2,1)-category give equivalent groupoids sort of by definition.

Assume X is connected, thus there is only one 0-cell $X^{(0)} = \{x\}$.
Also we would like to replace $\overline{B}C$ by an Eilenberg-MacLane space $K(G,1)$.

To prove that the functor is essentially surjective and full

$$\begin{tikzcd}
 & \Pi_1\overline{BC} = \Pi_1 K(G,1) \\
\Pi_1 X \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\eta"] & BG \arrow[u, "\phi"']
\end{tikzcd}$$

F_1 and F_2 are objects in $HOM_{\mathsf{Gpd}}(\Pi_1 X, C)$ and η is a 1-morphism in this groupoid. We have an equivalence $BG \xrightarrow{\phi} \Pi_1\overline{B}C$.

We would like to construct $f_1, f_s : X \to K(G,1)$.
Inside $\Pi_1 X$ there is a point and $B\pi_1(X,x) \hookrightarrow \Pi_1 X$ is an equivalence.
If we find $\Pi_1 f_1$ and $\Pi_1 f_{"}$ such that this diagram

$$\begin{tikzcd}
 & \Pi_1\overline{BC} = \Pi_1 K(G,1) \\
B\pi_1(X,x) \arrow[ur, dashed, bend left=60, "\Pi_1 f_1"{name=A}] \arrow[ur, dashed, bend right, "\Pi_1 f_2"'{name=B}] \arrow[d, hook] & \\
\Pi_1 X \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\eta"] & BG \arrow[uu, "\phi"']
\arrow[from=A, to=B, Rightarrow, "\pi_1 J"]
\end{tikzcd}$$

is 2-commutative, it follows that Π_1- is essentially surjective and full.
For again, to show that this is essentiall surjective and full one can replace in $HOM_{\mathsf{Gpd}}(\Pi_1 X, \Pi_1\overline{B}C)$ the groupoid $\Pi_1 X$ by $B\pi_1(X,x)$ and look at the essential surjectivities there.

To show faithfulness, suppose you have a map $f : X \to K(G,1)$ and a homotopy $H : f \Rightarrow f$ such that $\Pi_1 H = id_{\pi_1 f}$.
What we want to conclude here, is that H is homotopic via a homotopy between homotopies to id_f.

$$\begin{tikzcd}
X \times I \arrow[r, bend left, "H"] \arrow[r, bend right, "id_f"'] \arrow[r, phantom, "\Downarrow"] & K(G,1)
\end{tikzcd}$$

where if_f is the constant homotopy from f to f.
We know by assumption that after applying Π_1 we have an equality

$$\begin{tikzcd}
\Pi_1(X \times I) \arrow[r, bend left, "\Pi_1 H"] \arrow[r, bend right, "\Pi_1 id_f"'] \arrow[r, phantom, "\|"] & \Pi_1 K(G,1)
\end{tikzcd}$$

■

Theorem 2.12.2 (Seifert van Kampen).
If

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, shift left] \\ C \arrow[r, "v"'] & D \arrow[u, shift left, "u"'] \end{tikzcd}$$

is a homotopy pushout in **Top**, *then*

$$\begin{tikzcd} \Pi_1 A \arrow[r] \arrow[d] & \Pi_1 B \arrow[d] \arrow[dl, phantom, "\swarrow"] \\ \Pi_1 C \arrow[r] & \Pi_1 D \end{tikzcd}$$

is a weak(=strict) 2-pushout in **Gpd**.

Proof. (sketch)
check the universal property:
WLOG, take first a connected groupoid, because if it is not connected, again you can take the connected components seperately. But because any connected component is equivalent to some BG, we can take as a test object BG.

$$\begin{tikzcd} \Pi_1 A \arrow[r] \arrow[d] & \Pi_1 B \arrow[d, shift left] \arrow[ddr, bend left, "\beta"] & \\ \Pi_1 C \arrow[r] \arrow[drr, bend right, "\gamma"'] & \Pi_1 D \arrow[u, shift left] & \\ & & BG \end{tikzcd}$$

$$\begin{tikzcd} HOM(C, \overline{B}G) \arrow[r, "\simeq"] & HOM(\Pi_1 C, \Pi_1 \overline{B}G) \ni \pi \circ \gamma \\ & HOM(\Pi_1 C, BG) \ni \gamma \arrow[u, "\simeq"] \end{tikzcd}$$

So we get

$$\begin{tikzcd} \Pi_1 A \arrow[r, "\Pi_1 f"] \arrow[d, "\Pi_1 g"'] & \Pi_1 B \arrow[d, shift left] \arrow[ddr, bend left, "\beta"] & & \\ \Pi_1 C \arrow[r] \arrow[drr, bend right, "\gamma"'] & \Pi_1 D \arrow[u, shift left] \arrow[dr, phantom, "\xi"] & & \\ & & BG \arrow[dr, "\phi"', "\simeq"] & \\ & & & \Pi_1 \overline{B}G \end{tikzcd}$$

Since the map on HOM groupoids is an equivalence, we get γ, such that

$$\begin{tikzcd} HOM(C, \overline{B}G) \arrow[r, "\simeq"] & HOM(\Pi_1 C, \Pi_1 \overline{B}G) \ni \pi \circ \gamma \\ \in & \\ \Gamma \mapsto \Pi_1 \gamma \overset{\simeq}{\Rightarrow} \pi \circ \gamma & HOM(\Pi_1 C, BG) \ni \gamma \arrow[uu, "\simeq"] \end{tikzcd}$$

Similarly, we get $B : B \to \overline{B}G$ and an isomorphism $\Pi_1 B \xrightarrow{\simeq} \phi \circ \beta$. This only uses surjectivity of those functor so far.

But now we have ξ

$$HOM(A, \overline{B}G) \to HOM(\Pi_1 A, \Pi_1 BG)$$

$$\begin{tikzcd}
\Gamma \circ g \arrow[r, dashed, "h"] & B \circ f \arrow[r, mapsto] & \Pi_1\Gamma \circ \Pi_1 g \arrow[r, dashed] \arrow[d] & \Pi_1 B \circ \Pi_1 f \arrow[d] \\
& & \phi \circ \gamma \circ \Pi_1 g & \phi \circ \beta \circ \Pi_1 f \arrow[l, "\phi\circ\xi"]
\end{tikzcd}$$

What you get is the space $\overline{B}G$ and a homotopy h which was constructed out of $\phi \circ \xi$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, shift left, "u"] \arrow[ddr, bend left, "B"] & \\
C \arrow[r, "v"'] \arrow[drr, bend right, "\Gamma"'] & D \arrow[u, shift left] \arrow[dr, phantom, "h"] & \\
& & \overline{B}G
\end{tikzcd}$$

By the universal property we get a map $D \to \overline{B}G$ (and homotopies making everything commutative).
Applying Π_1 we get $\Pi_1 D \to \Pi_1 \overline{B}G$ we can translate it back to BG using some homotopy inverse of ϕ. Thus we have a morphism $\Pi_1 D \to BG$.

For uniqueness assume one is given two arrows $\Pi_1 D \to BG$. Translate them to the first square, find a homotopy between them there and translate it back. ■

2-pushouts and 2-pullbacks in `Gpd`

Exercise (Criterion for 2-pushout)
A square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, dashed, bend left] \arrow[dl, phantom, "\Leftarrow"] & \\
C \arrow[r] \arrow[drr, dashed, bend right] & D & \\
& & T
\end{tikzcd}$$

is a strict 2-pushout
$\Leftrightarrow$

$$\begin{tikzcd}
HOM(D,T) \arrow[r] \arrow[d] & HOM(B,T) \arrow[d] \arrow[dl, phantom, "\Leftarrow"] \\
HOM(C,T) \arrow[r] & HOM(A,T)
\end{tikzcd}$$

is a strict 2-pullback in `Gpd`.

Proposition 2.12.3. - *construction:*

$$\begin{tikzcd} & \mathscr{H}_2 \arrow[d, "g"] \\ \mathscr{H}_1 \arrow[r, "f"'] & \mathscr{H}_3 \end{tikzcd}$$

in Gpd*. We define some category* $\mathscr{H}_1 \times h_{\mathscr{H}_3}\mathscr{H}_2$ *which we will see is the strict 2-pullback by*

- *on objects:* $(\underset{\in\mathscr{H}_1}{h_1}, \underset{\in\mathscr{H}_2}{h_2}, f(h_1) \underset{\simeq}{\xrightarrow{\alpha}} g(h_1))$
- *on morphisms* $(h_1, h_2, \alpha) \xrightarrow{(a,b)} (h_1', h_2', \alpha'$ *where* $a : h \to h' \in \mathscr{H}_1$ *and* $b : h_2 \to h_2' \in \mathscr{H}_2$ *which are compatible with the isomorphisms:*

$$\begin{tikzcd} f(h_1) \arrow[r] \arrow[d, "\alpha"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & f(h_1') \arrow[d, "\cong"', "\alpha'"] \\ g(h_2) \arrow[r] & g(h_2') \end{tikzcd}$$

Then this category makes

$$\begin{tikzcd} \mathscr{H}_1 \times h_{\mathscr{H}_3}\mathscr{H}_2 \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathscr{H}_2 \arrow[d, "g"] \arrow[dl, Rightarrow, "\alpha"'] \\ \mathscr{H}_1 \arrow[r, "f"'] & \mathscr{H}_3 \end{tikzcd}$$

a strict 2-pullback square

Proof. (proof)
Given $(T, u, v, \eta : g \circ v \Leftarrow f \circ u)$, we want to construct ω

$$\begin{tikzcd} T \arrow[dr, dashed, "\omega"] \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "u"'] & & \\ & \mathscr{H}_1 \times h_{\mathscr{H}_3}\mathscr{H}_2 \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathscr{H}_2 \arrow[d, "g"] \arrow[dl, Rightarrow, "\alpha"'] \\ & \mathscr{H}_1 \arrow[r, "f"'] & \mathscr{H}_3 \end{tikzcd}$$

On objects we define it by $t \mapsto (u(t), v(t), \eta_t)$. Now the triangles commute without any homotopy. This gives the existence part of the 2-pullback.

To prove uniqueness we take ω which does not need homtopies and $\tilde{\omega}$ that has homotopies. Then we would like to find a unique natural transformation $\xi : \omega \Rightarrow \tilde{\omega}$.

$$\begin{tikzcd} T \arrow[dr, bend left, "\tilde{\omega}"] \arrow[dr, bend right, "\omega"'] \arrow[dr, phantom, "\xi"] \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "u"'] & & \\ & \mathscr{H}_1 \times h_{\mathscr{H}_3}\mathscr{H}_2 \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathscr{H}_2 \arrow[d, "g"] \arrow[dl, Rightarrow, "\alpha"'] \\ & \mathscr{H}_1 \arrow[r, "f"'] & \mathscr{H}_3 \end{tikzcd}$$

As soon as we have proven that for ω and $\tilde{\omega}$, we have proven uniqueness for all pairs, because we work with groupoids were natural transformations are invertible.

Given $(\tilde{\omega}, h_U, h_V)$, to construct ξ we have to find for all $t \in T$ a map $\omega(t) \to \tilde{\omega}(t)$. But

what is either of them? For ω we know that $\omega(t) = (u(t), v(t), \eta_t)$ for $\tilde{\omega}$ we don't really know but it has to be $\tilde{\omega}(t) = (pr_1 \circ \tilde{\omega}(t), pr_2 \circ \tilde{\omega}(t), \tilde{\eta}_t)$.
So to construct a morphism between the two, we need to find $u(t) \to pr_1 \circ \tilde{\omega}(t)$ but this is precisely the homotopy that is given by $\tilde{\omega}$, h_u. Similarly, $v(t) \xrightarrow{h_V} pr_2 \circ \tilde{\omega}(t)$.
We have to check that this gives a natural transformation and makes everything 2-commutative, but we can see that we did not choose anything. The construction was completely out of the given data and this sort of explains the uniqeness. ■

CHAPTER 3

Brown Representability Theorem

Goal: study conditions that a functor $F : \mathtt{CW}_*^{0,op} \to \mathtt{Set}_*$ is representable.
$\mathtt{CW}_*^{0,op}$ is the category of pointed but also connected CW-complexes.

Definition 3.0.1.
A functor is representable, if there is some K such that $[-, K] \xrightarrow{\cong} F(-)$

We have already seen one condition of representability in AT I which is that

$$\tilde{H}^n(X, A) \cong [X, K(A, n)]$$

One application will be to construct classifying spaces of topological groups.

So how does one go about finding conditions for representability? Well, the natural answer is to study representable functors and their properties and find out which of these properties actually determine the functor and allow to construct the object back out of the functor.

Lemma 3.0.2. *Suppose we have $K \in \mathtt{CW}_*^0$, then $h_K(-) := [-, K]$ satisfies:*

1. *(homotopy invariance)*
   Any two homotopic maps $f_0, f_1 : X \to Y$, induce the same map after applying this functor: $h_K(f_0) = h_K(f_1) : h_K(Y) \to h_K(X)$

2. *(Mayer-Vietoris or excision property)*
   If we have a pushout square

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, hook] & A \arrow[d] \\ B \arrow[r, hook] \arrow[ur, phantom, "\square"] & X \end{tikzcd}$$

   (Since everything is connected, one can just assume $C = A \cap B$)
   Then

$$\begin{tikzcd} h_K(X) \arrow[r] \arrow[d] & h_K(C) \arrow[d] \\ h_K(B) \arrow[r] & h_K(A) \end{tikzcd}$$

   is weakly 1-cartesian, i.e. $h_K(X) \twoheadrightarrow h_K(B) \times_{h_K(A)} h_K(C)$ is surjective (only existence is given, no uniqueness)

3. *(additivity)*
   For any family of pointed, connected CW-complexes $\{X_\alpha\}_{\alpha \in J}$, we have a canonical inclusion

$X_\alpha \overset{i_\alpha}{\hookrightarrow} \bigvee_{\alpha \in J} X_\alpha$. *So we get an induced map (after applying* h_K*)*

$$h_K\left(\bigvee_{\alpha \in J} X_\alpha\right) \xrightarrow[h_k(i_\alpha)]{\cong} \prod_{\alpha \in J} h_K(X_\alpha)$$

which is an isomorphism.

Remark 3.0.3.

- 1. and 3. imply that $h_k(*) \cong *$.
  This is because

$$\begin{tikzcd} h_K(X \vee *) \arrow[rr, "\cong"] \arrow[dr, "\cong"'] & & h_K(X) \times h_K(*) \arrow[dl] \\ & h_K(X) & \end{tikzcd}$$

  Thus we get that the map on the right also needs to be an isomorphism. The only space, however, for which the product with some space is isomorphic to that space is the one point space.

- Under the condition of 1., 2. is equivalent to 2.' where in 2.' we have an arbitrary homotopy pushout square. (every homotopy pushout square is equivalent to the one as in 2.)
  In particular,

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & X \arrow[d] \\ * \arrow[r] & X/A \end{tikzcd}$$

$$h_K(X/A) \to h_K(X) \to h_K(A)$$

  is exact.

Proof.

3.
We have to compare $[\bigvee_{\alpha \in J} X_\alpha, K] \to \prod_{\alpha \in J}[X_\alpha, K]$.
This is not completely obvious, $[\bigvee_{\alpha \in J} X_\alpha, K] = Hom_{\mathsf{Top}_*}(\bigvee_{\alpha \in J} X_\alpha, K)/\sim$.
The wedge sum of X_α is of course a coproduct in the category of pointed topological spaces, but not in the homotopy category. Thus

$$\left(\prod_{\alpha \in J} Hom(X_\alpha, K)\right)/\sim \cong Hom_{\mathsf{Top}_*}\left(\bigvee_{\alpha \in J} X_\alpha, K\right)/\sim$$

On the other hand

$$\prod_{\alpha \in J}[X_\alpha, K] = \prod_{\alpha \in J}\left(Hom_{\mathsf{Top}_*}(X_\alpha, K)/\sim\right)$$

So one has to compare the homotopy relation on the product to the one on each component.
We know that the map $[\bigvee_{\alpha \in J} X_\alpha, K] \to \prod_{\alpha \in J}[X_\alpha, K]$ exists and it is obvious that it is surjective, so we have to show that this map is injective.
Take $(f_\alpha)_\alpha, (g_\alpha)_\alpha \in \prod Hom(X_\alpha, K)$ such that $f_\alpha \overset{h_\alpha}{\simeq} g_\alpha$ as maps $X_\alpha \to K$.

We need to construct $(\bigvee_\alpha X_\alpha) \times I \to K$ between $\vee f_\alpha$ and $\vee g_\alpha$.
An important thing to note here is that h_α is a map $X_\alpha \times I \to K$ but it is a pointed homotopy, so it factors through the quotient:

$$\begin{tikzcd}
 & X_\alpha \times I / x_\alpha \times I \arrow[dr] & \\
X_\alpha \times I \arrow[ur, two heads] \arrow[rr, "h_\alpha"'] & & K
\end{tikzcd}$$

This allows us to define the map

$$\left(\bigvee_\alpha X_\alpha\right) \times I \to K$$
$$(\underset{\in X_\alpha}{x}, t) \mapsto h_\alpha(x,t)$$

2.' Let

$$\begin{tikzcd}
C \arrow[r, "i"] \arrow[d, "j"'] & A \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\
B \arrow[r] & X
\end{tikzcd}$$

be a homotopy pushout in the (2,1)-category of CW-complexes.
We get a square of groupoids

$$\begin{tikzcd}
HOM(X,K) \arrow[r] \arrow[d] & HOM(A,K) \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\
HOM(B,K) \arrow[r] & HOM(C,K)
\end{tikzcd}$$

We get a functor

$$F : HOM(X,K) \to HOM(B,K) \times^h_{HOM(C,K)} HOM(A,K)$$

Claim: The functor F is essentially surjective.
In other words, since these are groupoids, if you apply π_0 on them, the map is surjective. This is really relevant to what we want to discuss, because $\pi_0(HOM(X,K))$ is precisely the set of morphisms $X \to K$ in the homotopy category of pointed CW-complexes.

Recall that there is a really nice model for this strict 2-pushout $HOM(B,K) \times^h_{HOM(C,K)} HOM(A,K)$:

- Objects: $(B \xrightarrow{\phi} K, A \xrightarrow{\psi} K, h : \psi \circ i \implies \phi \circ j)$
- Morphisms: $(\phi, \psi, h) \xrightarrow[(h_\phi, h_\psi)]{} (\phi', \psi', h')$ such that they "commute" with h and h'.

So we need to take an object in this category and find one in $HOM(X,K)$. This, however, is just the definition of weak 2-pushout:

$$\begin{tikzcd}
C \arrow[r, "i"] \arrow[d, "j"'] & A \arrow[d, "u"] \arrow[dl, phantom, "\Swarrow H"] \arrow[ddr, bend left, "\psi"] & \\
B \arrow[r, "v"'] \arrow[drr, bend right, "\phi"'] & X \arrow[dr, dashed, "\omega"] & \\
 & & K
\end{tikzcd}$$

By the universal property for (ϕ, ψ, h) we get to fill in the dashed line $\omega : X \to K$ and homotopies $h_u : \omega \circ u \implies \psi$, $h_v : \omega \circ v \implies \phi$ such that we get a 2-commutative diagram.

$$F(\omega) = (\omega \circ v, \omega \circ u, \omega \circ H)$$

This object is clearly isomorphic to (ϕ, ψ, h) via (h_u, h_v):

$$F(\omega) = (\omega \circ v, \omega \circ u, \omega \circ H) \xrightarrow[\cong]{(h_u, h_v)} (\phi, \psi, h)$$

Hence the claim.

We need to show that

$$[X, K] \to [B, K] \times_{[C,K]} [A, K]$$

is surjective. But

$$\begin{tikzcd}
{[X,K]} \arrow[r] \arrow[d, equal] & {[B,K] \times_{[C,K]} [A,K] := \{(\phi : B \to K, \psi : A \to K) \mid \exists h : \psi \circ i \implies \phi \circ j\}} \\
\pi_0(HOM(X,K)) \arrow[r, two heads] & \pi_0(HOM(B,K) \times^h_{HOM(C,K)} HOM(A,K)) \arrow[u, two heads]
\end{tikzcd}$$

Since this square is commutative, the map $[X, K] \to [B, K] \times_{[C,K]} [A, K]$ is surjective as well.

■

Theorem 3.0.4 (Brown)**.**
Let $h : (CW^0_*)^{op} \to \mathsf{Set}_*$ satisfy 1.,2., 3. in the previous lemma.
Then $\exists K \in CW^0_*$ and a natural transformation $[-, K] \xrightarrow[\cong]{T_u} h(-)$ which is an isomorphism of functors.
In explicit, by Yoneda lemma T_u is given by an element $u \in h(K)$.

$$\begin{array}{ccc}
[X, K] & \longrightarrow & h(X) \\
\in & & \in \\
f & \longmapsto & f^*(u) = h(f)(u)
\end{array}$$

Corollary 3.0.5.

- *There exists a space $K(A, n)$ that represents reduced cohomology $\tilde{H}^n(X, A)$*
- *It is uniquely (up to homotopy equivalence) determined by*

$$\pi_i K(A, n) = \begin{cases} A & i = n \\ 0 & \text{otherwise} \end{cases}$$

Proof. (of corollary)
$\tilde{H}^n(-, A)$ satisfies 1.,3. what about the second, the Mayer-Vietoris property? The property states that for

$$\begin{tikzcd}
C \arrow[r, hook] \arrow[d] & N \arrow[d] \\
M \arrow[r] \arrow[ur, phantom, "\square"] & X
\end{tikzcd}$$

$\tilde{H}^n(X,A) \twoheadrightarrow \tilde{H}^n(M,A) \times_{\tilde{H}^n(C,A)} \tilde{H}^n(N,A) = \ker(\tilde{H}^n(M,A) \oplus \tilde{H}^n(N,A) \to \tilde{H}^n(C,A))$ where $C = M \cap N$
So this is indeed part of the Mayer-Vietoris sequence and 2. is fulfilled.

Hence $\exists K, u$. We would like to compute the homotopy groups of this space K, but these homotopy groups are nothing more than the value of this functor T_u on the spheres:

$$\pi_i K \cong \tilde{H}^n(S^i, A) = \begin{cases} A & i = n \\ 0 & \text{else} \end{cases}$$

So we see that this space K that represents the cohomology indeed is an Eilenberg-MacLane space. Now an interesting part is of course to show that it is uniqely determined by these properties. Therefore the notation $K(A,n)$ is un-ambiguous, just take any space with these homotopy groups, it will represent the cohomology and be unique up to homotopy equivalence.

Suppose we have K' such that its homotopy groups are the same as for the space that represents cohomology:

$$\pi_i K' = \begin{cases} A & i = n \\ 0 & \text{else} \end{cases}$$

Now we need to go from the homotopy groups to the cohomology groups. By Hurewicz $H_n K' \cong A$, thus $\tilde{H}^n(K', A) \cong Hom_{\mathsf{Ab}}(A, A) \ni id_A$.
Thus $id_A \in \tilde{H}^n(K', A)$. Since K represents cohomology, we get $f : K' \to K$, induced by id_A.
The question is now: what does this map induce on the homotopy groups? Because if it induces an isomorphism, then by the Whitehead theorem this will be an homotopy equivalence (because we assume K' to be a pointed CW-complex).

$$\begin{tikzcd} \pi_n K' \arrow[r] \arrow[d, equal] & \pi_n K \arrow[d, equal] \\ A = H_n K' \arrow[r, "f_*"] & H_n K = A \end{tikzcd}$$

$$\begin{tikzcd} \tilde{H}^n(K, A) \arrow[r, "f^*"] \arrow[d, "\cong"'] & \tilde{H}^n(K', A) \arrow[d, "\cong"] \\ Hom(A, A) \arrow[r, "\circ f_*"] & Hom(A, A) \end{tikzcd}$$

We have a canonical element $u \in \tilde{H}1n(K, A)$ which is being sent by f^* to the element we started with, which is the identity id_A. Because K is to represent cohomology one can check via Yoneda that u corresponds to id_A in $Hom(A, A)$. Thus, by commutativity, id_A is sent onto itself by $\circ f_*$.

$$\begin{tikzcd} u \arrow[r, mapsto] \arrow[d, mapsto, "\text{Yoneda}"'] & id_A \arrow[d, mapsto, "\text{Yoneda}"] \\ id_A \arrow[r, mapsto] & id_A \end{tikzcd}$$

Therefore f_* is an isomorphism.

To summarise:
Brown representability theorem gives us some space K and an element u that represents cohomology. Just by checking the cohomology on the spheres you get the homotopy groups of K, which turn out to be A in degree n and 0 everywhere else.
Suppose we have a different space with the same homotopy groups. Now we can, using the homotopy groups, for this very particular space, compute its cohomology groups. We can compute the n-th homology group via Hurewicz to be A and from that follows that the n-th cohomology

group with coefficients in A which is just the morphisms of abelian groups A to A, $Hom_{\mathsf{Ab}}(A, A)$ because there are no Ext groups with $H_{n-1}K'$ homology because this is 0.
We have a canonical element id_A in this n-th cohomology group, so we can apply the isomorphism of functors. So the canonical element id_A in $h(K')$ should correspond to a map $K' \to K$. The final thing to check is that this map induces an isomorphism on homology groups, actually the identity using Yoneda. ■

Proof. (of Brown representability)
IDEA: construct a space K, an element $u \in h(K)$ such that the natural transformation $h_i : [S^i, K] \to h(S^i)$ is an isomorphism for all $i \geq 1$.
If the functor h is indeed representable by some space K' then similar to before you get a morphism $K \to K'$ and because the homotopy groups of both K and K' are $h(S^i)$ this morphism will be the identity on the homotopy groups and therefore a homotopy equivalence between these CW-complexes.
Thus if h is representable, then K is the representing space.

It is reasonable to expect that K represents h on all CW-complexes, because CW-complexes are by definition obtained by the homotopy pushout of spheres (usually there are disks in the bottom left corner but for the homotopy pushout it doesn't matter)

$$\begin{tikzcd} \bigvee_n S^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X^{(n)} \arrow[d] \\ * \arrow[r] & X^{(n+1)} \end{tikzcd}$$

Lemma 3.0.6.
Let $Z \in CW_*^0$, $z \in h(Z)$.
Then there exists $K \in CW_*^0$, $u \in h(K)$ such that

1. $[S^m, K] \xrightarrow[\cong]{T_u} h(S^m)$, $m \geq 1$
2. $\exists f : Z \to K$ *such that* $h(f)(u) = f^*u = z$.

Proof.

- $K_1 := Z \vee \bigvee_\alpha S^1$, $\alpha \in h(S^1)$; note that $[S^1, K_1] \twoheadrightarrow h(S^1)$. To define this map, we have to choose an element $u_1 \in h(K^1) \cong h(Z) \times \prod_\alpha h(S^1)$. This u_1 corresponds to $(z, (\alpha)_\alpha)$ where $z \in h(Z)$ and $\alpha \in h(S^1)$ is in the α-th component of the product.
  Now we have the element u_1 inducing the natural transformation $T_{u_1} : [S^1, K_1] \twoheadrightarrow h(S^1)$ which we claim to be surjective. It is defined by $\{S^1 \xrightarrow{i_\alpha} K_1\} \mapsto i_\alpha^*(u_1) = \alpha$

  We also have $f_1 : Z \hookrightarrow K_1$ which is just the inclusion, such that $f_1^*(u_1) = z$.
- We construct by induction K_n, u_n such that
  1. $[S^m, K_n] \to h(S^m)$ is an isomorphism for $m < n$ and surjective for $m = n$.
  2. $f_n : Z \to K_n$ such that $f_n^*(u_n) = z$

  $n \to n+1$
  $U_{n+1} = \{g_\beta : S^n \to K_n \mid g_\beta \not\simeq const, g_\beta^*(u_n) = *\}$.
  So any element g_β is not null-homotopic, but by the second property, that $g_\beta^*(u_n) = *$, it lies in the kernel of $[S^m, K_n] \to h(S^m)$ $(g_\beta \mapsto *)$

  These maps, so the set U_{n+1} are the problem of the non-isomorphism in degree n. So we kill them.

$$\begin{tikzcd}
\bigvee_{\beta \in U_{n+1}} S^n \arrow[r, "g_\beta"] \arrow[d] & K_n \arrow[d, "q_n"] \\
* \arrow[r] & C_n \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

$$K_{n+1} := C_n \vee \bigvee_\alpha S^{n+1} \quad \alpha \in h(S^{n+1})$$

There is a canonical map $K_n \to K_{n+1}$. Using the already existing map $f_n : Z \to K_n$, we get a map f_{n+1}

$$\begin{tikzcd}
K_{n+1} & \\
K_n \arrow[u] & Z \arrow[l, "f_n"] \arrow[ul, "f_{n+1}"']
\end{tikzcd}$$

Now we have to construct the element u_{n+1} and check the two properties.

$$u_{n+1} \leftrightarrow (\overline{u_n}, (\alpha)_\alpha)$$

What's the element $\overline{u_n} \in C_n$? Applying h to the pushout square, we get

$$\begin{tikzcd}
h(C_n) \arrow[r] & h(K_n) \arrow[r] & h(\bigvee S^n) \cong & \prod_\beta h(S^n) \\
\in & \in & & \in \\
\overline{u_n} \arrow[r, mapsto] & u_n \arrow[rr, mapsto] & & * = (*_\beta)
\end{tikzcd}$$

The Mayer-Vietories property tells us that this is an exact sequence of pointed sets (that's where $\overline{u_n}$ comes from - since u_n is mapped onto 0, there has to be a preimage)

Now check the two conditions:

1. $[S^m, K_{n+1}] \to h(S^m)$ is surjective for $m = n+1$. This is clear because the elements $S^m \to K_{n+1} = C_n \vee \bigvee_\alpha S^m$ are just the inclusions and the map maps these inclusions the the corresponding $\alpha \in h(S^{n+1})$.
   C_n is obtained as the pushout along the $n+1$-connected arrow, which is $\bigvee S^n \to *$. Therefore the inclusion $K_n \hookrightarrow K_{n+1}$ induces an isomorphism for $m < n$:

   $$\begin{tikzcd}
   {[S^m, K_n]} \arrow[rr, "\cong"] \arrow[dr, "\cong"'] & & {[S^n, K_{n+1}]} \arrow[dl, "\cong"] \\
   & h(S^n) &
   \end{tikzcd}$$

   because the restriction of u_{n+1} to K_n along the canonical map is first $\overline{u_n}$ and then u_n which makes the triangle commute.
   For $m = n$:

   $$\begin{tikzcd}
   & h(S^n) & \\
   {[S^n, K_n]} \arrow[ur] \arrow[rr] & & {[S^n, K_{n+1}]} \arrow[ul]
   \end{tikzcd}$$

   By the induction assumption the map $[S^n, K_n] \to h(S^n)$ is surjective and by the construction because we attach only at least $n+1$-cells to K_n, the map $[S^n, K_n] \to [S^n, K_{n+1}]$ is surjective. Therefore so is $[S^n, K_{n+1}] \to h(S^n)$.

This proves surjectivity, what about injectivity? Take an element g in $[S^n, K_{n+1}]$ and suppose it goes to zero, so the distinguished element in $h(S^n)$. Because the horizontal map is surjective, we can lift this element to an element $g' \in [S^n, K_n]$. By commutativity g' is also sent onto the distinguished element and thus lies in the kernel of $[S^n, K_n] \to h(S^n)$. Hence $g' \in U_{n+1}$. But by construction, the map $K_n \to K_{n+1}$ factors through the inclusion $K_n \to C_n$ where we kill all these maps, so $g \simeq *$

2. By the commutativity of

$$\begin{tikzcd} K_{n+1} & \\ K_n \arrow[u] & Z \arrow[l, "f_n"] \arrow[ul, "f_{n+1}"'] \end{tikzcd}$$

we can regard $f_n^*(u_{n+1})$ first as the preimage of u_{n+1} by the map $K_n \to K_{n+1}$ and then the restriction to Z. The latter, however, is by induction assumption the element z, with which we started.

- WLOG, $K_n \hookrightarrow K_{n+1}$ is a relative CW-complex of relative dimension $\geq n+1$.
  Define the CW-complex

$$K := \bigcup_i K_i = \operatorname*{colim}_i K_i$$

We know from AT I, that $\pi_n K \xleftarrow{\cong} \operatorname{colim} \pi_m K_i \xrightarrow{\cong} h(S^m)$.
We still need $u \in h(K)$.
claim: The following square is a homotopy pushout square

$$\begin{tikzcd} \bigvee_i K_i \arrow[r] \arrow[d] & \bigvee_i K_{2i} \arrow[d] \\ \bigvee_i K_{2i+1} \arrow[r] & K \end{tikzcd}$$

By introducing the actual pushout into this diagram we get a map from it to K:

$$\begin{tikzcd} \bigvee_i K_i \arrow[rr] \arrow[dd] & & \bigvee_i K_{2i} \arrow[dl] \arrow[dd] \\ & Q \arrow[dr, dashed] & \\ \bigvee_i K_{2i+1} \arrow[ur] \arrow[rr] & & K \end{tikzcd}$$

To compare Q with K it's reasonable to look at the skeletal filtration, because every next step of the skeletal filtration is obtained as a pushout and Q is also obtained as a pushout and pushouts commute with each other.
$Q^{(n)}$ is a pushout of a much simpler diagram:

$$\begin{tikzcd} \bigvee_i K_i^{(n)} \arrow[r] \arrow[d] & \bigvee_i K_{2i}^{(n)} \arrow[d] \\ \bigvee_i K_{2i+1}^{(n)} \arrow[r] & Q^{(n)} \end{tikzcd}$$

It's much simpler because as we have discussed, the higher indexed K_i are obtained by higher and higher dimensional cells. So at some point, if we forget about them, these spaces stabilise. You have to check that the finite number of non-stabilised K_i's does not change the pushout. Assuming everything is stabilised, you have a pushout diagram

$$\begin{tikzcd}
\bigvee_i X \arrow[r] \arrow[d] & \bigvee_{2i} X \arrow[d] \\
\bigvee_{2i+1} X \arrow[r] \arrow[ur, phantom, "\square" near end] & X
\end{tikzcd}$$

Thus $Q^{(n)} \cong \operatorname{colim}_i K_i^{(n)} = K_i^{(n)}$, for $i >> 0$. Hence, because Q is a colimit of its skeletons and K is a colimit of its K_i's it follows that $Q \cong K$.

This justifies the claim. One needs this claim to use the Mayer-Vietoris property for h to say something about how to construct an element $u \in h(K)$.
We get that

$$h(K) \twoheadrightarrow \left(\prod h(K_{2i})\right) \times_{\prod h(K_i)} \left(\prod h(K_{2i+1})\right)$$

On the right hand side we have an element $((u_{2i}), (u_{2i+1}))$ and surjectiveness gives us some lift $u \in h(K)$ of this element.
In particular, $u \mapsto u_n$ for $K_n \to K$.
The map $f : Z \to K$ will be the map that factors through all K_n's

$$\begin{tikzcd}
Z \arrow[rr, "f"] \arrow[dr, "f_n"'] & & K \\
& K_n \arrow[ur] &
\end{tikzcd}$$

Therefore $f^*(u) = z$.

∎

We apply the lemma to $Z = *$, $z = * \in h(*)$. This gives us $K, u \in h(K)$.
Take $X \in CW_*^0$ and look at the natural transformation applied to X

$$[X, K] \to h(X)$$

We have to prove surjectivity and injectivity. Both will use a "trick" which could also be out aside as a lemma, but we will only do it once for surjectivity.

- surjectivity:
  Take $\alpha \in h(X)$, we would like to construct a map $X \to K$ such that the restriction, the pullback of $u \in K$ to $h(X)$ is precisely α.
  We construct a homotopy pushout by attaching X to K:

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & K \arrow[d] \\
X \arrow[r] \arrow[ur, phantom, "\Leftarrow"] & W
\end{tikzcd}$$

  Now $h(W) \cong h(X) \vee h(K)$ there is an element ω which corresponds to (α, u) in the product.
  We use the lemma again for (W, ω) to get some space K' and a map $q : W \to K'$ such that

$$q^*(u') = (\alpha, u)$$

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & K \arrow[d, "i"] \arrow[dr, bend left, "q\circ i"] & \\
X \arrow[r] \arrow[ur, phantom, "\Leftarrow"] & W \arrow[r, "q"'] & K'
\end{tikzcd}$$

So $(q \circ i)^*(u') = u$. From this follows that we have a commutative diagram

$$\begin{tikzcd}
{[S^m, K]} \arrow[rr, "T_u", "\cong"'] \arrow[dr] & & h(S^m) \\
& {[S^m, K']} \arrow[ur, "\cong", "T_{u'}"'] &
\end{tikzcd}$$

Hence $[S^m, K] \to [S^m, K']$ is an isomorphism and therefore $K \to K'$ is a weak equivalence and therefore a homotopy equivalence. Now we just take the inverse

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & K \arrow[d, "i"] \arrow[dr, bend left, "q\circ i"] & \\
X \arrow[r, "j"'] & W \arrow[r, "q"'] & K' \arrow[ul, bend right, "(q\circ i)^{-1}"']
\end{tikzcd}$$

and we define the map $X \to K$ to be the composition: $(q \circ i)^{-1} \circ q \circ j : X \to K$ such that the pullback of u is α.

- injectivity:
  Given two maps $f_0, f_1 : X \to K$ such that $x = f_0^*(u) = f_1^*(u)$. We have to prove that they are homotopic to each other.

$$\begin{tikzcd}
X \vee X \arrow[r, "f_0 \vee f_1"] \arrow[d] & K \arrow[d] \\
X \times I \arrow[r] & W
\end{tikzcd}$$

By Mayer-Vietoris we will now find an element in W:

$$h(W) \to \underbrace{h(X \times I)}_{\cong h(X)} \times_{h(X \vee X)} h(K)$$

On the right side we find a pair (x, u) which indeed lies in the fiber product because of the condition $x = f_0^*(u) = f_1^*(u)$. So we get an element in $h(W)$ which is mapped onto (x, u).

Now, as before, we use the lemma to construct a space K'

$$\begin{tikzcd}
X \vee X \arrow[r, "f_0 \vee f_1"] \arrow[d] & K \arrow[d] \arrow[dr, bend left] & \\
X \times I \arrow[r] & W \arrow[r, "q"'] & K'
\end{tikzcd}$$

Similarly, $q \circ i$ is a homotopy equivalence and therefore we can define the map $h : X \times I \to K$ using the inverse of the map $K \to K'$

$$\begin{tikzcd}
X \vee X \arrow[d, hook] \arrow[dr, "f_1 \vee f_2"] & \\
h : X \times I \arrow[r, "(q\circ i)^{-1} \circ q \circ j"'] & K
\end{tikzcd}$$

This will not be commutative, but there is a homotopy

$$\begin{tikzcd} X \vee X \arrow[d, hook] \arrow[dr, "f_1 \vee f_2"] & \\ h : X \times I \arrow[r, "(q\circ i)^{-1}\circ q\circ j"'] \arrow[ur, phantom, "\Rightarrow"] & K \end{tikzcd}$$

It implies that $f_0 = f_1$ in $[X, K]$.

