Algebraic Topology 2 - Lecture notes

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Contents

Preliminary warning: These notes are not proof-read by anyone, and are thus very likely full of formal and conceptual errors. It is highly advised to use them with caution and scepticism.

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Conventions

All maps are assumed to be continuous. The set $\mathbb{N}$ of natural numbers contains 0 (usually). Notation:

- $*$: Single-point space
- $I := [0,1]$: Standard interval; I^n: standard cube
- D^n: n-disk, S^{n-1}: $n-1$-sphere
- For a category $\mathcal{C}$, we let $\mathrm{Ob}(\mathcal{C})$ be the class of its objects and for $x, y \in \mathrm{Ob}(\mathcal{C})$ we let $\mathrm{Mor}_{\mathcal{C}}(x,y)$ be the class of morphisms between x and y. Sometimes we will abbreviate $x, y \in \mathcal{C}$ for $x, y \in \mathrm{Ob}(\mathcal{C})$.
- Top: Category of topological spaces.
- Top^2: Category of pairs of spaces (i.e. objects in Top^2 are of the form (A,B), where $B \subset A$, and a morphism $f : (A,B) \to (C,D)$ is a map $f : A \to C$ with $f|_B : B \to D$. Similarly define triples of spaces, etc. We will omit the curly brackets if the subspace is just a point: $(X,x) := (X,\{x\})$. The category Top_* of pointed topological spaces is (in a suitable sense) a subcategory of Top^2.
- Sets, Sets_*: Categories of sets and pointed sets.
- Grp, Ab: Categories of groups and abelian groups.
- We use the notation "$f \sim g$" if two maps f, g are homotopic. If they are homotopic through a homotopy H, we indicate this by $f \sim^H g$. We use the notation "$X \simeq Y$" if two spaces X, Y are homotopy equivalent. We use "$X \cong Y$" if they are homeomorphic.
- If $F : \mathcal{C} \to \mathcal{D}$ is a covariant functor from a category $\mathcal{C}$ to a category $\mathcal{D}$, $x, y \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Mor}_{\mathcal{C}}(x,y)$, then we denote the induced map by $f_* \in \mathrm{Mor}_{\mathcal{D}}(F(x), F(y))$, or by $F(f)$ or Ff. If $F : \mathcal{C} \to \mathcal{D}$ is instead a contravariant functor, we denote the induced map by $f^* \in \mathrm{Mor}_{\mathcal{D}}(F(y), F(x))$ (or by $F(f)$ or Ff).
- For a point $x \in X$ in some topological space X, we abbreviate the constant map $Y \to X$, $y \mapsto x$ (for any other topological space Y), simply by x.
- We denote equivalence classes by square brackets, e.g. a map $f : (S^n, *) \to (X, x)$ defines an element $[f] \in \pi_n(X, x)$. However sometimes we will simply write f and mean "f up to some equivalence".
- For a homotopy $H : X \times I \to Y$ between two maps $f, g : X \to Y$, we write $H_t(x) := H(x,t)$ for $x \in X$.
- We denote the neutral element in a group sometimes by 1, sometimes by e and sometimes by 0; there should not arise confusions.

– If X, Y are topological spaces, then we denote the set of maps from X to Y modulo homotopy by $[X, Y]$. If we consider pointed spaces (X, x), (Y, y) and pointed maps modulo pointed homotopy, we denote the corresponding set by $[(X, x), (Y, y)]_*$ or sometimes abbreviatedly by $[X, Y]_*$. Thus e.g. $\pi_n(X) = [S^n, X]_*$.

– $A \times_B^h C$: homotopy pullback, $A \cup_B^h C$: homotopy pushout.

– $\mathrm{St}(\cdot)$: Stabilizer (for group action).

– $\mathrm{Gr}^{\mathbb{K}}(k, n)$: Grassmann manifold (or Grassmannian), $V^{\mathbb{K}}(k, n)$: Stiefel manifold.

– $\underline{\mathbb{R}}, \underline{\mathbb{R}}^n, \underline{\mathbb{C}}, \underline{\mathbb{C}}^n, \ldots$: Trivial bundles.

– $\mathcal{O}(-1)$: Tautological (real resp. complex) line bundle on $\mathbb{R}P^n$ (resp. $\mathbb{C}P^n$), $\mathcal{O}(1)$: its dual.

1 Elementary methods in homotopy theory

Literature:

– A. Hatcher, *Algebraic Topology*

– T. tom Dieck, *Algebraic Topology*

1.1 Homotopy groups of pairs, long exact sequence

Recall the definition of homotopy groups:

Definition 1.1.1 (Homotopy group)**.** Let (X, x) be a pointed topological space and $n \in \mathbb{N}$. We define the n'th *homotopy group* of (X, x) by

$$\pi_n(X, x) := \{f : (I^n, \partial I^n) \to (X, x)\}/\sim,$$

where

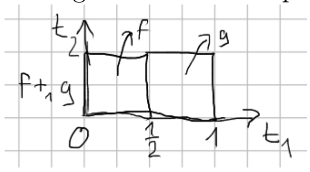
$$f \sim g :\Leftrightarrow \exists H : I^n \times I \to X \text{ such that} H|_{I\times\{0\}} = f, H|_{I\times\{1\}} = g,$$
$$\forall t \in I : H_t := H|_{I^n\times\{t\}} : (I^n, \partial I^n) \to (X, x) \text{ (is a map between pairs of spaces).}$$

[1]

Remark 1.1.2. Recall that $\pi_n : \mathrm{Top} \to \mathrm{Sets}_*$ is a functor. For $n \geq 1$, even $\pi_n(X, x) \in \mathrm{Grp}$, and for $n \geq 2$, $\pi_n(X, x) \in \mathrm{Ab}$: For $f, g : (I^n, \partial I^n) \to (X, x)$, one can define a composition

$$f +_1 g : (I^n, \partial I^n) \to (X, x), \qquad t \mapsto \begin{cases} f(2t_1, t_2, \dots, t_n), & t \leq \frac{1}{2} \\ g(2t_1 - 1, t_2, \dots, t_n), & t \geq \frac{1}{2}. \end{cases}$$

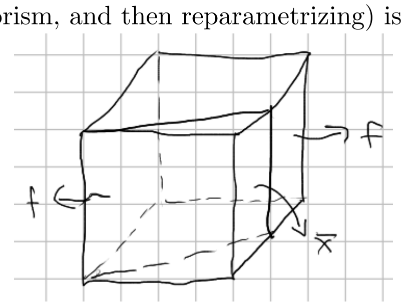
We can glue these two maps because the boundary ∂I^n goes to X. Picture:



(Remark: We define $(I^0, \partial I^0) = (*, \emptyset)$.)

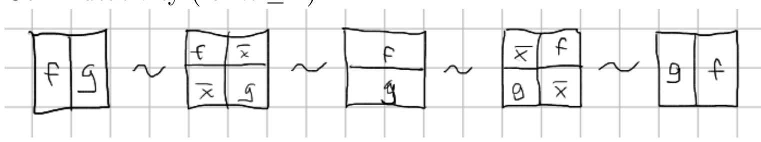
The operation $+_1$ is well-defined, i.e. compatible with $\sim$. Proof: Let $H : I^n \times I \to X : f_1 \sim f_2$ be a homotopy. Since $H|_{\partial I^n \times I} = \{x\}$, the "side" faces are sent by H to x. We get a homotopy $f_1 +_1 g \sim f_2 +_1 g$ by glueing to H the trivial homotopy $g \sim g$ and reparametrizing (in the t_1-direction).

Neutral element: $\overline{x} : (I^n, \partial I^n) \to (\{x\}, x)$. (Often it will also be denoted simply x.) Proof: The cube below (which is obtained as the glueing of the cube $f \sim f$ and the constant map to x in the prism, and then reparametrizing) is a homotopy from f to $f +_1 \overline{x}$.



[1]In this case, we say that f and g are homotopic as maps between pairs of spaces.

Commutativity (for $n \geq 2$):



Now define relative homotopy groups:

Definition 1.1.3. Let (X, A, x) be a pointed pair of topological spaces, i.e. $x \in A \subset X$, and let $n \geq 1$. Let $J^{n-1} := \partial I^{n-1} \times I \cup I^{n-1} \times \{0\} \subset \partial I^n$. We define the n'th *relative homotopy group* of (X, A, x) by

$$\pi_n(X, A, x) := \{f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x)\}/\sim,$$

where

$$f \sim g :\Leftrightarrow \exists H : I^n \times I \to X \,\text{such that}\; H_0 = f, H_1 = g,$$
$$\forall t \in I : H_t(I^n, \partial I^n, J^{n-1}) \to (X, A, x) \,\text{(is a map between triples of spaces).}$$

Remark 1.1.4.

- The definition of $\pi_n(X, A, x)$ is equivalent to what we obtained if we used

$$\{(D^n, S^{n-1}, *) \to (X, A, x)\}/\sim .$$

- We have an isomorphism $\pi_n(X, \{x\}, x) \cong \pi_n(X, x)$.
- For $n \geq 2$, $\pi_n(X, A, x)$ has a group structure, and this group is Abelian if $n \geq 3$.
- A map $f : (X, A, x) \to (Y, B, y)$ between triples of spaces induces a map $f_* : \pi_n(X, A, x) \to \pi_n(Y, B, y)$, $\phi \mapsto f_*(\phi) := f \circ \phi$.[2]
- We have a diagram of inclusions

$$(A, \{x\}, x) \xrightarrow{i} (X, \{x\}, x) \xrightarrow{j} (X, A, x) .$$

  Let us define a map $\partial : \pi_n(X, A, x) \to \pi_{n-1}(A, x) \cong \pi_{n-1}(A, \{x\}, x)$ as follows: For $f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x)$, let

$$\partial f := f|_{I^{n-1} \times \{1\}} : (I^{n-1}, \partial I^{n-1}) \to (X, x).$$

  (This yields a well-defined map between equivalence classes of maps in π_n.)

Theorem 1.1.5 (The long exact sequence of homotopy groups)**.** *The following sequence of pointed sets and groups is exact:*

$$\ldots \longrightarrow \pi_n(A, x) \xrightarrow{i_*} \pi_n(X, x) \xrightarrow{j_*} \pi_n(X, A, x) \xrightarrow{\partial} \pi_{n-1}(A, x) \longrightarrow \ldots .$$

$$\ldots \longrightarrow \pi_1(X, A, x) \xrightarrow{\partial} \pi_0(A, x) \xrightarrow{i_*} \pi_0(X, x).$$

Proof.