$\blacksquare$

CHAPTER 4

Principal G-bundles and vector bundles

4.1 Principal G-bundles

We will assume G to be a topological group (we assume $G \in CGHaus$ (compactly generated Hausdorff) and has homotopy type of a CW-complex)
e.g. discrete group, $GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$ and so on.

Definition 4.1.1.
A principal G-bundle over a topological space X is a topological space P together with a map $\pi : P \to X$ where G acts on $P \to X$ freely and transitively on fibers fulfilling:

- $$\begin{tikzcd} G \times P \arrow[rr, "a"] \arrow[dr, "\pi \circ pr_2"'] & & P \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$ The commutativity of this triangle means that G acts fiberwise.

- $\forall x \in X$
$$G \circlearrowleft \underset{p \in}{\pi^{-1}(x)} = P_x$$
$$G \times \{p\} \xrightarrow[a]{\cong} P_x$$
where the choice of the point p gives the isomorphism. (not really necessary for the definition but rather an explanation as to how G acts on the fibers)

- Most importantly $\forall x \in X$, $\exists U \subset X$ open neighbourhood of x and an isomorphism
$$\begin{tikzcd} G \times U \arrow[rr, "\rho_U", "\cong"'] \arrow[dr, "pr_2"'] & & P|_{\pi^{-1}(U)} \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$
where ρ_U is G-equivariant, G acts on $G \times U$ by $(g, (\tilde{g}, u)) \mapsto (g\tilde{g}, u)$

Such an U is often called a trivialising neighbourhood of x.

Lemma 4.1.2.

1. *π is a fiber bundle with fiber G*
2.

$$\begin{tikzcd} P \arrow[dd, "\pi"'] \arrow[dr, "pr"] & \\ & P/G \arrow[dl, "\cong"] \\ X & \end{tikzcd}$$

Proof.

1. obvious
2. suffices to check locally. Then $G \times X/G \xrightarrow{\cong} X$

■

Example 4.1.3.

- If G is discrete, the notion of principal G-bundle over X = Galois covering of X with the automorphism group $\cong G$.
- $\begin{array}{c} \mathbb{C}^{n+1} \setminus \{0\} \\ \downarrow{\scriptstyle p_n} \\ \mathbb{C}P^n \end{array}$ canonical map which is the projection for all n.

  Also the infinite variant: $\begin{array}{c} \mathbb{C}^\infty \setminus \{0\} = \operatorname{colim}_n \mathbb{C}^n \setminus \{0\} \\ \downarrow{\scriptstyle p} \\ \mathbb{C}P^\infty = \operatorname{colim}_n \mathbb{C}P^n \end{array}$

  p and p_n are principal $\mathbb{C}^*$-bundles.

  Proof.
  We first define the action of $\mathbb{C}^*$:

  $$\begin{aligned} \mathbb{C}^* \times \mathbb{C}^\infty \setminus \{0\} &\to \mathbb{C}^\infty \setminus \{0\} \\ \tau, (z_0, z_1, ...) &\mapsto (\tau z_0, \tau z_1, ...) \end{aligned}$$

  These two elements $(z_0, z_1, ...)$ and $(\tau z_0, \tau z_1, ...)$ define the same element in $\mathbb{C}P^\infty$ and are thus in the same fiber.
  It is also clear that this action is free and transitive on fibers because each fiber is $\mathbb{C}^*$.
  What you really have to check is that the map $\mathbb{C}^\infty \setminus \{0\} \to \mathbb{C}P^\infty$ is locally trivialisable with that action of $\mathbb{C}^*$.
  For that we find a trivialising cover $\{U_i\}$ of $\mathbb{C}P^\infty$

  $$U_i = \{(z_0 : z_1 : ... : z_i : ...) \mid z_i \neq 0\}$$

  So the preimage by π is

  $$\begin{tikzcd} \mathbb{C}^* \times U_i \arrow[rr, "\rho"] \arrow[dr, "pr_2"'] & & \mathbb{C}^\infty \setminus \{z_i = 0\} \arrow[dl, "\pi"] \\ & U_i & \end{tikzcd}$$

  Such a map ρ is uniquely determined by

$$\begin{tikzcd} \rho|_{\{e\}\times U_i} : \{e\} \times U_i \arrow[rr] \arrow[dr, "\cong"'] & & \mathbb{C}^\infty \arrow[dl, "\pi"] \\ & U_i & \end{tikzcd}$$

So this is basically just a section. Then you extend it to a $\mathbb{C}^*$-equivariant map by using the action of $\mathbb{C}^*$.

Such $\rho|_{\{e\}\times U_i}$ is defined by

$$(z_0 : z_1 : ... : z_i : ...) \mapsto \left(\frac{z_0}{z_i}, \frac{z_1}{z_i}, ..., \frac{z_i}{z_i} = 1, ... \right)$$

■

- $\eta : S^3 \to S^2 = \mathbb{C}P^1$ with fiber $S^1 \cong U(1) = \{z \in \mathbb{C}^* \mid |z| = 1\} = GL_1(\mathbb{C})$ is a principal $U(1)$-bundle.
- fact: $H \subset G$ closed (but not necessarily normal) subgroup where H and G are Lie-groups (topological group which is itself a smooth manifold and the multiplication is a smooth map) In this case $G \to G/H$ is a principal H-bundle (G/H is not necessarily a group)

Pullbacks of principal G-bundles

Suppose we are given a principal G-bundle $\pi : P \to X$ and a map of topological spaces $f : Y \to X$

$$\begin{tikzcd} f^*P \arrow[r] \arrow[d, "f^*\pi"'] \arrow[dr, phantom, "\square"] & P \arrow[d, "\pi"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

Then $f^*\pi : f^*P \to Y$ is a principal G-bundle:

- $f^*P = \{(y, p) \mid \pi(p) = f(y)\}$

$$G \times f^*P \to f^*P$$
$$(g, (y, p)) \mapsto (y, gp)$$

- The fibers of $f^*\pi : f^*P \to Y$ over $y \cong$ fiber of π over x (G-equivariance)
- For the trivialising cover take the pullback of a trivialising cover $\{U_i\}$ of π: $\{f^{-1}(U_i)\}$ is a trivialising cover of $f^*\pi$

Definition 4.1.4.
Let P_1, P_2 be two principal G-bundles over X.
A morphism between them is

$$\begin{tikzcd} P_1 \arrow[rr, "\rho"] \arrow[dr, "\pi_1"'] & & P_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

such that

- $\pi_2 \circ \rho = \pi_1$
- ρ is G-equivariant

A section of P over X is

$$\begin{tikzcd} P \arrow[d, "\pi"] \\ X \arrow[u, bend left, "s"] \end{tikzcd}$$

such that $\pi \circ s = id_X$

Lemma 4.1.5.
Every morphism of principal G-bundles is an isomorphism.

Proof.
Suppose we are given a morphism ρ:

$$\begin{tikzcd} P_1 \arrow[rr, "\rho"] \arrow[dr, "\pi_1"'] & & P_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

To prove that it is a homeomorphism it suffices to look locally on X:

$$\begin{tikzcd} G \times U \arrow[rr, "\rho_U"] \arrow[dr, "pr_2"'] & & G \times U \arrow[dl, "pr_2"] \\ & U & \end{tikzcd}$$

That ρ_U is G-equivariant and the triangle commutes is equivalent to the fact that we have to define a map $\overline{\rho} : U \to G$ and $\rho_U(g, u) = (g\overline{\rho}(u), u)$ ($\overline{\rho} = pr_1 \circ \rho_U|_{\{e\} \times U}$).
Then an inverse ρ_U^{-1} is given by $(\overline{\rho})^{-1}$ (in terms of multiplication by G)

$$(g, u) \mapsto (g\overline{\rho}(u), u) \mapsto (g, u)$$
$$(h, u) \mapsto (h\overline{\rho}(u)^{-1}, u)$$

$\implies \rho$ is a homeomorphism $\implies \rho^{-1}$ is also G-equivariant. ■

Corollary 4.1.6.
Suppose we have a square

$$\begin{tikzcd} \tilde{P} \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{\pi}"'] & P \arrow[d, "\pi"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

where $P, \tilde{P}$ are principal G-bundles, $\tilde{f}$ is G-equivariant.

$\implies$ *it is a pullback square, i.e.*

$$\begin{tikzcd} \tilde{P} \arrow[rr, "\cong"] \arrow[dr] & & f^*P \arrow[dl] \\ & Y & \end{tikzcd}$$

Why is this a corollary of the lemma? We have a canonical map from $\tilde{P}$ to the pullback. They are both principal G-bundles over Y and this map is G-equivariant because $\tilde{f}$ was G-equivariant and therefore by the lemma it is an isomorphism.

Corollary 4.1.7.
A principal G-bundle over X is trivial if and only if it has a section.

Proof.
A principal G-bundle is trivial if it is isomorphic to the product:

$$\begin{tikzcd} G \times X \arrow[rr, "\rho"] \arrow[dr, "pr_2"'] & & P \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

Defining ρ as G-equivariant is the same as defining $\rho|_{\{e\}\times X}$ which is a section of π. ■

All this allows us to define the following functor:

$$\begin{aligned} Bun_G : \mathtt{CW}^{op} &\to \mathtt{Set} \\ X &\mapsto \left\{ \begin{matrix} \text{classes of isomorphisms of} \\ \text{principal } G\text{-bundles over } X \end{matrix} \right\} \\ (f : X \to Y) &\mapsto (p \in Bun_G(Y) \mapsto f^*P \in Bun_G(X)) \end{aligned}$$

Also the functor of pointed principal bundles

$$\begin{aligned} Bun_G^* : CW_*^{op} &\to \mathtt{Set}_* \\ (X, x) &\mapsto \left\{ \begin{matrix} \text{classes of isomorphisms of} \\ \text{pointed principal bundles} \\ (\pi : P \to X,\ p \in \pi^{-1}(x) \end{matrix} \right\} \end{aligned}$$

Those are of the form

$$\begin{tikzcd} (P_1, p_1) \arrow[rr, "\rho", "\cong"'] \arrow[dr, "\pi_1"'] & & (P_2, p_2) \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

ρ is an isomorphism principal G-bundles $\rho(p_1) = \rho(p_2)$.
Morphisms are again defined via the pullback.

There is a difference between pointed and unpointed principal G-bundles.
We always have the forgetful functor

$$Bun_G^*(X) \to Bun_G(X)$$

which is surjective but in general not injective.

Goal: Use Brown representability to get $BG \in \mathtt{CW}_*^0$ which classifies pointed principal G-bundles: $[X, BG]_* \cong Bun_G^*(X)$ for all $X \in \mathtt{CW}_*^0$.
Then $[X, BG] \cong Bun_G(X)$.
Such an BG will be called a classifying space.

4.2 Existence of classifying spaces

The following lemma is an easy step towards showing the homotopy invariance of the functor Bun_G. It is a consequence of what we know about fiber bundles and Serre fibrations.

Lemma 4.2.1.
$Bun_G(I^n) \cong \{*\}$, $Bun_G^*(I^n) \cong \{*\}$ for all n.

Proof.
Let $\pi : P \to I^n$ be a principal G-bundle. We want to show that it is trivial, so we have to construct a section.
This we intend to do by induction on n:

$$\begin{tikzcd}
P_{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & P \arrow[d] \\
I^{n-1} \times \{0\} \arrow[r, hook] & I^n
\end{tikzcd}$$

By induction there is a section s_{n-1}

$$\begin{tikzcd}
P_{n-1} \arrow[r, "\rho"] \arrow[d] \arrow[dr, phantom, "\square"] & P \arrow[d] \\
I^{n-1} \times \{0\} \arrow[r, hook] \arrow[u, bend left, "s_{n-1}"] & I^n
\end{tikzcd}$$

What we want to do is extend this section to the whole cube I^n.

$$\begin{tikzcd}
I^{n-1} \times \{0\} \arrow[r, "\rho_n s_{n-1}"] \arrow[d, hook] & P \arrow[d, "\pi"] \\
I^n \arrow[r] \arrow[ur, dashed, "s_n"] & I^n
\end{tikzcd}$$

We would like to find a lift s_n that makes the lower triangle commutative and so it will be a section of the map π. So why does this lifting exist? It does, because π is a fiber bundle and therefore it is a Serre fibration. ■

Lemma 4.2.2.
Let

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "\bar{j}"] \\
C \arrow[r, "\bar{i}"'] & X
\end{tikzcd}$$

be a pushout in **Top**.
Then if P is a principal G-bundle, and you take the pushout then it will be isomorphic to P:

$$\overset{P|_C}{\bar{i}^* P} \cup_{\bar{i}^* j^* P = i^* j^* P} \overset{P|_B}{\bar{j} * P} \xrightarrow{\cong} P$$

If $P_1 \xrightarrow{\cong} P_2$ is a principal G-bundle over X, then the induced square

$$\begin{tikzcd}
P_1 \arrow[r, "\cong"] & P_2 \\
P_1|_C \cup_{P_1|_A} P_1|_B \arrow[r, "\cong"] \arrow[u, "\cong"] & P_2|_C \cup_{P_2|_A} P_2|_B \arrow[u, "\cong"']
\end{tikzcd}$$

commutes.

Proof.

- $\tilde{P} := P|_C \cup_{P|_A} P|_B$ is a principle G-bundle and $P|_C \cup_{P|_A} P|_B \to P$ is G-equivariant.

- G acts on $\tilde{P}$, $G \in CGHaus$, so $\times G$ commutes with pushouts. Thus $G \times \tilde{P} \to \tilde{P}$ is a pushout of $G \times P|_A \to P|_A$

$$\begin{tikzcd}
P|_A \arrow[rr] \arrow[ddr] \arrow[ddd] & & P|_C \arrow[d] \arrow[ddr] & \\
& & \tilde{P} \arrow[dd] \arrow[dr] \arrow[dddr] & \\
& P|_B \arrow[ur] \arrow[rr] \arrow[dd] & & P \arrow[dd] \\
A \arrow[rr] \arrow[dr] & & C \arrow[dr] & \\
& B \arrow[rr] & & X
\end{tikzcd}$$

- $\tilde{P}$ is locally trivial, $x \in X$, then there exists an open neighbourhood U of x trivialising P, therefore if we look at

$$\tilde{P}|_U = P_U|_A \cup_{P_U|_A} P_U|_C$$

  Now the claim is, that if $P_U := P|_U$ itself is trivial, the pushout we can compute looks like

$$\begin{tikzcd}
G \times A \cap U \arrow[rr, "id_G \times -"] \arrow[dr, "id_G \times -"'] \arrow[drrr] \arrow[dd] & & G \times C \cap U \arrow[dr] \arrow[dd] \arrow[d, phantom, "\square"] & \\
& G \times B \cap U \arrow[rr] \arrow[dd] & {} & P_U \\
A \cap U \arrow[rr] \arrow[dr] & & C \cap U & \\
& B \cap U & &
\end{tikzcd}$$

  So because $\tilde{P}$ is the pushout of the upper square, it is isomorphic

$$P_U \cong G \times B \cap U \cup_{G \times A \cap U} G \times C \cap U = \tilde{P}|_U$$

■

Theorem 4.2.3 (homotopy invariance for $Bun_G/\ Bun_G^*$).
Let $X \in$ `CW` (or $X \in$ `CW`$_*$), denote $pr : X \times I \to X$, then

$$Bun_G(X) \xrightarrow[pr^*]{\cong} Bun_G(X \times I)$$
$$Bun_G^*(X) \xrightarrow[pr^*]{\cong} Bun_G^*(X \times I)$$

unpointed case, pointed similar.

- pr^* is always injective, because pr has a section $i_0 : X \hookrightarrow X \times I$, $x \mapsto (x, 0)$

$$\begin{tikzcd}
Bun_G(X) \arrow[r, "pr^*"] \arrow[rr, bend right, "id"'] & Bun_G(X \times I) \arrow[r, "i_0^*"] & Bun_G(X)
\end{tikzcd}$$

- P is a principal G bundle over $X \times I$. We need to show $P \cong pr^* i_0^* P$. We need to construct f in the following square

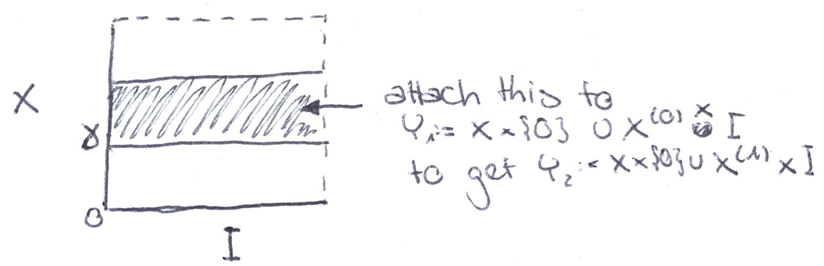
$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & P_0 := i_0^* P \arrow[d] \\ X \times I \arrow[r, "pr"'] & X \end{tikzcd}$$

f has to satisfy the commutativity of this square and be G-equivariant.
We construct f by induction on "half-skeletal" filtration of $X \times I =: Y$, $Y_0 := X \times \{0\}$, $Y_n = X \times \{0\} \cup X^{(n-1)} \times I$ for all n. $Y_n \subset X \times I = Y$ and $Y = \operatorname{colim}_n Y_n$.

To do this by induction we have to understand how we get Y_n from the previous one by attaching cells.
If we are given n-cells $\{I^n \xrightarrow{\phi} X^{(n)}\}$ of X. Then we have the following pushout square:

$$\begin{tikzcd} \coprod(\partial I^n \times I \cup I^n \times \{0\}) \arrow[r] \arrow[d] & Y_n \arrow[d] \\ \coprod(I^n \times I) \arrow[r] \arrow[ur, phantom, "\square"] & Y_{n+1} \end{tikzcd}$$

In the case of $n = 1$ (to understand it better) we have an attaching map $\gamma : I \to X^{(1)}$



We now define $P_n := P|_{Y_n}$

$$\begin{tikzcd} P_{n+1} \arrow[r, "f_{n+1}"] \arrow[d] & P_n \arrow[d] \\ Y_n \arrow[r] & Y_0 = X \end{tikzcd}$$

$$P_{n+1} = P_n \cup_{P|_{\coprod(\partial I^n \times I \cup I^n \times \{0\})}} P|_{\partial I^n \times I}$$

We have a map $f_n : P_n \to P_0$, we also get a map from $P|_{\coprod(\partial I^n \times I \cup I^n \times \{0\})}$ and we want to extend it to $P|_{\partial I^n \times I}$.
$P|_{I^n \times I}$ is trivial, so P restricted to $\partial I^n \times I \cup I^n \times \{0\}$ is also trivial. So we need to find ψ making this commute:

$$\begin{tikzcd} P|_{\partial I^n \times I \cup I^n \times \{0\}} \arrow[r] \arrow[d] \arrow[dr, "\tilde{f}_n"] & P_n \arrow[d, "f_n"] \\ P|_{I^n \times I} \arrow[r, dashed, "\exists \psi ?"'] & P_0 \end{tikzcd}$$

Because both $P|_{\partial I^n \times I \cup I^n \times \{0\}}$ and $P|_{I^n \times I}$ are trivial, we actually have

$$\begin{tikzcd}
(\partial I^n \times I \cup I^n \times \{0\}) \times G \arrow[r, "\tilde{f}_n"] \arrow[d, hook] & P_0 \\
I^n \times I \times G \arrow[ur, dashed, "\psi"'] &
\end{tikzcd}$$

ψ is G-equivariant $\implies$

$$\begin{tikzcd}
I^n \times \{0\} \arrow[d, hook] & \cong & \partial I^n \times I \cup I^n \times \{0\} \arrow[r, "\tilde{f}_n|"] \arrow[d, hook] & P_0 \arrow[d, "\pi"] \\
I^n \times I & \cong & I^n \times I \arrow[r] \arrow[ur, dashed, "\psi"] & X
\end{tikzcd}$$

ψ exists because π is a Serre fibration.
Thus we find f_{n+1}. This is the step of the induction.

Similarly $P \cong \operatorname{colim} P_n \xrightarrow{\operatorname{colim} f_n} P_0$ is G-equivariant map of principal G-bundles, hence an isomorphism.

■

Corollary 4.2.4.

$$Bun_G : h\mathtt{CW}^{op} \to \mathtt{Set}$$
$$Bun_G : h\mathtt{CW}_*^{op} \to \mathtt{Set}_*$$

Proposition 4.2.5.
Let $i : A \hookrightarrow X$ be a closed cofibration, P a principal G-bundle over X. There is a section $s : A \to P$ which is a section of P over A:

$$\begin{tikzcd}
 & P \arrow[d, "\pi"] \\
A \arrow[ur, "s"] \arrow[r, "i"] & X
\end{tikzcd}$$

It is the same as saying that if we pullback P to A, we get a principal G-bundle over A. A section of this π_A is the same as the section $A \to P$ making this triangle commutative by the universal property of the pullback square.

Then there exists an open neighbourhood $U \subset X$ of A, $A \subset U$ over which we have section $s_u : U \to P$ extending that on A: $\pi \circ s_U = j$, $s_U|_A = s$

Proof.
$i : A \hookrightarrow X$ is a neighbourhood deformation retract: $\exists h : X \times I \to X$ such that $h_0 = id_X$, $h(a, t) = a$ for all $a \in A$, all t and there exists an open $U \subset X$ such that $h_1(U) \subset A$.

You can construct this homotopy h from the retraction $X \times I \to A \times I \cup X \times \{0\}$.

Denote $r = h_1|_U : U \to A$. So there is a homotopy

$$h|_U : U \times I \to X$$
$$h_0|_U = id_U \quad h_1|_U = i \circ r$$

We get that for $\begin{tikzcd} i_A : A \arrow[rr] \arrow[dr, "i"'] & & U \arrow[dl, "j"] \\ & X & \end{tikzcd}$

$$r \circ i_A = id_A$$

Now we have two maps $U \to X$: the identity and $i \circ r$.
By homotopy invariance we get that there exists an isomorphism between the pullback of P along $h_0|_U = id_U$ and along $h_1|_U$.
But starting from the first this is just $P_U = P|_U \cong r^* P_A := r^*(i^* P)$.

$$\begin{tikzcd}
P_U \arrow[r, "\psi"] \arrow[d] \arrow[dr, phantom, "\square"] & P_A \arrow[d] \\
U \arrow[r, "r"] & A \arrow[u, bend right, "s"']
\end{tikzcd}$$

ψ is G-equivariant. $r^*(s) = s_U := (id_U, s \circ r)$:

$$\begin{tikzcd}
U \arrow[dr, "s_U"] \arrow[drr, bend left, "s \circ r"] \arrow[ddr, bend right, "id_U"'] & & \\
& P_U \arrow[r, "\psi"] \arrow[d] \arrow[dr, phantom, "\square"] & P_A \arrow[d] \\
& U \arrow[r, "r"] & A \arrow[u, bend right, "s"']
\end{tikzcd}$$

But now we have a map $i : A \to U$.

$$\begin{tikzcd}
U \arrow[dr, "s_U"] \arrow[drr, bend left, "s \circ r"] \arrow[ddr, bend right, "id_U"'] & & \\
& P_U \arrow[r, "\psi"] \arrow[d] \arrow[dr, phantom, "\square"] & P_A \arrow[d] \\
A \arrow[r, "i"'] & U \arrow[r, "r"] & A \arrow[u, bend right, "s"']
\end{tikzcd}$$

We could either construct a pullback square over it (which would now be the left square of two glued together pullback squares) or we could construct the outer square. We have seen that both are equivalent.
The composition $r \circ i$ is however the identity. By the property of pullback squares in 1-category theory we can write P_A in the upper left corner and that there exists an G-equivariant map $P_A \to P_U$.

$$\begin{tikzcd}
U \arrow[dr, "s_U"] \arrow[drr, bend left, "s \circ r"] \arrow[ddr, bend right, "id_U"'] & & \\
P_A \arrow[r, dashed] \arrow[d] & P_U \arrow[r, "\psi"] \arrow[d] \arrow[dr, phantom, "\square"] & P_A \arrow[d] \\
A \arrow[r, "i"'] & U \arrow[r, "r"] & A \arrow[u, bend right, "s"']
\end{tikzcd}$$

Thus

$$s_U|_A = s$$

is equivalent to $i^*(s_U) = i^*(r^*(s)) = id^*(s) = s$ ■

Corollary 4.2.6.

$$Bun_G^*\left(\bigvee_\alpha X_\alpha\right) \xrightarrow{\cong} \prod_\alpha Bun_G^*(X_\alpha)$$

for $(X_\alpha, x_\alpha) \in \mathtt{CW}_*$.

Proof.
Recall that the map is canonical, so given an element of the set $Bun^*_G(\bigvee_\alpha X_\alpha)$ we can restrict it, so take the pullback of the inclusion of X_α in the wedge-sum. This will give an element in $Bun^*_G(X_\alpha)$. Thus we get a tupel of such elements and this is how this map was constructed in the first place.

We just have to check that it is an isomorphism.

- surjectivity:

  We are given a collection of pointed principle G-bundles
  $$\begin{tikzcd} (P_\alpha, p_\alpha) \arrow[d, "\pi_\alpha"] \\ (X_\alpha, x_\alpha) \end{tikzcd}$$

  We construct a new principle G-bundle over the wedge sum which is given by $\coprod_\alpha P_\alpha$ and then we have to identify fibers over the given point of all these principle G-bundles

  $$\begin{tikzcd} (\coprod_\alpha P_\alpha)/g \cdot p_\alpha \sim g \cdot p_\beta \arrow[d, "\pi"] & \forall \alpha, \beta \forall g \in G \\ \bigvee_\alpha X_\alpha \end{tikzcd}$$

  The claim is that this is again a principal G-bundle.
  So the only non-trivial thing happening is over the distinguished point of all the spaces that we glue together and we identify the fibers of this principle G-bundle of all these distinguished points with each other.
  It is easy to see that for the distinguished point $* \in \bigvee_\alpha X_\alpha$: $\pi^{-1}(*) \cong G$.
  It is also clear how G acts on this space. Away from this distinguished fiber

  $$\begin{tikzcd} P_\alpha \setminus G \arrow[r] \arrow[d] & P \arrow[d] \\ X_\alpha \setminus \{x_\alpha\} \arrow[r] & \bigvee X_\alpha \end{tikzcd}$$

  So something non–trivial that we need to check happens only over this distinguished point $*$ and basically the main thing is to check that there is a neighbourhood of $*$ that trivialises the principal G-bundle π. For that of course we need the proposition we have just proven and the fact that they are well-pointed, that is that the inclusion of x_α into X_α is a neighbourhood deformation retract.

  We need to show that P is locally trivial in the neighbourhood of $*$:
  by proposition there exists $x_\alpha \in U_\alpha \subset X_\alpha$ open such that $\exists s_\alpha : U_\alpha \to P_\alpha$, $s(x_\alpha) = p_\alpha$.
  Then $U := \bigvee_\alpha U_\alpha \subset \bigvee_\alpha X_\alpha$ open and

  $$\begin{tikzcd} P|_U \arrow[d, "\pi_U"] & G \times U \arrow[l, "\cong"'] \arrow[dl, "pr_2"] & \{e\} \times U \arrow[l, hook'] \arrow[ll, bend right, "\vee s_\alpha"'] \\ U \end{tikzcd}$$

  The fact that $s_\alpha(x_\alpha) = p_\alpha$ is what allows us to glue them all together.

- Injectivity
  Suppose we are given

$$\begin{tikzcd}
(P_1, p_1) \arrow[dr, "\pi_1"'] & & (P_2, p_2) \arrow[dl, "\pi_2"] \\
& \bigvee_\alpha X_\alpha &
\end{tikzcd}$$

We know that restricted to X_α they are all isomorphic

$$\begin{tikzcd}
P_1|_{X_\alpha} \arrow[rr, "\phi_\alpha", "\cong"'] \arrow[dr, "\pi_1"'] & & P_2|_{X_\alpha} \arrow[dl, "\pi_2"] \\
& X_\alpha &
\end{tikzcd}$$

$\phi_\alpha(p_1) = p_2$.
Define $\phi : P_1 \to P_2$ such that $\phi(p) = \phi_\alpha(p)$ if $\pi_1(p) = x_\alpha$.
Of course one has to check that this is well-defined and again the important thing is to regard the happenings in the distinguished point in $\bigvee_\alpha X_\alpha$.
But since $\phi_\alpha(p_1) = p_2$ for all $\alpha \implies \phi_\alpha(gp_1) = gp_2\ \forall g$. Hence ϕ is well-defined on $\pi_1^{-1}(*) \cong Gp_1$.

■

Corollary 4.2.7 (Mayer-Vietoris property for Bun_G^*)**.**
Suppose we are given a pushout square of pointed **CW***-complexes:*

$$\begin{tikzcd}
A \arrow[r, hook, "i"] \arrow[d, "j"] & B \arrow[d] \\
C \arrow[r] & X
\end{tikzcd}$$

i, j are inclusions of subcomplexes so in particular they are closed cofibrations.
Then given three principal G-bundles P_A, P_B, P_C over A, B, C respectively. Also we assume that we are given isomorphisms $j^*P_C \xleftarrow{\cong} P_A \xrightarrow{\cong} i^*P_B$.

Then the pushout

$$P := P_B \cup_{P_A} P_C$$

is a principle G-bundle on X such that $P|_C \cong P_C$, $P|_B \cong P_B$.

In particular,

$$Bun_G^* \twoheadrightarrow Bun_G^*(B) \times_{Bun_G^*(A)} Bun_G^*(C)$$

Proof.
Earlier we saw how G acts on P (use that G is compactly generated Hausdorff and thus product with G commutes with pushouts)

$$\begin{tikzcd}
P_A \arrow[rr] \arrow[dr] \arrow[dd] & & P_C \arrow[dr] \arrow[dd] & \\
& P_B \arrow[rr] \arrow[dd] & & P \arrow[dd] \\
A \arrow[rr, hook] \arrow[dr, hook] & & C \arrow[dr, hook] & \\
& B \arrow[rr, hook] & & X
\end{tikzcd}$$

It is obvious that on each fiber G acts freely and transitively.
What we need to check is local triviality of P:

- Suppose we start with $x \in X \setminus B$ (or $x \in X \setminus C$) then $P|_{X\setminus B} \cong P_C|_{C\setminus A}$ is locally trivial.
- $x \in A$. We need to find a section of P in a neighbourhood of x.
  Start with $x \in U_C \subset C$ trivialising $P_C \rightsquigarrow s_C : U_C \to P_C$ section of π_C. Then we take the pullback.
  Recall that in the cube above we have pullbacks in the upper left corner of the top and left face. Those are due to the isomorphisms that are required in the assumption.
  $$j^* s_C : \underset{x\in}{U_C} \cap A \to P_A$$
  is a section. $U_C \cap A \subset A$ is open.
  fact*: $\exists W \subset B$ open such that $W \cap A = U_C \cap A \hookrightarrow W$ is a closed cofibration.

  By proposition we extend $j^* s_C$ to an open neighbourhood $U_B \subset B$ where $U_C \cap A \subset U_B$. It follows that $U_B \cap A = U_C \cap A$ and we have a section $U_B \xrightarrow{s_B} P_B$

  Finally, $U_C \cup_{U_C \cap A = U_B \cap A} U_B \xrightarrow{s_C \cup s_B} P$ is a section of π by $G \times (U_C \cup U_B) \xrightarrow{\cong} P|_{U_C \cup U_B}$

■

Theorem 4.2.8.
There exists $BG \in \mathtt{CW}^0_*$ and a principle G-bundle $EG \to BG$ such that for all $X \in \mathtt{CW}^0_*$

$$[X, BG] \xrightarrow{\cong} Bun^*_G(X)$$
$$f : X \to BG \rightsquigarrow f^* EG$$

Proof. Apply Brown representability theorem. ■

4.3 Properties and Construction of Classifying Spaces

Proposition 4.3.1 ("clutching construction")**.**
Let $X \in \mathtt{CW}^0_*$.
There exists a natural bijection (isomorphism of functors)

$$Bun^*_G(\Sigma' X) \cong [X, G]_*$$

Proof.

$\to$ Let P be a principle G-bundle over $\Sigma' X$

$$\begin{tikzcd} X \arrow[r, hook, "i_+"] \arrow[d, hook, "i_-"'] & C^+X \arrow[d] \\ C^-X \arrow[r] & \Sigma' X \end{tikzcd}$$

Since C^+X and C^-X are contractible, $Bun^*_G(C^\pm X) = \{*\}$.
Thus if we restrict P there exists an isomorphism

$$P|_{C^-X} \underset{\cong}{\xrightarrow{\rho_-}} G \times C^-X, \qquad P|_{C^+X} \underset{\cong}{\xrightarrow{\rho_+}} G \times C^+X$$
$$P|_X = (P|_{C^-X})|_X \underset{\cong}{\xrightarrow{\rho_-|_X}} G \times X$$
$$= (P|_{C^+X})|_X \underset{\cong}{\xrightarrow{\rho_+|_X}} G \times X$$

we get

$$\begin{tikzcd} G \times X \arrow[rr, "\rho=(\rho_+|_X)\circ(\rho_-|_X)", "\cong"'] \arrow[dr, "pr_2"'] & & G \times X \arrow[dl, "pr_2"] \\ & X & \end{tikzcd}$$

This is uniquely determined by $\rho|_{\{e\}\times X} : X \to G$.

ρ_- is defined up to an automorphism on the trivial G-bundle $G \times C^-X \xrightarrow{\cong} G \times C^-X$ determined by $\kappa_- : C^-X \to G$
Similarly, ρ_+ is determined up to $\kappa_+ : C^+X \to G$.
If we change ρ_- to $(\rho'_-)^{-1}(g,x) = g \cdot \kappa(x) \cdot \rho_-(e,x)$. In the end we get

$$\rho'|_{\{e\}\times X}(x) = \kappa_+|_X(x) \cdot \rho|_{\{e\}\times X}(x) \cdot \kappa_-^{-1}|_X(x)$$

we would like to show that this is homotopic to the previous one. This is obvious because

$$\kappa_+ : C^+X \to G$$

is a homotopy from $\kappa_+|_X$ to $e \in G$ (constant map).
Similarly for κ_-.
Therefore $\rho' \simeq \rho \implies$ the map $Bun_G^*(\Sigma' X) \to [X,G]_*$ is well-defined.

$\leftarrow$
Let $f : X \to G$, $f(x_0) = e$ be a pointed map.

$$\begin{tikzcd} & & G \times C^+X \arrow[dd] \\ G \times X \arrow[urr] \arrow[r, "id\times i_-"'] \arrow[dd] & G \times C^-X \arrow[dd] & \\ & & C^+X \\ X \arrow[urr] \arrow[r] & C^-X & \end{tikzcd}$$

$G \times X \to G \times C^+X, \quad (g,x) \mapsto (gf(x), i_+(x))$
Now take the pushout to get a pointed principal G-bundle $P(f) \to \Sigma' X$.

We have to check that homotopic maps go to isomorphic principle G-bundles. So let $f_0 \simeq f_1 : X \to G$. Thus there is a homotopy $h : X \times I \to G$.
We get

$$\begin{tikzcd} P(h) \arrow[d] & P(f_0) \arrow[l] \arrow[d] \\ \Sigma'(X \times I) & \Sigma' X \arrow[l, "\Sigma' j_0"] \end{tikzcd}$$

By the naturality of all construction one gets the map $P(f_0) \to P(h)$. This induces an isomorphism $P(f_0) \xrightarrow{\cong} (\Sigma' j_0)^* P(h)$. Similarly we get an isomorphism from $P(f_1)$. Thus there is an isomorphism $P(f_0) \cong P(f_1)$.
Hence the map is well-defined.

■

Proposition 4.3.2.
EG is contractible and $\Omega BG \simeq G$.

Proof.
$EG \to BG$ is a fiber bundle and therefore a Serre fibration over a `CW`-complex. From this follows that it is a quasi-fibration.
The fact that it is a quasi-fibration means that if we take the distinguished point and look at the fiber

$$\begin{tikzcd} G \arrow[r] \arrow[d] & EG \arrow[d] \\ * \arrow[r] & BG \end{tikzcd}$$

then this is not just a pullback square but also a homotopy pullback square.
Note that now we can get a canonical map from the loop space of BG to G by taking the pullback:

$$\begin{tikzcd} \Omega BG \arrow[r] \arrow[d, "\phi"'] \arrow[dr, phantom, "\square"] & * \arrow[d] \\ G \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & EG \arrow[d] \\ * \arrow[r] & BG \end{tikzcd}$$

If EG is contractible then G is homotopy equivalent to ΩBG but not abstractly but rather real concrete via the concrete map between them.
Vice versa if we show that this particular map is an homotopy equivalence it follows that EG is contractible. For example by looking at the LES of homotopy groups we would get that the homotopy groups of G are the same as the homotopy groups of BG shifted by one such that the connecting homomorphism of this LES is always an isomorphism. The rest are the homotopy groups of EG and thus 0.

Claim: ϕ is a homotopy equivalence with homotopy inverse the map ψ described below
We have just proven that we have an isomorphism of functors

$$\begin{tikzcd} Bun_G^*(\Sigma' X) \arrow[r, phantom, "\cong"] \arrow[d, equal] & {[X, G]_*} \arrow[d, dashed, "\cong"', "\psi"] \\ {[\Sigma' X, BG]} \arrow[r, phantom, "\cong"] & {[X, \Omega BG]_*} \end{tikzcd}$$

By Yoneda the dashed map is given by a map $G \xrightarrow{\psi} \Omega BG$ which is given by a principle G-bundle over $\Sigma' G$ that is given by the clutching construction with respect to $G \xrightarrow[\cong]{id} G$.
ψ is a homotopy equivalence.

What is $\phi(\gamma)$, $\gamma : S^1 \to BG$?

$$\begin{tikzcd} & & EG \arrow[d, "\pi"] \\ I \arrow[r, two heads] \arrow[urr, "\tilde{\gamma}"] & S^1 \arrow[r, "\gamma"'] & BG \end{tikzcd}$$

If you go through the definitions of homotopy pullback and how we defined the map ϕ by abstract nonsense we can get that to define this map from the loops of BG to the fiber of BG what you can do is take the surjection of the unit interval and then construct the lift. If we were talking about abstract homotopy pullbacks we would have to construct a map from I to the fiber up to some homotopies but here it is just a Serre fibration so we can actually just define a lift $\tilde{\gamma}$ as we did.

Say the distinguished point of S^1 is 1 and we regard S^1 as the unit circle of complex numbers, so $\gamma(1) = *$ with $*$ the distinguished point in BG. 0 is the distinguished point in I.

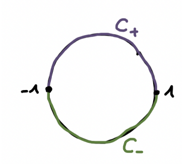
$$\tilde{\gamma}(0) = *' \text{distinguished point of the fiber } \pi^{-1}(*)$$

because this is a pointed principle G-bundle. Finally the other point the end of the interval is

$$\tilde{\gamma}(1) = g^{*'} \in \pi^{-1}(*) \cong G$$

So $\phi(\gamma) = \tilde{\gamma}(1)$

What is $\psi(g)$?