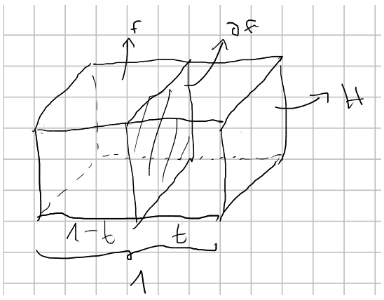
- Exactness at $\pi_n(X, A, x)$: $\partial \circ j_* = 0$, since for $f : (I^n, \partial I^n) \to (X, x)$, we observe that $j \circ f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x)$ satisfies $\partial(j \circ f) = x$ by definition of ∂.

  Furthermore $\ker \partial \subset \mathrm{im} j_*$: Let $f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x)$ with $\partial f = 0$, i.e. $\partial f = f|_{I^{n-1} \times \{1\}} \sim x$. I.e.

$$\exists H : I^{n-1} \times I \to A \subset X \,\text{with}\, H_0 = \partial f, H_1 = x, H(\partial I^{n-1} \times I) = \{x\}.$$

  Note $I^{n-1} \times I = I^n$. Define $F : I^n \times [0, 1/2] \to X$ by "glueing increasingly large parts of H to f". Picture of F_t (for $n = 3$, using the same notation as for homotopies):

[2] I.e. π_n as defined above is again a functor.



Thus

$$F_0 = f, \qquad \forall t \in (0,1/2] : F_t : I^n \to X, \begin{cases} F_t|_{I^{n-1}\times[0,1-t]}(x_1,\dots,x_n) = f(x_1,\dots,x_{n-1},x_n/(1-t)), \\ F_t|_{I^{n-1}\times[1-t,1]}(x_1,\dots,x_n) = H(x_1,\dots,x_{n-1},2(x_n-t+1)) \end{cases}.$$

I.e. we speed up f in the n'th coordinate by a factor of $1/(1-t)$ for $t > 0$, and then we glue H to it, which we speed up by a constant factor of 2.

Then by construction $F_{1/2} \in \pi_n(X,x)$ and $j_*(F_{1/2}) = [f]$ (i.e. they lie in the same homotopy class): $F_{t/2}$ is a homotopy in t between f and $F_{1/2}$.

- Exactness at $\pi_n(A,x)$: $i_* \circ \partial = 0$: Consider $f : (I^n, \partial I^n, J^{n-1}) \to (X,A,x)$. We need to show $\partial f = f|_{I^{n-1}\times\{1\}} : I^{n-1} \sim x$. The required homotopy is just given by f itself (up to inverting the parametrization): We have $H = f : I^{n-1} \times I \to X$ with $H_1 = \partial f$, $H_0 = x$ and $H_t : (I^{n-1}, \partial I^{n-1}) \to (X,x)$ for all $t \in I$.

  Furthermore $\ker i_* \subset \mathrm{im}\partial$: Let $f : (I^n, \partial I^n) \to (A,x)$ with $i \circ f \sim^H x$, i.e. $H : I^n \times I \to X$ with $H_0 = i \circ f$, $H_1 = x$, $H(\partial I^n \times I) = x$. Consider then the "mirrored homotopy" $\overline{H}(x_1,\dots,x_{n+1}) := H(x_1,\dots,1-x_{n+1})$. Then $\overline{H} : I^{n+1} \to X$ can be regarded also as an element of $\pi_n(X,A,x)$ which defines a preimage of f under ∂, since $\overline{H}$ sends ∂I^{n+1} to A and J^n to x.

- Exactness at $\pi_n(X,x)$: Exercise. It is helpful to imagine the case $n = 2$, when the homotopy can be imagined in 3D.

□

1.2 Serre fibrations

Definition 1.2.1. A map $p : E \to B$ satisfies the *homotopy lifting property* (HLP) with respect to a space X if for every commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook, "{(\mathrm{id},0)}"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "h"] & B \end{tikzcd}$$

there exists a map $\tilde{h} : X \times I \to E$ (a "lift") such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook, "{(\mathrm{id},0)}"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "h"] \arrow[ur, dashed, "\tilde{h}"] & B \end{tikzcd}$$

is commutative. In other words, $\tilde{h}$ satisfies $\tilde{h} \circ (\mathrm{id}, 0) = g$, $p \circ \tilde{h} = h$.

Definition 1.2.2. A map $p : E \to B$ is called a *Serre fibration*, if it has the HLP with respect to I^n for all $n \geq 0$.

Remark 1.2.3. – The HLP is stable under pullbacks and homeomorphisms.

– In Algebraic Topology 1, we saw that every covering map $p : E \to B$ with fiber F is a Serre fibration: We know that

$$1 \longrightarrow \pi_1(E, e) \longrightarrow \pi_1(B, b) \longrightarrow F \cong \pi_0 F \longrightarrow \pi_0 E \longrightarrow \pi_0 B$$

is exact for any $e \in E$, $b \in B$ (the map $\pi_1(B, b) \to \pi_0 F$ is given by lifting loops in B to paths in E and looking at their endpoints). In other words, $\pi_1(E, e) \to \pi_1(B, b)$ is injective, implying the HLP for $n = 1$ (at least when considering pointed maps). For $n = 0$ the required statement is just the path-lifting property. For $n \geq 2$ we had even an isomorphism $\pi_n(E, e) \cong \pi_n(B, b)$.