$\psi(g)$ is constructed by identifying C^+ with C^- over the point -1 by multiplication with g. This is our clutching construction: we have on C^+ and C^- a trivial G-bundle when we glue them together we do not have a choice on the distinguished point - we have to glue the distinguished points together. On the point -1, however, we have a choice and this choice is precisely the choice of an element in the fiber and this is g which is the element we started with. So we glue those two things together and you get a principle G-bundle which gives you a map from S^1 to BG such that the pullback is $P(g)$ constructed by this clutching construction:

$$\begin{tikzcd}
P(g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & EG \arrow[d] \\
S^1 \arrow[r] & BG
\end{tikzcd}$$

Now the question is what would be the composition of those two maps ϕ and ψ? So we would like to see that if we went from the unit interval to S^1 surjectively what kind of lift $\tilde{\psi}$ would we get and where would it end?

$$\begin{tikzcd}
 & P(g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & EG \arrow[d] \\
I \arrow[r, two heads] \arrow[ur, dashed, "\tilde{\gamma}"] & S^1 \arrow[r] & BG
\end{tikzcd}$$

We would like to lift the circle to a path in the principle G-bundle. So we would start with the distinguished point 1 and then we would start constructiong a lift.
So if we look at the upper half circle C^+ our $P(g)$ is trivial over that. Not just abstractly trivial but precisely trivial because by construction there is a section. So we can use that to lift the upper half of the circle to a trivial map in the fiber

$$\begin{tikzcd}
 & P(g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & EG \arrow[d] \\
I \arrow[r, two heads] \arrow[ur, dashed, "\tilde{\gamma}"] & S^1 \arrow[r] & BG \\
\cup & & \\
C^+ \cong [0,1/2) \arrow[r] & G \times C^+ & \\
t \arrow[r, mapsto] & (e,t) &
\end{tikzcd}$$

And then at the point -1 we glue this identity e with an element g over which we multiply. After that we again go trivially.

In the end what you get is that $\tilde{\gamma}(1) = g$. This shows that the map ϕ is a homotopy inverse to ψ which is a homotopy equivalence. So ϕ is a homotopy equivalence and thus EG is contractible. ∎

Corollary 4.3.3.
Let E be a contractible space with free action of G such that $E \to E/G$ is a principle G-bundle.

> *For example we discussed in AT I that when the group is discrete that when the group is discrete then the sufficient condition for this thing to be a principle G-bundle that is a Galois covering with the group G was the proper discontinuous action.*

Then

$$\begin{tikzcd} E \arrow[r, "\simeq"] \arrow[d] & EG \arrow[d] \\ E/G \arrow[r, "\simeq"] & BG \end{tikzcd}$$

i.e. we can take E/G to be the classifying space.

Proof.
We get this square by the fact that $E \to E/G$ is a principle G-bundle (it is a pullback square in fact)

$$\begin{tikzcd} E \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "\square"] & EG \arrow[d] \\ E/G \arrow[r, "\simeq"] & BG \end{tikzcd}$$

It is a homotopy pullback square. We look at the LES of the homotopy fibers and get

$$\begin{tikzcd} 0 = \pi_i(E) \arrow[r, "\cong"] \arrow[d] & \pi_i(EG) = 0 \arrow[d] \\ \pi_i(E/G) \arrow[r] \arrow[d] & \pi_i(BG) \arrow[d] \\ \pi_{i-1}(G) \arrow[r, "id", "\cong"'] & \pi_{i-1}(G) \end{tikzcd}$$

Both $\pi_i(E/G)$ and $\pi_i(BG)$ go to the homotopy fibers which are both G. The homotopy fibers are in fact the fibers because we have quasi-fibrations. The map on the fiber is the identity because it is a map of pointed principal G-bundles.

So by the 5-lemma and induction we get

$$\pi_i(E/G) \xrightarrow{\cong} \pi_i(BG) \quad \forall i$$
$$\implies E/G \xrightarrow{\simeq} BG \text{ is a homotopy equivalence}$$

∎

Example 4.3.4. $\mathbb{C}P^\infty \cong BU(1)$ (Exercise)
From this corollary one just have to check that the canonical $U(1)$-bundle over $\mathbb{C}P^\infty$ that we introduced last time has the total space $\mathbb{C}P^\infty \setminus \{0\}$ and that this space is contractible.

Later we will describe $BU(n)$

Why are these called principle G-bundles?

Proposition 4.3.5.
Suppose G acts on some space F on the right.
Suppose we have a principle G-bundle $P \to A$.
Then $\pi_F : F \times_G P \to X$ (where $F \times_G P := F \times P/_\sim$ with the equivalence relation given by $(f \cdot g, P) \sim (f, g \cdot P)$) is a fiber bundle with fiber F.

Proof.
Suffices to look locally over X: Say $U \subset X$ is an open in X such that it trivialises P.
So we get that

$$\begin{tikzcd} F \times_G (G \times U) \arrow[rr, "\cong"] \arrow[dr] & & F \times U \arrow[dl, "pr_2"] \\ & U & \end{tikzcd}$$

where the map $F \times_G (G \times U) \to F \times U$ is given by $(f, g, u) \mapsto (fg, u)$, so one can easily check that this map is the map on the product that identifies the quotient over this equivalence relation with $F \times U$. ∎

Example 4.3.6.

- $U(1)$ acts on $\mathbb{C}$: $(z, \tau) \mapsto \tau \cdot z$.
  From a $U(1)$-bundle we get a fiber bundle with fiber $\mathbb{C}$.
- Σ_n acts on $\{1, 2, ..., n\}$ gives us

$$\begin{tikzcd} Bun_{\Sigma_n}(X) \arrow[r] \arrow[d, equal] & \left\{ \begin{array}{c} \text{coverings of } X \\ \text{of degree } n \end{array} \right\} \\ \left\{ \begin{array}{c} \text{Galois coverings of } X \\ \text{with group } \Sigma_n \end{array} \right\} & \end{tikzcd}$$

  Exercise: this is almost an isomorphism.

4.4 Pointed vs Unpointed (part 2)

Here we are again going to prove a rather general statement as to how the morphisms in the pointed and unpointed category are related to one another.

Proposition 4.4.1.
Let $(X, x_0), (Y, y_0) \in \mathtt{CW}_*$ be pointed CW-complexes. Assume that Y is connected.
Then there exists an action of the fundamental group of Y, $\pi_1(Y, y_0)$, on the homotopy classes of maps $X \to Y$, $[X, Y]_*$ and

$$[X, Y]_*/\pi_1(Y, y) \xrightarrow{\cong} [X, Y]$$

Proof.
$Map(X, Y) := Y^X$, consider the following pullback square

$$\begin{tikzcd} Map_*(X, Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Map(X, Y) \arrow[d, "ev_{x_0}"] \\ * \arrow[r, "y_0"'] & Y \end{tikzcd}$$

What is the universal property of this pointed mapping space?:

$$\begin{tikzcd}
T \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Map_*(X,Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Map(X,Y) \arrow[d, "ev_{x_0}"] \\
& * \arrow[r, "y_0"'] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
Hom_{\mathsf{Top}}(T, Map_*(X,Y)) \arrow[r, "\cong"] \arrow[dr, hook] & Hom(T, Map(X,Y)) \arrow[d, phantom, "\cong"] \\
& Hom(T \times X, Y)
\end{tikzcd}$$

So what is the image of this inclusion?:

$$Hom_{\mathsf{Top}}(T, Map_*(X,Y)) = \{f : T \times X \to Y \mid f|_{T\times\{x_0\}} = \{y_0\}\}$$

In particular, points in this mapping space $Map_*(X,Y)$ are pointed maps $(X,x_0) \to (Y,y_0)$ and paths in $Map_*(X,Y)$ are pointed homotopies.
Thus $\pi_0 Map_*(X,Y) \cong [X,Y]_*$.

So what is the action of $\pi_1(Y,y_0)$ on $[X,Y]_*$? Actually this was defined in a more general context in Exercise 5.2. We claim that the action which we will see now is a special case of what we did there.
First of all, ev_{x_0} is a fibration as we once have proven (maybe for $X = I$ the unit interval but this is true for any X). Therefore this is not only a pullback square but also a homotopy pullback square:

$$\begin{tikzcd}
T \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Map_*(X,Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Map(X,Y) \arrow[d, "ev_{x_0}"] \\
& * \arrow[r, "y_0"'] & Y
\end{tikzcd}$$

Now we can also make it a pointed homotopy pullback square because this is a fibration and we can make it a pointed fibration by choosing a point above y_0. A point above y_0 in $Map(X,Y)$ is a point in $Map_*(X,Y)$, so if we choose a pointed map $f : X \to Y$, then this is a homotopy pullback square in the pointed category.
We can then write the long exact sequence of homotopy groups:

$$\pi_1(Y,y_0) \xrightarrow{\partial_f} \underset{=[X,Y]_*}{\pi_0(Mor_*(X,Y))} \xrightarrow{\pi} \underset{=[X,Y)]}{\pi_0(Map(X,Y))} \to \underset{=*}{\pi_0 Y}$$

The map ∂_f depends on the choice of f. The claim that we have to check (what we won't do in detail) is that the image of some $\gamma \in \pi_1(Y,y_0)$

$$\partial_f(\gamma) = f \cdot \gamma$$

is the action of γ on $[X,Y]_*$.
So from the exactness we get that $\pi^{-1}(f) = f \cdot \pi_1(Y,y_0)$. This proves the claim that

$$[X,Y]_*/\pi_1(Y,y) \underset{\cong}{\xrightarrow{\overline{\pi}}} [X,Y]$$

with $\overline{\pi}$ induced by the map π does not depend on the choice of the point f. This is just the forgetful map, given a pointed map in $[X,Y]_*$ we get an unpointed map in $[X,Y]$. ■

Corollary 4.4.2.
$[K(G_1,1),K(G_2,1)] \cong Hom(G_1,G_2)/G_2$ *where* G_2 *acts by conjugation on* G_2.

Proof.
We have computed that before:

$$[K(G_1,1),K(G_2,1)] \cong [\Pi_1 K(G_1,1),\Pi_1 K(G_2,1)] \cong [BG_1,BG_2]_{h\mathsf{Gpd}} = Hom(G_1,G_2)/G_2$$

where we regard BG_1 and BG_2 as before as groupoids in the homotopy category of groupoids which we haven't studied in detail but groupoids are a (2,1)-categories. There is a homotopy category which is just inverting all the equivalences between groupoids.
Then you can easily compute that what you get is precisely $Hom(G_1,G_2)/G_2$. ■

Remark 4.4.3.
One can show that when you look at the pointed case, you get

$$[K(G_1,1),K(G_2,1)]_* \cong Hom(G_1,G_2)$$

So by the previous proposition you get the corollary as a corollary of this statement here and the fact that G_2 acts by conjugation is also not surprising because this is how the fundamental group acts.

Corollary 4.4.4.
$\forall X \in \mathsf{CW}^0$: $[X,K(A,n)] \cong H^n(X,A)$

Proof.
If $n \geq 2$, then $\pi_1(K(A,n)) = 0$ and thus the pointed maps are the same as the unpointed maps and the pointed maps are by Brown representability the cohomology.
If $n=1$, since A is abelian, $\pi_1 = A$ acts trivially on $Hom(\pi_1(X),A) = [X,K(A,1)]_*$ ■

The following proposition states that BG also represents unpointed principal G-bundles.

Proposition 4.4.5.
$(X,x_0) \in \mathsf{CW}^0_*$.
Then

$$\begin{tikzcd}
{[(X,x_0),BG]_*} \arrow[r, two heads] \arrow[d, "\cong"'] & {[(X,x_0),BG]_*/\pi_0 G} \arrow[r, "\cong"] \arrow[d, "\cong"] & {[X,BG]} \arrow[d, "\cong"] \\
Bun^*_G(X) \arrow[r, two heads] & Bun^*_G(X)/G \arrow[r, "\cong"] & Bun_G(X)
\end{tikzcd}$$

here G *acts on* $Bun^*_G(X)$ *by* $(P_1,p_1) \xrightarrow{g} (P_1,gp_1)$.

Proof.
If suffices to check

1. $[(X,x_0),BG]_* \to Bun^*_G(X)$ is G-equivariant.
2. $Bun^*_G(X)/G \xrightarrow{\cong} Bun_G(X)$

Why are these two enough? The first implies in particular that the action of G on $Bun^*_G(X)$ factors through $\pi_0 G$. Since we have the same set on the left column of the diagram with the same G-action, then of course the quotients under this actions are isomorphisms and thus the middle vertical arrow is an isomorphism.
Now all the smaller squares are commutative, that the arrow $[(X,x_0),BG]_*/\pi_0 G \to [X,BG]$ is an isomorphism follows from the previous proposition. We have just concluded that the middle vertical arrow is an isomorphism and by the second property we want to check, the map

$Bun_G^*(X)/G \to Bun_G(X)$ is an isomorphism, too. By these three properties, the right vertical arrow also is an isomorphism.

2. is almost obvious:

   - *Surjectivity* is clear, because we already have a surjective map $Bun_G^*(X) \twoheadrightarrow Bun_G(X)$ since to lift a principal G-bundle to a pointed principal G-bundle one only has to choose a point in the fiber.
   - *Injectivity*:
     Suppose we have two pointed principal G-bundles $(P_1, p_1), (P_2, p_2)$ such that they become isomorphic as unpointed ones: $P_1 \xrightarrow[\cong]{\phi} P_2$.
     So ϕ of the distinguished point p_1 goes to some point in the fiber of P_2. But on the fiber G acts transitively and freely. Thus there exists a unique element $g \in G$ such that $\phi(p_1) = gp_2$. But then it means precisely that in the quotient set Bun_G^*/G the elements $(P_1, p_2) \sim (P_2, gp_1)$ are now equivalent and thus there is an isomorphism $(P_2, gp_1) \xrightarrow{\cong} (P_2, p_2)$.

1. The first claim is not as easy as the second one because we have to understand the action $\pi_1(BG)$ on the set $[(X, x_0), BG]_*$ which is done with some not super concrete constructions. We also have to identify $\pi_1(BG)$ with $\pi_0(G)$.

   Let $\gamma \in \pi_1(BG)$ that corresponds to a principal G-bundle on S^1 that is obtained by the clutching construction

   > (gluing two principal G-bundles over two half-circles of S^1 over the distinguished point and in this distinguished fiber we glued these trivial G-bundles by shifting one to the other by multiplication with some element $g \in G$)

   So $\pi_1(BG) \cong \pi_0(G)$ and thus γ corresponds to some class of elements $[g] \in \pi_0 G$.
   So when we act by γ on $[(X, x_0), BG]_*$ we actually have to check that the action of γ on sort of the space BG which we obtained by exercise 5.2 it also acts on EG and on this EG we just shift in the distinguished fiber the distinguished point by g, by the action of this element.
   $\overset{\text{(without proof)}}{\rightsquigarrow}$ the action of γ on BG acts on EG: $* \to g*$.
   Therefore this map $[(X, x_0), BG]_* \to Bun_G^*(X)$ that is obtained by taking the universal principal G-bundle and then the pullback, after we have acted by γ on BG and have taken the pullback, we have just acted on the distinguished point in the distinguished fiber of this principal G-bundle. So if we go back and take the pullback we get the same principal G-bundle but the distinguished point is different. The distinguished point is obtained exactly by multiplication of the old distinguished point by g.

■

Corollary 4.4.6.
$\forall X \in \texttt{CW}$

$$[X, BG] \cong Bun_G(X)$$

Proof.
Take X to be the disjoint union of its connected components: $X = \coprod X_i$.
For each connected component choose a point and then apply the proposition.

$$[X, BG] \cong \prod_i [X_i, BG] \cong \prod_i Bun_G(X_i) \xleftarrow{\cong} Bun_G(X)$$

where the last isomorphism is due to the fact that given principal G-bundles on each component their disjoint union gives you a principal G-bundle on the whole space X, thus an element in $Bun_G(X)$ and vice versa. ■

Finally we have proven that the BGs are the classifying spaces not only the pointed principal G-bundles but also for the unpointed ones.

4.5 Functoriality of BG in G

Let $\phi : G_1 \to G_2$ be a homomorphism between topological groups.

Then for a principal G_1-bundle $\begin{array}{c} P \\ \pi\downarrow \\ X \end{array}$ we can change the fiber of the principal G_1-bundle from G_1 to G_2 using ϕ:

$$\begin{tikzcd} G_2 \times_{G_1} P \arrow[d, "\pi_2"'] \\ X \end{tikzcd}$$

is a principal G_2-bundle:
The action of G_2 is given by multiplication on the left: $g \cdot (h, p) = (gh, p)$.
Note that since it is a action on the left it does not interfere with the quotient where we identify $(hg_1, p) \sim (g, g_1 p)$:

$$\begin{tikzcd} (hg_1, p) \arrow[d, "\cdot g"'] & \sim & (g, g_1 p) \arrow[d, "\cdot g"] \\ (ghg_1, p) & \sim & (gh, g_1 p) \end{tikzcd}$$

Thus this action of G_2 which we have only described on the product factors factors through that change of fibers to G_2.

Last time we have explained that this is a a locally trivial G_2-bundle and by exactly the same proof with the action of G_2 tells you that the trivialising cover for P, is also a trivialising cover for the G_2 action.

This induces a natural transformation of functors

$$\begin{tikzcd} Bun_{G_1}(-) \arrow[r] \arrow[d, "\cong"'] & Bun_{G_2}(-) \arrow[d, "\cong"] \\ {[-, BG_1]} & {[-, BG_2]} \end{tikzcd}$$

By Yoneda this corresponds to a map $\overline{\phi} : BG_1 \to BG_2$

Proposition 4.5.1.
$\overline{\phi}$ is a homotopy equivalence if and only if ϕ is a homotopy equivalence (iff $Bun_{G_1}(-) \to Bun_{G_2}(-)$ is an isomorphism)

Proof.
We have this map $\overline{\phi} : BG_1 \to BG_2$ by Yoneda lemma but how is it related to universal principal G-bundles?
First, let's check how exactly $\overline{\phi}$ is obtained by Yoneda. We have the natural transformation of functors as above which we can apply e.g. on BG_1:

$$\begin{tikzcd} Bun_{G_1}(BG_1) \arrow[r] \arrow[d, "\cong"'] & Bun_{G_2}(BG_1) \arrow[d, "\cong"] \\ {[BG_1, BG_1]} & {[BG_1, BG_2]} \end{tikzcd}$$

Thus we have the identity in $[BG_1, BG_1]$. The identity corresponds to $EG_1 \in Bun_{G_1}(BG_1)$. Under all these isomorphisms $id \mapsto \overline{\phi} \in [BG_1, BG_2]$. Also, by our construction, EG_1 maps to $G_2 \times_{G_1} EG_1$. Thus $\overline{\phi}$ corresponds to the principal G-bundle $G_2 \times_{G_1} EG_1$.

$$\begin{tikzcd}
EG_1 \arrow[r, mapsto] & G_2 \times_{G_1} EG_1 \arrow[d, mapsto] \\
id \arrow[u, mapsto] \arrow[r, mapsto] & \overline{\phi}
\end{tikzcd}$$

So it means that the pullback of EG_2 to BG_1 is the following G_2-bundle:

$$\begin{tikzcd}
G_2 \times_{G_1} EG_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & EG_2 \arrow[d, "\pi_2"] \\
BG_1 \arrow[r, "\overline{\phi}"'] & BG_2
\end{tikzcd}$$

and of course there is a canonical map $EG_1 \to G_2 \times_{G_1} EG_1$ because it is the quotient of $G_2 \times EG_1$ and thus the canonical map is given by $t \mapsto (e, t)$.

$$\begin{tikzcd}
EG_1 \arrow[r] \arrow[dr, "\pi_1"'] & G_2 \times_{G_1} EG_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & EG_2 \arrow[d, "\pi_2"] \\
& BG_1 \arrow[r, "\overline{\phi}"'] & BG_2
\end{tikzcd}$$

It's not that important to us that the square is homotopy cartesian. What is important though, is that the outer square commutes and that we know that what happens on the fiber is precisely this map ϕ:

$$\begin{tikzcd}
G_1 \arrow[r] \arrow[d, hook] \arrow[rr, bend left, "\phi"] & G_1 \times_{G_1} G_1 \arrow[r, "\cong"] & G_2 \arrow[d, hook] \\
EG_1 \arrow[r] \arrow[d] & G_2 \times_{G_1} EG_1 \arrow[r] \arrow[dl] & EG_2 \arrow[dl] \\
BG_1 \arrow[r, "\overline{\phi}"'] & BG_2 &
\end{tikzcd}$$

G_1 is the fiber over the distinguished point $*_1$.

Now we look at the LES of homotopy groups. G_1 is the homotopy fiber of $EG_1 \to BG_1$ over the point $*_1$ and G_2 is the homotopy fiber of $EG_2 \to BG_2$ of the image of that point. So we get

$$\begin{tikzcd}
\pi_i(BG_1) \arrow[r, "\overline{\phi}_*"] \arrow[d, "\cong"'] & \pi_i(BG_2) \arrow[d, "\cong"] \\
\pi_{i-1}(G_1) \arrow[r, "\phi_*"'] & \pi_{i-1}(G_2)
\end{tikzcd}$$

So if ϕ_* is an isomorphism for all $i \geq 1$, then $\overline{\phi}_*$ is an isomorphism for all $i \geq 1$ and vice versa. This is precisely the claim that these two maps are simultaneously homotopy equivalences. ■

Example 4.5.2.
Let $\{\pm 1\} = O_1 \hookrightarrow GL_1(\mathbb{R}) = \mathbb{R}^*$ and $S^1 = U_1 \hookrightarrow GL_1(\mathbb{C}) = \mathbb{C}^*$ be homotopy equivalences. By the previous proposition this implies that the classifying spaces are homotopy equivalent and that the functors of principal G-bundles are isomorphic. In particular, any principal $GL_1(\mathbb{R})$-bundle can be given a structure of principal O_1-bundle.

Lemma 4.5.3.
$O_n \hookrightarrow GL_n(\mathbb{R})$ is a homotopy equivalence.

Recall: $GL_n(\mathbb{R})$ has a topology induced by the inclusion $GL_n(\mathbb{R}) \hookrightarrow Mat_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$

Proof.
$A \in GL_n(\mathbb{R})$, denote $P(A) := A^T A$. This has two nice properties: it is symmetric and positive definite ($v^T A^T A v = (Av)^T (Av) > 0$ for $v \neq 0$)

$$\mathcal{P}_n := \{B \in GL_n(\mathbb{R}) \mid B \text{ is symmetric and pos def}\}$$

A fact from linear algebra: on this space we have a sort of square root of unity

$$\mathcal{P}_n \xrightarrow{\cong} \mathcal{P}_n$$
$$B \mapsto B^2$$

is a homeomorphism. In fact you can explicitly write a global inverse map, so you can kind of explicitly write the square root of a matrix in terms of its values in some power series.
Let $Q(A) \in \mathcal{P}_n$ be such that $Q(A)^2 = P(A) = A^T A$.

Consider

$$GL_n(\mathbb{R}) \to O_n(\mathbb{R}) \times \mathcal{P}_n$$
$$A \mapsto (A \cdot Q(A)^{-1}, P(A))$$

Why is $A \cdot Q(A)^{-1}$ an orthogonal matrix? For this one has to check $(A \cdot Q(A)^{-1})^T (A \cdot Q(A)^{-1}) = Q(A)^{-1} A^T A Q(A)^{-1} = Q(A)^{-1} Q(A)^2 Q(A)^{-1} = I$.
The claim is that this is in fact a homeomorphism. We can give an inverse by

$$GL_n(\mathbb{R}) \leftarrow O_n \times \mathcal{P}_n$$
$$B \cdot \sqrt{P} \leftarrow\!\shortmid (B, P)$$

The next claim is that $\mathcal{P}_n$ is contractible.
This is because it is $*$-convex: Given $A \in \mathcal{P}_n$ we can connect A to the identity matrix by straight line homotopy:

$$tA + (1-t)I \in \mathcal{P}_n$$

Finally,

$$\begin{tikzcd} O_n \arrow[r, hook] \arrow[drr, "id"'] & GL_n(\mathbb{R}) \arrow[r, phantom, "\cong"] & O_n \times \mathcal{P}_n \arrow[d, "pr_1"] \\ & & O_n \end{tikzcd}$$

pr_1 is a homotopy equivalence because $\mathcal{P}_n$ is contractible, the identity is a homotopy equivalence in any case and since $GL_n \to O_n \times \mathcal{P}_n$ is a homeomorphism it is also a homotopy equivalence.
Therefore $O_n \hookrightarrow GL_n(\mathbb{R})$ is a homotopy equivalence (actually a deformation retract) ■

<u>Exercise:</u> $U_n \hookrightarrow GL_n(\mathbb{C})$ is a homotopy equivalence.

Corollary 4.5.4.

$$BU_n \xrightarrow{\simeq} BGL_n(\mathbb{C}), \quad BO_n \xrightarrow{\simeq} BGL_n(\mathbb{R})$$

Our next goal will be to construct these spaces. Not up to homotopy but actually as really nice manifolds.

4.6 Grassmann and Stiefel Manifolds

There are two parallel stories here, over the real and the complex numbers. Sometimes there is also a third, over the quaternion numbers. But these two stories we try to say simultaneously at least up to some point.

W is a vector space of dimension n	over $\mathbb{R}$	over $\mathbb{C}$
the set of k-dimensional subspaces of W	$Gr_k(W) \cong Gr_k(\mathbb{R}^n)$	$Gr_k(W) \cong Gr_k(\mathbb{C}^n)$

Let W possess an inner product.

the set of orthonormal sequences of length k in W	$V_k(W) \subset W^{\times k}$ $\cong V_k(\mathbb{R}^n) \subset (\mathbb{R}^n)^{\times k}$	$V_k(W) \subset (\mathbb{C}^n)^{\times k}$ $\cong V_k(\mathbb{C}^n)$

Note that there is a map of sets

$$\begin{aligned} p : V_k(W) &\to Gr_k(W) \\ (u_1, ..., u_k) &\mapsto \langle u_1, ..., u_k \rangle \end{aligned}$$

Definition 4.6.1.
$V_k(W)$ with the induced topology from $W^{\times k}$ is called <u>real/complex Stiefel manifold</u> of orthonormal k-frames in W.
$Gr_k(W)$ with the quotient topology defined by p is called the <u>real complex Grassmann manifold</u> of k-dimensional subspaces in W.

Lemma 4.6.2.
We have a commutative diagram with $\cong$ denoting homeomorphisms:

$$\begin{tikzcd} O_n/O_{n-k} \arrow[r, "\cong"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & V_k(\mathbb{R}^n) \arrow[d] \\ O_n/O_k \times O_{n-k} \arrow[r, "\cong"'] & Gr_k(\mathbb{R}^n) \end{tikzcd} \qquad \begin{tikzcd} U_n/U_{n-k} \arrow[r, "\cong"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & V_k(\mathbb{C}^n) \arrow[d, "p"] \\ U_n/U_k \times U_{n-k} \arrow[r, "\cong"'] & Gr_k(\mathbb{C}^n) \end{tikzcd}$$

So these topological spaces V_k and Gr_k which we have just defined can be defined as the quotient of those nice Lee-groups O_n/O_{n-k}, $O_n/O_k \times O_{n-k}$ (same for the complex case).

Proof.
O_n acts on $V_k(\mathbb{R}^n)$ continuously: $A, (u_1, ..., u_k) \mapsto (Au_1, ..., Au_k)$.
It also acts on $Gr_k(\mathbb{R}^n)$: $(A, W' \hookrightarrow \mathbb{R}^n) \mapsto (A(W') \hookrightarrow \mathbb{R}^n)$.

O_n also acts transitively on $V_k(\mathbb{R}^n)$ (and also $Gr_k(\mathbb{R}^n)$) because given two orthonormal sequences $(u_1, .., u_k)$ you can always find a matrix sending one to the other. In the case of $k = n$ this matrix A is unique and the claim that it sends an orthonormal sequence to another is precisely the claim that it is othogonal.
Hence $V_k(\mathbb{R}^n)$, $Gr_k(\mathbb{R}^n)$ are homogenous spaces (in fact, manifolds).
If $G \circlearrowleft X$ acts on X transitively, then $G/Stab(x) \xrightarrow{\cong} X$.
So to finalise this lemma we have to compute the stabilisers.
The stabiliser of $(e_1, ..., e_k) \in V_k(\mathbb{R}^n)$ is

$$\left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \\ 0 & 1 & & & \star \\ & & \ddots & & \\ \hline & & 0 & & * \end{array}\right)$$

where $\star$ is the zero matrix and $* \in O_{n-k}$. Also the top left matrix is I_k.
The stabilizer of $\langle e_1, ..., e_k \rangle$ is

$$\left(\begin{array}{c|c} O_k & 0 \\ \hline 0 & O_{n-k} \end{array}\right)$$

The complex situation is similar. ■

Lemma 4.6.3.

$$\begin{tikzcd} V_k(\mathbb{R}^n) \arrow[d, "p"] \\ Gr_k(\mathbb{R}^n) \end{tikzcd}$$ *is a principal O_k-bundle.*

$$\begin{tikzcd} V_k(\mathbb{C}^n) \arrow[d, "p"] \\ Gr_k(\mathbb{C}^n) \end{tikzcd}$$ *is a principal U_k-bundle.*

Moreover, $Gr_k(\mathbb{R}^n)$ is locally homeomorphic to $\mathbb{R}^{k(n-k)}$ and $Gr_k(\mathbb{C}^n)$ to $\mathbb{C}^{k(n-k)}$.

Proof.
O_k acts on $V_k(\mathbb{R}^n)$: $O_k \hookrightarrow O_n$ given by $\left(\begin{array}{c|c} O_k & 0 \\ 0 & I_{n-k} \end{array}\right)$
O_n acts on $V_k(\mathbb{R}^n)$.
It is really easy to see that this action of O_n does not change the subspaces but only changes the bases of the subspaces. This is what means that it acts on the fibers. Moreover, this action on the fibers of p is transitive and free.

So we would like to see that locally this is a trivial O_k-bundle.

- $W \in Gr_k(\mathbb{R}^n)$ and choose a complement $V \subset \mathbb{R}^n$ such that $W \oplus V \cong \mathbb{R}^n$
  Define $U \subset Gr_k(\mathbb{R}^n)$, $U := \{\tilde{W} \mid \tilde{W} \cap V = 0\} \Leftrightarrow \tilde{W} \oplus V \cong \mathbb{R}^n$

  The claim is that $U \overset{\rho}{\cong} \mathbb{R}^{n(n-k)}$ is isomorphic as sets.

$$\begin{tikzcd} \tilde{W} \arrow[r, hook, "s"] \arrow[dr, "\cong"'] & \mathbb{R}^n \arrow[d, "\pi"] \\ & \mathbb{R}^n/V \end{tikzcd}$$

  The isomorphism is due to both $\tilde{W}$ and $\mathbb{R}^n/V$ being k-dimensional spaces and the kernel of the map $\tilde{W} \to \mathbb{R}^n$ is precisely $\tilde{W} \cap V$.
  Therefore defining $\tilde{W}$ is the same as defining a linear section of π. But the map π is fixed depends only on the choice of V and $\mathbb{R}^n/V \cong \mathbb{R}^k$. We can also assume wlog that $V = \langle e_{k+1}, ..., e_n \rangle$.

$$\begin{tikzcd} \tilde{W} \arrow[rr, hook, "s"] \arrow[drr, "\cong"'] & & \mathbb{R}^n \arrow[d, "\pi"'] \\ & & \mathbb{R}^n/V \arrow[d, "\cong"'] \\ \langle \underset{\pi(e_1)}{\overline{e_1}}, ..., \underset{\pi(e_k)}{\overline{e_k}} \rangle & \cong & \mathbb{R}^k \arrow[uu, dashed, bend right, "s"'] \end{tikzcd}$$

  So a section s has to send $\overline{e_i} \mapsto e_i + \sum_{j=k+1}^{n} \lambda_i^j e_j$.
  Thus we can identify

$$\tilde{W} \leftrightarrow (\lambda_i^j)_{\substack{i \in \{1, ..., k\} \\ j \in \{k+1, ..., n\}}} \in \mathbb{R}^{k(n-k)}$$

That's how we define the identification of U with $\mathbb{R}^{k(n-k)}$ via the map ρ

- U is open, ρ is a homeomorphism $\Leftrightarrow \rho^{-1}(U)$ is open in $V_k(\mathbb{R}^n)$
  $\rho^{-1}(U) \subset (\mathbb{R}^n)^{\times k}$ consists of $\{(u_1, ..., u_k)$ orthonormal seq $\mid \langle u_1, ..., u_k\rangle \cap V = 0\}$.
  Again assuming that $V = \langle e_{k+1}, ..., e_n\rangle$, the condition on the set boils down to $\det(u_1, ..., u_k, e_{k+1}, ..., e_n) \neq 0$.
  From this follows that $\rho^{-1}(U)$ is open.

  ρ is a homeomorphism.

  Remark 4.6.4.
  A smooth manifold is something that is locally homeomorphic to $\mathbb{R}^{k(n-k)}$ and then there are gluing functions which have to be smooth. In this case we have covered the space $Gr_k(\mathbb{R})$ by these sets and now one has to look at the transition functions between the open sets. It is not hard to show that these are in fact smooth.
  Therefore $Gr_k(\mathbb{R})$ is a smooth manifold.

- Finally, to find a section $\begin{tikzcd} \rho^{-1}(U) \arrow[d, "p"'] \\ U \arrow[u, bend right, "s"'] \end{tikzcd}$ we have to choose given any subspace $W \subset U$ an orthonormal basis in there.
  But we are already given a canonical basis $\tilde{W} = \langle e_1 + \sum \lambda_1^i e_i, e_2 + ...\rangle$. What we need is an orthonormal basis. To do that we apply the Gram-Schmidt procedure to get one for $\tilde{W}$. This gives us a section.
  Thus, by the standard argument, an isomorphism

$$\begin{tikzcd} U \times O_k \arrow[rr, phantom, "\cong"] \arrow[dr] & & p^{-1}(U) \arrow[dl] \\ & U & \end{tikzcd}$$

■

Example 4.6.5.

$$\begin{tikzcd} V_1(\mathbb{R}^n) \arrow[r, phantom, "\cong"] \arrow[d, "p"'] & S^{n-1} \arrow[d, "\pi"] \\ Gr_1(\mathbb{R}^n) \arrow[r, phantom, "\cong"] & \mathbb{R}P^{n-1} \end{tikzcd} \qquad \begin{tikzcd} V_1(\mathbb{R}^n) \arrow[r, phantom, "\cong"] \arrow[d, "p"'] & S^{2n-1} \arrow[d, "\pi"] \arrow[r, hook] & \mathbb{C}^n \setminus \{0\} \arrow[dl] \\ Gr_1(\mathbb{C}^n) \arrow[r, phantom, "\cong"] & \mathbb{C}P^{n-1} & \end{tikzcd}$$

In general, $Gr_k(\mathbb{R}^n),\ Gr_k(\mathbb{C}^n)$ are compact (closed manifolds), as well as $V_k(\mathbb{R}^n),\ V_k(\mathbb{C}^n)$.

Remark 4.6.6.
There are versions of Stiefel manifolds of k-frames:

$$\begin{tikzcd} \tilde{V}_k(W) \arrow[r, phantom, ":="] \arrow[d] & \{(u_1, ..., u_k) \mid \dim\langle u_1, ..., u_k\rangle = k\} \\ Gr_k(W) & \end{tikzcd}$$

is a principal GL_k-bundle where $\tilde{V}_k(W)$ is not compact (so not a closed manifold but just any manifold).

Proposition 4.6.7.
Let $n < m$, then

$$O_n \hookrightarrow O_m \text{ is } (n-1)\text{-connected}$$
$$U_n \hookrightarrow U_m \text{ is } 2n\text{-connected}$$

Proof.
O_n acts transitively on $S^{m-1} \subset \mathbb{R}^m$ with the stabiliser of $(0,0,...,1)$ being $O_{m-1} \hookrightarrow O_m$ embedded via $\left(\begin{array}{c|c} O_{m-1} & 0 \\ \hline 0 & 0 \end{array}\right)$
We have

$$\begin{tikzcd} O_{m-1} \arrow[r] \arrow[d] & O_m \arrow[d] & \ni & A \arrow[d, mapsto] \\ * \arrow[r] & S^{m-1} & \ni & A \cdot \underbrace{(0,...,0,1)}_{*} \end{tikzcd}$$

It is not hard to check that π is a principal O_{m-1}-bundle, hence a Serre fibration and thus we can apply the LES of homotopy groups. This is what gives the result. Of course you have to apply it here several times, this is the step where the difference between O_n and O_m is just 1 (that's why we directly wrote O_{m-1}). If the difference were greater than one, one has to apply this several times.

Similarly, U_m acts on $S^{2m-1} \subset \mathbb{C}^m$ with the stabiliser U_{m-1}. ■

Corollary 4.6.8.
$\pi_i(V_k(\mathbb{R}^n)) = 0$ *for* $i \leq n-k-1$
$\pi_i(V_k(\mathbb{C}^n)) = 0$ *for* $i \leq 2n-2k$

Proof.
The claim follows from the LES for the following Serre fibration that we are going to show to be a principal bundle.
O_n naturally acts on the Stiefel manifold of orthogonal k-frames in the n-dimensional vector space. The stabiliser of one k-frame is O_{n-k}

$$\begin{tikzcd} O_{n-k} \arrow[r, hook] \arrow[d] & O_n \arrow[d, "\pi"] \\ * \arrow[r] & V_k(\mathbb{R}^n) \end{tikzcd}$$

π is a principal O_{n-k}-bundle on this space of the orthogonal group O_n. Of course we have the action of O_{n-k} on O_n given by multiplication (on the left or on the right - depending on the action of O_n on $V_k(\mathbb{R}^n)$).

It suffices to find a section locally on $V_k(\mathbb{R}^n)$.
Since O_n acts transitively on $V_k(\mathbb{R}^n)$ we can take any point, so let's take $(e_1,...,e_k) \in V_k(\mathbb{R}^n)$. As explained before, there exists an open neighbourhood $U \subset V_k(\mathbb{R}^n)$ such that $(u_1,...,u_k) \in U$ the vectors $(u_1,...,u_k,e_{k+1},...,e_n)$ form a basis of $\mathbb{R}^n$. Applying Gramm-Schmidt we get $(u_1,...,u_k,\overline{u_{k+1}},...,\overline{u_n})$ an orthonormal basis.

What is a section of the map π? Given a k-frame in $V_k(\mathbb{R}^n)$ we should find an orthogonal matrix that sends the chosen k-frame $(e_1,...,e_k)$ to the given k-frame

$$\begin{tikzcd} O_{n-k} \arrow[r, hook] \arrow[d] & O_n \arrow[d, "\pi"] & \ni & A \arrow[d, mapsto] \\ * \arrow[r] & V_k(\mathbb{R}^n) & \ni & (Ae_1,...,Ae_k) \end{tikzcd}$$

Now define $s: U \to O_n$ by $(u_1,...,u_k) \mapsto (u_1,...u_k,\overline{u_{k+1}},...,\overline{u_n})$ where we take the vectors u_i and $\overline{u_i}$ as columns of the orthogonal matrix.
Clearly this matrix sends e_1 to u_1, e_2 to u_2 and so on. ■

Theorem 4.6.9.
Let $Gr_k(\mathbb{R}^\infty) = \operatorname{colim}_n(Gr_k(\mathbb{R}^n) \hookrightarrow Gr_k(\mathbb{R}^{n+1}) \hookrightarrow \cdots)$ induced by $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1},\ (x_1, ..., x_n) \mapsto (x_1, ..., x_n, 0)$.
This is called the infinite real Grassmanian of subspaces of dimension k.

Let $V_k(\mathbb{R}^\infty) := \operatorname{colim}_n(V_k(\mathbb{R}^n)$ infinite real Stiefel manifold.

Then $V_k(\mathbb{R}^\infty) \to Gr_k(\mathbb{R}^\infty)$ is a principal O_k-bundle and defines

$$\begin{tikzcd} V_k(\mathbb{R}^\infty) \arrow[r, "heq", "\simeq"'] \arrow[d] & EO_k \arrow[d] \\ Gr_k(\mathbb{R}^\infty) \arrow[r, "\simeq", "heq"'] & BO_k \end{tikzcd}$$

Similarly,

$$\begin{tikzcd} V_k(\mathbb{C}^\infty) \arrow[r, "heq", "\simeq"'] \arrow[d] & EU_k \arrow[d] \\ Gr_k(\mathbb{C}^\infty) \arrow[r, "\simeq", "heq"'] & BU_k \end{tikzcd}$$

Remark 4.6.10.
Also recall that the inclusion of O_k in GL_k is a homotopy equivalence and therefore

$$BO_k \simeq BGL_k(\mathbb{R})$$
$$BU_k \simeq BGL_k(\mathbb{C})$$

The universal GL_k bundle over the Grassmanian is a Stiefel manifold of all k-frames not only of orthogonal k-frames.

All proofs are very similar. As explained before the advantage with the orthogonal k-frames is that everything is compact.

Proof.
We need to show that $\pi : V_k(\mathbb{R}^\infty) \to Gr_k(\mathbb{R}^\infty)$ is a principal O_k-bundle and that $V_k(\mathbb{R}^\infty)$ is contractible.
These two claims are sufficient by the proposition before.

Let's start with the latter, that $V_k(\mathbb{R}^\infty)$ is contractible. Assume without proof that it is a CW-complex. (For this one has to check that $V_k(\mathbb{R}^n)$ is a CW-complex)
It suffices to show that the homotopy groups are zero. We have kind of computed them before in the previous corollary because

$$\pi_n(V_k(\mathbb{R}^\infty)) = \operatorname*{colim}_m \pi_n V_k(\mathbb{R}^m)$$

We have seen that the number at which this groups vanish grows linearly with m, thus at some point this sequence stabilises and becomes the 0 sequence.

$\pi : V_k(\mathbb{R}^\infty) \to Gr_k(\mathbb{R}^\infty)$ is a principal O_k-bundle:

- action is clear
- to get a trivialising cover of the infinite Grassmanian you basically just need to take the union of the trivialising covers of the finite Grassmanians.
  Denote $\{U_n^\alpha\}$ the trivialising cover for $Gr_k(\mathbb{R}^n)$. This we have described before when showing that the map $V_k(\mathbb{R}^n) \to Gr_k(\mathbb{R}^n)$ in the finite case is a principal O_k-bundle.

This can be extended to a trivialising cover on the next Grassmanian $Gr_k(\mathbb{R}^{n+1})$. This is because of how the trivialising cover was constructed:
We had a point and a subspace $W \subset \mathbb{R}^n$, chose an orthogonal complement to W, s.t. $W^\perp \oplus W = \mathbb{R}^n$ and then took $U_n^W := \{W' \mid W' \oplus W^\perp = \mathbb{R}^n\}$
By inclusion the element $W \in Gr_k(\mathbb{R}^n) \hookrightarrow Gr_k(\mathbb{R}^{n+1})$ can be regarded as element of $Gr_k(\mathbb{R}^{n+1})$ and thus

$$U_n^W \hookrightarrow U_{n+1}^W := \{W' \mid W' \oplus (W^\perp \oplus e_{n+1}) \cong \mathbb{R}^{n+1}\}$$

Clearly, every $W' \in U_n^W$ lands in the open subset U_{n+1}^W.

Thus we can define $U_\infty^W := \mathrm{colim}_n U_n^W$ which builds up a trivialising cover $\{U_\infty^W\}$ of π.

■

One might ask: "Why exactly are we doing all of this? We already know the existence of this homotopy types of the classifying space BO_k of these groups. Why do we care about any particular space that is homotopy equivalent to that?". The answer to that is that now we can use the geometry of this space for the study of the abstractly defined classifying space.
For example we can study the cohomology of this space using this model. We can use different methods, e.g. that the grassmanian and the Stiefel manifold are closed manifolds so you can use Poincaré duality to compute the multiplication in there. This was not easily done without this particular geometric notions.

We will do that later on. Before we introduce the notion of vector bundles, which are related to these U_k and GL_k-bundles.

4.7 Vector Bundles: Definition and Properties

Definition 4.7.1.
A real vector bundle of rank n over a topological space B is a continuous map $\pi : V \to B$ that

- has a structure of a real vector space of dimension n on all fibers $V_b := \pi^{-1}(b)$ for all $b \in B$ ($V_b \xrightarrow{\cong} \mathbb{R}^n$ fixed homeomorphisms which induce a structure of a vector space on V_b such that this map is a linear isomorphism)
- is locally trivial: there should exist a trivialising cover / atlas of B, i.e. for all $b \in B$ there exists an open neighbourhood $b \in U \subset B$ and isomorphisms

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi_U", "\cong"'] \arrow[dr] & & U \times \mathbb{R}^n \arrow[dl] \\ & U & \end{tikzcd}$$

where ϕ_U has to be linear over all fibers.

Similarly, you can define a complex vector bundle of rank n by replacing $\mathbb{R}$ by $\mathbb{C}$ everywhere.

Remark 4.7.2.
If B is not connected, $B = B_1 \amalg B_2$, then it is reasonable to speak of a vector bundle $V = V_1 \amalg V_2$ over B where V_1 is a vector bundle over B_1, V_2 a vector bundle over B_2 but have different ranks.

Definition 4.7.3.
A morphism of vector bundles $V \to W$ over B is the map

$$\begin{tikzcd} V \arrow[rr, "\phi"] \arrow[dr, "\pi_V"'] & & W \arrow[dl, "\pi_W"] \\ & B & \end{tikzcd}$$

such that

- it is a continuous map, s.t. $\pi_W \circ \phi = \pi_V$
- ϕ is linear in all fibers, i.e. $V_b \xrightarrow{\phi_b} W_b$ is a linear map for all $b \in B$.

Lemma 4.7.4.
Let $\pi : V \to B$ be a vector bundle of rank n, $f : B' \to B$.
Then $\pi' : B' \times_B V = V' \to B'$ has a unique structure of a vector bundle s.t. there is a canonical map $V'_{b'} \xrightarrow{\cong} V_{f(b')}$ which is linear (V' is often denoted f^*V)

Proof.
So to proof the claim, first note that the canonical map is indeed canonical. The structure of a vector bundle is the structure of real vector spaces on fibers. Here we are given it by the assumption of the lemma. Thus the only thing we need to check is that π' is indeed locally trivial.

Of course we shall prove now that if we have an trivialising atlas over V over B then we just take the pullback over the map $f : B' \to B$. Then it should be the trivialising atlas for the pullback vector bundle.

Suppose we have $U \subset B$ trivialising open for V, $U' := f^{-1}(U)$

$$\begin{tikzcd} V'|_{U'} \arrow[r, "f'"] & V|_U \arrow[rr, "\rho_U"] \arrow[dl, squiggly] & & U \times \mathbb{R}^n \arrow[dl, squiggly] \\ V'|_{U'} \arrow[r, phantom, "\cong"] \arrow[rrr, bend right, "\rho'_{U'}"'] & U' \times_U V|_U \arrow[r, phantom, "\cong"] & U' \times_U (U \times \mathbb{R}^n) \cong U' \times \mathbb{R}^n & \end{tikzcd}$$

f' and ρ_U are linear fiberwise. ρ_U by the definition of a trivialisation and f' by the definition of the structure of a real vector space.
Thus the composition $\rho'_{U'}$ is also fiberwise linear. ∎

Remark 4.7.5.
$Vect_n(B)$ is a 1-category.
Unfortunately $\texttt{Top}^{op} \xrightarrow{Vect_n} \texttt{Cat}$ is not a functor. This is because $f^*(g^*V) \cong (g \circ f)^*V$ is not a trivial isomorphism.
This is however not a big issue. One can fix it by saying it is not a 1-functor between 1-categories but rather a weak functor from a 1-category to a 2-category.

$Vect_n^{\mathbb{R}}(B)$ is an additive category (in fact $\mathbb{R}$-linear), but not abelian: if $\phi_1, \phi_2 : V \to W$ are morphisms of vector bundles one can define their sum $\phi_1 + \phi_2 : V \to W$ fiberwise as $(\phi_1 + \phi_2)(b) = \phi_1(b) + \phi_2(b)$ for all $b \in V$.

Definition 4.7.6.
For a vector bundle $\pi : E \to B$, $E' \subset E$ is a subbundle of rank k, if there exists a trivialising atlas for E such that

$$\begin{tikzcd} E'|_U \arrow[d, "\cong"'] & \subset & E|_U \arrow[d, "\cong"] \\ U \times \mathbb{R}^k \arrow[rr, hook] & & U \times \mathbb{R}^n \end{tikzcd}$$

$$(u, (x_1, ..., x_k)) \longmapsto (u, (x_1, ..., x_k, 0, ...0))$$

From this follows that E' is a vector bundle of rank k.

Proposition 4.7.7.

Let $\begin{tikzcd} V \arrow[rr, "\phi"] \arrow[dr, "\pi_V"'] & & W \arrow[dl] \\ & B & \end{tikzcd}$ *be a morphism of vector bundles.*

Assume that over each point $b \in B$ the linear map of real vector spaces $\phi_b : V_b \to W_b$ has rank not depending on b (we say ϕ is of constant rank)

Then we can define

1. $\ker(\phi) := \{v \in V \mid \phi(v) = 0\}$ *where the* 0 *is the zero of the vector space of* $W_{\pi_V(v)}$.
2. $\mathrm{im}(\phi) := \{w \in W \mid \exists v : \phi(v) = w\}$
3. $\mathrm{coker}(\phi) := \coprod_{b \in B} W_b/\phi(v_b)$ *(as a set) with the quotient topology from* $W \to \mathrm{coker}\,\phi$.

which are vector bundles.

Proof.
2. and 3. will be exercises.
To 1.:

- structure of vector spaces is induced from V.
- find to each $b \in B$ an open $U \ni b$ that trivialises V and W.
  With these trivialisations we can think of ϕ as the following map:

  $$\begin{tikzcd} U \times \mathbb{R}^n \arrow[rr, "\phi_U"] \arrow[dr, "pr_1"'] & & U \times \mathbb{R}^m \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

  which is given by $(b, v) \mapsto (b, \phi_b(v))$. So this map ϕ_b clearly depends on b. Therefore if we just look at the kernel of ϕ_U pointwise, it will not be clear that over this open subspace U this is still isomorphic to $U \times \mathbb{R}^k$ because the kernel still depends on the base point.

  Denote $\mathbb{R}^n \supset K_x := \ker(\phi_x)$, $K := K_b$. Choose $\mathbb{R}^n = K \oplus K'$. This is equivalent to choosing a map $p : \mathbb{R}^n \to K$ which is a linear projection on K (K' is the kernel of p).
  Now there exists an open subset $b \in V \subset U$ such that $K_b \oplus K' = \mathbb{R}^n$. This open exists because it is defined by $K' \cap \ker(\phi) = \{0\}$

  We can define by choosing bases $\{e_1, .., e_n\}$ of $\mathbb{R}^n$ and $\{f_1, ..., f_m\}$ of $\mathbb{R}^m$ the morphism ϕ_U more explicitly taking b to be $b = \sum_{i=1}^n \lambda_i e_i$:

  $$\left(\sum_{i=1}^n \lambda_i e_i, v\right) \mapsto (b, \phi_b(v)) = \sum_{i=1}^n \lambda_i \phi_b(e_i) = \sum_{i,j} \lambda_i ???$$

Now we can choose a basis of K'. The fact that the intersection $k' \cap \ker \phi = \{0\}$ is equivalent to the fact that the rank of ϕ on K' is the dimension of K' which in turn is equivalent that in the bases minors of the corresponding matrix have ton be non-zero.
So what are these minors? They can be computed in terms of the the coefficients of $\phi_U(b, v)$ which we have just described in more detail.

So we will get some subset V where the continuous function which is a polynomial in this continuous function does not vanish and of course this is an open subset. This explains why we have V open such that $K_b \oplus K' = \mathbb{R}^n$.

We can now explicitly write an isomorphism

$$\begin{tikzcd} & & V \times K \\ \ker \phi_V \arrow[urr, "\cong"] & \subset & V \times \mathbb{R}^n \arrow[u, "id \times p"'] \end{tikzcd}$$

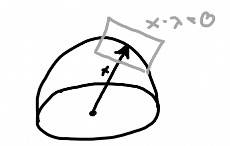
where p is the projection from before. So because over each fiber over this open set V the kernel of ϕ_V does not intersect with K' therefore it does not intersect with the kernel of the projection p and thus the map $\ker \phi_V \to V \times K$ is fiberwise isomorphism and therefore a homeomorphism which is also fiberwise linear.

■

Example 4.7.8.
Tangent vector bundle

$$\begin{tikzcd} TS^n \arrow[d] & := \{(x, \lambda) \mid x \in S^n, \lambda \in \mathbb{R}^{n+1} : \ x \cdot \lambda = 0\} \subset S^n \times \mathbb{R}^{n+1} \\ S^n \end{tikzcd}$$

where $x \cdot \lambda$ is the scalar product.



This is indeed a vector bundle:

- The structure of vector spaces on $\{(x, \lambda)\}$ is induced by $\mathbb{R}^{n+1} \ni \lambda$
- We show the trivialising property on an open set. Since S^n is really symmetric one can rotate everything to get the cover.
  $U_{n+1} := \{(x_1, ..., x_{n+1}) \mid x_{n+1} \neq 0\} \subset S^n$

$$\begin{tikzcd} TS^n|_{U_{n+1}} \arrow[rr, "\rho", "\cong"'] \arrow[dr] & & U_{n+1} \times \mathbb{R}^n \arrow[dl] \\ & U_{n+1} & \end{tikzcd}$$

$\rho(x, \lambda_1, ..., \lambda_{n+1}) = (x, \lambda_1, ..., \lambda_n)$.
$\rho^{-1}(x, \lambda_1, ..., \lambda_n) = (x, \lambda_1, ..., \lambda_n, \lambda_{n+1} = -\sum_{i=1}^n \lambda_i \frac{x_i}{x_{n+1}})$
where we get back the λ_{n+1} from the equation $0 = x \cdot \lambda = \sum_{i=1}^{n+1} x_i \lambda_i$.
Clearly ρ and its inverse are fiberwise linear because for some fixed x ρ just forgets just one of the coordinates of λ and the inverse is also clearly linear for a fixed x.

<u>Tautological vector bundles on Grassmanians</u>
In order to be able to treat the real and complex case at the same time, take V to be an open subset of either $\mathbb{R}^n$ or $\mathbb{C}^n$.

$$\begin{tikzcd} E_k(V) := \arrow[d, "\gamma_k"] & \left\{(x,v) \,\middle|\, \begin{array}{c} x \in Gr_k(V) \ k\text{-dim subspace of } V, \\ v \in V,\ v \in x \end{array}\right\} \subset Gr_k(V) \times V \\ Gr_k(V) & \end{tikzcd}$$

- The fiber of γ_k over x_0 is a collections of pairs (x,v) where $x = x_0$ and v is an element in x_0. So the fiber is x_0 as a vector space.
  This is why this is called the tautological vector bundle: over each point we have tautologically the same vector space.
- local triviality (over $\mathbb{R}$, $V = \mathbb{R}^n$)
  Given $x \in Gr_k(\mathbb{R}^n)$ define the decomposition of $x \oplus x^\perp = \mathbb{R}^n$. This defines the orthogonal projection $p_x : \mathbb{R}^n \twoheadrightarrow x$.
  You can write it down explicitly by using some bases of x. Then you can explicitly find some bases of $x^\perp$ and write this x in terms of these bases. Then in these bases it might be clearer that this depends continuously on x. But it might not be so clear that this is independent of the choice of the bases. This is why we decided to define it without any choices.
  So in summary, this depends on x continuously and defines the following morphism of vector bundles

$$\begin{tikzcd} Gr_k(\mathbb{R}^n) \times \mathbb{R}^n \arrow[rr] \arrow[dr] & & Gr_k(\mathbb{R}^n) \times \mathbb{R}^n \arrow[dl] \\ & Gr_k(\mathbb{R}^n) & \end{tikzcd}$$

  which is given by $(x,v) \mapsto (x, p_x(v))$ (this is clearly fiberwise linear, because then it is just the projection) and $\operatorname{im}\phi = E_k(V)$.
  By the proposition before this is also a morphism of constant rank k. (Because the rank is equal to k the rank of this map is the dimension of the image which is the dimension of x). Thus the image of this map is a vector bundle.

  In fact we will show that all vector bundles over all topological spaces are obtained as pullbacks of this tautological vector bundles on grassmanians (at least for compact topological spaces otherwise one would have to use the infinte grassmanian of course).

4.8 Čech cocycle presentation

We will now explain how one can construct principal G-bundles by some local data.
The following lemma is part of the construction

Lemma 4.8.1.
Let $B \in$ Top, $\{U_i\}$ an open cover of B, $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$.
Given $\{\phi_{ij} : U_{ij} \to G\}_{i,j}$ (called <u>Čech 1-cocycle with values in G</u> such that $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ (cocyle equality / condition).
There exists a right principal G-bundle P over B such that

$$\begin{tikzcd} P|_{U_i} \arrow[rr, "\cong"] \arrow[dr] & & U_i \times G \arrow[dl] \\ & U_i & \end{tikzcd}$$

and these ϕ_{ij} are obtained as follows:

$$\begin{tikzcd}
 & U_{ij} \times G \arrow[dd, "\cong"', "\rho_j \circ \rho_i^{-1}"] \\
P|_{U_{ij}} \arrow[ur, "\rho_i|_{U_{ij}}", "\cong"'] \arrow[dr, "\rho_j|_{U_{ij}}"', "\cong"] & \\
 & U_{ij} \times G
\end{tikzcd}$$

this is G-equivariant and on U_{ij} $\rho_j \circ \rho_i^{-1}(u,g) := (u, \phi_{ij}(u) \cdot g)$.

Moreover, every principal G-bundle trivialised over $\{U_i\}$ can be defined this way and one can compute $Hom_B(P,Q)$ in terms of these cocycles.

Proof.
Define

$$P := \coprod_i U_i \times G / \underset{\in U_i \times G}{(u,g)} \sim \underset{\in U_i \times G}{(u, \phi_{ij}(u) \cdot g)} \text{ if } u \in U_{ij}$$

Basically we glue two trivial G-bundles on U_i and U_j and on the intersection we glue them according to these transition functions ϕ_{ij}.

- There is a map

$$\begin{aligned} P &\to B \\ (u_i, g) &\mapsto u_i \end{aligned}$$

- G acts on P on each $U_i \times G$ by multiplication on the right: $(u,g) \overset{\cdot h}{\mapsto} (u, gh)$
-

$$\begin{tikzcd}
U_i \times G \arrow[rr, "\cong"] \arrow[dr] & & P|_{U_i} \arrow[dl] \\
 & U_i &
\end{tikzcd}$$

with the inverse

$$\begin{tikzcd}
U_{ik} \times G \subset U_k \times G \arrow[dr] & & (u,g) \arrow[dr, mapsto] & \\
 & U_i \times G & & (u, \phi_{ik}^{-1} \cdot g)
\end{tikzcd}$$

So you get that P is a principal G-bundle.
The other claim is that if $Q \in Bun_G(B)$ is trivialised on $\{U_i\}$

$$\begin{tikzcd}
 & U_{ij} \times G \arrow[dd, "\cong"', "\Phi_{ij} = \rho_j \circ \rho_i^{-1}"] \\
Q|_{U_{ij}} \arrow[ur, "\rho_i|_{U_{ij}}", "\cong"'] \arrow[dr, "\rho_j|_{U_{ij}}"', "\cong"] & \\
 & U_{ij} \times G
\end{tikzcd}$$

It follows that $\Phi_{ik} = \Phi_{jk} \circ \Phi_{ij}$. So these are the G-equivariant maps between $U_{ijk} \times G \to U_{ijk} \times G$ and writing down a G-equivariant that is constant in the first component and some multiplication by some $\phi(u)$ in the second component you get the cocycle condition.
$\implies \phi_{ik} = \phi_{jk} \circ \phi_{ij}$ ■

Next time:

$$\begin{aligned} Bun_{GL_n(\mathbb{R})}(-) &\xrightarrow{\cong} Vect_n^{\mathbb{R}}(-) \\ P &\mapsto P \times_{GL_n(\mathbb{R})} \mathbb{R}^n \end{aligned}$$

In particular,

$$\begin{aligned} Vect_n^{\mathbb{R}}(X) &\overset{\cong}{\longleftrightarrow} [X, Gr_n(\mathbb{R}^\infty)] \\ f^*\gamma_n &\leftarrow f \end{aligned}$$

So if you want to construct some vector bundles on a space you can just choose a map from X to the infinite Grassmanian.
Similarly over $\mathbb{C}$.

Theorem 4.8.2.
Let $B \in$ `CW`, then there exists a natural isomorphism of sets

$$Bun_{GL_n(K)}(B) \xrightarrow{\cong} \pi_0(Vect_n^{iso,K}$$

where the latter are the classes of isomorphisms of vector bundles of rank n and K is either $\mathbb{R}$ or $\mathbb{C}$.

Proof. (for $\mathbb{R}$)
If we have a principle GL_n-bundle $P \in Bun_{GL_n(\mathbb{R})}(B)$, then as we have discussed before we get a vector bundle by changing the fibers, so we get $P \times_{GL_n(\mathbb{R})} \mathbb{R}^n$ which is a vector bundle over B.
If we have a vector bundle $V \in Vect_n(B)$, we get a principle GL_n-bundle $E = E(V)$ such that E_b is naturally isomorphic to $Iso(\mathbb{R}^n, V_b)$. In other words, those are choices of bases of V_b.

We define E by cocycle: Let $\{U_i\}$ be a trivialising cover for V, thus

$$V|_{U_i} \underset{\cong}{\xrightarrow{\rho_i}} U_i \times \mathbb{R}^n$$

Then, when we restrict to U_{ij}, we get two isomorphisms:

$$\begin{tikzcd} V|_{U_{ij}} \arrow[r, "\rho_i|_{U_{ij}}", "\cong"'] \arrow[dr, "\cong", "\rho_j|_{U_{ij}}"'] & U_{ij} \times \mathbb{R}^n \arrow[d, dashed, "\psi_{ij}", "\cong"'] \\ & U_{ij} \times \mathbb{R}^n \end{tikzcd}$$

$\psi_{ij}(u, v) = (u, \underbrace{\psi_{ij}^u}_{\in GL_n\mathbb{R}}(v)$. Moreover, since ψ_{ij} is continuous, we also get that the map

$$\begin{aligned} \phi_{ij} : U_{ij} &\to GL_n(\mathbb{R}) \\ u &\mapsto \psi_{ij}^u \end{aligned}$$

is continuous. In particular, $\{\phi_{ij}\}$ is a cocycle. By definition $\psi_{ji} \circ \psi_{ij} = \psi_{ik}$.

By these cocycles we get the principal $GL_n(\mathbb{R})$-bundle E.
As an exercise:

- $E(P \times_{GL_n(\mathbb{R})} \mathbb{R}^n) \xrightarrow{\cong} P$

- $E(V) \times_{GL_n(\mathbb{R})} \mathbb{R}^n \xrightarrow{\cong} V$
  Again the key in finding this isomorphism is defining the natural map. You can do that by using that the cocycle defining E comes from the cocycle defining V.

■

Corollary 4.8.3.
For $B \in \texttt{CW}$, the following sets are naturally isomorphic

- $[B, Gr_n(\mathbb{R}^\infty)]$
- $Bun_{GL_n(\mathbb{R})}(B)$
- $Bun_{O_n}(B)$
- *classes of isomorphisms of real vector bundles of rank n over B*
- *(if $B = \bigcup_i U_i$) has an open cover, where the open sets have homotopy types of $\texttt{CW}$-complexes) every principle GL_n-bundle is obtained from a cocycle over this coverings, so*

$$\{\phi_{ij} : U_{ij} \to GL_n(\mathbb{R}) \mid \text{ cocycle condition}\}/isom$$

as defined in Excercise 8.4 a

4.9 Linear Algebra of vector bundles

Proposition 4.9.1 (Linear operations of vector spaces can be transferred to vector bundles)**.**

$$\begin{aligned}
\bigoplus : Vect_n(B) \times Vect_m(B) &\to Vect_{m+n}(B) \\
Hom(-,-) : Vect_n(B)^{op} \times Vect_m(B) &\to Vect_{nm}(B) \\
\bigotimes : Vect_n(B) \times Vect_m(B) &\to Vect_{nm}(B) \\
Sym^k, \bigwedge^k : Vect_n &\to Vect_{\dots}()
\end{aligned}$$

where $\cdots$ is either the $\dim_{\mathbb{R}} Sym^k(\mathbb{R}^n)$ *or* $\dim_{\mathbb{R}} \bigwedge^k(\mathbb{R}^n)$

Proof. - just the construction
Suppose $V \in Vect_n(B)$ which corresponds to the cocycles $\{\phi_{ij} : U_{ij} \to GL_n\}$ and $W \in Vect_n(B)$ corresponding to $\{\psi_{ij} : U_{ij} \to GL_m\}$.

For the direct sum $V \oplus W$ (and similarly for other operations) what we basically do is applying this operation to an n-dimensional vector space and an m-dimensional vector space and then see how it becomes a GL_n- or GL_m-torsor after this operation.
So for example here we have GL_n acting on $\mathbb{R}^n$ and GL_m on $\mathbb{R}^m$. Therefore they act on their direct sum $\mathbb{R}^n \oplus \mathbb{R}^m$ which is the same as saying that we give a map $\rho_+ : GL_n \times GL_m \to GL_{n+m}$ given by $(A, B) \mapsto \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$
so the vector bundle $V \oplus W$ corresponds to a cocycle $\{\rho_+ \circ (\phi_{ij}, \psi_{ij}) : U_{ij} \to GL_{n+m}\}$. This $V \oplus W$ comes with

$$V \underset{p_V}{\overset{i_V}{\longleftrightarrow}} V \oplus W \underset{i_W}{\overset{p_W}{\longleftrightarrow}} W$$

i_V, i_W subbundles, p_V, p_W quotient bundles.

Similarly, for $Hom(\mathbb{R}^n, \mathbb{R}^m)$ we again have the action of GL_n on $\mathbb{R}^n$ and GL_m on $\mathbb{R}^m$ and so we get again a map $\rho_{Hom} : GL_n^{op} \times GL_m \to GL_{nm}$ but here we have the opposite group because GL_n acts on $\mathbb{R}^n$ which is the first argument in Hom, which is contravariant in the first argument. That's how you define the internal hom between to vector bundles $Hom(V, W)$.

Similarly, we have $GL_n \otimes GL_m \xrightarrow{\rho_\otimes} GL_{nm}$ which gives $V \otimes W$ and analogously for $\bigwedge^k$ and Sym^k.

We cheated here a little bit because we assumed the map $\bigoplus : Vect_n(B) \times Vect_m(B) \to Vect_{m+n}(B)$ is natural construction, so is a functor, which is not generally true because it depends on how we define the direct sum. It could also be something close to a functor (also some other things are unprecise). ■

Example 4.9.2.
Recall that we have a tautological line bundle over $\mathbb{C}P^n$ which is usually denoted by $\mathbb{O}(-1)$ ($\mathbb{O}, \mathbb{O}_B, \mathbb{B}_B, \mathbb{C}_B$ often denote the trivial line bundle).
We can define

$$\mathbb{O}(1) := Hom(\mathbb{O}(-1), \mathbb{O}) = \mathbb{O}(-1)^*$$

and define

$$\mathbb{O}(k) := \begin{cases} \mathbb{O}(1)^{\otimes k} & \text{if } k > 0 \\ \mathbb{O}(-1)^{\otimes(-k)} & \text{if } k < 0 \end{cases}$$

<u>Exercise:</u> Show that $\{\mathbb{O}(k)\}_{k \in K}$

- are all non-isomorphic.
- represent all classes of isomorphisms of complex line bundles (any complex line-bundle is isomorphic to one of them)
- over S^n, $TS^n \oplus \mathbb{O} \cong \mathbb{O}^{\oplus(n+1)}$ trivial real vector bundle of rank $n+1$
  The map defining the isomorphism is given by
  $$\begin{aligned} \{(x, \lambda) \mid x \cdot \lambda = 0\} \times \{(x, v) \mid v \in \mathbb{R}\} &\to \{(x, y) \mid x \in S^n, y \in \mathbb{R}^{n+1}\} \\ (x, \lambda), (x, v) &\mapsto (x, \lambda + v \cdot x) \end{aligned}$$
- $\mathbb{C}P^n = Gr_1(\mathbb{C}^{n+1}) \cong Gr_n(\mathbb{C}^{n+1}) \rightsquigarrow \tau_n$ is the tautological complex vector bundle of rank n pullback to $\mathbb{C}P^n$.
  Then $\mathbb{O}(-1) \oplus \tau_n \cong \mathbb{O}^{\oplus(n+1)}$.

4.10 A glimpse into tangent and normal vector bundles of manifolds

Let $h : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map (i.e. $(x_1, ..., x_n) \to (h_1(\vec{x}), ..., h_n(\vec{x}))$, $h_i \in C^\infty(\mathbb{R}^n)$)
Define $T\mathbb{R}^n$, $T\mathbb{R}^m$ as trivial vector bundles of rank n and m respectively.
The <u>differential</u> is a map of vector bundles

$$\begin{aligned} dh : \underbrace{T\mathbb{R}^n}_{=\mathbb{R}^n \times \mathbb{R}^n} &\to \underbrace{T\mathbb{R}^m}_{=\mathbb{R}^m \times \mathbb{R}^m} \\ (x, v) &\mapsto (h(x), \underbrace{(\partial_i h_j(x))}_{m \times n\text{-matrix in } \mathbb{R}} (v)) \end{aligned}$$

Suppose $M \subset \mathbb{R}^n$ is $h^{-1}(0)$ for some $h : \mathbb{R}^n \to \mathbb{R}^m$ smooth. If dh is of constant rank r in the neighbourhood of M, then M is an <u>embedded smooth manifold of dimension $n - r$</u>.

The tangent bundle $TM := \ker(j^*T\mathbb{R}^n \xrightarrow{dh} j^*h^*T\mathbb{R}^m)$ is a vector bundle of rank $\dim M = n - r$.

Fact: TM does not depend on the choice of h.

Example 4.10.1.

$$S^n := \{(x_1, ..., x_{n+1}) \mid \sum x_i^2 = 0\} \subset \mathbb{R}^{n+1}$$

where we now take $h : \mathbb{R}^{n+1} \to \mathbb{R}$ to be $\sum x_i^2$. Then the differential is

$$d_x h = (2x_1, ..., 2x_n)$$

The tangent space can now be redefined as

$$\begin{aligned} TS^n &= \ker(T\mathbb{R}^{n+1}|_{S^n} \xrightarrow{dh} T\mathbb{R}|_{S^n}) \subset S^n \times \mathbb{R}^{n+1} \\ &= \{(x, y) \mid \sum 2x_i y_i = 0\} \Leftrightarrow x \cdot y = 0 \end{aligned}$$

So this is how you can define the tangent bundle to an embedded smooth manifold and embedded means subset of $\mathbb{R}^n$. Then if we believe the fact that this does not depend on the choice of h, you actually don't need this M to be defined as $h^{-1}(0)$ for one h, you just need it to be true locally. Then locally you can define the canonical tangent bundle. You can now glue those together to obtain the tangent bundle on the whole smooth manifold.

We also get a cotangent bundle.

$$0 \to TM \to T\mathbb{R}^n|_M \to N \to 0$$

where the quotient N is the normal bundle of $M \subset \mathbb{R}^n$.
Basically we have just checked in the previous example that the normal bundle for the inclusion of the sphere $S^n \hookrightarrow \mathbb{R}^{n+1}$ is trivial.

More generally, M is a smooth manifold of dimension d if M is a topological space such that $M = \bigcup_i U_i$, $U_i \overset{p_i}{\cong} \mathbb{R}^d$, U_i open such that on the intersection of any two of these open sets

$$\begin{tikzcd} U_{ij} \arrow[r, "\cong", "\phi_i"'] \arrow[dr, "\cong", "\phi_j"'] & \phi_i(U_{ij}) \overset{\text{open}}{\subset} \mathbb{R}^d \arrow[d, "\cong"', "f_{ij}"] \\ & \phi_j(U_{ij}) \overset{\text{open}}{\subset} \mathbb{R}^d \end{tikzcd}$$

f_{ij} is a smooth map $\implies f_{jk} \circ f_{ij} = f_{ik}$.

We can now define the tangent bundle TM using the cocycle

$$\begin{aligned} U_{ij} &\to GL_n(\mathbb{R}) \\ u &\mapsto d_u f_{ij} = (\partial_k (f_{ij})_m(u))_{k,m} \end{aligned}$$

Also we have to check that as a vector bundles this does not depend on the choices we made.

Example 4.10.2.
$\mathbb{C}P^n$ is a smooth manifold of dimension $2n$.
$\mathbb{C}P^n = \{U_i\}$

$$\begin{tikzcd} U_i \arrow[dr, "\phi_i", "\cong"'] & = \{(z_0 : ... : z_n) \mid z_i \neq 0\} \arrow[dr, mapsto] & \\ & \mathbb{C}^n \cong \mathbb{R}^{2n} & \left(\frac{z_0}{z_i}, \frac{z_1}{z_i}, ..., 1, ..., \frac{z_n}{z_i}\right) \end{tikzcd}$$

We need to determine what the cocycles are. For $j < i$:

$$\begin{aligned} U_{ij} &\xrightarrow{\cong} \phi_i(U_{ij}) = \mathbb{C}^n \setminus \{w_j = 0\} \\ \phi_j(U_{ij}) &= \mathbb{C}^n \setminus \{w_{i-1} = 0\} \end{aligned}$$

Thus

$$\begin{aligned} f_{ij} : \mathbb{C}^n \setminus \mathbb{C}^{n-1} &\xrightarrow{\cong} \mathbb{C}^n \setminus \mathbb{C}^{n-1} \\ (\underset{\frac{z_0}{z_i}}{w_1}, ..., \underset{\frac{z_i}{z_j}}{w_{i+1}} ..., \underset{\frac{z_0}{z_n}}{w_1}) &\mapsto (w_1, ..., w_{i+1}, ..) \end{aligned}$$

Exercise: Check $T\mathbb{C}P^1$ is a real vector bundle of rank 2, thus is isomorphic to $\mathcal{O}(-2)$. There can also be defined a complex analytic structure.
More generally $\bigwedge^n T\mathbb{C}P^n \cong \mathcal{O}(-n-1)$ as real vector bundles.

So we have this new geometric structure of vector bundles over topological spaces and smooth manifolds and one questions why we need that. One reason is that vector bundles appear naturally in the context of smooth manifolds which are reasonable objects to study e.g. we have just introduced embedded manifolds as the preimage of the zero point of some smooth function.
One can ask whether one can use these vector bundles that appear naturally in some geometry to study other topological invariants and yes, one can use them for the construction of the classes in the cohomology groups. These are called characteristic classes.

4.11 Characteristic classes of vector bundles

Recall that we have classified all real and complex line bundles over CW-complexes

$$\begin{tikzcd} \pi_0(Vect_1^{iso,\mathbb{R}}(B)) \arrow[rr, bend left, "w_1"] & = [B, \underset{K(\mathbb{Z}/2,1)}{\mathbb{R}P^\infty}] & \cong H^1(B, \mathbb{Z}/2) \\ \pi_0(Vect_1^{iso,\mathbb{C}}(B)) \arrow[rr, bend right, "c_1"'] & = [B, \underset{K(\mathbb{Z},2)}{\mathbb{C}P^\infty}] & \cong H^2(B, \mathbb{Z}/2) \end{tikzcd}$$

w_1 is the first Stiefel-Whitney class of a line bundle.
c_1 is the first Chern class of a complex line bundle.

By Yoneda, natural transformations

$$\pi_0(Vect_n^{iso}(-)) \to H^m(-, A)$$

because both functors are representable in the homotopy category ($\pi_0(Vect_n^{iso}(-))$ by Grassmanian, $H^m(-, A)$ by Eilenberg-MacLane space) are in one-to-one correspondence with $H^m(Gr_n, A)$.
These natural transformations are called characteristic classes because we have a vector bundle and its characteristic class is an element in the cohomology group.

So to classify all of them and get a way to compute them we just need to compute cohomology groups of some very nice spaces.

Theorem 4.11.1. *(real case, $A = \mathbb{Z}/2$)*
There exist unique $w_i : \pi_0(Vect^{iso,\mathbb{R}}) \to H^i(-, \mathbb{Z}/2)$ (note $\pi_0(Vect^{iso,\mathbb{R}})$ describes all ranks) called

the i-th Stiefel Whitney class for $i \geq 0$, $w_0 = 1$.
These satisfy the Whitney sum/product formula

$$w_k(V \oplus W) = \sum_{i=0}^{k} w_i(V) \cup w_{k-i}(W)$$

and for a line bundle $w_i(L)$ is defined as before.

Moreover, $H^*(Gr_k(\mathbb{R}^\infty), \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, ..., w_k]$. So the cohomology of the grassmanian with $\mathbb{Z}/2$-coefficients of k-dimensional subspaces is generated as a ring by $w_1, ..., w_k$, so there are no relations between them. Thus all characteristic classes are, in fact, some polynomials in the Stiefel-Whitney classes. This isomorphism is preserved under inclusions of the Grassmanians.

Remark 4.11.2. Also one can write the formula for when you have an exact sequence of vector bundles

$$0 \to V \to U \to W \to 0$$

where V is a subbundle and W a quotient bundle. Then the same formula holds

$$w_k(U) = \sum_{i=0}^{k} w_i(V) \cup w_{k-i}(W)$$

This might seem as first to be a more general case to where we have seen the formula first because there we had the direct sum which is nought but a special case of exact sequence. However, in the compact case (and we will work with CW-complexes) in which we always will work in, all such sequences are split.