Proposition 1.2.4. *Being a Serre fibration is local on the base, i.e.: If $p : E \to B$ is a map and there exists an open cover $\{U_\alpha\}_\alpha$ of B such that $p|_{p^{-1}(U_\alpha)} : p^{-1}(U_\alpha) \to U_\alpha$ is a Serre fibration for all α, then p is a Serre fibration.*

The proof uses similar constructions as the proof of proposition 1.3.2 below.

Remark 1.2.5. 1. Fibrations in general are *not* local on the base.

2. In the category of pointed CW-complexes, a Serre fibration is the same as a fibration (without proof here).

Theorem 1.2.6. *If $p : E \to B$ is a Serre fibration, then for $n \geq 1$, $e \in E$, $b = p(e)$, $B_0 \subset B$, $E_0 := p^{-1}(B_0)$, we have an isomorphism*

$$p_* : \ \pi_n(E, E_0, e) \xrightarrow{\cong} \pi_n(B, B_0, b) \ .$$

Before proving it, we note that due to the long exact sequence of homotopy groups, we get:

Corollary 1.2.7 (Long exact sequence of a Serre fibration)**.** *If we choose $B_0 := \{b\}$ and define $E_0 =: F = p^{-1}(b)$, then there is a long exact sequence*

$$\ldots \longrightarrow \pi_n(F, e) \longrightarrow \pi_n(E, e) \longrightarrow \pi_n(B, b) \cong \pi_n(E, F, e) \longrightarrow \pi_{n-1}(F, e) \longrightarrow \ldots$$

$$\ldots \longrightarrow \pi_0(F, e) \longrightarrow \pi_0(E, e) \longrightarrow \pi_0(B, b) \longrightarrow 0 \ .$$

The isomorphism $\pi_n(B, b) \cong \pi_n(E, F, e)$ is precisely given by p_*.

[3]

Lemma 1.2.8. *If $p : E \to B$ is a Serre fibration, then if the solid arrows in the diagram*

$$\begin{tikzcd} J^{n-1} \arrow[r] \arrow[d, hook, "\mathrm{incl}"'] & E \arrow[d, "p"] \\ I^n \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

define a commutative diagram, there exists a lift as indicated by the dashed arrow.

Proof. There exists a homeomorphism ϕ such that the diagram

$$\begin{tikzcd} I^{n-1} \times \{0\} \arrow[r, "\cong"] \arrow[d, hook] & J^{n-1} \arrow[r] \arrow[d, hook, "\mathrm{incl}"'] & E \arrow[d, "p"] \\ I^{n-1} \times I \arrow[r, "\cong", "\phi"'] & I^n \arrow[r] & B \end{tikzcd}$$

commutes. Then use the HLP for I^{n-1}. □

[3]From the requirement of the HLP for I^0 it becomes clear that the map $\pi_0(E, e) \to \pi_0(B, b)$ must be surjective.

Proof of theorem 1.2.6. p_* is surjective: Let $f : (I^n, \partial I^n, J^{n-1}) \to (B, B_0, b)$. By the lemma, we get $g : (I^n, \partial I^n, J^{n-1}) \to (E, E_0, e)$ such that the diagram

$$\begin{tikzcd} J^{n-1} \arrow[r] \arrow[d, hook] & \{e\} \arrow[r, hook] & E \arrow[d, "p"] \\ I^n \arrow[urr, dashed, "g"] \arrow[rr, "f"] & & B \end{tikzcd}$$

commutes. I.e. we have $p \circ g = f$. Note that this is a map of triples, also because the diagram commutes. (We must have $g(J^{n-1}) = e$ directly from the commutativity, and furthermore $g(\partial I^n) \subset p^{-1}(B_0) = E_0$ because $f(\partial I^n) \subset B_0$.) Thus $p_* g = f$, i.e. p_* is surjective.

p_* is injective: Let $f_0, f_1 : (I^n, \partial I^n, J^{n-1}) \to (E, E_0, e)$ with $p \circ f_0 \sim^H p \circ f_1$. I.e. $H : I^n \times I \to B$, $H_0 = p \circ f_0$, $H_1 = p \circ f_1$, $H_t : (I^n, \partial I^n, J^{n-1}) \to (B, B_0, b)$ is a map of triples for all t. Let $T := I^n \times \partial I \cup J^{n-1} \times I$ and define

$$G : T \to E, \qquad G(u,t) := \begin{cases} f_0(u), & t = 0 \\ f_1(u), & t = 1 \\ e, & u \in J^{n-1} \end{cases}$$

We have a commutative diagram

$$\begin{tikzcd} J^n \arrow[r, "\cong"] \arrow[d, hook] & T \arrow[d] \\ I^{n+1} \arrow[r, "\Psi"] & I^n \times I \end{tikzcd}$$

with $\Psi(t_1, \ldots, t_n, t_{n+1}) = (t_1, \ldots, t_{n+1}, t_n)$. Similar to the lemma, therefore we obtain (since p is a Serre fibration) a lift $\tilde{H}$ in the diagram

$$\begin{tikzcd} T \arrow[r, "G"] \arrow[d, hook] & E \arrow[d, "p"] \\ I^n \times I \arrow[ur, dashed, "\tilde{H}"] \arrow[r, "H"] & B \end{tikzcd} .$$

Then $\tilde{H}$ is a homotopy between f_0 and f_1 of pointed pairs: Since

$$\begin{tikzcd} \partial I^n \arrow[r, "\tilde{H}_t"] \arrow[dr, "H_t"] & E \arrow[d, "p"] \\ & B_0 \end{tikzcd}$$

commutes for all t, $\tilde{H}_t|_{\partial I^n}$ must go to E_0. Thus p_* is injective. $\square$