So to see that we could have assumed this case in the theorem instead and would not have lost any generality, we take as fact
Fact: Any (exact) sequence is split over any paracompact topological space.

Theorem 4.11.3. *(complex case, $A = \mathbb{Z}$)*
$\exists! c_i : \pi_0 Vect^{iso,\mathbb{C}} \to H^{2i}(-, \mathbb{Z})$ called the i-th Chern classes, satisfying Cartan's / Whitney sum formula

$$c_k(V \oplus W) = \sum_{i=0}^{k} c_i(V) \cup c_{k-i}(W)$$

with $c_0 = 1$, c_1 is defined as before on line bundles.
This are again all the characteristic classes you can get with integral coefficients and in fact with any coefficients because

$$H^*(Gr_k(\mathbb{C}^\infty), \mathbb{Z}) \cong \mathbb{Z}[c_1, ..., c_k]$$

Again, by the discussion before using the Yoneda lemma, any characteristic class of a complex vector bundle of rank k is in fact a polynomial in the first k Chern classes.
Also when computing $c_k(V \oplus W)$ one could again have demanded to only have an exact sequence

$$0 \to V \to U \to W \to 0$$

such that

$$c_k(U) = \sum_{i=0}^{k} c_i(V) \cup c_{k-i}(W)$$

Examples of computations

1. $w_i(\mathbb{O}^n) = 0$, $c_i(\mathbb{O}^n) = 0$ for all $i > 0$.
   The trivial bundle on the base B is obtained as the pullback from a point $B \xrightarrow{p} \bullet$

$$
\begin{aligned}
p^*\mathbb{R}^n &= \mathbb{O}^{\oplus n} \\
w_i(p^*\mathbb{R}^n) &\underset{\text{nat trans}}{=} p^*(\underbrace{w_i(\mathbb{R}^n)}_{H^i(\bullet,\mathbb{Z}/2)=0}) = 0
\end{aligned}
$$

   The natural transformation is with respect to what? So these were vector bundles and the cohomology are contravariant functors from CW-complex so that for each morphism we have a pullback. Thus natural transformation commutes with pullback.

2. $w_i(V) = 0$ if $i > rk(V)$
   This does not fall out from the axioms but rather from the computation that cohomology of the Grassmanian of k-dimensional subspaces do not have any Stiefel-Whitney classes apart from the first k ones.
   So if $k = rank(V)$, let's spell out what this is in terms of the Yoneda lemma

$$
\begin{aligned}
B &\xrightarrow{f} Gr_k(\mathbb{R}^\infty) \\
f^*\gamma_k &= V \\
w_i(V) &= f^*w_i, \quad w_i \in H^i(Gr_k(\mathbb{R}^\infty), \mathbb{Z}/2)
\end{aligned}
$$

   We are not claiming that this latter cohomology class is zero but that there are these natural maps between Grassmanians, in fact embeddings

$$
\begin{aligned}
Gr_k(\mathbb{R}^\infty, \mathbb{Z}/2) &\hookrightarrow Gr_{k+1}(\mathbb{R}^\infty, \mathbb{Z}/2) \\
H^*(-, \mathbb{Z}/2) : \mathbb{Z}/2[w_1, ..., w_k] &\leftarrow \mathbb{Z}/2[w_1, ..., w_{k+1}] \\
w_i &\leftarrow\!\shortmid w_i, \quad i \leq k \\
0 &\leftarrow\!\shortmid w_{k+1}
\end{aligned}
$$

   So if V is a vector bundle of rank $k < i$, to get the pullback of the w_i we can factor it through the inclusion of the Grassmanians

$$
\begin{aligned}
&B \xrightarrow{f} Gr_k(\mathbb{R}^\infty) \hookrightarrow Gr_i(\mathbb{R}^\infty) \\
&0 \leftarrow\!\shortmid w_i
\end{aligned}
$$

   Analogously $c_i(V) = 0$ if $i > rank(V)$.

total Stiefel-Whitney class is defined as $w = \sum_i w_i$, naturally this lives in the direct product of all the cohomology groups with $\mathbb{Z}/2$-coefficients.
total Chern class is defined as $c = \sum_i c_i$.

Then the Whitney sum formula can be rewritten as

$$
\begin{aligned}
w(V \oplus W) &= w(V) \cup w(W) \\
c(V \oplus W) &= c(V) \cup c(W)
\end{aligned}
$$

3. $w_i(V \oplus \mathbb{O}^n) = w_i(V)$, $c_i(V \oplus \mathbb{O}^n) = c_i(V)$.
   This is of course due to the Whitney sum formula.

4. $w(TS^n) = 1$ because $TS^n \oplus \mathbb{O} \cong \mathbb{O}^{\oplus(n+1)}$

5. $w(\mathbb{O}(1)) = 1 + a$ with $\mathbb{O}(1)$ on $\mathbb{R}P^n$ and where a is the notation we used in AT I for $\mathbb{R}P^\infty$ $H^*(\mathbb{R}P^n, \mathbb{Z}/2) \cong \mathbb{Z}/2[a]/a^{n+1}$ where $\deg a = 1$.

   So this a, when $n = \infty$, a is precisely the first Stiefel-Whitney Chern class, thus it is clear that there first one should be a rather than 0.

6. $w(T\mathbb{R}P^n) = (1 + a)^{n+1}$.
   This is because there exists an exact sequence of vector bundles

$$0 \to \mathbb{O} \to \mathbb{O}(1)^{\oplus(n+1)} \to T\mathbb{R}P^n \to 0$$

   This can be split because $T\mathbb{R}P^n$ is para-compact, so we can use the Whitney sum formula. So $w()$ of the sum of $T\mathbb{R}P^n$ and $\mathbb{O}$ is just $w(\mathbb{O}(1)^{\oplus(n+1)})$. We have just computed $w(\mathbb{O}(1)) = 1+a$ and the total Whitney class sends sums to products.

   Assume that $\mathbb{R}P^n \hookrightarrow \mathbb{R}^m$ is a smooth embedding. Then we get

$$0 \to T\mathbb{R}P^n \to \mathbb{O}^{\oplus m} \to N \to 0$$

   where N is the quotient bundle with rank $m - n$. Now by the Whitney sum formula

$$w(T\mathbb{R}P^n) \cdot w(N) = w(\mathbb{O}^{\oplus m}) = 1$$

   Take $n = 2^k$, then $w(T\mathbb{R}P^n) = (1 + a^{2^k})(1 + a) = 1 + a + a^{2^k}$.
   From this follows that $w(N) = 1 + a + a^2 + a \cdots + a^{2^k-1}$. But this can only happen, that we have a non-trivial Stiefel-Whitney class in degree $2^k - 1$, if $rank(N) = m - n \geq 2^k - 1$, so $\underline{m > 2^{k+1} - 1}$.

   This is a very non-trivial result on embeddings of projective spaces: you cannot embed $\mathbb{R}P^n$ in small spaces.
   E.g. $\mathbb{R}P^4 \rightsquigarrow \mathbb{R}^7$, $\mathbb{R}P^8 \rightsquigarrow \mathbb{R}^{15}$.

4.12 Cohomology of projective bundles and Chern classes

Definition 4.12.1.

Let $\begin{tikzcd} V \arrow[d] \\ B \arrow[u, bend right, dashed, "z"'] \end{tikzcd}$ be a complex vector bundle of rank n. Take V^0 to be the image of the zero-section which is in each fiber V_b just the point 0, so $z(B) = V^0 \subset V$. $\mathbb{C}^\times$ acts freely on $V \setminus V^0$ by

$$(\underset{\in V}{v}, \underset{\in \mathbb{C}}{\lambda}) \mapsto \lambda \cdot v$$

where the multiplication happens in the fiber over $p(v)$.
Then we can take the quotient of $V \setminus V^0$ under the action of $\mathbb{C}^\times$ and this quotient $(V \setminus V^0)/\mathbb{C}^\times = \mathbb{P}(V) \to B$ is a <u>projective bundle associated to V</u> ($\mathbb{P}(V)_n$ is $\mathbb{P}(V_b) \cong \mathbb{CP}^{n-1}$)

Remark 4.12.2.
If $\{U_i\} \subset B$ is a trivialising cover for V, so $V|_{U_i} \cong U_i \times \mathbb{C}^n$, then we can glue $\mathbb{P}(V)$ from $\mathbb{CP}(V|_{U_i}) \cong U_i \times \mathbb{CP}^{n-1}$. So V is glued from this trivial bundles over the U_i's according to some cocycle and we can do the same for the projectivisation if we find some corresponding cocycle (which we will not do here).
In particular, this map $\mathbb{P}(V) \to B$ is a fiber bundle.

Properties 4.12.3.

1. *$\exists \mathcal{O}(-1) \to \mathbb{P}(V)$ which is the relative tautological line bundle. This can be defined similarly to the definition of the tautological line bundle on a projective space, but we want to define it using the original vector space directly: $V \setminus V^0 \to \mathbb{P}(V)$. Since $\mathbb{C}^\times$ acts freely on this space $V \setminus V^0$ you can locally write explicitly this action. So it is not hard to show that $V \setminus V^0 \to \mathbb{P}(V)$ is a principle U_1-bundle.*

   Thus we can associate to it the line bundle by the change of fiber $$\begin{array}{c}(V \setminus V^0) \times_{U_1} \mathbb{C} =: \mathcal{O}(-1) \\ \downarrow \\ \mathbb{P}(V)\end{array}$$

   As in the case of the usual tautological line bundle on the projective space the fiber over a point, so a line in $\mathbb{P}(V)$ consists of the points of this line.

2. *Another property of this construction is that it is functorial. Suppose we are given a vector bundle $W \to C$ and a map $f : B \to C$. Then you can take the pullback*

   $$\begin{tikzcd} f^*W \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & W \arrow[d] \\ B \arrow[r, "f"'] & C \end{tikzcd}$$

   Then you can take the projectivisation:

   $$\begin{tikzcd} \mathbb{P}(f^*W) \arrow[r, "\overline{f}"] \arrow[d] & \mathbb{P}(W) \arrow[d] \\ B \arrow[r, "f"'] & C \end{tikzcd}$$

   and the $\overline{f}(\mathcal{O}(-1) = \mathcal{O}(-1)$.

The key to defining the Chern classes is the following computation:

Theorem 4.12.4 (Projective Bundle Formula)**.**

Suppose we have a complex vector bundle of rank n $$\begin{array}{c} V \\ \downarrow \\ B \end{array}$$

Then we can associate to it the projective bundle $$\begin{array}{c} \mathbb{P}(V) \\ \downarrow \pi \\ B \end{array}$$

On this projective bundle there is $\mathcal{O}(-1)$ which is a line bundle over $\mathbb{P}(V)$ and therefore we can associate to it the first Chern class which we have already defined. This is because the first Chern class comes from the fact that $\mathbb{C}P^\infty$ turns out to be the Eilenberg-Mac Lane space $K(\mathbb{Z}, 2)$. So we have the associated element $c_1(\mathcal{O}(-1)) =: \xi \in H^2(\mathbb{P}(V))$ (with integral coefficients).

Then $H^*(\mathbb{P}(V))$ is a free module over $H^*(B)$ with the basis $1, \xi, \xi^2, ..., \xi^{n-1}$ where the structure of the $H^*(B)$-module is also given by the cup product and the pullback: $\underset{\in H^*(B)}{\alpha} \cdot \underset{H^*(\mathbb{P}(V))}{\beta} = \pi^*(\alpha) \cup \beta = \pi^*(\alpha) \cdot \beta$ (we denote starting here, $\cdot$ to mean $\cup$)

The proof to this theorem follows from

Theorem 4.12.5 (Leray-Hirsch)**.**

Let $\begin{array}{c} E \\ \big\downarrow{\scriptstyle \pi} \\ B \end{array}$ *be a Serre-fibration over a connected CW-complex* B*.*

Choose $b \in B$*, then the fiber over this point* $F := E_b$ *(which is actually a homotopy fiber, because Serre fibrations are fibrations).*
Assume

- $H^k(F)$ *is a free finitely generated module (the coefficients are here ignored because they are* $\mathbb{Z}$*, also the same construction can be translated for the Stiefel-Whitney classes where we would need coefficients* $\mathbb{Z}/2$*)*
- $\exists$ *elements* $\{t_i\} \subset H^*(E)$ *such that if we pull them back to the fiber using the canonical inclusion map* $i : F \hookrightarrow E$*, then* $\{i^* t_i\}$ *is a basis of the cohomology of* $H^*(F)$*. So you can in particular do that in any degree (but also simultaneously in all degrees as we did here) in some degree we have that* $H^*(F)$ *is a free finitely generated module and there is a finite number of these* t_i*'s in this degree that restrict to the basis of the module* $H^*(F)$ *after we apply the pullback along* i*.*

Then

$$H^*(B) \otimes H^*(F) \xrightarrow{\cong} H^*(E)$$
$$\alpha \otimes i^* t_i \mapsto \pi^*(\alpha) \cdot t_i$$

So $H^*(B) \otimes H^*(F)$ *is a free finitely generated module over* B *and thus this map makes* $H^*(E)$ *too with basis* $\{t_i\}$*.*

Proof. that $H^*(\mathbb{P}(V))$ is a free module over $H^*(B)$ with the basis $1, \xi, \xi^2, ..., \xi^{n-1}$

We have to check that $1, \xi, \xi^2, ..., \xi^{n-1}$ restrict to free generators of $H^*(\mathbb{P}(V)_n) \cong H^*(\mathbb{P}^{n-1})$. This is because $\xi = c_1(\mathcal{O}(-1))$ and the pullback $i^*(\underset{\mathbb{P}^n(V)}{\mathcal{O}(-1)}) = \underset{\mathbb{P}^{n-1}}{\mathcal{O}(-1)}$
Therefore the assumptions of the Leray-Hirsch theorem are satisfied: the fiber of the fiber bundle $\mathbb{P}(V) \to B$ (which is thus a Serre fibration) is $\mathbb{P}^{n-1}$ and here we have universal elements $1, \xi, \xi^2, ..., \xi^{n-1}$ in the cohomology of the total space $H^*(\mathbb{P}(V))$, that restrict to the generators. Thus $H^*(\mathbb{P}(V))$ is a free module as we wanted ■

A corollary of the projective bundle formula is the definition of Chern classes

Corollary 4.12.6. $\exists ! c_j(V) \in H^{2j}(B)$, $n \geq j \geq 1$, *such that in* $H^*(\mathbb{P}(V))$*:* ξ^n *is no element of the basis, but exists as an element and therefore has to be represented by some sum of elements in the base with some coefficients. Those coefficients are the* <u>*Chern classes of* V</u>*:*

$$\xi^n - c_1(v) \cdot \xi^{n-1} + c_2(v) \cdot \xi^{n-2} - ... + (-1)^n c_n(V) = 0$$

Also take $c_0 = 1$ *and* $c_m(V) = 0$ *for* $m > rank(V)$*.*

So here we got by definition that the Chern classes are 0 for $m > rank(V)$ which we argued to be the case beforehand.

Properties 4.12.7.

1. *functoriality:*

$$\begin{tikzcd} f^*W \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & W \arrow[d] \\ B \arrow[r, "f"'] & C \end{tikzcd} \qquad \begin{tikzcd} \mathbb{P}(f^*W) \arrow[r, "\overline{f}"] \arrow[d] & \mathbb{P}(W) \arrow[d] \\ B \arrow[r, "f"'] & C \end{tikzcd}$$

The element $\xi_{\mathbb{P}(W)} = c_1(\underset{\mathbb{P}(W)}{\mathcal{O}(-1)})$, *but as claimed the pullback* $\overline{f}^* \underset{\mathbb{P}(W)}{\mathcal{O}(-1)} = \underset{\mathbb{P}(f^*W)}{\mathcal{O}(-1)}$.
Hence c_1 *is defined as a natural isomorphism of functors we get that* $\xi_{\mathbb{P}(f^*W)} = \overline{f}^* \xi$.
Now consider the equality defining the Chern classes of W *where* $n = rank(W)$*:*

$$0 = \sum_{j=0}^{n} (-1)^j c_j(W) \cdot \zeta^{n-j}$$

$$\overset{PB}{\rightsquigarrow} 0 = \overline{f}^* \left(\sum_{j=0}^{n} (-1)^j c_j(W) \cdot \zeta_{\mathbb{P}(W)}^{n-j} \right) \overset{(*)}{=} \sum_{j=0}^{n} (-1)^j) f^*(c_j(W))) \cdot \xi_{\mathbb{P}(f^*W)}^{n-j}$$

where in $(*)$ *we use additivity of the PB and that the commutativity of*
$$\begin{tikzcd} \mathbb{P}(f^*W) \arrow[r, "\overline{f}"] \arrow[d] & \mathbb{P}(W) \arrow[d] \\ B \arrow[r, "f"'] & C \end{tikzcd},$$
which tells us that instead of pulling first up to the base of $\mathbb{P}(W)$ *and then to* $\mathbb{P}(f^*W)$, *we could also first pull it to the base of* B *and then up to* $\mathbb{P}(f^*W)$. *So the pullback is always emitted in* $c_j(W) \cdot \zeta^{n-j}$ *in that we have the action on the base by the cohomology of the projective bundle.*

Thus by

$$0 = \sum (-1)^j f^*(c_j(W))) \cdot \xi_{\mathbb{P}(f^*W)}^{n-j}$$

follows that $f^*(c_j(W)))$ *are* $c_j(f^*W)$ *from uniqueness.*

2. *The new definition* $c_1^{new}(L) = \xi = c_{1\mathbb{P}(L)}(\mathcal{O}(-1))$ *coincides with the old definition. So we have to understand what* $c_{1\mathbb{P}(L)}(\mathcal{O}(-1))$ *is.*

$$\begin{tikzcd} & \mathcal{O}(-1) = (L \setminus L^0) \times_{U_1} \mathbb{C}^* \xrightarrow{\cong} L \arrow[dl] \\ \mathbb{P}(L) \arrow[dr, "\cong"'] & \\ & B \end{tikzcd}$$

Thus $\xi = c_{1\mathbb{P}(L)}^{old} = c_1^{old}(L)$.

3. *Whitney sum formula*
We want to show that

$$c_k(V \oplus W) = \sum_{i+j=k} c_i(V) \cdot c_j(W)$$

for V, W *vector bundles over* B. *Let's consider the projectivisation of the direct sum*

$$\begin{tikzcd} \mathbb{P}(V) \arrow[r, hook] \arrow[dr] & \mathbb{P}(V \oplus W) \arrow[d] & \mathbb{P}(W) \arrow[l, hook'] \arrow[dl] \\ & B & \end{tikzcd}$$

All these projections are the canonical maps of the projective bundles. So for instance the map $\mathbb{P}(V) \hookrightarrow \mathbb{P}(V \oplus W)$ *sends a line in the fiber of* V *to the fiber of* V *in* $V \oplus W$.
In fact, both $\mathbb{P}(V)$ *and* $P(W)$ *both lie in an open* U_V *respectively* U_W *in* $\mathbb{P}(V \oplus W)$.

$$U_V := \mathbb{P}(V \oplus W) \setminus \mathbb{P}(W)$$

$$U_W := \mathbb{P}(V \oplus W) \setminus \mathbb{P}(V)$$

We claim that the map $s_V : \mathbb{P}(V) \hookrightarrow U_V$, $s_V(x) = [x, 0]$ *is a deformation retract.*
For this we define the map in the other direction r_V *by* $r_V([x, y]) = x$ *where* x *is a line in* V *and* y *a line in* W. x *cannot be 0 in* U_V *because we have deleted* $\mathbb{P}(W)$ *which is exactly the pairs* $(0, y)$, *so the map* $r_V([x, y]) = x \neq 0$ *is well-defined.*
$r_v \circ s_v = id$ *and* $s_v \circ r_v : [x, y] \mapsto [x, 0] \simeq id$ *by* $[x, y] \mapsto [x, ty]$, $t \in [0, 1]$.

Thus s_V *is a deformation retract. In particular, it induces an isomorphism* $H^*(U_V) \xrightarrow{\cong} H^*(\mathbb{P}(V))$.

Denote $k = rank(V)$, $l = rank(W)$.
Consider $x = \sum_{j=0}^{k} c_j(V) \cdot \zeta^{k-j}$ *where* $\xi := c_1(\mathcal{O}_{\mathbb{P}(V \oplus W)}(-1))$. *In particular, it is not clear that* $x = 0$ *but restricted to* U_V *(and* $\mathbb{P}(V)$*) is* 0.
Similarly $y = \sum_{i=0}^{l} c_i(V) \cdot \zeta^{l-i}$ *is restricted to* U_W *zero.*
There exist corresponding elements $x' \in H^*(\mathbb{P}(V \oplus W), U_V)$ *and* $y' \in H^*(\mathbb{P}(V \oplus W), U_W)$.
The multiplication $x' \cdot y' \in H^*(\mathbb{P}(V \oplus W), U_V \cup U_W)$. *But* $U_V \cup U_W$ *is an open cover and thus* $U_V \cup U_W = \mathbb{P}(V \oplus W)$ *and thus* $H^*(\mathbb{P}(V \oplus W), U_V \cup U_W) = 0 \implies x \cdot y = 0$.

But what is $x \cdot y$*?*

$$\sum_{i=0}^{k+l} (-1)^i \left(\sum_{r+s=i} c_r(V) c_s(W) \right) \cdot \xi^i = 0$$

4.13 Cohomology of BU_n

Lemma 4.13.1.
There is a map $BU_{n-1} \times BU_1 \xrightarrow{\beta} BU_n$ *such that*

- $\beta^*(\gamma_n) = pr_1^* \gamma_{n-1} \oplus pr_2^* \gamma_1$
- $$\begin{tikzcd} BU_{n-1} \times BU_1 \arrow[rr, "\cong"] \arrow[dr, "\beta"'] & & \mathbb{P}(\gamma_n) \arrow[dl, "\pi"] \\ & BU_n & \end{tikzcd}$$

Here one can see why we said "there is a map" because the first property determined the map uniquely but up to homotopy. Here, however, the claim is that you can choose the space homeomorphic to the projective bundle and the map as projection.

Proof. (sketch)
We have $U_{n-1} \times U_1 \hookrightarrow U_n$ given by $(A, B) \mapsto \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right)$.
Consider the universal U_n-bundle:

$$\begin{tikzcd} & & EU_n \arrow[dl] \arrow[dd] \\ & EU_n/U_{n-1} \times U_1 \arrow[dl, equal] \arrow[d, equal] & \\ EU_n \times_{U_n} (U_n/(U_{n-1} \times U_1)) & B(U_{n-1} \times U_1) \arrow[r, "\alpha"] & BU_n = EU_n/U_n \end{tikzcd}$$

$$U_n/(U_{n-1} \times U_1) \cong \mathbb{C}P^{n-1}$$

This is because U_n acts on $\mathbb{C}^n$ and therefore it acts on $\mathbb{C}P^{n-1}$ and the stabiliser of some point in $\mathbb{C}P^{n-1}$ is the set of matrices $\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$.
Thus $EU_n \times_{U_n} (U_n/(U_{n-1} \times U_1)) = EU_n \times_{U_n} \mathbb{C}P^n \cong \mathbb{P}(\gamma_n)$.
$\gamma_n := EU_n \times_{U_n} \mathbb{C}^n$, $\gamma_n^* = EU_n \times_{U_n} \{0\}$, $(\gamma_n \setminus \gamma_n^0/U_1) = EU_n \times_{U_n} Ç^n \setminus \{0\}/U_1$.

Now to get the map β, we use an exercise that there is

$$BU_{n-1} \times BU_1 \xrightarrow{heq} \underbrace{B(U_{n-1} \times U_1)}_{=\mathbb{P}(\gamma_n} \to BU_n$$

So we can take $BU_{n-1} \times BU_1$ as $\mathbb{P}(\gamma_n)$.

We have now constructed β such that it fulfils the second property required by the lemma. We now have to check that if we take the pullback $\beta^*(\gamma_n) = pr_1^*\gamma_{n-1} \oplus pr_2^*\gamma_1$.
Let's recall how we defined the classifying space:

$$\begin{array}{ccc} EU_{n-1} \times EU_1 & \xrightarrow{U_{n-1}\times U_n\text{-equiv}} & EU_n \\ \downarrow & & \downarrow \\ BU_{n-1} \times BU_1 & \longrightarrow & BU_1 \end{array}$$

So what is $\gamma_{n-1} \oplus \gamma_1$ (ignoring the pullbacks)?

$$\gamma_{n-1} \oplus \gamma_1 = (EU_{n-1} \times EU_1) \times_{U_{n-1}\times U_1} (\mathbb{C}^{n-1} \times \mathbb{C}^1) \to EU_n \times_{U_{n-1}\times U_1} \mathbb{C}^n \to EU_n \times_{U_n} \mathbb{C}^n = \gamma_n$$

This induces $\gamma_{n-1} \oplus \gamma_1 \xrightarrow{\cong} \beta^*\gamma_n$. ∎

Theorem 4.13.2.
Let $(BU_1)^{\times n} \xrightarrow{f} BU_n$ be the map classifying $pr_1^*\gamma_1 \oplus pr_2^*\gamma_1 \oplus ... \oplus pr_n^*\gamma_1$.
Then $f^* : H^*(BU_n) \to H^*(BU_1^{\times n}) \cong \mathbb{Z}[t_1, ..., t_n]$ of $\deg t_i = 2$

- *is injective*
- *its image consists of symmetric polynomials*
- *$f^*(c_i(\gamma_n))$ is the i-th symmetric polynomial in $t_1, ..., t_n$. In particular, $H^*(BU_n) \cong \mathbb{Z}[c_1(\gamma_n), c_2(\gamma_n), ..., c_n(\gamma_n)]$*

Proof.

- $\sigma \in \Sigma_n$ acts on $(BU_1)^{\times n}$ by permutations

  $$(BU_1)^{\times n} \xrightarrow{\sigma} (BU_1)^{\times n} \xrightarrow{f} BU_n$$

  This composition classifies a vector bundle of rank n. Which is that? The pullback of the canonical, the tautological vector bundle of rank n in BU_n, is by definition of f the direct sum $pr_1^*\gamma_1 \oplus pr_2^*\gamma_1 \oplus ... \oplus pr_n^*\gamma_1$. Then the pullback over σ is just the permutation of these summands:

  $$\sigma^* \circ f^*(\gamma_n) \cong f^*(\gamma_n) \cong pr_1^*\gamma_1 \oplus ... \oplus pr_n^*\gamma_1$$

  So by the property of the classifying space, since both these maps, f and the composition with σ yields isomorphic pullbacks, they have to be homotopic

  $$\sigma \circ f \simeq f$$

  and in particular induce the same map on cohomology H^*.
  This explains why the image of that map f^* lies inside of symmetric polynomials. Although we still have to check that all symmetric polynomials lie in the image of the map and that it is injective.

- injectivity
  f can be obtained as a sequence
  $$BU_1^{\times n} \to BU_2 \times (BU_1)^{\times(n-1)} \to BU_3 \times (BU_1)^{\times(n-2)} \to \cdots \to BU_n$$
  at each stage it will be a projective bundle and by the projective bundle formula the map from the cohomology of the base to the cohomology of the projective bundle is injective and thus so is this map.

- $f^*(c_i(\gamma_n)) = c_i(f^*\gamma_n) = c_i(pr_1^*\gamma_1 \oplus ... \oplus pr_n^*\gamma_1)$.
  How do these relate to the variables t_i. One should remark that t_i is a generator in the cohomology of the i-th BU_1 and $BU_1 \cong \mathbb{C}P^\infty$. The generator of the cohomology of $\mathbb{C}P^\infty$ is by definition the first Chern class of the tautological line bundle $\mathcal{O}(-1)$, so $t_i = pr_i^*(c_1(\gamma_1))$ (where $\gamma_1 = \mathcal{O}(-1)$).

  Thus we get
  $$f^*(c_i(\gamma_n)) = c_i(f^*\gamma_n) = c_i(pr_1^*\gamma_1 \oplus ... \oplus pr_n^*\gamma_1) \overset{\text{sum formula}}{=} i\text{-th symmetric polynomial}$$
  Perhaps this is easier explained, if we look at the total Chern class
  $$c(\gamma_1^{\oplus n}) = (1+t_1)(1+t_2)\cdots(1+t_n)$$
  If we open the brackets and look at the polynomial at degree $2i$ (because $\deg t_i = 2$) this will be precisely the i-th symmetric polynomial in the variables $t_1, t_2, ..., t_n$.

Now everything follows because the i-th symmetric polynomials in the variables $t_1, t_2, ..., t_n$ generate freely the ring of symmetric polynomials inside the ring $\mathbb{Z}[t_1, ..., t_n]$.
So $H^*(BU_n) \cong \mathbb{Z}[c_1(\gamma_n), c_2(\gamma_n), ..., c_n(\gamma_n)]$ follows from the proven three items by the theorem of symmetric polynomials. ■

Remark 4.13.3.
Cohomology of complex Grassmanians $Gr_n(\mathbb{C}^\infty) = BU_k$ can be computed by using cells only in even dimensions.
Classically by using Schubert cells (which are parametrised by Young diagrams). The relation between this basis and $c_1, ..., c_e$ is non-trivial.

4.14 Splitting principle

In general, there are no relations between $c_1(V), ..., c_r(V)$. But it turns out that there are relations between $c_i(V), c_j(\bigwedge^k V)$ and $c_r(Sym^m V)$.

Theorem 4.14.1 (splitting principle).
*$B \in$ `CW`, $V_1, ..., V_k$ complex vector bundles over B.
Then $\exists X \xrightarrow{f} B$ such that*

1. *$f^* : H^*(B) \to H^*(X)$ is split-injective. (split in the sense of modules)*
2. *f^*V_j is split in line bundles, so is isomorphic to a direct sum of line bundles.*

Corollary 4.14.2.
Let V be of rank r on X, then $c_1(V) = c_1(\bigwedge^r V)$ ($\bigwedge^r V$ is a line bundle)

Proof. of corollary
WLOG we can assume $V \cong L_1 \oplus ... \oplus L_r$ is the direct sum of r line bundles. We can do so because we want to prove some equality between two classes in the cohomology and if we take this equality by some pullback to another cohomology group where this map is split-injective, then if the equality holds after the pullback, it should also hold before the pullback. After the pullback V will be a

direct sum of line bundles.

We can now apply Whitney sum formula to see that

$$c_1(V) = \sum_{i=1}^{r} c_1(L_i)$$

On the other hand, from the isomorphism $V \cong L_1 \oplus ... \oplus L_r$ follows that

$$\bigwedge^r V \overset{Ex}{\cong} L_1 \otimes ... \otimes L_r$$

and again by some exercise

$$c_1(L_1 \otimes ... \otimes L_r) = \sum_{i=1}^{r} c_1(L_i)$$

■

Proof. of splitting principle

Works by the following construction: Let $\begin{matrix} V \\ \downarrow \\ B \end{matrix}$ be a vector bundle of rank n.

Then we can look at the projectivisation of this vector bundle $\mathbb{P}(V) \to B$ and take the pullback:

$$\begin{tikzcd} \pi^*V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d] \\ \mathbb{P}(V) \arrow[r, "\pi"] & B \end{tikzcd}$$

The claim is that we can split off a direct summand which is in fact $\mathcal{O}(-1)$:

$$\pi^*V \cong \overset{\mathbb{P}(V)}{\mathcal{O}}(-1) \oplus V'$$

where V' is of rank $n-1$. This is all we need to prove for the splitting principle because for one vector bundle we can do this construction several times, first for V, then for the left over V' and so on until we get the decomposition in the sum of line bundles. Of course if you already have the decomposition in some line bundles, it does not matter if you take the pullback because you will also have the decomposition there.

So why did we already prove that? We have the map $B \to BU_n$ which corresponds to the vector bundle V, so the pullback of the canonical tautological line bundle is $g^*\gamma_n = V$.
We can also look at the projectivisation $\mathbb{P}(\gamma_n) \cong BU_{n-1} \times BU_1$. By the functoriality of the projective bundle construction the top left spot can be filled in by $\mathbb{P}(V)$:

$$\begin{tikzcd} \mathbb{P}(V) \arrow[r, "g'"] \arrow[d, "\pi"'] & \mathbb{P}(\gamma_n) \cong BU_{n-1} \times BU_1 \arrow[d, "f"] \\ B \arrow[r, "g"'] & BU_n \end{tikzcd}$$

So we get that $(g \circ \pi)^*\gamma_n = \pi^*V$ but this map $g \circ \pi$ factors through $BU_{n-1} \times BU_1$.
Thus

$$f^*\gamma_n = \gamma_{n-1} \oplus \gamma_1$$

and therefore

$$(g \circ \pi)^* \gamma_n = \pi^* V = g'(\gamma_{n-1} \oplus \gamma_1) = \underbrace{g'^* \gamma_1}_{\mathcal{O}(-1)} \oplus g'^* \gamma_{n-1}$$

■

Remark 4.14.3.
Stiefel-Whitney classes can be constructed in absolutely the same way.

There were two main claims in the construction of Chern classes and the proof that the cohomology of BU_n is the way it is. The first claim was that we used the Leray-Hirsch theorem for the projective bundle and that was based on the cohomology of the projective space being generated by the first Chern class of $\mathcal{O}(-1)$.
This is also true for the real projective space with $\mathbb{Z}/2$-coefficients

$$H^*(\mathbb{R}P^n, \mathbb{Z}/2) = \mathbb{Z}/2[a]$$

where $a = w_1(\mathcal{O}(-1))$ with $\deg(a) = 1$.

All the other considerations were completely universal.

CHAPTER 5

Spectral Sequences

5.1 What is a spectral sequence?

Fix an abelian category $\mathcal{A}$, e.g. (and for us almost always) $\mathtt{Ab}$.

Definition 5.1.1.
A sequence of objects $E_r \in \mathcal{A}$, where $r \in \mathbb{Z}$, $r \geq r_0$ for some $r_0 \in \mathbb{Z}$ (most often $r_0 = 0, 1, 2$) together with endomorphisms $d_r : E_r \to E_r$ such that $d_r \circ d_r = 0$ and $E_{r+1} \cong \ker d_r / \operatorname{im} d_r$ is called a <u>spectral sequence</u>.

The object E_r is called the <u>r-th sheet / page</u> and d_r the <u>r-th differential</u>.

Remark 5.1.2.
We will always assume the isomorphisms $E_{r+1} \cong \ker d_r / \operatorname{im} d_r$ to be equalities.

Definition 5.1.3.
A <u>bigraded spectral sequence</u> (E_r, d_r) where every page is bigraded, so

$$E_r \cong \bigoplus_{p,q \in \mathbb{Z}} E_r^{p,q}$$

In that case, the differential is assumed to be bigraded, so it has to change the grading somehow (so be of non-zero degree):
<u>cohomological convention:</u>

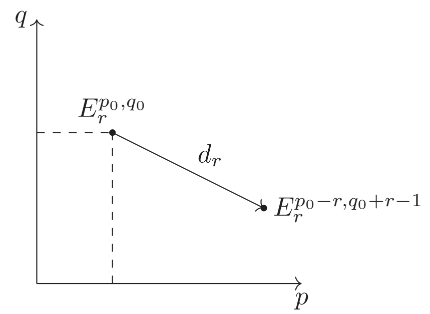
$$d_r^{p,q} := d_r|_{E_r^{p,q}} : E_r^{p,q} \to E_r^{p+r,q-p+1}$$

The isomorphism $E_{r+1} \cong \ker d_R / \operatorname{im} d_r$ is now a bigraded isomorphism, so

$$E_{r+1}^{p,q} \cong (\ker d_r|_{E_r^{p,q}})/(\operatorname{im} d_r \cap E_r^{p,q}) = (\ker d_r|_{E_r^{p,q}})/(\operatorname{im} d_r^{p-r,q+r-1})$$

(and here r_0 matters !)

Bigraded spectral sequences are often drawn as follows:



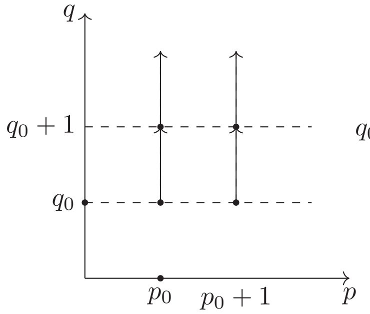
Recall that a point in the E_1-page is given by $E_1^{p_0,q_0} \cong \ker d_0^{p_0,q_0} / \operatorname{im} d_0^{p_0,q_0-1}$.
The general formula for the bigraded differential is

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

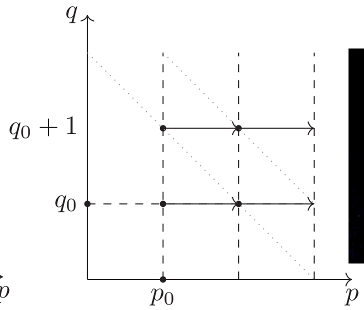
Note that for each page r

$$d_r : \bigoplus_{p+q=n} E_r^{p,q} \to \bigoplus_{p'+q'=n+1} E_r^{p',q'}$$

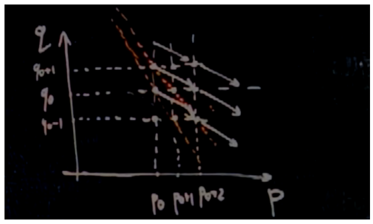
This means that the next point is always on the next diagonal turning clockwise. So it shifts one object at diagonal at a time clockwise.



E_0-page



E_1-page



E_2-page

Definition 5.1.4.
Spectral sequence degenerates (at page m) if $d_r = 0$ for $r \geq m$.

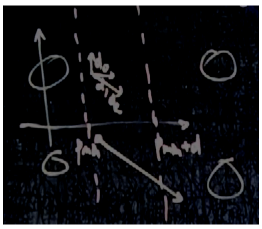
Note that if $d_r = 0$, then $E_{r+1} \cong E_r$. Because the kernel of d_r is the whole E_r and the image is 0. For degenerate spectral sequences we have that the differential is 0 at some point and from there onwards, so the objects do not change anymore and nothing happens anymore.

Lemma 5.1.5.
Assume that for a bigraded spectral sequence $E_{r_0}^{p,q} = 0$

1. *if $p \notin [p_{min}, p_{min} + d]$*
2. *if $q \notin [q_{min}, q_{min} + d]$*

Then the spectral sequence degenerates (at page $d+1$).