1.3 Hopf fibration (Hopf bundle)

Recall the definition of a fiber bundle: $p : E \to B$ is a fiber bundle with fiber F if there exists an open cover $\bigcup_\alpha B_\alpha = B$, such that for all α, there exists a commutative diagram

$$\begin{tikzcd} p^{-1}(B_\alpha) \arrow[d, "p"'] & B_\alpha \times F \arrow[l, "\cong"'] \arrow[dl, "\mathrm{pr}_1"] \\ B_\alpha & \end{tikzcd} .$$

Definition 1.3.1. The *Hopf map* $\eta : S^3 \to S^2$ is a fiber bundle with fiber S^1. It is defined by the commutative diagram

$$\begin{tikzcd}
S^3 \arrow[d, "\eta"'] & \subset & \mathbb{R}^4\backslash\{0\} \cong \mathbb{C}^2\backslash\{0\} \arrow[d, "\pi"] \\
S^2 & \cong & \mathbb{C}P^1 = \{[z_1 : z_2] \,|\, (z_1, z_2) \neq 0, (z_1, z_2) \in \mathbb{C}^2/\mathbb{C}^*\}
\end{tikzcd},$$

where $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$. The fibers are S^1, since $\pi^{-1}([z_0 : z_1]) = \{(\mu z_0, \mu z_1) \,|\, \mu \in \mathbb{C}^*\}$ for $[z_0 : z_1] \in \mathbb{C}P^1$. Without loss of generality we have chosen a representative such that $|z_0|^2 + |z_1|^2 = 1$, then $\eta^{-1}([z_0 : z_1])$ consists of $(\mu z_0, \mu z_1)$ with $\mu \in \mathbb{C}^*$ and such that $|\mu z_0|^2 + |\mu z_1|^2 = 1$, i.e. $|\mu|^2 = 1$ resp. $\mu \in S^1 \subset \mathbb{C}^*$.

Proposition 1.3.2. *Every fiber bundle is a Serre fibration.*

Remark 1.3.3. From the proposition, we get a long exact sequence, for $n \geq 3$:

$$\ldots \longrightarrow \pi_n(S^1) = 0 \longrightarrow \pi_n(S^3) \xrightarrow[\cong]{\eta_*} \pi_n(S^2) \longrightarrow \pi_{n-1}(S^1) = 0 \longrightarrow \ldots .$$

Furthermore we get

$$\ldots \longrightarrow \pi_2(S^1) = 0 \longrightarrow \pi_2(S^3) = 0 \longrightarrow \pi_2(S^2) \xrightarrow[\cong]{} \pi_1(S^1) \cong \mathbb{Z} \longrightarrow \pi_1(S^3) = 0 \longrightarrow \ldots,$$

since it is a known fact that $\pi_k(S^n) = 0$ for $k < n$.[4] In particular we obtain

$$\begin{tikzcd}
\pi_3(S^3) \arrow[r, "\eta_*", "\cong"'] & \pi_3(S^2) \neq 0 . \\
{[\mathrm{id}_{S^3}]} \arrow[r, mapsto] & {[\eta]}
\end{tikzcd}$$

[5] Thus $\pi_3(S^2) \cong \mathbb{Z} \cdot [\eta]$, i.e. it is not zero, but isomorphic to $\mathbb{Z}$ with the generator $[\eta]$.

For the proof of the proposition, we will need the Lebesgue lemma:

Lemma 1.3.4 (Lebesgue lemma)**.** *Let X be a compact metric space. Let $\mathcal{A} := \{V_\alpha\}_\alpha$ be an open cover of X. Then ther exists $\epsilon > 0$ such that for all $x \in X$, there exists α such that $B_\epsilon(x) \subseteq V_\alpha$.*

Remark 1.3.5. Application of the Lebesgue lemma: Let $\mathcal{A} = \{V_\alpha\}_\alpha$ be an open cover of the cube I^n. Consider subdivisions of I^n of the form $I^n = \bigcup_{0\leq j\leq N} \prod_j [\frac{k_j}{N}, \frac{k_j+1}{N}]$, where k_j, $j = 0, \ldots, N$ such that $[\frac{k_j}{N}, \frac{k_j+1}{N}]$ form a subdivision of I. By Lebesgue's lemma, there is some such subdivision such that each cube is entirely contained in some V_α.

Proof of proposition 1.3.2. There exists an open cover $\{U_\alpha\}$ of B such that

$$\begin{tikzcd}
U_\alpha \times F \cong p^{-1}(U_\alpha) \arrow[r, hook] \arrow[d, "\mathrm{pr}"'] & E \arrow[d, "p"] \\
U_\alpha \arrow[r, hook] & B
\end{tikzcd}$$

commutes for all α.

1. At first, note that every trivial fiber bundle is a Serre fibration: In

$$\begin{tikzcd}
I^n \times \{0\} \arrow[r, "g"] \arrow[d, hook] & U \times F \arrow[d, "\mathrm{pr}_1"] \\
I^n \times I \arrow[r, "h"] \arrow[ur, dashed, "H"] & U
\end{tikzcd},$$

[4]It follows e.g. from cellular approximation, theorem 1.6.8 later.

[5]$\pi_3(S^3) \cong \mathbb{Z} \cdot [\mathrm{id}_{S^3}]$ due to the Hurewicz theorem 1.9.8 later.

the lift can be defined by $H(u,t) = (h(u,t), g(u,0))$.

2. Now consider

$$\begin{tikzcd} I^n \times \{0\} \arrow[r, "g"] \arrow[d, hook] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "h"] \arrow[ur, dashed, "H"] & B \end{tikzcd},$$

where the solid arrows are given and we want to find the lift H. Let $\{U_\alpha\}$ be a trivializing open cover for p. Then $\{h^{-1}U_\alpha\}$ is an open cover of $I^n \times I$. Thus by the Lebesgue lemma (resp. remark 1.3.5) there exist $0 = t_1 \le \cdots \le t_N = 1$ such that there is a subdivision of $I^n \times I$ into cuboids $C \times [t_j, t_{j+1}]$, where C are certain cubes and we have $C \times [t_j, t_{j+1}] \xrightarrow{h} U_\alpha \subset B$ for all such cuboids.

Let $V^k \subset I^n$ be the union of k-dimensional faces of the cubes C. By induction the solid arrows in the diagram

$$\begin{tikzcd} I^n \times \{0\} \cup V^{k-1} \times [0,t_1] \arrow[r, "H_{(k-1)}"] \arrow[d, hook] & E \arrow[d, "p"] \\ I^n \times \{0\} \cup V^k \times [0,t_1] \arrow[r, "h"] \arrow[ur, dashed, "H_{(k)}"] & B \end{tikzcd}$$

exist. We want to perform the induction step, thus we need to define $H_{(k)}$. To this end, consider a cube W of V^k. Then we have $\partial W \subset V^{k-1}$. Therefore, by lemma 1.2.8, there exists a lift H_W in the commutative diagram

$$\begin{tikzcd} J^k \arrow[r, "\cong"] \arrow[d, hook] & W \times \{0\} \cup \partial W \times [0,t_1] \arrow[r, "H_{(k-1)}"] \arrow[d, hook] & U_\alpha \times F \arrow[r, hook] \arrow[d] & E \arrow[d, "p"] \\ I^{k+1} \arrow[r, "\cong"] & W \times [0,t_1] \arrow[r, "h"] \arrow[ur, dashed, "H_W"] & U_\alpha \arrow[r, hook] & B \end{tikzcd}.$$

Glue all these H_W together to obtain $H_{(k)}$. For $k = n$, we obtain

$$\begin{tikzcd} & E \arrow[d, "p"] \\ I^n \times [0,t_1] \arrow[r, "h"] \arrow[ur, "H_{(n)}"] & B \end{tikzcd}.$$

The same argument for $[t_1, t_2], \ldots,$ leads to the map $I^n \times I \to E$ (after finitely many steps).

□

1.4 Cofibrations

Definition 1.4.1. A map $i : A \to X$ satisfies the *homotopy extension property* (HEP) for a space T, if for $f : A \times I \to T$, $f : X \to T$ such that $f \circ i = h|_{A \times \{0\}}$, there exists $H : X \times I \to T$ such that $H|_{X \times \{0\}} = f$, $H \circ (i \times \mathrm{id}) = h$. In diagrams: If the solid arrows in the diagram below form a commutative square, then there exists the dashed arrow making the whole diagram commutative.

$$\begin{tikzcd}
A \arrow[rr, hook, "{(\mathrm{id},0)}"] \arrow[dd, hook, "i"'] & & A \times I \arrow[dl, "\forall h"'] \arrow[dd, hook, "{(i,\mathrm{id}_I)}"] \\
& Y & \\
X \arrow[ur, "\forall f"] \arrow[rr, hook, "{(\mathrm{id},0)}"'] & & X \times I \arrow[ul, dashed, "\exists H"]
\end{tikzcd} .$$

Remark 1.4.2. The HEP is equivalent to the existence of a lift H in the (commutative) diagram

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "i"'] & T^I \arrow[d, "\mathrm{ev}_0"] & \ni & (g : I \to T) \arrow[d, mapsto, "\mathrm{ev}_0"] \\
X \arrow[r, "f"'] \arrow[ur, dashed, "H"] & T & \ni & g(0)
\end{tikzcd} ,$$

where $T^I = \{g : I \to T\}$ is the space of maps $I \to T$ (equipped with the compact-open topology) and ev_0 denotes the evaluation at zero. Indeed, $H|_{X\times\{0\}} = f \Leftrightarrow \mathrm{ev}_0 \circ H = f$, $H \circ (i \times \mathrm{id}) = h \Leftrightarrow H \circ i = h$.

Definition 1.4.3. $i : A \to X$ is a *cofibration* if it satisfies the HEP for all spaces.

We define $i : A \to X$ to be a *Serre cofibration*, if it satisfies the HEP for cubes, but this is usually not of much interest.

Exercise 1.4.4. If $i : A \to X$ is a cofibration, then

- it is injective and a homeomorphism onto its image.
- if X is Hausdorff, then $A \subset X$ is a closed subset.

Definition 1.4.5. The pushout $\mathrm{Cyl}(i)$ in the diagram

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, hook, "{(\mathrm{id},0)}"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "j"] \\
A \times I \arrow[r, "\hbar"'] & \mathrm{Cyl}(i)
\end{tikzcd}$$

is called the *mapping cylinder* of i.