Proof.
Case 1)



As soon as d_r goes outside of this non-trivial line (indicated as dashed), it is zero.
You should look at the objects on the line $p_{\min}$ where the differential goes outside of the line $p_{\min} + d$ at the same time, if the differential goes inside an object on the line $p_{\min}+d$, it comes from an object outside of the line $p_{\min}$.
You can calculate when that happens and it should be something like $d + 1$. ■

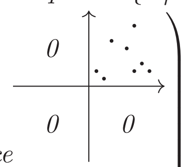
Definition 5.1.6.
A bigraded spectral sequence $\{E_r^{p,q}, d_r\}$ <u>converges</u> to $E_\infty^{p,q}$ if for all (p,q) there exists m such that for $r \geq m$

$$d_r^{p,q} = d_r|_{E_r^m} = 0 \quad d_r^{p-r,q+r-1} = 0$$

(In other words: when we compute cohomology of the r-th page in the point (p,q) nothing happens because the differential going out of it as well as the one going in is 0)
and then $E_\infty^{p,q} \cong E_m^{p,q}$.

Remark 5.1.7.
Convergence is weaker than degeneration.
Degeneration means that at some page everything stops, no differentials appear as they are all zero, everything is stable. Convergence means that if we look at some particular point in this bigraded spectral sequence then starting from some page everything going in and out of it is going to be 0. But for different points the number of page at which this starts converging against the infinite group $E_\infty^{p,q}$ can be different and it does not have to be bounded, so at some higher page things can happen.

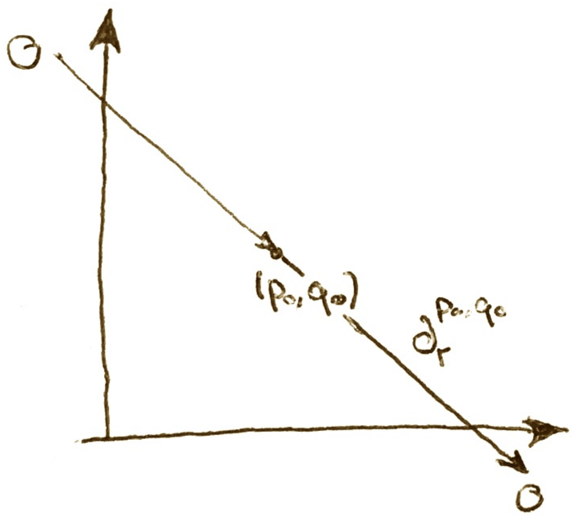
Lemma 5.1.8.
Assume that for a bigraded spectral sequence $\{E_r^{p,q}, d_r\}_{r\geq r_0}$, $E_{r_0}^{p,q} = 0$ unless $p \geq 0$ and $q \geq 0$
(*a <u>first quadrant spectral sequence</u>*



)

Then it converges.

Proof.
You just have to look at how the differentials behave at some particular point (p_0, q_0) - you do not have to do it universally.
At some point, differentials going out of it will go to 0 for sufficiently big r and differentials going to it will also come from 0 for $r \gg 0$.



As soon as it goes out of this quadrant, all the next ones, which will be on the next diagonal (the differential turns clockwise on this diagonal) they will be also out of the quadrant. ■

Definition 5.1.9.
If a spectral sequence $\{E_r^{p,q}, d_r\}$ converges, then $E_\infty^{p,q}$ is called infinite page / sheet.

This is what spectral sequence computes!

A spectral sequence (bigraded, cohomological) is a collection of these pages where each page is such a diagram of abelian groups (/objects in the abelian category) and differentials on this page which are differentials ($d^2 = 0$) and they go in specific directions.
On the next page we put on this sheet the homology of the previous page. We are somehow given the new differentials on this next page which go slightly different and we again compute homology. At some point this process stops (at least locally for each point) and we get the infinite term. Altogether, the infinite term on the whole page is what we compute by spectral sequence.

The reason why they are called pages, is because some people think of spectral sequences as a book. You have the first page with differentials, you turn the page of the book you get the next page.

Typically, one relates E_∞ (bigraded) (which is defined only for convergent bigraded spectral sequences) to some (resp. graded) object $G \in \mathcal{A}$ (in most cases an abelian group) with a descending filtration $\cdots \subseteq F^{s+1} \subseteq F^sG \subseteq \cdots$, $s \in \mathbb{Z}$:

Recall that a a filtration on an abelian group is called exhaustive if $\bigcup_{s\in\mathbb{Z}} F^sG = G$.
It is called Hausdorff if $\bigcap_{s\in\mathbb{Z}} F^sG = 0$ and complete if "Cauchy sequences converge":
A Cauchy sequence consists of $X_n \in G$ such that $\forall s$ $\exists N$ such that $X_n - X_m \in F^sG$ for all $n \geq N$ $\rightsquigarrow$ it converges if $\exists x_0 \in G$ such that $\forall s$ exists N such that $X_n - X_0 \in F^sG$ for $n \geq N$.

For example, if a filtration on G is finite ($F^sG = F^{s+r}G$ for $s \gg 0$, $r > 0$ as well as $F^kG = F^{k-r}$ for $k \ll 0$), then it is automatically complete

$$0 \overset{\text{Hausdorff}}{=} F^{\min}G \subseteq ... \subseteq F^{n+1}G \subseteq F^{\max}G \overset{\text{exhaustive}}{\quad G}$$

Definition 5.1.10.
A bigraded spectral sequence converges strongly to a graded G with a descending filtration F^sG if

- it converges (so the infinite page is defined)
- the filtration on G is complete exhaustive Hausdorff
- there exist isomorphisms

$$E_\infty^{p,q} \cong F^pG^{p+q}/F^{p+1}G^{p+q} = gr_F^p(G^{p+q})$$

If $E^{p,q}$ strongly converges to G one writes $E_r^{p,q} \implies G^{p+q}$. (the sum is due to the fact that $E_r^{p,q}$ at infinity is an subquotient of G^{p+q}).
Recall: $E_\infty^{p,q} \cong F^pG^{p+q}/F^{p+1}G^{p+q}$.

5.2 Examples of Spectral Sequences with Applications (but without constructions)

double complex $C^{\bullet,\bullet}$ in $\mathcal{A}$

A double complex is a diagram of objects, bigraded, with differentials that are invertible and vertical maps

$$\begin{array}{ccccccc} & & \vdots & & \cdots & & \\ & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & C^{p,q} & \xrightarrow{d} & C^{p+1,q} & \longrightarrow & \cdots \\ & & \partial\uparrow & & \partial\uparrow & & \\ \cdots & \longrightarrow & C^{p,q-1} & \xrightarrow{d} & C^{p+1,q-1} & \longrightarrow & \cdots \end{array}$$

$$\begin{aligned} d \circ \partial &= \partial \circ d \\ d^2 &= 0 \\ \partial^2 &= 0 \end{aligned}$$

and let's assume that at each diagonal $\bigoplus_{p+q=n} C^{p,q}$ there are only finitely many non-zero terms.

To the double complex $C^{\bullet,\bullet}$ one can assign its totalisation of a double complex $Tot(C^{\bullet,\bullet})$ which is a complex in $\mathcal{A}$:

$$\begin{aligned} Tot(C^{\bullet,\bullet})^n &= \bigoplus_{p+q=n} C^{p,q} \\ d^n_{Tot} &= \sum_{p+q=n} (d^{p,q} + (-1)^p \partial^{p,q}) \end{aligned}$$

Of course, one has to check that this is in fact a differential. For this, if one regards the composition with the next one, one uses the commutativity of d and ∂, as well as that their composition with itself are 0. Then with some play with the signs all that is left, cancels out.

Theorem 5.2.1.
There exist two strongly convergent bigraded spectral sequences:

1. *The first one has as 0-th term ${}^I E_0^{p,q} = C^{p,q}$ and the differential is given by just forgetting about the horizontal differentials, so $d_0 = \partial$.*
   So the first page is given by forgetting about the horizontal differentials. Then after computing the homology of the vertical differentials you can define a new differential on the cohomology of the vertical arrows.

   ${}^I E_r^{p,q} \implies H^{p+q}(Tot(C^{\bullet,\bullet}))$

2. ${}^{II} E_0^{p,q} = C^{q,p}$, $d_0 = d$
   So this is basically similar to before but just the mirrored image.

   ${}^{II} E_r^{p,q} \implies H^{p+q}(Tot(C^{\bullet,\bullet}))$.
   This is because the totalisation does not depend on the mirroring. What does not stay the same is the filtration one gets.

Example 5.2.2 (Proof of Snake lemma)**.**
We suppose the following commutative diagram to have exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E & \longrightarrow & F & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

We can see it as a double complex. Also we assume that all downward maps are differentials, so we actually have

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, "f_1"] & B \arrow[r] \arrow[d, "f_2"] & C \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & E \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

So if we start by looking at ${}^{II}E^{p,q}$ the 0-th page would be this diagram where we forgot all about the vertical differentials. Then we have to compute the homology with respect to the horizontal differentials. These are, of course 0, because the rows are exact.
Thus the first page

$${}^{II}E_1^{p,q} = 0 \quad \forall p, q$$

In particular, because it strongly converges against the homology of the totalisation, it means that

$$H^n(Tot(C^{\bullet,\bullet})) = 0 \quad \forall n$$

This group should have a complete, exhaustive Hausdorff filtration with the graded quotients being ${}^{II}E_1^{p,q}$ which are 0. Therefore the filtration is 0 because if the graded filtration is 0 then so is the filtration.

Now look at ${}^{I}E_1^{p,q}$ the first page of the first spectral sequence. This looks on objects like

$$\begin{array}{ccc}
\ker(f_1) & \ker(f_2) & \ker(f_3) \\
 & & \\
\operatorname{coker}(f_1) & \operatorname{coker}(f_2) & \operatorname{coker}(f_3)
\end{array}$$

So how does the differential look like? The differential on the first page comes from the other differential, the one we haven't used to compute homology, so it comes from the horizontal arrows. We now get as the first page ${}^{I}E_1^{p,q}$:

$$\begin{tikzcd}
\ker(f_1) \arrow[r, "d_1"] & \ker(f_2) \arrow[r, "d_2"] & \ker(f_3) \\
\operatorname{coker}(f_1) \arrow[r, "d_1"] & \operatorname{coker}(f_2) \arrow[r, "d_2"] & \operatorname{coker}(f_3)
\end{tikzcd}$$

Now we compute the second page of the same spectral sequence. We do not compute the objects, but only want to point out, where the differentials go

$$\begin{array}{ccc}
\bullet & \bullet & \bullet \\
 & & \\
\bullet & \bullet & \bullet
\end{array}$$

Again, you either have to check through old notation or you have to remember the rule over thumb that the differential always goes to the next diagonal and goes clockwise (here, however, counter-clockwise because we have mirrored at some point).
So on the 0-th page, the differential started by going down, then it went to being horizontal, so now it will go up on the next horizontal:

$$\begin{tikzcd}
\bullet & \bullet & \overset{(p_0+2,q_0-1)}{\bullet} \\
\underset{(p_0,q_0)}{\bullet} \arrow[urr, "d_2"] & \bullet & \bullet
\end{tikzcd}$$

(same actually for the other objects, but they go to 0. One can also see from here, that $d_r = 0$ for $r \geq 3$.

Because $H^n(Tot(C^{\bullet,\bullet})) = 0$, so ${}^I E_\infty^{p,q} = 0$ for all p, q. But ${}^I E_\infty^{p,q} = E_2^{p,q}$ if (p, q) is neither (p_0, q_0) nor $(p_0 + 2, q_0 - 1)$. Thus all other dots have to be 0s.

$$\begin{tikzcd}
0 & 0 & \underset{}{\bullet}^{(p_0+2,q_0-1)} \\
\underset{(p_0,q_0)}{\bullet} \arrow[urr, "d_2"] & 0 & 0
\end{tikzcd}$$

What about these two left-over dots? Starting from the next page all differentials will be 0 and what's going to be on the next page is the homology of this differential d_2. Thus both, the kernel and the cokernel of the map d_2 have to be 0 which means that d_2 is an isomorphism and

$$E_\infty^{p_0,q_0} = E_3^{p_0,q_0} = \ker d_2 = 0 = \operatorname{coker} d_2 = E_3^{p_0+2,q_0-1} = E_\infty^{p_0+2,q_0-1} \implies d_2 \text{ is an iso}$$

This means that

$$\ker f_1 \to \ker f_2 \to \ker f_3$$

is exact on the left,

$$\operatorname{coker} f_1 \to \operatorname{coker} f_2 \to \operatorname{coker} f_3$$

is exact on the right and

$$\operatorname{coker} d_1 = \ker f_3 / \ker f_2 \xleftarrow[\cong]{d_3} \ker d_1 := (\operatorname{coker} f_1 \to \operatorname{coker} f_2)$$

This glues to the "snake" exact sequence.

Exercise:
To obtain the long exact sequence for homology of a short exact sequence of complexes.

Čech spectral sequence

Theorem 5.2.3.
Let $\{U_i\}_{i\in I}$ be an open covering of a CW-complex X, such that $U_{i_1} \cap \cdots \cap U_{i_k} = \emptyset$ for $i_1 \neq \cdots \neq i_k$ for $k \gg 0$. (Assume I to be ordered)
Then there exists a strongly convergent spectral sequence (and in this case it does not start with a 0-th but rather a first page):

$$E_1: \ q\text{-th row} \overset{p=0}{\bigoplus_{i\in I}} H^q(U_i) \xrightarrow{d_2} \overset{p=1}{\bigoplus_{i<j}} H^q(U_{ij}) \to \overset{p=2}{\bigoplus_{i<j<k}} H^q(U_{ijk}) \to \cdots$$

$$E_r^{p,q} \implies H^{p+q}(X)$$

then $d_1(\{x_i\})_{kj} = x_i|_{U_{ij}} - x_j|_{U_{ij}} \in H^q(U_{ij})$

Remark 5.2.4.
Mayer-Vietoris is a particular case of Čech spectral sequence: $|I| = 2$.
Then $E_1^{p,q} = 0$ for $p > 1$.

E_1:

$$H^q(U) \oplus H^q(V) \longrightarrow H^q(U \cap V)$$

$$H^{q-1}(U) \oplus H^{q-1}(V) \longrightarrow H^{q-1}(U \cap V)$$

E_2:

$$\begin{tikzcd}
 & & 0 \\
\ker_q \arrow[urr] & \mathrm{coker}_q & 0 \\
\ker_{q-1} \arrow[urr] & \mathrm{coker}_{q-1} &
\end{tikzcd}$$

thus if $X = U \cup V$ this spectral sequence degenerates at page 2. There is a filtration of two steps of $H^{p+q}(X)$:

$$0 \to \mathrm{coker}_{q-1} \to H^q(X) \to \ker_q \to 0$$

For two quotients of the filtration of $H^{p+q}(X)$ they both come from the same diagonal of the spectral sequence. So what appears on this diagonal will go to the same cohomology group of X, namely $H^q(X)$.
Regarding the map $H^q(X) \to \ker_q$: $\ker_q$ is a subgroup in $H^q(U) \oplus H^q(V)$. So this map is the usual restriction map. The map $H^q(X) \to H^q(U) \oplus H^q(V)$ is surjective on the kernel and the kernel of the map $H^q(X) \to \ker_q \to 0$ is precisely the cokernel coker_{q-1} of $H^{q-1}(U) \oplus H^{q-1}(V)$.
If you glue that all together, you get the Mayer-Vietoris sequence.
In general, if you have a covering with more than two covering opens, you get the Čech spectral sequence.

Remark 5.2.5.
Suppose that $U_{i_1,\ldots,i_k}$ is contractible or empty for all k (a good covering), then

$$H^q(X) = H^q(\bigoplus_i \mathbb{Z}\pi_0 U_i \to \bigoplus_{i<j} \mathbb{Z}\pi_0 U_{ij} \to \cdots \to \bigoplus_{i_1<\ldots<i_k} \mathbb{Z}\pi_0 U_{i_1\ldots i_k} \to \cdots)$$

In particular, if $H^q(X) = 0$, then there exists no good cover of cardinality $< q+1$.

$H^n(S^n) = \mathbb{Z}$, so S^n cannot be covered with n opens U_i such that $U_{i_1,\ldots,i_k}$ is contractible or empty for all $i_1 < \ldots < i_k$.

Serre Spectral Sequence

Theorem 5.2.6.

Let B be a simply connected CW-complex and let $\begin{array}{c} X \\ \downarrow \\ B \end{array}$ be a Serre fibration with fiber F over some point b.
Then for every abelian group A, there exist two strongly convergent spectral sequences (one is for homology, the other for cohomology), $r \geq 1$:

- *homological:*

$$E^r_{p,q} \implies H_{p+q}(X, A), \quad d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$$

- *cohomological:*

$$E_r^{p,q} \implies H^{p+q}(X, A), \quad d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

$$E^2_{p,q} = H_p(B, H_q(F, A)) \qquad E_2^{p,q} = H^p(B, H^q(F, A))$$

Remark 5.2.7.
These are first quadrant spectral sequences because either homology or cohomology $(H^p(\cdots)H^q(F, A) = 0)$ is 0 unless $p \geq 0$, $q \geq 0$

Remark 5.2.8.
If $\begin{array}{c} X \cong B \times F \\ {\scriptstyle pr}\downarrow \\ B \end{array}$ and A is a PID, then

$$H_m(X, A) \overset{\text{Künneth}}{\cong} \bigoplus_{p+q=m} H_p(B, A) \otimes_A H_q(F, A) \bigoplus_{p'+q'=m-1} Tor_1(H_{p'}(B, A), H_{q'}(F, A))$$

We can also compute what is happening on the second page:

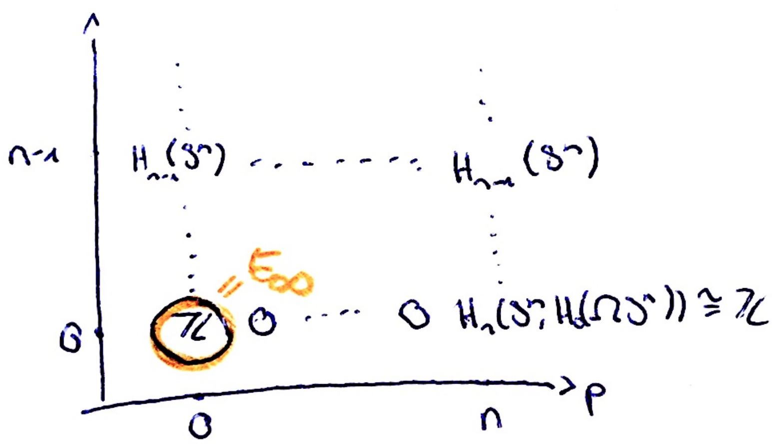
$$H_p(B, H_q(F, A)) \cong H_p(B, A) \otimes_A H_q(F, A) \oplus Tor_1^A(H_p(B, A), H_q(B, A))$$

Thus E_r degenerates at the second page.
In this way, the spectral sequence tells you, how the homology of the total space of a general Serre fibration differs from the homology of the trivial Serre fibration, because on the second page what stands there is absolutely the same for both cases but in one case you can show that all differentials become 0 and you get basically the Künneth formula. If the Serre fibration is non-trivial there are non-trivial differentials that kill some parts of the homology, so something non-trivial happens.

Example 5.2.9 (of application)**.**
Consider the Serre fibration of paths in a sphere: $\begin{array}{c} X = \Pi_* S^n \\ \downarrow \\ * \in S^n \end{array}$ with fiber over $*$ being the loop space $\Omega_* S^n$.
We know that $X \simeq *$, but we don't know nothing about the homology of the loop space.
If $n = 1$, $\Omega_* S^1 \cong \coprod_{\pi_1(S^1)} *$.
If $n \geq 2$, then S^n is simply-connected and we can apply Serre spectral sequence. In that we have on the second page

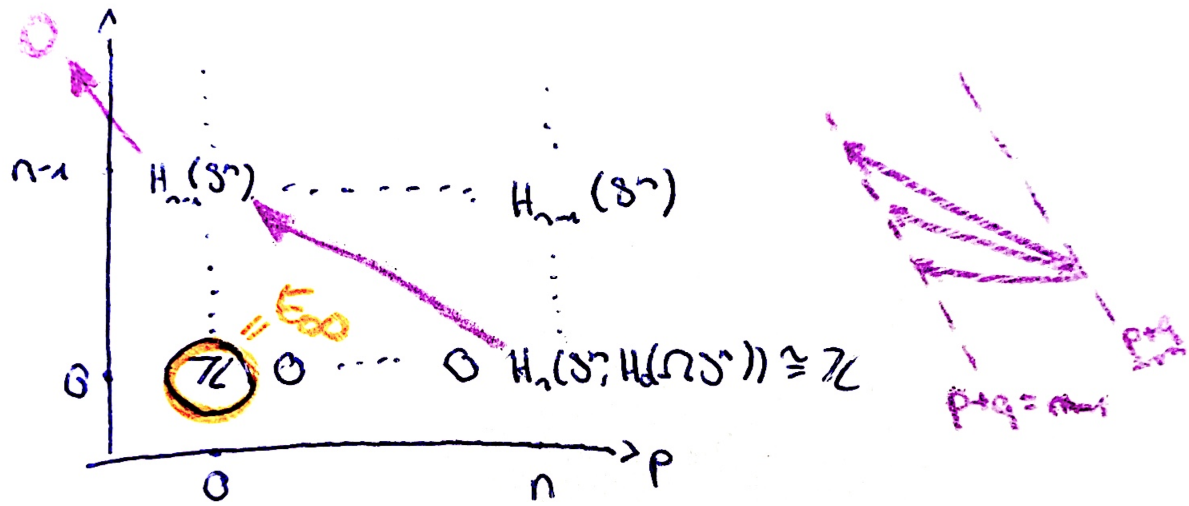
$$E^2_{p,q} = H_p(S^n, H_q(\Omega S^n)) \neq 0$$

only for $p = 0, n$ because you can compute the homology of S^n with any coefficients using for example the universal coefficient theorem and because it is the homology of S^n with free abelian groups as coefficients, there will be no Tor-groups appearing, so this is just the product of $H_p(S^n)$ with $H_q(\Omega S^n)$



This strongly converges to $H_*(\Pi S^n) = \mathbb{Z}$ in degree 0.
Therefore $E^\infty_{p,q} = 0$ unless $(p,q) = 0$.

But now, because there are only two columns here, there are not so many differentials that can possibly be non-trivial. First, how do differentials go in this homological spectral sequence? Well, they go from the diagonal $p+q=m$ to the previous diagonal $p+q=m-1$, going one object at the time and also turning clockwise.
Only something interesting can appear when the differential can go from the n-th column to a group in the 0-th column. So if we want to go from the group $H_n(S^n, H_0(\Omega S^n)) \cong \mathbb{Z}$ which lives on the $0+n=n$-th diagonal, we have to go to the $n-1$-th diagonal. The only element of the 0-th column lying on this diagonal, however, is $H_{n-1}(\Omega S^n)$.



Correction: in the $(n-1)$-th row, instead of the homology of S^n, there should be the homologies of the loop space ΩS^n. Also the indicated diagonal should be labeled d^n.

The only non-trivial differentials can happen on $E^n_{p,q}$ and they go

$$\begin{tikzcd} \overset{\text{total degree } n+m-1}{H_{n+m-1}(\Omega S^n)} & & \\ & \underset{\text{total degree } n+m}{H_m(\Omega S^n)} \arrow[ul, "d^n"'] & \cong H_n(S^m, H_m(\Omega S^n)) \end{tikzcd}$$

Therefore it is always an isomorphism.
So no differential goes into $H_k(\Omega S^n)$ for $0 < k < n-1$. But if nothing goes in there and this is the only non-trivial differential that can happen, this group should survive to the infinite page E^∞. But we have said, that this infinite page only contains the group $\mathbb{Z}$ in $(0,0)$, thus $H_k(\Omega S^m) = 0$ for $0 < k < n-1$.
But there is also the group $H_n(S^m, H_0(\Omega S^n)) \cong \mathbb{Z}$ in $(n,0)$ which should also vanish in the infinite page. The only way for this to vanish, however, is for the differential d^n to have no kernel, thus is injective.
Therefore

$$\mathbb{Z} \cong H_0(\Omega S^n) \xrightarrow{d^n} H_{n-1}(\Omega S^n)$$

has to be an iso, so that $E^\infty_{0,n-1} = E^\infty_{n,0} = 0$, thus $H_{n-1}(\Omega S^n) \cong \mathbb{Z}$.

Similarly, $H_k(\Omega S^n) = 0$ for $n-1 < k < 2(n-1)$ and $H_{2(n-1)}(\Omega S^n) \cong \mathbb{Z}$.

Doing the same indefinitely we get

$$H_n(\Omega S^n) = \begin{cases} \mathbb{Z} & m = k(n-1),\ k \geq 0 \\ 0 & \text{else} \end{cases}$$

5.3 Construction and convergence of spectral sequences

Spectral sequences rise from exact couples

Definition 5.3.1.
An exact couple in an abelian category $\mathcal{A}$ is (A, E, i, j, k) that form the triangle

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

where every corner is exact, i.e. $\ker i = \operatorname{im} k$ and so on.

To an exact couple one can associate

1. a differential on E which is

   $$d := j \circ k : E \to E$$

   it is a differential because

   $$d \circ d = j \circ \underbrace{k \circ j}_{0} \circ k = 0$$

2. a derived couple (A', E', i', j', k')

   $$E' := \ker d / \operatorname{im} d$$
   $$A' := \operatorname{im} i = \ker j$$

   $$\begin{tikzcd} i' : A' \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & A' \arrow[d, hook] \\ A \arrow[r, "i"] & A \end{tikzcd}$$

   i.e. $i'(a) = i(a) \overset{?}{\in} A' = \operatorname{im} i = \ker j$, but $j(i(a)) = 0$, hence $i(a) \in A'$.
   To define $j' : A' \to E'$. This is not just the restriction of the map j but a bit more complicated.
   Take $a \in A'$ such that $a = i(a')$. Define $j'(a) := [j(a')]$ (denotes the canonical map $[-] : \ker j \twoheadrightarrow E'$). To check is that

   - $j(a') \in \ker(d)$. $d(j(a')) = j \circ k \circ j(a') = 0$
   - we have chosen some a' such that $i(a') = a$, but we could have chosen another one, so $j(\ker i) = j(\operatorname{im} k) = \operatorname{im} d$. Now $[j(a')] = [j(a'')]$ if $a' - a'' \in \ker i$.

   Define $k' : E' \to A'$ by $[e] \mapsto k(e)$ where $e \in \ker d$. Check:

   - $k(e) \in A' = \ker j$
     $j(k(e)) = d(e) = 0$
   - $k(\operatorname{im} d) = 0$: $k \circ d = k \circ j \circ k = 0$

Lemma 5.3.2.
A derived couple of an exact couple is an exact couple.

Proof. by diagram chasing:
We have the original exact couple and the new couple:

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

We won't do all parts of the exactness, but for example in one corner one checks:

- $j' \circ i' = 0$:
  Let $a \in A'$ such that $a = i(a')$, then
  $$j'(i'(a)) = j'(i(a)) = [j(a)] = [j(i(a'))] = [0] = 0$$
- $\ker j' \subset \operatorname{im} i'$:
  Let $a \in A'$ be in the kernel, so $j'(a) = 0$. a is again $a = i(a')$ for some $a' \in A'$ and $j'(a) = [j(a')]$ thus $j(a') \in \operatorname{im} d$. By definition $j(a') = d(e) = j(k(e))$.
  Consider $a' - k(e)$. It is in $\ker j = \operatorname{im} i$. Thus there is a'' such that $i(a'') = a' - k(e)$.
  Now $a = i(a') = i(i(a'') + k(e)) = i^2(a'')$.
  $\operatorname{im} i' = i(A') = i^2(A')$. Thus we are finished.

■

Corollary 5.3.3.
If

$$\begin{tikzcd} A_{r_0} \arrow[rr, "i_{r_0}"] & & A_{r_0} \arrow[dl, "j_{r_0}"] \\ & E_{r_0} \arrow[ul, "k_{r_0}"] & \end{tikzcd}$$

is an exact couple for some $r_0 \in \mathbb{Z}$, then we define for $r \geq r_0$

$$\begin{aligned} E_{r+1} &:= E'_r \\ A_{r+1} &:= A'_r \\ i_{r+1} &= i'_r \text{ and so on...} \end{aligned}$$

$\{E_r, d_r = j_r \circ k_r\}_{t \geq r_0}$ is a spectral sequence.

Obtaining (bigraded) exact couples

Let $\cdots \to X_p \to X_{p+1} \to \cdots$ be a sequence of CW-complexes, $p \in \mathbb{Z}$.
Then define

$$\begin{aligned} A^1 &:= \bigoplus_{n,p} A^1_{n,p} := \bigoplus_{n,p} H_n(X_p) \quad \text{(with some coefficients)} \\ E^1 &:= \bigoplus_{n,p} E^1_{n,p} := \bigoplus_{n,p} \tilde{H}_n(X_p/X_{p-1}) \end{aligned}$$

where the latter, X_p/X_{p-1}, is the homotopy quotient given by

$$\begin{tikzcd} X_{p-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X_p \arrow[d] \\ * \arrow[r] & X_p/X_{p-1} \end{tikzcd}$$

Also, define the maps:

$$i^1 : \underbrace{H_n(X_{p-1})}_{=A^1_{n,p-1}} \xrightarrow{\alpha_*} \underbrace{H_n(X_p)}_{=A^1_{n,p}}$$

$$j^1 : \underbrace{H_n(X_p)}_{=A^1_{n,p}} \xrightarrow{\beta_*} \underbrace{H_n(X_p/X_{p-1})}_{=E^1_{n,p}}$$

$$k^1 : \underbrace{H_n(X_p/X_{p-1})}_{=E^1_{n,p}} \xrightarrow{\gamma_*} H_n(\Sigma X_{p-1}) \cong \underbrace{H_{n-1}(X_{p-1})}_{=A^1_{n-1,p-1}}$$

Where we have used:

$$\begin{tikzcd} X_{p-1} \arrow[r, "\alpha"] \arrow[d] & X_p \arrow[r] \arrow[d, "\beta"] \arrow[dl, phantom, "\square"] & * \arrow[d] \arrow[dl, phantom, "\square"] \\ * \arrow[r] & X_p/X_{p-1} \arrow[r, "\gamma"] & \Sigma X_{p-1} \end{tikzcd}$$

Puppe LES for homology implies that $\begin{tikzcd} A^1 \arrow[rr, "i^1"] & & A^1 \arrow[dl, "j^1"] \\ & E^1 \arrow[ul, "k^1"] & \end{tikzcd}$ is an exact couple.

Proposition 5.3.4.
These give rise to a bigraded spectral sequence $(E^r_{n,p}, d_r)$ with

$$E^r_{n,p} \xrightarrow{d_r} E^r_{n-1,p-r}$$

and in the derived exact couples

$$i^r : A^r_{n,p} \to A^r_{n,p+1}$$

The latter is actually really easy to see because the map i^1 does absolutely the same as i^r and this does not change by derivation because that is just the restriction to some subgroups. So it has exactly the same grading.

Notation 5.3.5.
Renumber: $\tilde{E}^r_{p,q} := E^r_{p+1,p} \xrightarrow{d_r} E^r_{p+q-1,p-r} = \tilde{E}^r_{p-r,q+r-1}$
This is the "standard" homological convention for the differentials.

Strong Convergence of the above constructed spectral sequence

We need to make more assumptions (then later one can regard what we will do in more general settings than just with the above constructed sequence):

1. For each n, $E^1_{n,p} \neq 0$ only for finitely many p's.
   $\Leftrightarrow$ $E^1_{p,q}$ has only finitely many non-zero groups on the diagonal $n = p + q$.

   In the exact couple obtained above $\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$, the LES looks like

   $$\cdots \to A^1_{n+1,p} \to E^1_{n+1,p} \to A^1_{n,p-1} \to A^1_{n,p} \to E^1_{n,p} \to \cdots$$

   if $(*)$: $E^1_{n+1,p} = 0 = E^1_{n,p}$, then $A^1_{n,p-1} \xrightarrow{\cong} A^1_{n,p}$.

   Consider a sequence $(**)$

   $$\cdots \to A^1_{n,p-1} \to A^1_{n,p} \to A^1_{n,p+1} \to \cdots$$

$$\left.\begin{array}{l}\text{If } p \gg 0, \text{ then } (*) \text{ holds}\\ \text{If } p \gg 0, \text{ then } (*) \text{ holds}\end{array}\right\} \implies \text{in } (**) \text{ there is a maximal value } A^1_{n,+\infty} \text{ and minimal values}$$

$A^1_{n,-\infty}$, so there are stable values on the right and on the left.

2. $A^1_{n,-\infty} = 0$. Define the increasing, exhaustive, finite Hausdorff filtration on $A^1_{n,\infty}$.

$$F^p A^1_{n,\infty} = \operatorname{im}(A^1_{n,p} \to \cdots \to A^1_{n,\infty})$$

Theorem 5.3.6.
Assume in the in 3.2 constructed exact couples (and spectral sequences)

1. *For each n, $E^1_{n,p} \neq 0$ only for finitely many p's.*
2. *$A^1_{n,-\infty} = 0$ for all n*

Then the spectral sequence $E^r_{n,p} \implies A^1_{n,\infty}$ strongly converges with filtration F^p defined as $F^p A^1_{n,\infty} = \operatorname{im}(A^1_{n,p} \to A^1_{n,\infty})$. I.e. there exists isomorphisms

$$E^\infty_{n,p} \cong F^p A^1_{n,\infty}/F^{p+1} A^1_{n,\infty}$$

where $E^1_{n,p} = H_n(X_p/X_{p-1})$ and $A^1_{n,p} = H_n(X_p)$.

Proof.

- $E^r_{n,p}$ converges (i.e. $E^\infty_{n,p}$ for the same reason as last time using condition 1. (last time we have talked about spectral sequences lying in some strip or in the first quadrant but of the essence actually was that on each diagonal there were only finitely many non-trivial values. E.g. for $\tilde{E}^1_{p,q}$ there are only finitely many non-zero values on each diagonal))
- Consider the LES of derived couples

  $$A^r_{n,p+r-2} \xrightarrow{i} A^r_{n,p+r-1} \to E^r_{n,p} \to A^r_{n-1,p-1} \to A^r_{n-1,p} \to E^r_{n-1,p-r+1} \to \cdots$$

  Fix n, p and let r grow. For sufficiently big r that $E^r_{n-1,p-r+1} = 0$ because for fixed n, p the second index grows to $-\infty$.
  By the second condition

  $$A^r_{n-1,p-1} = \operatorname{im}(A^2_{n-1,p-1-r} \twoheadrightarrow A^2_{n-1,p-r} \to \cdots)$$

  hence $A^r_{n-1,p-1} = 0$ for $r \gg 0$.

  Then $E^r_{n,p} = A^r_{n,p+r-1}/A^r_{n,p+r-2} = i^{n-1}(A^1_{n,p})/i^r(A^1_{n,p-1})$, so when $E^\infty_{n,p} = F^p A^1_{n,\infty}/F^{p-1} A^1_{n,\infty}$.

■

Example 5.3.7.
Suppose $X_p = \emptyset$ for $p < 0$, let

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots$$

be CW-subcomplexes, $X = \operatorname{colim}_p X_p$ be a CW complex.
Then

1. $H_n(X_p/X_{p-1}) \neq 0$ for fixed n and only finally many p's

is satisfied if $X_p \hookrightarrow X_{p+1}$ is $f(p)$-connected $f(p) \to \infty$ for $p \to \infty$ (e.g. if $f(p)$ is linear in p) and

2. $A^1_{n,-\infty} = H_n(X_p) = 0$ for $p \ll 0$

is satisfied .

Then $A^1_{n,\infty} = \operatorname{colim}_p H_n(X_p) = H_n(X)$, so the spectral sequence strongly converges to $H_n(X)$.

Remark 5.3.8.
If we take X_p to be the p-th skeleton of X, then $E^1_{p,p} \to E^1_{p-1,p-1} \to E^1_{p-2,p-2} \to \cdots$ can be identified with the cellular complex of X.

Then this spectral sequence actually degenerates at the second page and computes the homology by using the cellular complex.

Quick recap of last lecture:

1. given a sequence of CW-complexes $\cdots \to X_p \to X_{p+1} \to \cdots$ we constructed a spectral sequence $E^r_{p,q}$, $d_r : E^r_{p,q} \to E^r_{p-r,q+r+1}$ and $E^1_{p,q} \cong \tilde{H}_{p+q}(X_p/X_{p-1}) \xrightarrow{d_1} \tilde{H}_{p+q-1}(X_{p-1}/X_{p-2})$. This map d_1 was defined via the exact couple

$$\begin{tikzcd} E^1_{p,q} \cong \tilde{H}_{p+q}(X_p/X_{p-1}) \arrow[rr, "d_1"] \arrow[dr] & & \tilde{H}_{p+q-1}(X_{p-1}/X_{p-2}) \\ & H_{p+q}(\Sigma X_{p-1}) \cong H_{p+q-1}(X_{p-1}) \arrow[ur] & \end{tikzcd}$$

   This is a very simple setting in which we get a spectral sequence. The only thing is, though, we have no idea of what it computes. It computes something, under some assumptions on the CW-complexes it might as well degenerate and have thus an infinite page, but we do not know what that in turn relates to.

2. if furthermore is assumed that $X_{-1} = \emptyset$, $X_p \hookrightarrow X_{p-1}$ is p-connected and is an inclusion of a CW-subcomplex. Let $X = \operatorname{colim}_p X_p$.
   Then $E^1_{p,q} = 0$ unless $p \geq 0, q \geq 0$ (for p this is obvious, for q note that $\tilde{H}_{<p}(X_p/X_{p-1}) = H_{<p}(X_p, X_{p-1}) = 0$ because $X_{p-1} \hookrightarrow X_p$ is $(p-1)$-connected), so this is a first quadrant spectral sequence.
   It comes from an exact couple, where we had

$$\begin{aligned} A^1_{n,-\infty} &\overset{p\leqslant 0}{\cong} H_n(X_p) \overset{p\leqslant 0}{\cong} 0 \\ A^1_{n,\infty} &\overset{p\geqslant 0}{\cong} H_n(X_p) = H_n(X) \end{aligned}$$

   so this spectral sequence that we have defined for arbitrary CW-complexes strongly converges

$$E^r_{p,q} \implies H_{p+q}(X)$$

   with the filtration given by images of $H_{p+q}(X_p) \to H_{p+q}(X)$.

Functoriality

Lemma 5.3.9.
Assume that we have a commutative up to homotopy diagram of CW-complexes

$$\begin{tikzcd} \cdots \arrow[r] & Y_p \arrow[r] \arrow[d, "f_p"] & Y_{p+1} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & X_p \arrow[r] & X_{p+1} \arrow[r] & \cdots \end{tikzcd}$$

One could also assume that it commutes (without the homotopies) because the homotopies will not matter.