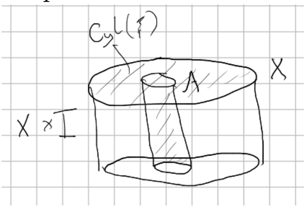
(The symbol "$\ulcorner$" denotes the pushout diagram.) By the universal property of the pushout, for any T, if the solid arrows in the diagram below commute, then there exists a map as indicated by the dashed arrow (still making the diagram commutative).

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, hook, "{(\mathrm{id},0)}"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "j"] \arrow[ddr, "f"] & \\
A \times I \arrow[r, "\hbar"'] \arrow[drr, "h"'] & \mathrm{Cyl}(i) \arrow[dr, dashed] & \\
& & T
\end{tikzcd} .$$

In particular we obtain therefore a map $s : \mathrm{Cyl}(i) \to X \times I$ by the diagram

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, hook, "{(\mathrm{id},0)}"'] & X \arrow[d, "j"] \arrow[ddr, "{(\mathrm{id},0)}"] & \\
A \times I \arrow[r, "\hbar"] \arrow[drr, "i\times\mathrm{id}"'] & \mathrm{Cyl}(i) \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "s"] & \\
 & & X \times I
\end{tikzcd}.$$

Example: If $A \subset X$ is a subspace, this looks like



.

Proposition 1.4.6. *For $i : A \to X$, the following are equivalent (TFAE):*

1. *i is a cofibration.*
2. *i satisfies the HEP for* $\mathrm{Cyl}(i)$*.*
3. *$s : \mathrm{Cyl}(i) \to X \times I$ has a retraction $r : X \times I \to \mathrm{Cyl}(i)$.*

Proof.
- 1. $\Rightarrow$ 2. by definition.
- 2. $\Rightarrow$ 3. $(\hbar, j)$ can be extended to $H : X \times I \to \mathrm{Cyl}(i)$. Verify commutativity in the diagram

$$\begin{tikzcd}
X \arrow[d, "j"] \arrow[dr, "{(\mathrm{id},0)}"] \arrow[drr, "j"] & & \\
\mathrm{Cyl}(i) \arrow[r, "s"] & X \times I \arrow[r, "H"] & \mathrm{Cyl}(i) \\
A \times I \arrow[u, "\hbar"] \arrow[ur, "i\times\mathrm{id}"] \arrow[urr, "\hbar"'] & &
\end{tikzcd}$$

Thus $H \circ s = \mathrm{id}_{\mathrm{Cyl}(i)}$ (since they agree on the images of $A \times I$ and X in $\mathrm{Cyl}(i)$) and we have found the desired retraction.

- 3. $\Rightarrow$ 1. Given $h : A \times I \to T$, $f : X \to T$ with $f \circ i = h|_{A\times\{0\}}$, consider

$$\begin{tikzcd}
X \arrow[d, "j"] \arrow[ddrrr, "f"] & & & \\
\mathrm{Cyl}(i) \arrow[dr, "s"] \arrow[drr, "\mathrm{id}"] \arrow[drrr, "{(h,f)}"] & & & \\
A \times I \arrow[u, "\hbar"] \arrow[r, "i\times\mathrm{id}"] & X \times I \arrow[r, "r"] \arrow[rr, bend right, "H"'] & \mathrm{Cyl}(i) \arrow[r, "{(h,f)}"'] & T
\end{tikzcd}$$

where (h, f) is a suggestive notation for the push-out map from $\mathrm{Cyl}(i)$ to T induced by h and f and $H := (h, f) \circ r$ is defined as the composition. Verify commutativity in this diagram, and then note that the restriction of H to X gives f and the restriction to $i(A) \times I$ gives h. Thus H extends (f, h).

$\square$

Corollary 1.4.7. *If A is a closed subset in X and there exists a retraction $r : X \times I \to X \times \{0\} \cup A \times I$, then $A \hookrightarrow X$ is a cofibration.*

Proof. Consider

$$\begin{tikzcd}
A \arrow[r, hook, "i"] \arrow[d, hook, "{(\mathrm{id},0)}"'] & X \arrow[d, hook] \arrow[ddr, "\eta"] & \\
A \times I \arrow[r] \arrow[drr, "{(0,\mathrm{id})}"'] & \mathrm{Cyl}(i) \arrow[ul, phantom, "\ulcorner"] \arrow[dr, dashed, "\exists! \phi"] & \\
& & X \times \{0\} \cup A \times I \subseteq X \times I
\end{tikzcd}.$$

ϕ is a continuous bijection. Furthermore, since $A \subset X$ is closed, ϕ is a homeomorphism. (ϕ is open because an open set in $\mathrm{Cyl}(A \hookrightarrow X)$ is of the form (U, W) with $U \subseteq X$ open, $W \subseteq A \times I$ open. This gets mapped onto something open in $X \times \{0\} \cup A \times I$, since $U \backslash A$ is open and $W \cap X \times \{0\} \subseteq X \times \{0\} \cong X$ must be contained in U. Then every point in $\phi(U, W)$ can be shown to have a small neighbourhood contained in $\phi(U, W)$ directly from the definitions.)