Then there exist maps

$$^{Y}E^r_{p,q} \to {}^{X}E^r_{p,q}$$

that commute with differentials, reduces the map on E^{r+1} *and on* $r = 1$ *this is the canonical map on the quotient*

$$H_{p+q}(Y_{p+1}/Y_p) \to H_{p+q}(X_{p+1}/X_p)$$

And moreover, if 2. is satisfied (so the assumptions for strong convergence) for X *and for* Y*, then*

$$H_{p+q}(Y) \to H_{p+q}(X)$$

preserves the filtration and on the graded factors this is the map $^{Y}E^\infty_{p,q} \to {}^{X}E^\infty_{p,q}$

Proof. (Idea)
There is a morphism between the exact couples

$$\begin{tikzcd} {}^{Y}A \arrow[rr] & & {}^{Y}A \arrow[dl] \\ & E \arrow[ul, leftarrow] & \end{tikzcd} \qquad \begin{tikzcd} {}^{X}A \arrow[rr] & & {}^{X}A \arrow[dl] \\ & E \arrow[ul, leftarrow] & \end{tikzcd}$$

given by

$$^{Y}A \to {}^{X}A$$
$$^{Y}E \to {}^{X}E$$

which commute with structure maps.
Out of this morphism of exact couples you get a morphism on the derived exact couples and so you continue to prove the first part of the lemma.
For the second part you have to look through the proof of how the filtration here is related to this exact couple to get the claim. ■

5.4 Serre spectral sequence

Theorem 5.4.1.

Let B *be a simply-connected CW-complex, let* $\begin{array}{c} X \\ \big\downarrow{\scriptstyle \pi} \\ B \end{array}$ *be a Serre fibration with (homotopy) fiber* F *which we assume to be a CW-complex and* X *a CW-complex.*
Then there exist a strongly convergent spectral sequence $E^r_{p,q} \implies H_{p+q}(X, A)$, $r \geq 1$ *with* $d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ *and* $E^2_{p,q} \cong H_p(B, H_q(F, A))$ *(where* A *is some coefficient group we will ignore).*

Moreover, it is functorial:
If

$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ B' \arrow[r] & B \end{tikzcd}$$

is a pullback diagram with B' *simply-connected, then there are maps* $'E^r_{p,q} \to E^r_{p,q}$ *that is on* $r = 2$ *the canonical map* ($F' = F$) *and which are compatible with* $H_{p+q}(X') \to H_{p+q}(X)$ *that preserves filtrations (morphism of spectral sequences).*

Proof. (sketch)

- construction
  Let $\cdots \hookrightarrow B_p \hookrightarrow B_{p+1} \hookrightarrow \cdots$ be the skeletal filtration on B, let $X_p := \pi^{-1}(B_p) \subset X$, so this is a filtration $\cdots \hookrightarrow X_p \hookrightarrow X_{p+1} \hookrightarrow \cdots$ inside X which satisfies the assumptions that $X_{-1} = \emptyset$ and $X = \operatorname{colim}_p X_p$.

- $X_p \hookrightarrow X_{p+1}$ is p-connected.
  Since $B_p \to B_{p+1}$ is part of a skeletal filtration, B_{p+1} is obtained from B_p by attaching cells

$$
\begin{tikzcd}
\coprod S^p \arrow[rr] \arrow[dr] & & \coprod D^{p+1} \arrow[dr] & \\
& B_p \arrow[rr] \arrow[urr, phantom, "\square"] & & B_{p+1}
\end{tikzcd}
$$

Over B_{p+1} we get a Serre fibration with total space X_{p+1} and can then take the pullback over B_p

$$
\begin{tikzcd}
& X_p \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\square"] & & X_{p+1} \arrow[dd] \\
\coprod S^p \arrow[rr] \arrow[dr] & & \coprod D^{p+1} \arrow[dr] & \\
& B_p \arrow[rr] \arrow[urr, phantom, "\square"] & & B_{p+1}
\end{tikzcd}
$$

We can also take the pullback over the side faces, so the spheres and disks

$$
\begin{tikzcd}
\coprod \tilde{S}^p \arrow[rr] \arrow[dr] \arrow[dd] \arrow[ddr, phantom, "\square"] & & \coprod \tilde{D}^{p+1} \arrow[dr] \arrow[dd] \arrow[ddr, phantom, "\square"] & \\
& X_p \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\square"] & & X_{p+1} \arrow[dd] \\
\coprod S^p \arrow[rr] \arrow[dr] & & \coprod D^{p+1} \arrow[dr] & \\
& B_p \arrow[rr] \arrow[urr, phantom, "\square"] & & B_{p+1}
\end{tikzcd}
$$

So all side faces of this cube are by definition pullbacks and since $X_{p+1} \to B_{p+1}$ is a Serre fibration, they are also homotopy pullbacks.
By the 2nd Mathe Cube Theorem, we get that the top face is a homotopy pushout:

$$
\begin{tikzcd}
\coprod \tilde{S}^p \arrow[r] \arrow[d] & \coprod \tilde{D}^{p+1} \arrow[d] \\
X_p \arrow[r] & X_{p+1}
\end{tikzcd}
$$

But $\tilde{D}^{p+1} \to D^{p+1}$ is a Serre fibration, therefore the inclusion of the fiber

$$
\begin{tikzcd}
\tilde{D}^{p+1} \arrow[d] & F \arrow[l, hook', "i"'] \arrow[dl] \\
D^{p+1} &
\end{tikzcd}
$$

is a homotopy equivalence because D^{p+1} is contractible, for example you can look at the LES of homotopy groups of a Serre fibration to see that this map is a weak equivalence and thus because we work with CW-complexes this is a homotopy equivalence.
Thus $\tilde{D}^{p+1}$ is nothing but the fiber F, so what about $\tilde{S}^p$. It is obtained as a pullback in

$$\begin{tikzcd}
\tilde{S}^p \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \tilde{D}^{p+1} \arrow[d] & F \arrow[l, hook', "i"'] \arrow[dl] \\
S^p \arrow[r] & D^{p+1} &
\end{tikzcd}$$

Again, it is also a homotopy pullback. Therefore, to compute it, we can replace $\tilde{D}^{p+1}$ by something homotopy equivalent.
To get out of this map $F \to D^{p+1}$ a fibration in order to be able to compute the homotopy pullback as a pullback, we can replace F by $F \times D^p$. This is possible because D^p is contractible and thus $F \times D^p$ is again homotopy equivalent to $\tilde{D}^{p+1}$. Of the map i one can think as the projection onto F and then the inclusion of the fiber (it does not really matter).
So $\tilde{S}^p$ is homotopy equivalent to $F \times S^p$:

$$\begin{tikzcd}
\tilde{S}^p \arrow[r, "heq", "\simeq"'] \arrow[dr] & F \times S^p \arrow[r] \arrow[d] & F \times D^p \arrow[r, "\simeq"] \arrow[d] & \tilde{D}^{p+1} \arrow[dl] \\
& S^p \arrow[r] & D^{p+1} &
\end{tikzcd}$$

Thus

$$\begin{tikzcd}
\coprod \tilde{S}^p \arrow[rr] \arrow[dd] \arrow[dr, equal] & & \coprod \tilde{D}^{p+1} \arrow[dd] \\
& F \times S^p \hookrightarrow F \times D^{p+1} \arrow[ur, phantom, "\simeq"] \arrow[dr, phantom, "\square"] & \\
X_p \arrow[rr] & & X_{p+1}
\end{tikzcd}$$

and $F \times S^p \hookrightarrow F \times D^{p+1}$ is p-connected. Thus $X_p \hookrightarrow X_{p+1}$ is p-connected.

Hence we obtain by the last lecture a spectral sequence

$$\begin{aligned}
E^1_{p,q} &= \tilde{H}_{p+q}(X_p/X_{p-1}) \\
E^r_{p,q} &\implies H_{p+q}(X)
\end{aligned}$$

This part of the theorem is done - we have constructed some sequence that strongly converges against the homology of the total space of the Serre fibration.
And we have described the first page of that sequence. It depends, however, on the CW-structure of B and is not easily computed. Luckily, we will show now that the groups of the second page can be computed just by knowing the homology of B and that of the fiber.
This is what gives so much power to this spectral sequence. You can write down these proofs without knowing anything about the map π.

The question left is the computation of the 2nd page $E^2_{p,q}$.

$$
\begin{tikzcd}
\coprod \tilde{S}^p \arrow[rr] \arrow[dd] \arrow[dr, equal] & & \coprod \tilde{D}^{p+1} \arrow[dd] \\
 & F \times S^p \hookrightarrow F \times D^{p+1} \arrow[ur, phantom, "\simeq"] \arrow[dr, phantom, "\square"] & \\
X_p \arrow[rr] & & X_{p+1}
\end{tikzcd}
$$

This diagram being a homotopy pushout, implies that

$$\tilde{H}_{p+q}(X_{p+1}/X_p) \cong \bigoplus \tilde{H}_{p+q}(\tilde{D}^{p+1}/\tilde{S}^p) \cong \bigoplus H_{p+q}(F \times D^{p+1}, F \times S^p) \cong \bigoplus H_q(F)$$

The direct sum is indexed by $(p+1)$-cells of B.

Left is to compute the differential. <u>claim:</u> $d_1 : \bigoplus_{I^p} H_q(F) \to \bigoplus_{I^{p-1}} H_q(F) \to$ is $\partial \oplus id_{H_q(F)}$ where I^p denotes the set of p-cells in B and ∂ is the differential in the cellular complex of B. So the claim is that this is in fact isomorphic to $C_*^{cell}(B, H_q(F))$

Idea of proof of the claim:
we need to understand the map

$$
\begin{tikzcd}
H_q(F) \arrow[r, "?"] \arrow[d, hook, "e_\alpha"] & H_q(F) \arrow[d, hook, "e_\beta"] \\
\bigoplus H_q(F) \arrow[r, "d_\alpha"'] & \bigoplus H_q(F)
\end{tikzcd}
$$

where e_β is a $(p-1)$-cell of B and e_α a p-cell of B.
From the lemma before the functoriality of Serre spectral sequences is established. This is because if we have a map from some other CW-complex B' to B we can always make it cellular and when we take the pullback of the Serre-fibration to this B', the filtration on X' will be compatible with that on X because they are both the preimages of the cells on the base space and the map on the base space is cellular.
We can compute ? by restricting to the case of a CW-complex with two cells e_α and e_β, i.e. we have

$$
\begin{tikzcd}
S^{p-1} \arrow[r, hook] \arrow[d, "g"'] & D^p \arrow[d] \\
S^{p-1} \arrow[r] \arrow[ur, phantom, "\square"] & B^p
\end{tikzcd}
$$

if $\deg g = \pm 1$, we can assume that $B^p = D^p$ and then the computation is easy because then up to homotopy $X \simeq F \times D^p$.
if $\deg g = n$, we can factor it as

$$S^{p-1} \xrightarrow{\text{pinch}} \underbrace{S^{p-1} \vee S^{p-1} \vee \cdots \vee S^{p-1}}_{n} \xrightarrow{\text{fold}} S^{p-1}$$

and then again take the pullback of X over B^p to the space obtained from $\bigvee S^{p-1}$ by attaching one p-cell:

$$
\begin{tikzcd}
 & D^p \arrow[rr] & & ? \arrow[r] & B^p \\
S^{p-1} \arrow[ur, hook] \arrow[r, "\text{pinch}"'] & \underbrace{S^{p-1} \vee S^{p-1} \vee \cdots \vee S^{p-1}}_{n} \arrow[urr] \arrow[rr, "\text{fold}"'] \arrow[ur, phantom, "\square"] & & S^{p-1} \arrow[ur] \arrow[u, phantom, "\square"] &
\end{tikzcd}
$$

Thus ? is obtained by attaching one p-cell to $S^{p-1} \vee S^{p-1} \vee \cdots \vee S^{p-1}$ and then the inclusion of S^{p-1} has degree 1 by definition of the pinch map. Thus we can reduce to the case of $\deg g = 1$.

From this one gets that

$$E^2_{p,q} = H_{p+q}(B, H_q(F))$$

■

Remark 5.4.2.

1. If the base is not simply-connected, then the fundamental group of the base acts on the fiber of some distinguished point b by homotopy equivalences.
   This makes the homology $\underline{H}_q(F_b)$ a local system over B. There exists a corresponding Serre spectral sequence with homology with coefficients in a local system.

2. relative version:
   Assume $B' \subset B$ is given in the assumptions of the theorem, $X' = \pi^{-1}(B')$, then there is a Serre spectral sequence $E^2_{p,q} = H_p(B, B'; H_q(F)) \implies H_{p+q}(X, X')$.
   This allows to prove the relative Hurewicz theorem (similar to Ex 10.2)

3. cohomological version:
   If we have the sequence of spaces $\cdots \to X_p \to X_{p+1} \to \cdots$ we can construct an exact couple

$$A \cong \bigoplus H^n(X_p)$$
$$E \cong \bigoplus \tilde{H}^n(X_p/X_{p-1})$$

   that gives a spectral sequence with $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$.
   This yields the cohomological Serre spectral sequence.

5.5 Multiplicative structure on cohomological spectral sequences

Theorem 5.5.1.
In the setting of Serre spectral sequence for cohomological one with coefficients in a ring R there exist R-bilinear maps which are multiplication

$$\cdot : E_r^{p,q} \times E_r^{s,t} \to E_r^{p+s,q+t} \qquad r \geq 1$$

satisfying the Leibnitz rule ($x \in E_r^{p,q}$):

$$d_r(x \cdot y) = (d_r x) \cdot y + (-1)^{p+q} x \cdot d_r(y)$$

that induces multiplication on E_{r+1}
$x, y \in \ker d \rightsquigarrow d(x \cdot y) = 0$
$(x + \operatorname{im} d) \cdot y \subset x \cdot y + \operatorname{im} d,$ *bc* $dz \cdot y = d(z \cdot y)$

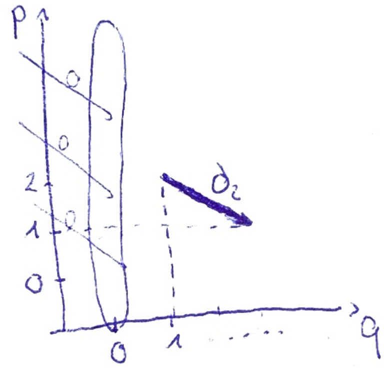
- *for E_2-page this multiplication is $(-1)^{qs}$ times cup-product*
- *the cup-product on $H^*(X, R)$ respects the filtration and on the graded quotients it is compatible with multiplications on $E_\infty^{p,q}$.*

Warning: the Leibnitz rule on E_{r+1} for multiplication does not follow from the Leibnitz rule on E_r

For proof see Hatcher's draft of chapter 5.

Leray-Hirsch theorem

Lemma 5.5.2.
Let $E_r^{p,q}$ be a cohomological Serre spectral sequence with coefficients R for $\pi : X \to B$. Assume F is connected



There are canonical maps

$$H^q(X,R) \twoheadrightarrow E_\infty^{0,q} = E_{q+1}^{0,q} \subset E_q^{0,q} \subset \cdots \subset E_2^{0,1} = H^0(B, H^q(F,R)) \cong H^q(F,R)$$

because $E_\infty^{0,q} \cong F^q H(X,R)/F^1 H^q(X,R)$ and $F^0 H^q(X,R) = H^q(X,R)$.

$$\underset{H^p(B,R)=H^p(B,H^0(F,R))=}{E_2^{p,0}} \twoheadrightarrow E_3^{p,0} \twoheadrightarrow \cdots \twoheadrightarrow \underset{F^pH^p/F^{p+1}H^p}{E_\infty^{p,0}} \hookrightarrow H^p(X,R) \text{ where } F^{p+1}H^p = 0$$

and these compositions are

$$\iota^* : H^q(X,R) \to H^q(F,R)$$
$$\pi^* : H^p(B,R) \to H^p(X,R)$$

Theorem 5.5.3 (Leray-Hirsch)**.**
Let $\begin{matrix} X \\ \downarrow \\ B \end{matrix}$ *be a fiber bundle with fiber F. There are two conditions:*

1. *$H^n(F,R)$ is a free finitely generated R-module*
2. *$\exists t_j \in H^{k_j}(X,R)$ such that $\iota^*(t_j)$ is a basis of $H^*(F,R)$*

Then

$$H^*(B,R) \otimes_R H^*(F,R) \xrightarrow[\cong]{\phi} H^*(X,R)$$
$$b_i \otimes \iota^*(t_j) \mapsto \pi^*(b_i) \cdot t_j \quad \textit{(cup-product)}$$

Proof.

1. implies that $H^p(B, H^q(F,R)) \overset{(*)}{\cong} H^p(B,R) \otimes_R H^q(F,R)$

<u>claim:</u> Serre spectral sequence degenerates at E_2

2. implies that $H^q(X,R) \twoheadrightarrow H^q(F,R)$. We have seen in the previous lemma that this map factors through

$$\begin{tikzcd} H^q(X,R) \arrow[rr, two heads] \arrow[dr] & & H^q(F,R) \arrow[drr, equal] & & \\ & E_\infty^{0,q} \arrow[r, hook] & E_q^{0,q} \arrow[r, hook] & \cdots \arrow[r, hook] & E_2^{0,q} \end{tikzcd}$$

it follows that all the inclusions have to be isomorphisms and thus $E_\infty^{0,q} = E_2^{0,q}$ i.e. $d_r|_{E_r^{0,q}} = 0$

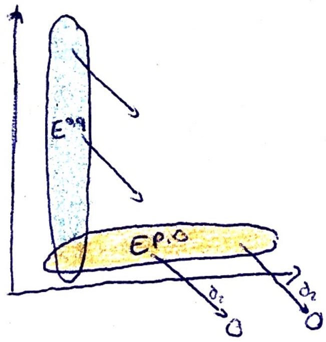
$$E_2^{0,q} \xleftarrow[\cong]{(*)} E_2^{p,0} \otimes E_2^{0,q}$$

both for $(*)$ and multiplication in E_2. What does the differential on $E_2^{p,0} \otimes E_2^{0,q}$ look like?

$$d_2(x,y) = \underbrace{d_2(x)}_{=0} \cdot y \pm x \cdot \overbrace{d_2(y)}^{=0} = 0$$

where $d_2(x) = 0$ because it goes out of the first quadrant and $d_2(y) = 0$ by above.

We have shown the $d_2 = 0$ on the whole page: the differential on $E^{p,0}$ is always zero, because it leaves the first quadrant, the differential on $E^{0,q}$ is zero because the assumptions of the Leray-Hirsch theorem tell you that the restriction map from the cohomology of the total space to the cohomology of the fiber should be surjective ($H^q(X,R) \twoheadrightarrow H^q(F,R)$) which means precisely that the spectral sequence $E_\infty^{0,q} \hookrightarrow E_q^{0,q} \hookrightarrow \cdots \hookrightarrow E_2^{0,q}$ does not have anything go out of the last group.



Using multiplicativity and the first assumption that $H^p(B, H^q(F,R)) \cong H^p(B,R) \otimes_R H^q(F,R)$ tells us that by applying Lebnitz-formula the differential on all objects in E_2 is 0.

Similarly for $d_3, d_4, ...$ and thus $E_\infty = E_2$.

To finish the claim, let's look at the map ϕ in more detail.

$$H^p(B,R) \otimes R\langle t_j \rangle_{t_j \in H^q(X,R)} \xrightarrow{\cong} H^{p+q}(X,R)$$

$$\begin{tikzcd}
& F^0H^q \arrow[d, equal] \\
R\langle t_j \rangle \arrow[ur, phantom, "\subset"] \arrow[r, phantom, "\subset"] \arrow[dr, "\cong"'] & H^q(X,R) \arrow[d] \\
& H^q(F,R)
\end{tikzcd}$$

Then $\pi^*(H^p(B,R)) = F^pH^p(X,R)$. Multiplying thus gives a commutative square

$$\begin{tikzcd}
F^0H^* \arrow[d] & \otimes & F^0H^q \arrow[d] \arrow[rr] & & F^pH^{p+q} \arrow[d, two heads] \\
H^p(B) & \otimes & H^q(F) \arrow[rr, "\cong", "id"'] & & E_2^{p,q} = H^p(B) \otimes H^q(F)
\end{tikzcd}$$

It follows that ϕ is surjective and injectivity follows from freeness of both left and right hand side as $H^*(B,R)$-modules.
The freeness is because on the LHS $R\langle t_j \rangle_{t_j \in H^q(X,R)}$ and on the right hand side is because the quotients of the filtration have been computed by the second page of the spectral sequence and there they are free as modules over $H^p(B)$. ∎

Example 5.5.4 ($H^*(K(\mathbb{Z},2),\mathbb{Z})$)**.**
We actually already know that $K(\mathbb{Z},2) \cong \mathbb{C}P^\infty$ which allows one to compute this cohomology geometrically (regard how the cohomology behaves with the inclusions of $\mathbb{C}P^n$)
But we do not want to use this possibility but rather Serre spectral sequences.
We can use the fibration of the pathspace P

$$\begin{tikzcd}
\underbrace{K(\mathbb{Z},1)}_{S^1} \simeq \underset{F=}{\Omega K(\mathbb{Z},2)} \arrow[r] & P \simeq * \arrow[d] \\
& \underbrace{K(\mathbb{Z},2)}_{\text{simply conn.}}
\end{tikzcd}$$

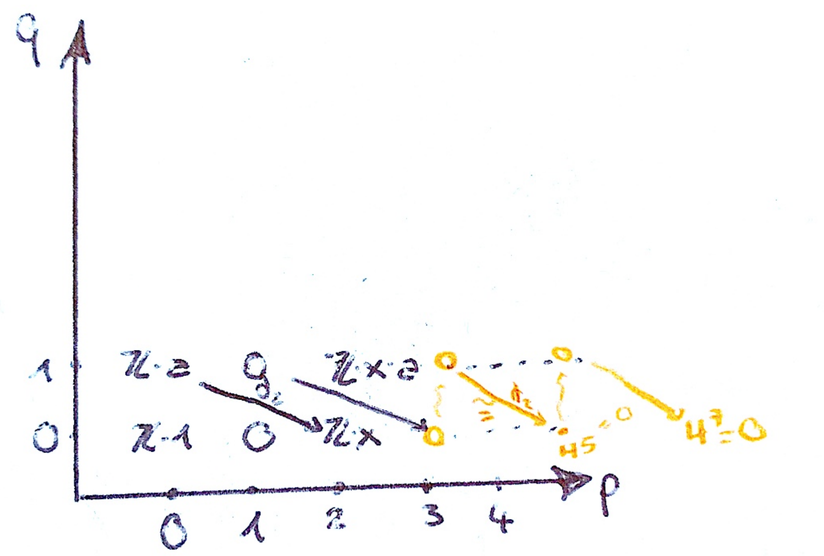
Because $K(\mathbb{Z},2)$ is simply connected we can apply the Serre spectral sequence.

$$H^q(F,\mathbb{Z}) = \begin{cases} \mathbb{Z}\cdot 1 & q=0 \\ \mathbb{Z}\cdot a & q=1 \\ 0 & \text{otherwise} \end{cases}$$

Now we have the Serre spectral sequence telling us that on the E_2-page

$$H^p(K(\mathbb{Z},2),H^q(F,\mathbb{Z})) \implies H^{p+q}(P,\mathbb{Z})$$

where the latter is 0 unless $p+q=0$.



Details:
The 0 at the bottom of the first column is 0 because by Hurewicz

$$\left.\begin{array}{r} H^1(K(\mathbb{Z},2)) = 0 \\ H^2(K(\mathbb{Z},2)) = \mathbb{Z}\cdot x \end{array}\right\} \Leftarrow \left.\begin{array}{r} H_1(K(\mathbb{Z},2)) = 0 \\ H_2(K(\mathbb{Z},2)) = \mathbb{Z} \end{array}\right\} \quad \text{Hurewicz theorem}$$

Also for the entry $E_2^{2,1}$:

$$\begin{array}{c} H^2(K(\mathbb{Z},2),H^1(K(\mathbb{Z},1))) \simeq \mathbb{Z}\cdot x\cdot a \\ \uparrow\cong \\ \underbrace{H^2(K(\mathbb{Z},2))}_{\mathbb{Z}\cdot x} \otimes_{\mathbb{Z}} \underbrace{H^0(K(\mathbb{Z},2),H^1(K(\mathbb{Z},1)))}_{\mathbb{Z}\cdot a} \end{array}$$

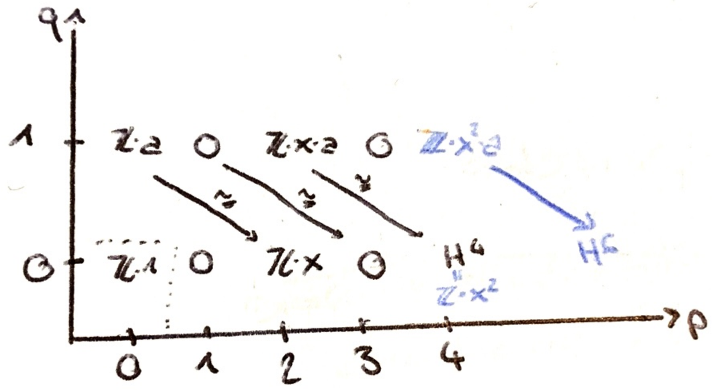
Since $E_2^{p,q}=0$ for $q>1$, $E_3=E_\infty$ but $E_\infty^{p,q} = gr^p H^{p+q}(P) = 0$ unless $p=q=0$.
Therefore d_2 is an isomorphism (because if it had a kernel, the kernel would stay on the next page and same for the cokernel) (except for $E^{0,0}$).

- $d_2(a) = x$
- $H^3(K(\mathbb{Z}, 2)) = 0$

Thus $E^{p,0} = 0$ and what stands directly above it, is too. But because this upper 0 goes to H^5 by an isomorphism, H^5 has to be 0 as well and so on:

$$d_2 : H^{2k-1}(K(\mathbb{Z}, 2), \underbrace{H^1(F, \mathbb{Z})}_{\mathbb{Z}}) \xrightarrow{\cong} H^{2k+1}(K(\mathbb{Z}, 2), \mathbb{Z})$$

By induction on k we get that $H^{2k+1}(K(\mathbb{Z}, 2), \mathbb{Z}) = 0$.



For the even degrees we start with $E_2^{2,0}$ which is $\mathbb{Z} \cdot x \cdot a$. This is mapped isomorphically onto H^4. This is of course good to know but what is more important is that the differential satisfies the Leibnitz rule:

$$d_2(x \cdot a) = d_2(x) \cdot a + x \cdot d_2(a)$$

(there is no sign because x is of degree 2)
$d_2(x) = 0$ because it leaves the first quadrant. Also $d_2(a) = x$. Thus

$$d_2(x \cdot a) = d_2(x) \cdot a + x \cdot d_2(a) = x^2$$

Thus $H^2(K(\mathbb{Z}, 2), \mathbb{Z}) = \mathbb{Z} \cdot x^2$. Again above that the group is generated by the generator of the below entry (here x^2) multiplied by a. Thus we have $\mathbb{Z} \cdot x^2 \cdot a$ which maps to H^6 and so forth - so we go by induction:

$$d_2(x^k(a) = x^k \overset{\text{induction}}{\Longrightarrow} H^{2k}(K(\mathbb{Z}, 2), \mathbb{Z}) \cong \mathbb{Z} \cdot x^k$$

Putting together everything we have seen so far we can conclude the example as we receive as result:

$$H^*(K(\mathbb{Z}, 2), \mathbb{Z}) \cong \mathbb{Z}[x]$$

where $\deg(x) = 2$.

Example 5.5.5 (Multiplication on E_∞ is not always enough to compute multiplication on $H^*(X)$)**.** For this we have to construct a very specific fiber bundle. We start with our favourite example which is the Hopf fibration $\eta : S^3 \to S^2$. Out of that we will construct a non-trivial fiber bundle $\pi : X \to S^2$ with fiber S^2.
The computation in this case will show that the multiplication on X cannot be reconstructed from

the Serre spectral sequence.

To construct this space X we first construct the space

$$X_0 := Cyl(\eta) = (S^3 \times I \amalg S^2)/\underset{\in S^3\times I}{(x,0)} \sim \underset{\in S^2}{\eta(x)}$$

There is a map $X_0 \to S^2$ which is glued from $\begin{matrix} S^3 \times I \\ \downarrow \\ S^2 \end{matrix}$ $\begin{matrix} (x,t) \\ \downarrow \\ \eta(x) \end{matrix}$ and the identity on $S^2 \to S^2$.

Claim: This is a fiber bundle with fiber $CS^1 \cong D^2$
Let $U \subset S^2$ such that $\eta^{-1}(U) \cong U \times S^1$. By definition of the mapping cylinder

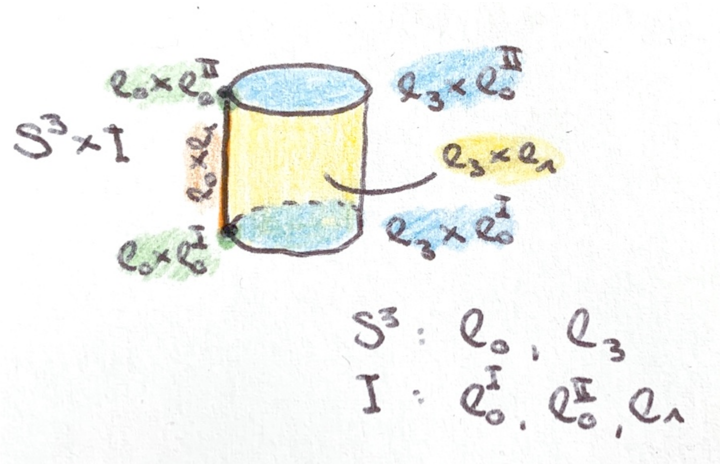
$$\begin{tikzcd} \pi_0^{-1}(U) \arrow[r, hook] \arrow[d] & X_0 \arrow[d] \\ U \arrow[r, hook] & S^2 \end{tikzcd}$$

so $\pi_0^{-1}(U) = Cyl(U \times S^1 \xrightarrow{pr} U)$. It suffices to show that

$$\begin{tikzcd} Cyl(U \times S^1 \to U) \arrow[rr, "\simeq"] \arrow[dr, "\pi_0"'] & & U \times CS^1 \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

The definition of cylinder (which is gluing point of $U \times S^1$ on the cylinder with points of U, this is sort of linear) $Cyl(U \times S^1 \to U) \cong U \times Cyl(S^1 \times *) \cong U \times CS^1 \cong U \times D^2$

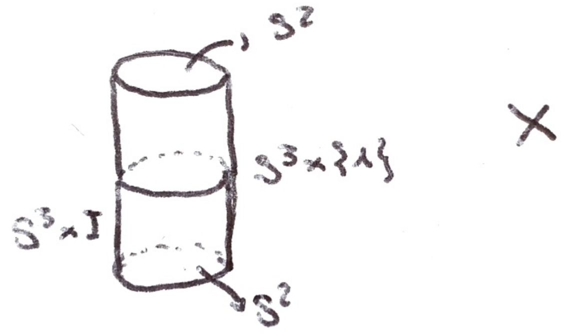
- X_0 has CW-structure with only one 4-cell and no k-cells for $k > 4$



  in X_0 we replace $e_3 \times e_0^I$ with $e_2 \times e_0^I$. So in the cylinder we have to glue the 3-dimensional sphere at the bottom which is $e_3 \times e_0^I$, where e_0^I is the point 0 in the unit interval I, so $S^3 \times \{0\}$ is precisely $e_3 \times e_0^I$. We glue that to S^2, so basically we replace e_3 with e_2 and then after we replace it, we exchange the gluing functions (the characteristic functions of cells). Now for example $e_3 \times e_1$ is glued to e_2 via the Hopf fibration map but is still a CW-complex because we glue e_3 to a lower dimensional cell.

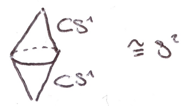
- $X_0/S^3 \times \{1\} \cong Come(\eta) \cong \mathbb{C}P^2$ (Exercise 5.3. a)

Finally $X = X_0 \cup_{S^3\times\{1\}} X_0$. One can picture X as



There is still a canonical map $\pi : X \to S^2$ which is glued from π_0 on X_0 for both ones.

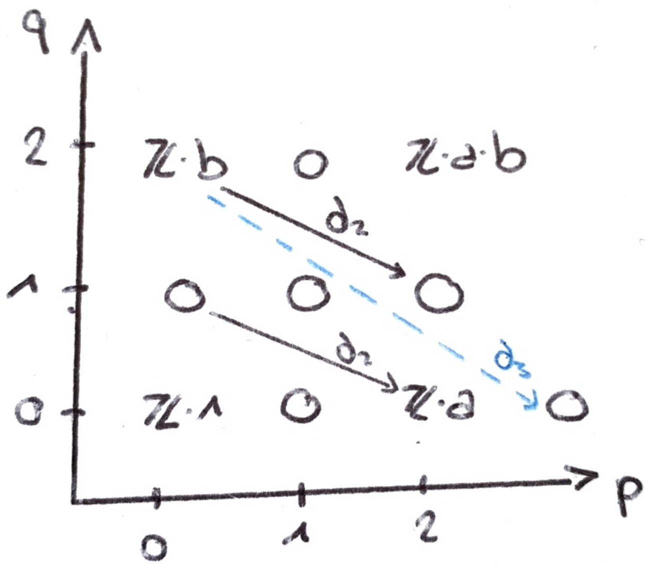
It is easy to check that π is a fiber bundle with fiber



Let's look now at the Serre spectral sequence for this fiber bundle π and see that it actually does not depend on X at all. It starts with $H^p(S^2, H^q(S^2, \mathbb{Z}))$ and should strongly converge against the cohomology of the total space:

$$H^p(S^2, H^q(S^2, \mathbb{Z})) \implies H^{p+q}(X, \mathbb{Z})$$

denote $H^2(S^2, \mathbb{Z}) = \mathbb{Z} \cdot a$ where S^2 is the base of the fibration and $H^2(F, \mathbb{Z}) = \mathbb{Z} \cdot b$



All the differentials are zero because they either start at 0 or end there. So for degree reasons $d_r = 0$, $r \geq 2$ and thus $E_\infty = E_2$. Thus it does not depend on X at all, but we have computed the graded quotient of the algebra of X and it is the same as for the trivial fiber bundle. So now we explain that for the non-trivial fiber bundle that we have just constructed, the cohomology ring of X is not the same as the cohomology ring of $S^2 \times S^2$.

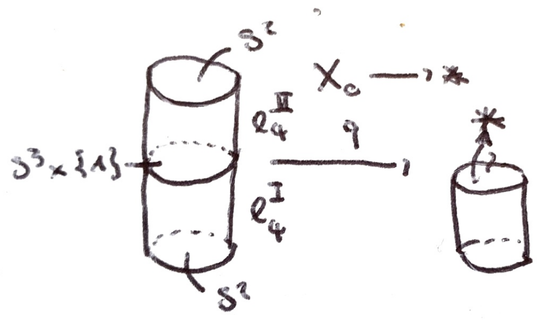
Well there is a map

$$X \xrightarrow{q} X_0/S^3 \times \{1\} \cong \mathbb{C}P^2$$

We will compute how it acts on the 4th hopmology or cohomology. X has only two 4-cells where one comes from the lower cylinder and one of upper one. Thus

$$d(e_4^I) = d(e_4^{II}) = [S^3 \times \{1\}]$$

The map q is then just contracting one of the X_0



$$d(e_4^I - e_4^{II}) = 0$$

is a free generator of $H_4(X)$. There are no higher dimensional cells which is why the fourth homology is just the subgroup of the free abelian group of 4-cells given by the kernel of the differential.
So the map q sends $q_*(e_4^{II}) = 0$, $q_*(e_4^I) = e_4$ which is the generator of $H_4(\mathbb{C}P^2)$. Going to cohomology it follows that

$$q^* : H^4(\mathbb{C}P^2) \to H^4(X)$$

is injective (in fact, an isomorphism). How does it help us? Well, we know the multiplication in $H^4(\mathbb{C}P^2) = \mathbb{Z} \cdot x^2$. So because the pullback is a morphism of rings

$$q^*(x) \in H^2(X), \text{ then } q^*(x)^2 = q^*(x^2) \neq 0$$

but $H^*(S^2 \times S^2) = \mathbb{Z}[a,b]/(a^2,b^2)$. There is no element in $H^2(S^2 \times S^2)$ that is non-zero squared.

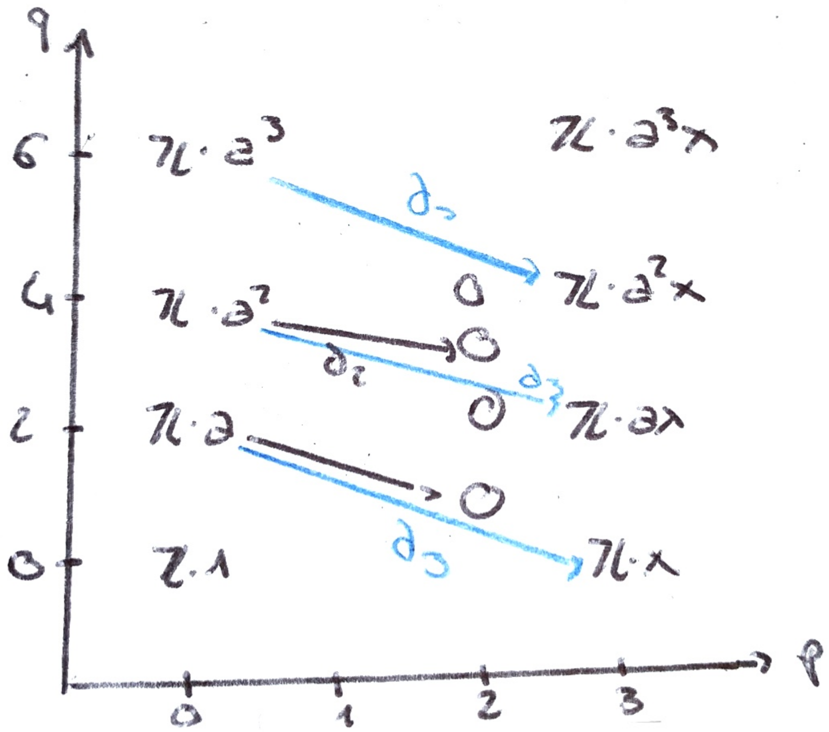
$$H^*(X) \ncong H^*(S^2 \times S^2)$$

Therefore it is impossible to compute the cohomology the whole ring multiplication here just from the Serre spectral sequence because the Serre spectral sequence in this instance does not care at all, whether the fiber bundle is trivial or not.

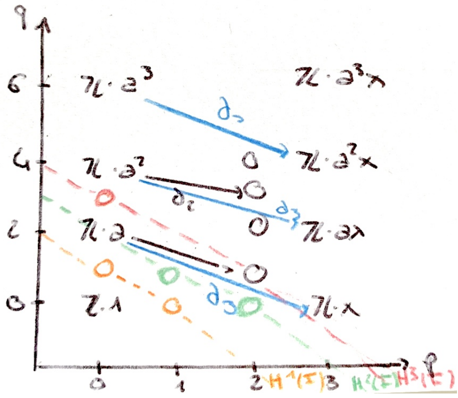
Example 5.5.6 ($\pi_4(S^3) \cong \mathbb{Z}/2$)**.**
Let $S^3 \to K(\mathbb{Z},3)$ be a map that induces isomorphism on π_3. Let F be its homotopy fiber, $f : D \to S^3$. The LES for homotopy groups yields that $\pi_{\geq 4} \xrightarrow[f_*]{\cong} \pi_{\geq 4}(S^3)$. Because F is 3-connected, $H_{\leq 3}(F) = 0$ and $H_4(F) \xleftarrow{\cong} \pi_4(F) \xrightarrow{\cong} \pi_4(S^3)$.
So now we have a space F and computing its 4th homology group gives us the homotopy group of the sphere. But the question is: "How do we know anything about the fiber F? It is a homotopy fiber of some map". The thing is that up to homotopy we replace it with a fibration and compute its fiber. What will it be?

$$\begin{tikzcd}
K(\mathbb{Z},2) \simeq \Omega K(\mathbb{Z},3) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & F \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & * \arrow[d] \\
* \arrow[r] & S^3 \arrow[r] & K(\mathbb{Z},3)
\end{tikzcd}$$

Replace f by a fibration then the fiber is $K(\mathbb{Z},2)$ and we get a Serre spectral sequence which $H^p(S^3, H^4(K(\mathbb{Z},2),\mathbb{Z})) \implies H^{p+q}(F,\mathbb{Z})$.
Recall that if $\tilde{H}_{\leq 3}(F) = 0$, then $H^{\leq 3}(F) = 0$ by the universal coefficient theorem.
Introduce the usual notation $H^3(S^3,\mathbb{Z}) = \mathbb{Z} \cdot x$ where $x^2 = 0$ and $H^*(K(\mathbb{Z},2),/Z) = \mathbb{Z}[a]$ with degree of a is 2.
Thus the E_2-page looks as follows:



Because on the second page all differentials go to 0, the third page is equal to the second one. For degree reasons $E_4 = E_\infty$.
What do we know where the spectral sequence converges? We don't know much. We know however, that the first three (reduced) homology groups are 0, so where do they come from?
The first comes from the diagonal drawn in orange, the second from the green one and the third in red



As all of them are zero, at the infinity page on these three lines we should have all zeros. But we have the group $\mathbb{Z} \cdot a$ and $\mathbb{Z} \cdot x$ on these lines and one differential d_3 between them. If d_3 were no isomorphism, then it would have a kernel and a cokernel which would stand in the places of $\mathbb{Z} \cdot a$ and $\mathbb{Z} \cdot x$ respectively on the next page and then on the infinity page in $E_\infty^{0,1} \neq 0$ or $E_\infty^{3,0} \neq 0$.
Thus d_3 has to be an isomorphism. Therefore d_3 has to send a to a generator of the group, so either x or $-x$

$$d_3(a) = x$$

Now by the Leibniz rule we can compute d_3 of all the other generators of the groups in the 0-th column

$$d_3(a^k) = k \cdot a^{k-1} \cdot d_3(a) = k \cdot a^{k-1} \cdot x$$

for $k > 1$ this is not an isomorphism, although it always is injective, but the cokernel is non-trivial. We can actually compute the cokernel on the next page which is the infinity page.

$$\begin{aligned} E_\infty^{0,q} &= 0 \quad \text{for } q > 0 \\ E_\infty^{3,2(k-1)} &= \mathbb{Z}/k \cdot a^{k-1} \cdot x \end{aligned}$$

But on each diagonal on the next page, in the 0-th column there will only be zeros and in the third some finite abelian groups. Thus on each diagonal will only be one abelian group. Therefore we do not need to glue these groups together to compute cohomology of the fiber

$$E_\infty^{3,2(k-1)} = \mathbb{Z}/k \cdot a^{k-1} \cdot x \cong H^{2k+1}(F, \mathbb{Z})$$

In particular, we get that

$$\left.\begin{array}{r} H^4(F) = 0 \\ H^5(F) \cong \mathbb{Z}/2 \end{array}\right\} \overset{\text{univ. coeff}}{\Longrightarrow} H_4(F) \cong \mathbb{Z}/2$$

This finishes the example.

An addition:
By Freudenthal's suspension theorem

$$\pi_4(S^3) \xrightarrow{\cong} \pi_5(S^4) \xrightarrow{\cong} \pi_6(S^5) \xrightarrow{\cong} \cdots \cong \mathbb{Z}/2$$

By Hopf fibration LES

$$\pi_4(S^3) \xrightarrow{\cong} \pi_4(S^2) \cong \mathbb{Z}/2$$

Example 5.5.7. $H^*(K(\mathbb{Z}, n), \mathbb{Q}) \cong \begin{cases} \mathbb{Q}[x], & n \text{ is even} \\ \mathbb{Q}[x]/x^2, & n \text{ is odd} \end{cases}$

(it suffices to remember only $n = 1$, where $K(\mathbb{Z}, 1) \cong S^1$ and $n = 2$, where $K(\mathbb{Z}, 2) \cong \mathbb{C}P^\infty$)

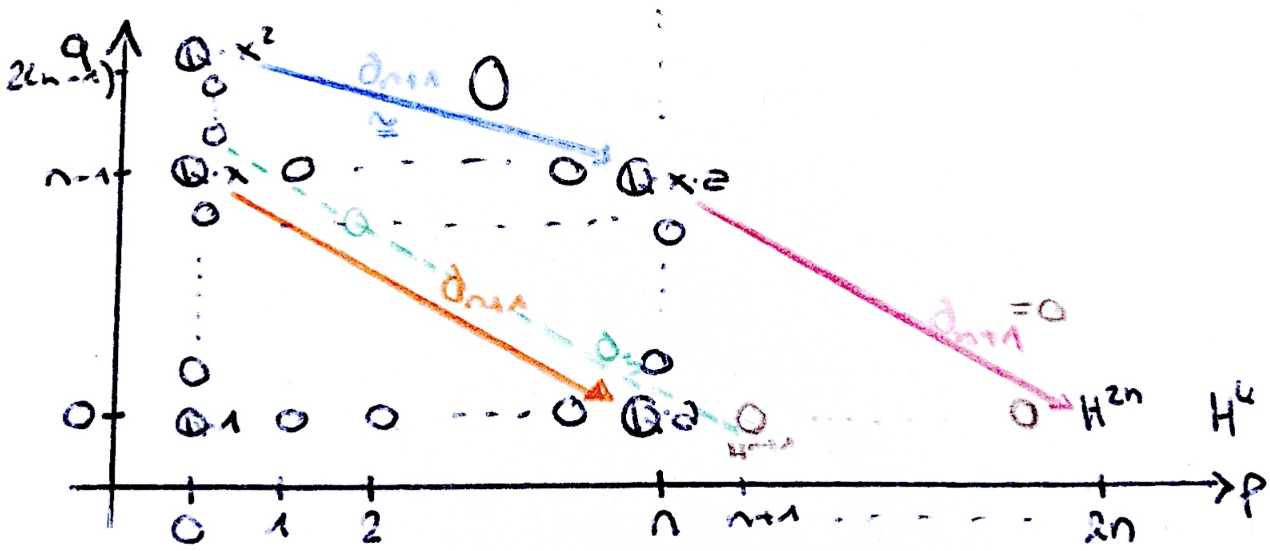
Prove by induction using the path fibration

$$\begin{tikzcd} \Omega K(\mathbb{Z}, n) \simeq K(\mathbb{Z}, n-1) \arrow[r] & P \simeq * \arrow[d] \\ & K(\mathbb{Z}, n) \end{tikzcd}$$

If n is even, then the computation is as in Example 1. By induction assumption we know that the cohomology of the fiber is like of a 1-dimensional sphere with rational coefficients and then we look at the two line spectral sequence and get out of it the claim that the cohomology of the space $K(\mathbb{Z}, n)$ is $\mathbb{Q}[x]$ absolutely in the same fashion.

But the case when n is odd is more tricky. We assume $H^*(F, \mathbb{Q}) \cong \mathbb{Q}[x]$ where $\deg x = n - 1$. We also know by Hurewicz that

$$\tilde{H}^{<n}(K(\mathbb{Z}, n), \mathbb{Q}) = 0 H^n(K(\mathbb{Z}, n), \mathbb{Q}) \qquad\qquad \cong \mathbb{Q} \cdot a$$



In the first n-columns the first non-trivial differential can only be d_{n+1}
On the next page E_{n+1}, the group $E_{n+2}^{0,n-1}$ survives to infinity because nothing ever hits it and the next differentials will leave the first quadrant, therefore because $\tilde{H}^*(P) = 0$

$$E_{n+2}^{0,n-1} = E_{\infty}^{0,n-1} = 0$$

The same is true for the group $E_{n+1}^{n,0}$: nothing can hit it anymore because the next differentials will start out of the first quadrant

$$E_{n+2}^{n,0} = E_{\infty}^{n,0} = 0$$

Therefore $d_{n+1}(x) = \lambda \cdot a$ where $\lambda \in \mathbb{Q} \setminus \{0\}$ (so a generator of $\mathbb{Q} \cdot a$).
We claim that the groups $E_{n+1}^{n+1,0}, ..., E_{n+1}^{2n-1,0}$ are all 0, i.e. there are no non-zero groups of total degree $p + q = k - 1$ (apart from $k = n + 1$).
The differential going from the n-th entry to the $n + 1$-th is the differential on the first page in the cohomological degree, so after the second group it will never appear.
So what hits H^{n+1} lies on the green diagonal, but everything on there is 0. Thus $E_{\infty}^{k,0} = H^k(K(\mathbb{Z}, n), \mathbb{Q}) = 0$. Continuing with the very same argument, all turn out to be zero until you reach H^{2n}.
And here is where we use the computation that $d_{n+1}(x) = \lambda \cdot a$. Also $E_{n+1}^{n,n-1} = \mathbb{Q} \cdot x \cdot a$ for degree reasons.

$$0 = d_{n+1}(d_{n+1}(\frac{x}{\lambda})) = d_{n+1}(x \cdot a = \lambda \cdot a^2$$

Thus we get that $a^2 = 0$ and thus $d_{n+1} = 0$ (red)

$$d_{n+1}(x^2) = \lambda \cdot xa$$
$$d_{n+1}(x^k) = \lambda \cdot x^{k-1} \cdot a$$

thus all d_{n+1} in the first n columns are isomorphisms.
So after this page, so starting from E_{n+2}, there is only $\mathbb{Q} \cdot 1$ non-zero in the first n columns. But this area is precisely the area from where we should get some differentials to H^{2n}. So $d_{n+1} = 0$, all previous are also 0 for degree reasons and all consequent are zero, because everything in this area will be zero.

$$\implies H^{2n} = 0$$

otherwise it survives.

Now by induction

$$H^k(K(\mathbb{Z}, n), \mathbb{Q}) = 0 \quad \text{for } k > 2n$$

because the differentials to these groups can be non-zero only from $\mathbb{Q} \cdot x^k a^n$ where $k \geq 0$, $a = 0, 1$, but they can only hit H^l at pages after E_{n+1}.

5.6 Serre’s finiteness results

(not part of the exam)

Let $\mathcal{C}$ be a full subcategory of `Ab`:

1. finitely generated abelian groups
2. (for fixed $P \subset \{\text{primes}\}$)
   torsion groups with elements having order divisible only by primes from P (e.g. $P = \{p\}$ $\rightsquigarrow$ $p^{\mathbb{Z}}$ torsion groups)
3. finite groups in 2.

We will be mostly interested in the case of 1. (finite free abelian groups) and 3. (finite torsion groups $P =$Primes)

Theorem 5.6.1.
Let X be a simply path-connected topological space.
Then $\pi_n(X) \in \mathcal{C}$ for all n ($n \geq 2$) if and only if $H_n(X) \in \mathcal{C}$ for all $n \geq 1$.

Remark 5.6.2.
WLOG one can assume that $X \in$ `CW`.
This is because we have proven the existence of `CW` approximation, so if we have a simply path-connected topological space, we can replace it with a simply path-connected (which is the same as simply-connected, then) CW-complexes. It will have by construction the same homotopy groups but we also have proved that the homology of CW-approximation does not change. They are the same as of the space.
Therefore in the proof we will always talk about CW-complexes.

Corollary 5.6.3.
Homotopy groups of finite (finitely many cells in each degree, thus finitely generated homology groups) simply connected CW-complexes are finitely generated in each degree.

Proof.
Take $\mathcal{C} =$ finitely generated groups. ∎

Lemma 5.6.4.
Categories $\mathcal{C}$ above satisfy following properties:

1. *[thick] If we have an exact sequence in* `Ab`

$$0 \to A \to B \to C \to D$$

   then $A, C \in \mathcal{C}$ iff $B \in \mathcal{C}$.
   (For example if we take an object in $\mathcal{C}$, then every subobject and every subquotient will also be in $\mathcal{C}$)

2. $A, B \in \mathcal{C} \implies \begin{cases} A \otimes B \in \mathcal{C} \\ Tor(A, B) \in \mathcal{C} \end{cases}$

Proof.
One might as an exercise prove these properties for the classes 2. and 3.
We are mostly interested in the first and for this, it is rather easy to see.
For two finitely generated abelian groups, their tensor product is finitely generated. Also if one recalls how to compute Tor: write resolution which is finitely generated in each degree, so Tor will also be finitely generated. ■

Lemma 5.6.5.

Suppose we have a fibration $X \to B$ with fiber F:
$$\begin{tikzcd} F \arrow[r, hook] & X \arrow[d] \\ & B \end{tikzcd}$$
Assume B is connected, $\pi_1(B)$ acts trivially on $H_*(F)$ (this is a sufficient condition for us to use the Serre spectral sequence, instead of simply connected)
If for two of these spaces F, X, B we have $H_n(-) \in \mathcal{C}$ for $n > 0$, then the same is true about the third space.

Proof.
There exists Serre spectral sequence $E^2_{p,q} = H_p(B, H_q(F)) \implies H_{p+q}(X)$. Recall that by universal coefficient formula $H_p(B, H_q(F)) \cong H_p(B) \otimes H_q(F) \oplus \operatorname{Tor}(H_{p-1}(B), H_q(F))$.

First case: $H_n(F), H_n(B) \in \mathcal{C}$, $\forall n > 0$
It implies, because we have in $H_p(B, H_q(F)) \cong H_p(B) \otimes H_q(F) \oplus \operatorname{Tor}(H_{p-1}(B), H_q(F))$ their tensor product and the Tor that

$$E^1_{p,q} \in \mathcal{C} \text{ except for } p = q = 0$$

By induction on r we show that $E^r_{p,q} \in \mathcal{C}$ (except for $p = q = 0$). Well, this is a first quadrant spectral sequence, there is something like $\mathbb{Z}$ (in each connected component) standing at $p = q = 0$. There are no differentials going anywhere or going in, starting from the second page, so this will survive to some $\mathbb{Z}$ to the power of connected components of X, but this can be not in $\mathcal{C}$. We are, however, only interested in homology in degree $n > 0$.

$$d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$$

by induction both these groups are in $\mathcal{C}$. Therefore $\ker d_r \in \mathcal{C}$ and $\operatorname{im} d_r \in \mathcal{C}$ and thus $E^{r+1}_{p,q} = \ker d_r / \operatorname{im} d_r \in \mathcal{C}$

Thus $E^\infty_{p,q} \in \mathcal{C}$ (for $(p,q) \neq (0,0)$). But then we have a strong convergence to $H_{p+q}(X)$ which has a finite filtration with subquotients $E^\infty_{p,q}$.
By induction (using property 1. in the lemma

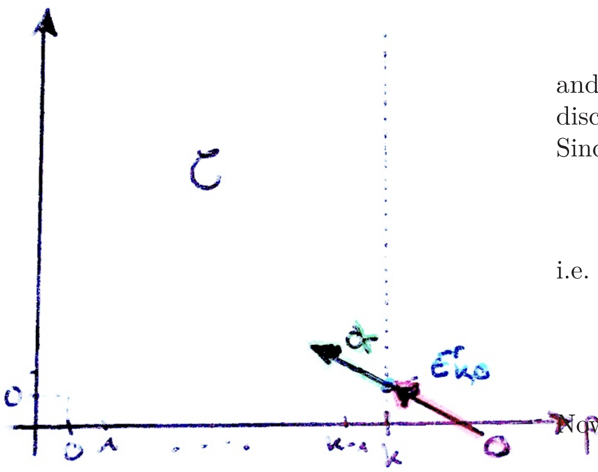
$$0 \to F_{p-1}H_{p+q}(X) \to F_pH_{p+q}(X) \to E^{p,q}_\infty \to 0$$

by induction $F_{p-1}H_{p+q}(X) \in \mathcal{C}$ and we have seen that $E^{p,q}_\infty \in \mathcal{C}$)
we prove that $H_{p+q}(X) \in \mathcal{C}$.

Second case: $H_n(F), H_n(X) \in \mathcal{C}$ for $n > 0$
We still use the spectral sequence, but the argument sort of goes in the other direction.
(Assume $(p,q) \neq (0,0)$)
$E^\infty_{p,q}$ is a subquotient of $H_{p+q}(X)$, hence it is in $\mathcal{C}$. So now we know that after many many differentials, what is left of the Serre spectral sequence lies in $\mathcal{C}$. We also know something about the second page of this spectral sequence.
Let's show by induction on k that

$$H_p(B) \in \mathcal{C} \quad \text{for } 0 < p < k$$

We will only prove the step of induction bu the base of induction is done absolutely similarly. By induction assumption the first k columns of the second page $E^2_{p,q} \in \mathcal{C}$, $p < k$. We have the formula $E^2_{p,q} = H_p(B, H_q(F))$ for $p < k$ where $H_q(F)$ always lies in $\mathcal{C}$ and $H_p(B)$ lies in $\mathcal{C}$ for $0 < p < k$ (for $p = 0$: because B is connected, the 0-th homology $H_0(B) = \mathbb{Z}$ and thus $E^2_{0,q} = H_q(F) \in \mathcal{C}$).
This implies (as in the first case) whatever we do with the differentials, $E^r_{p,q} \in \mathcal{C}$ for all $r \geq 2$, $p < k$. This is because $E^3_{p,q}$ is a subquotient of $E^2_{p,q}$, $E^4_{p,q}$ a subquotient of $E^3_{p,q}$ and the category $\mathcal{C}$ is closed under subquotients.



$$d_r : E^r_{k,0} \to E^r_{k-r,r-1}$$

and $E^r_{k-r,r-1}$ lies in $\mathcal{C}$ by what we have just discussed.
Since all differentials going in $E^r_{k,0}$ come from zero

$$E^{r+1}_{k,0} = \ker d_r|_{E^r_{k,0}}$$

i.e.

$$0 \to E^{r+1}_{k,0} \hookrightarrow E^r_{k,0} \xrightarrow{d_r} \underbrace{d_r(E^r_{k,0})}_{\in \mathcal{C}} \to 0$$

Now by the thickness of $\mathcal{C}$,

$$E^{r+1}_{k,0} \in \mathcal{C} \Leftrightarrow E^r_{k,0} \in \mathcal{C}$$

We know, however, that on the infinite page, $E^\infty_{p,q}$ is always in $\mathcal{C}$. Using the equivalence we can go back and get $E^2_{k,0} \in \mathcal{C}$ and $E^2_{k,0} = H_k(B)^{\oplus \pi_0(F)}$.
It is not hard to show that $\pi_0(F)$ is finite, because if it was not we would get infinitely many copies of the same thing in the homology of F which cannot happen because it is in $\mathcal{C}$.

$$\implies H_k(B) \in \mathcal{C}$$

This is precisely the induction step and thus concludes the second case.

Third case: $H_n(X), H_n(B) \in \mathcal{C}$
is done similarly

∎

Proposition 5.6.6.
If $A \in \mathcal{C}$, then $H_k((A, n)) \in \mathcal{C}$ for all $k > 0$.

Proof.

There is a path-fibration
$$\begin{array}{ccc} K(A, n-1) & \longrightarrow & P \simeq * \\ & & \downarrow \\ & & K(A, n) \end{array}$$

$H_{>0}(P) = 0 \in \mathcal{C}$ for any $\mathcal{C}$. Thus by the lemma before the claim is true for all n iff it is true for $n = 1$.
We are therefore only interested in $H_k(K(A, 1))$ (recall: $K(A, 1) = BA$). In general it can be computed algebraically as $\mathrm{Tor}^{\mathbb{Z}[A]}_k(\mathbb{Z}, \mathbb{Z})$. You can proof the claim of this proposition algebraically by writing a specific resolution of $\mathbb{Z}$ over this ring and then checking that after you tensor it with $\mathbb{Z}$, the homology will stay in the same category. One can assume that we have proven this proposition using this blackbox and some algebraic computations.

In the case where $\mathcal{C}$ is finitely generated or finitely generated or finite abelian groups, one can apply a different kind of argument. By classification of finitely generated abelian groups:

$A \cong \mathbb{Z}^{\oplus r} \oplus \bigoplus_i \mathbb{Z}/n_i$.
In this case

$$K(A,1) \cong \underbrace{\overset{\times r}{K(\mathbb{Z},1)}}_{S^1} \times \prod_i K(\mathbb{Z}/n_i,1)$$

We know the homology of a circle so we have to thing about what the homology of the spaces $K(\mathbb{Z}/n_i,1)$ is. One can construct those spaces explicitly

$$K(\mathbb{Z}/n,1) \simeq \underset{\text{(lens space)}}{S^\infty/\mathbb{Z}/n}$$

where the action is given by

$$\begin{aligned} \mathbb{Z}/n \times S^\infty &\to S^\infty \\ a, (x_1, ..., x_n, ...) &\mapsto (\mu^a x_0, \mu^a x_1, ...) \end{aligned}$$

where $\mu^n = 1$, $\mu^m \neq 1$ for $m < n$.
So it has a CW-structure (Hatcher, Ex 2.43) with one cell in each degree and cellular chain complex:

$$\cdots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{0} \cdots \underset{1}{\mathbb{Z}} \xrightarrow{0} \underset{0}{\mathbb{Z}}$$

Therefore for $k > 0$,

$$H_k(K(\mathbb{Z}/n,1)) = \begin{cases} 0 \\ \mathbb{Z}/n \end{cases}$$

■

Let X be a connected CW-complex. Then there exists a Postnikov system /tower which looks like

$$\begin{tikzcd} & X_n \arrow[d] \\ & X_{n-1} \arrow[d] \\ X \arrow[uur] \arrow[ur] \arrow[dr] & \vdots \arrow[d] \\ & X_1 \simeq K(\pi_1(X),1) \end{tikzcd}$$

that it as least commutative up to homotopy. It has the following properties

- $\pi_i(X_n) = 0$ for $i > n$
- $\pi_i(X) \xrightarrow{\cong} \pi_i(X_n)$ for $i \leq n$

To construct it (similar to Ex 7.2 (b)) X_n can be obtained by gluing $\geq (n+2)$-cells to X.
Then one can regard X as a subcomplex of X_n and X_{n+1}

$$\begin{tikzcd} & X_{n+1} \arrow[d, dashed] \\ X \arrow[ur, hook] \arrow[r, hook] & X_n \end{tikzcd}$$

where the dashed map exists because the pair (X_{n+1}, X_n) is $(n+1)$-connected and $\pi_i(X_n) = 0$ for $i > n$. The homotopy fiber of the map $X_{n+1} \to X_n$ is $K(\pi_{n+1}X, n+1)$ (by LES of homotopy groups)

$$\begin{tikzcd} K(\pi_{n+1}X, n+1) \arrow[r] & X_{n+1} \arrow[d] \\ & X_n \end{tikzcd} \qquad (*)$$

Moreover,

$$H_m(X) \xrightarrow{\cong} H_m(X_n) \qquad m \leq n$$

Now we have this claim about the homology of these Eilenberg-MacLane spaces and we have sort of a way as to how to glue X from these spaces. How to glue that, we will not make precise, but we have a way as to how to receive the homology groups, namely by the isomorphism above and the homotopy groups by $\pi_i(X) \xrightarrow{\cong} \pi_i(X_n)$.

The first theorem ($\pi_n(X) \in \mathcal{C} \Leftrightarrow H_n(X) \in \mathcal{C}$) is proved simultaneously with the following proposition:

Proposition 5.6.7 ("Hurewicz theorem modulo $\mathcal{C}$")**.**
Let X be path-connected, $\pi_1(X)$ acts trivially on $\pi_n(X)$ for $n \geq 1$ and $\pi_i(X) \in \mathcal{C}$ for $i < n$ ($n \geq 1$). Then

$$h : \pi_n(X) \to H_n(X)$$

has kernel and cokernel in $\mathcal{C}$

Proof. (only in the case of X being simply connected)
First we prove "$\Rightarrow$" of the first theorem:
Assume $\pi_i(X) \in \mathcal{C}$ for all i, then $H_k(K(\pi_i X, i)) \in \mathcal{C}$ for all $k > 0$. Now we can use the homotopy fiber of the map $X_{n+1} \to X_n$ to replace everything up to homotopy such that it is a fibration.
By induction on n and using $(*)$ we get by the proposition that $H_m(X_n) \in \mathcal{C}$ for $m > 0$. (For the fiber we derived the claim by assumption and we get by induction on n to get the claim for X_{n+1} form X_n. The base of the induction is $X_1 \simeq K(\pi_1 X, 1)$ which is actually trivial.
But $H_m(X_n) = X_n(X)$ for $n \geq m$

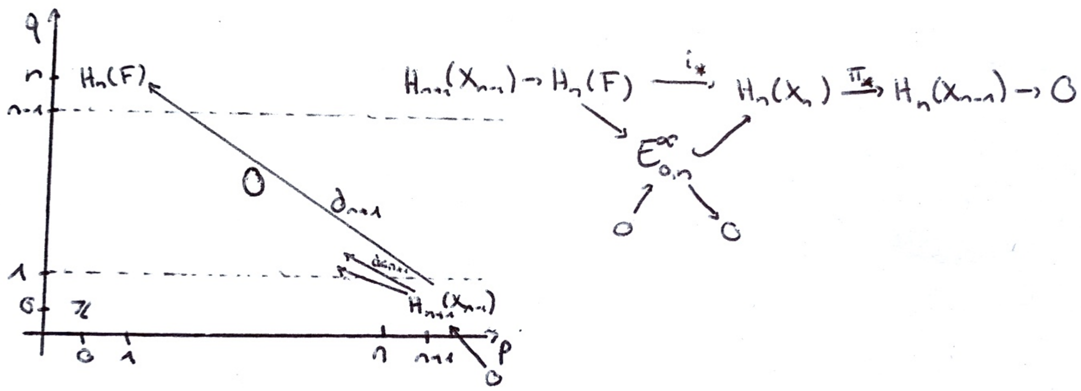
Let's prove the proposition. Note that

$$\begin{tikzcd} \pi_n(X) \arrow[r] \arrow[d, "\cong"'] & H_n(X) \arrow[d, "\cong"] \\ \pi_n(X_n) \arrow[r] & H_n(X_n) \end{tikzcd}$$

which commutes because the Hurewicz map is functorial.
So we look at the Serre spectral sequence for the fibration $(*)$:

$$\begin{tikzcd} F = K(\pi_n X, n) \arrow[r, hook] & X_n \arrow[d, "\pi"] \\ & X_{n-1} \end{tikzcd}$$

F is $(n-1)$-connected. Therefore $\tilde{H}_*(F) = 0$ for $* \leq n-1$.



d_{n+1} is the only non-trivial differential (on all previous pages the differentials go into the 0-belt) involving $H_{n+1}(X_{n-1})$ and $H_n(F)$.
So we get this exact sequence out of the Serre spectral sequence.

Assume that $\pi_i(X) \in \mathcal{C}$ for $i < n$, then $\pi_i(X_{n-1}) \in \mathcal{C}$ for all i. Then by "$\Rightarrow$" of the first theorem, $H_k(X_{n-1}) \in \mathcal{C}$ for all $k > 0$.
But now we can relate the homology of these spaces to the Hurewicz morphism:

$$\begin{tikzcd} H_n(F) \arrow[r, "i_*"] & H_n(X_n) \arrow[r, equal] & H_n(X) \\ \pi_n(F) \arrow[u, "\cong"] \arrow[r, "\text{by constr.}", "\cong"'] & \pi_n(X_n) \arrow[u, "h"'] \arrow[r, equal] & \pi_n(X) \end{tikzcd}$$

Thus the claim about h that it has kernel and cokernel in $\mathcal{C}$ is equivalent to the claim about i_* which we have just proved because the cokernel of i_* is $H_n(X_{n-1})$ which is in $\mathcal{C}$ and the kernel is a quotient of $H_{n+1}(X_{n-1})$ which in turn is in $\mathcal{C}$.
This finishes the proof of the proposition.

What is left to do is the other direction in the proof of the thoerem.
Finally, $\pi_1(X) = 0$, $H_n(X) \in \mathcal{C}$, $n > 0$.
Then by Hurewicz

$$\pi_2(X) \xrightarrow{\cong} H_2(X) \in \mathcal{C}$$

Then by Hurewicz mod $\mathcal{C}$

$$\pi_3(X) \to H_3(X)$$

with kernel and cokernel in $\mathcal{C}$. But $H_3(X) \in \mathcal{C}$ by assumption. So kernel, cokernel and $H_3(X)$ are all in $\mathcal{C}$ and thus $\pi_3(X)$ is glued out of objects from $\mathcal{C}$ and thus $\pi_3(X) \in \mathcal{C}$.
Then, however, we repeat the same argument and go so on by induction. ■

Theorem 5.6.8.
The groups $\pi_i(S^n)$ are finite for $i > n$, except for $\pi_{4k-1}(S^{2k}) = \mathbb{Z} \oplus$ finite abelian groups for $k > 0$.

Proof.
$n = 1$ is known, so let $n > 1$.
Consider the fibration $\begin{matrix} F \\ \downarrow \\ S_n \end{matrix}$ where F is the homotopy fiber of the map S^n to the Eilenberg-MacLane space

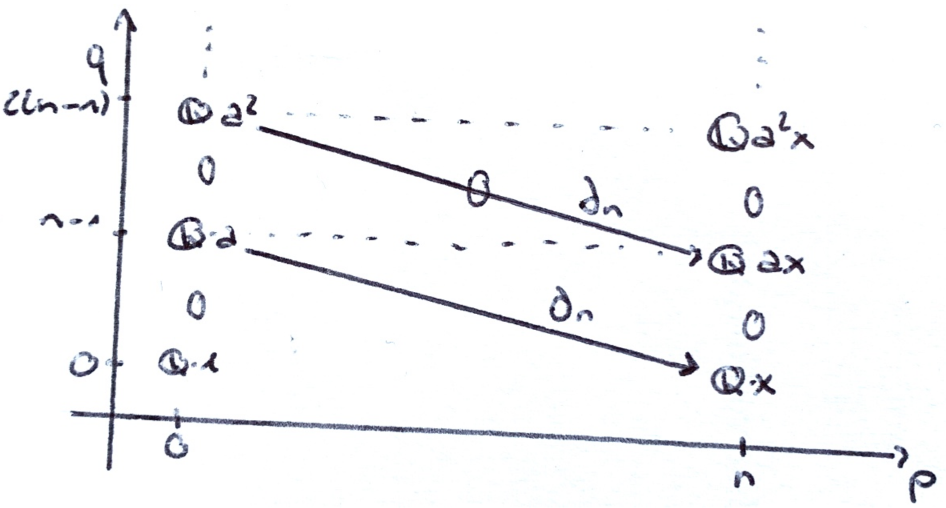
$$\begin{tikzcd}
K(\mathbb{Z}, n-1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & F \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & * \arrow[d] \\
* \arrow[r] & S^n \arrow[r] & K(\mathbb{Z}, n)
\end{tikzcd}$$

and look at the cohomological Serre spectral sequence, starting with $H^p(S^n, H^q(K(\mathbb{Z}, n{+}1), \mathbb{Q})) \implies H^{p+q}(F, \mathbb{Q})$.
There are two cases to consider, when n is even and when n is odd.

case: n is odd
In that case $n-1$ is even, thus

$$H^*(K(\mathbb{Z}, n-1), \mathbb{Q}) \cong \mathbb{Q}[a]$$

where $\deg(a) = n-1$.



$D_n : \mathbb{Q} \cdot a \to \mathbb{Q} \cdot x$ has to be an isomorphism (otherwise $H^{\leq n}(F, \mathbb{Q}) \neq 0$ but F is n-connected). Thus

$$d_n(a) = \lambda \cdot x \qquad \lambda \in \mathbb{Q}^\times$$
$$d_n(a^k) = k \cdot a^{n-1} \cdot x$$

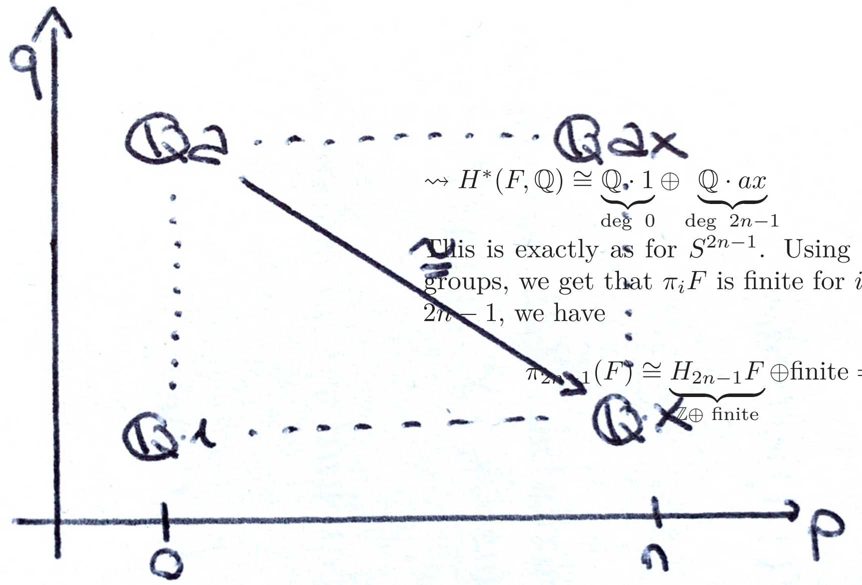
therefore $d_n : \mathbb{Q} \cdot a^k \to \mathbb{Q} \cdot a^{k-1} \cdot x$ is also an isomorphism for all k.
So, $E_3^{p,q} = E_\infty^{p,q} = 0$ unless $(p,q) = (0,0)$. Thus

$$H^m(F, \mathbb{Q}) = 0 \quad m > 0 \overset{\text{univ coeff}}{\implies} H_m(F, \mathbb{Q}) = 0 \quad m > 0$$

On the other hand, we know that $H_m(F)$ is finitely generated because we know that the the homology of S^n as well as the homology of $K(\mathbb{Z}, n)$ is finitely generated and then we can apply the lemma to get that the homology of F is finitely generated.
Thus $H_m(F)$ being finitely generated together with $H_m(F) \otimes \mathbb{Q} = H_m(F, \mathbb{Q}) = 0$ imply that $H_m(F)$ is finite.

But because $H_m(F)$ is finite for all n, it follows that $\pi_m(F)$ is finite. $\pi_m(F) = \pi_m S^n$ for $m > n$. Thus $\pi_m(S^n)$ are finite for $m > n$.

case: n is even
Therefore $n-1$ is odd and thus $H^*(K(\mathbb{Z}, n-1), \mathbb{Q}) \cong \mathbb{Q}[a]/a^2$ where $\deg(a) = n-1$.



$$\rightsquigarrow H^*(F,\mathbb{Q}) \cong \underbrace{\mathbb{Q}\cdot 1}_{\text{deg } 0} \oplus \underbrace{\mathbb{Q}\cdot ax}_{\text{deg } 2n-1}$$

This is exactly as for S^{2n-1}. Using Hurewicz mod finite groups, we get that $\pi_i F$ is finite for $i < 2n-1$. In degree $2n-1$, we have

$$\pi_{2n-1}(F) \cong \underbrace{H_{2n-1}F}_{\mathbb{Z}\oplus \text{ finite}} \oplus \text{finite} = \mathbb{Z} \oplus \text{ finite}$$

Thus $\pi_i S^n$ is finite, for $n < i < 2n-1$ because the homotopy groups of F are the same as the homotopy groups of S^n in degrees bigger than n.
Let's attach $(2n+1)$-cells to F to kill $\pi_i F$, for $i \geq 2n-1 \rightsquigarrow Y$. There is an inclusion $F \hookrightarrow Y$, we can change it to get a fibration $\begin{array}{ccc} Z & \longrightarrow & F \\ & & \downarrow \\ & & Y \end{array}$ where Z is the homotopy fiber.
So what can we say about the homotopy groups of these spaces? Y has the same homotopy groups as F in small degrees and then all are zero. Therefore

- $\pi_i Z \cong \pi_i F$ for $i \geq 2n-1$,
- Z is $(2n-2)$-connected and
- $\pi_i F \to \pi_i Y$ is an isomorphism for $i < 2n-1$

$\pi_i Y$ is finite for all i. Therefore $\tilde{H}_i Y$ is finite and thus $\tilde{H}^*(Y,\mathbb{Q}) = 0$.
Serre spectral sequence gives $H^p(Y, H^q(Z,\mathbb{Q})) \implies H^{p+q}(F,\mathbb{Q}) \rightsquigarrow H^*(Z,\mathbb{Q}) \cong H^p(F,\mathbb{Q}) \cong \mathbb{Q}[xa]/(xa)^2$

Finally, we repeat the argument of the case n is odd but not for the map from S^n to the Eilenberg-MacLane space but rather for

$$Z \to K(\mathbb{Z}, 2n-1)$$

So in this case, we know that Z has non-trivial homotopy group in the degree $2n-1$ and we will kill it by this map. This is a situation similar to what we started with: we had an odd dimensional sphere, we have killed Z in the homotopy groups of degree $2n-1$ and we have used the cohomological Serre spectral sequence because we knew the cohomology of the fiber with rational coefficients of the fiber and we still can do that here to get $\pi_i F$ is finite for $i > 2n-1$.

This is the last thing that we needed because for i in between n and $2n-1$ we have already shown that $\pi_i(S^n)$ is finite. ■

CHAPTER 6

What was not in this course but could be?

- Group (Co)Homology (and local coefficient systems)

$$H^*(K(G,1); A) \cong \mathrm{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z}, A)$$

  One can study this purely algebraically but you can also get an intuition and formulas for that by studying the space $K(G,1)$ geometrically, by studying the CW-complex $K(G,1)$ in some specific manner [Brown, Group cohomology(?)]

- Obstruction Theory
  It studies the question when one can continue a map from a subcomplex A of some CW-complex W to some space X

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & X \\ W \arrow[ur, dashed] & \end{tikzcd}$$

  Obstructions to the existence of the dashed map can be found in the relative cohomology of this pair $H^{n+1}(W, A, \pi_n X)$ [Hatcher]

  Also the characteristic classes that we have discussed of vector bundles are obstructions to the existence of r linearly independent sections of a vector bundle. [Milnor-Stasheft]

- Partitions of Unity in Homotopy Theory [tom Dieck]
  A result is for example that over a CW complex a Serre fibration is always a fibration.

- Steenrod Operations $H^*(-, \mathbb{Z}/p) \to H^*(\mathbb{Z}/p)$