Then

$$\begin{tikzcd}
\mathrm{Cyl}(A \hookrightarrow X) \arrow[r, "\phi", "\cong"'] \arrow[dr, "s"'] & X \times \{0\} \cup A \times I \arrow[d, hook] \\
& X \times I
\end{tikzcd}$$

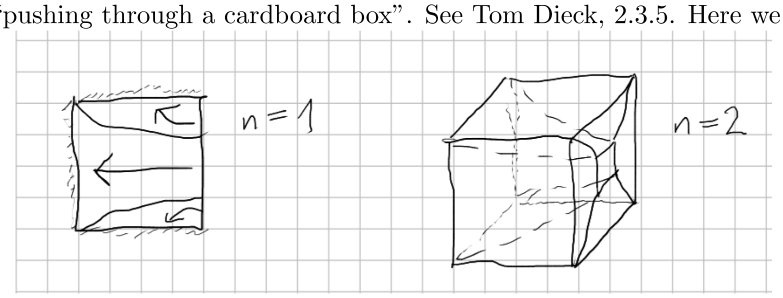
commutes; thus $\phi^{-1} \circ r$ yields a retraction of s. One applies the proposition before to obtain the claim. □

Proposition 1.4.8. *If $f : A \to X$ is a closed cofibration and Y is any topological space, then $f \times \mathrm{id}_Y : A \times Y \to X \times Y$ is a closed cofibration.*

Proof. There exists a retraction $r : X \times I \to X \times \{0\} \cup A \times I$. Then by multiplying with id_Y (and swapping factors in the product) we obtain a retraction $X \times Y \times I \to X \times Y \times \{0\} \cup A \times Y \times I$. So we can conclude by the corollary above. □

Example 1.4.9 (Main example)**.** $\partial I^n \subset I^n$ and $S^{n-1} \subset D^n$ are closed cofibrations.

To prove this, we need to construct a retraction $I^n \times I \to I^n \times \{0\} \cup \partial I^n \times I = J^n$. (For $S^{n-1} \subset D^n$, one can use the homeomorphism $(I^n, \partial I^n) \to (D^n, S^{n-1})$.) This is sometimes called "pushing through a cardboard box". See Tom Dieck, 2.3.5. Here we give just pictures:



Proposition 1.4.10. *Let*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] & B \arrow[d, "j'"] \\
X \arrow[r, "f'"'] & Y \arrow[ul, phantom, "\ulcorner"]
\end{tikzcd}$$

be a pushout diagram in Top*. If j has the HEP for a space T, then so does j'. If j is a cofibration, then j' is also.*

Proof. The second claim follows directly from the first one. For the first one, consider

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & B \arrow[r] \arrow[d] & T^I \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[r, "f'"'] \arrow[urr, dashed, "H"] \arrow[ur, phantom, "\ulcorner"] & Y \arrow[r] \arrow[ur, dashed, "\tilde{H}"'] & T \end{tikzcd} .$$

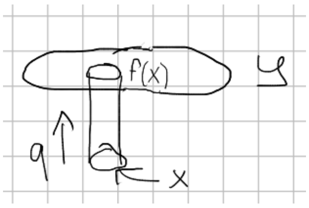
1. By the HEP for j, H exists.
2. By the universal property of the push-out, $\tilde{H}$ exists.
3. Again by the universal property of the push-out, $\mathrm{ev}_0 \circ \tilde{H} = (Y \to T)$ (they coincide on the images of X and B). The other triangle commutes by construction.

□

Exercise 1.4.11. Let (X, A) be a relative CW-complex (or a cubical subcomplex, to be defined later). Then $A \hookrightarrow X$ is a cofibration.

Proposition 1.4.12. *Let* $X \xrightarrow{f} Y$ *be a map in* Top. *Then there exist* $X \xrightarrow{j} \mathrm{Cyl}(f) \xrightarrow{q} Y$ *such that*

1. *j is a cofibration.*
2. *q is a homotopy equivalence, i.e.* $\pi \circ q \sim \mathrm{id}_{\mathrm{Cyl}(f)} \,\mathrm{rel}\, Y$ *(see below).*
3. *$q \circ j = f$.*



Proof. 3. Go through the diagram

$$\begin{tikzcd} & Y \arrow[d, "\pi"'] \arrow[dr, "\mathrm{id}_Y"] & \\ X \arrow[r, "j"] \arrow[dr, "(\mathrm{id}_X, 1)"'] & \mathrm{Cyl}(f) \arrow[r, "q"] & Y \\ & X \times I \arrow[u, "\hbar"] \arrow[r, "\mathrm{pr}_X"'] & X \arrow[u, "f"'] \end{tikzcd} .$$

(The map q exists by the universal property of the push-out and the diagram commutes then by construction.)

1. We verify that the left square in

$$\begin{tikzcd} X \times \partial I = X \sqcup X \arrow[r, "f \sqcup \mathrm{id}_X"] \arrow[d, hook, "\mathrm{id}_X \times \mathrm{incl}"'] & Y \sqcup X \arrow[d, "(\pi, j)"'] & X \arrow[l, "in_X"'] \arrow[dl, "j"] \\ X \times I \arrow[r] & \mathrm{Cyl}(f) & \end{tikzcd}$$

is a pushout diagram. Furthermore $\mathrm{id}_X \times \mathrm{incl}$ is a cofibration by proposition 1.4.8. Thus by the proposition before, (π, j) is a cofibration. Furthermore one can directly verify that in_X is a cofibration.

Then j is a cofibration as a composition of two cofibrations.

2. We need to construct the homotopy. Consider

$$\begin{tikzcd} Y \times I \arrow[d] \arrow[dr, hook] & \\ \mathrm{Cyl}(f) \times I \arrow[r, "H"] & \mathrm{Cyl}(f) \\ (X \times I) \times I \arrow[u] \arrow[ur] & \end{tikzcd}$$

defined by

$$H : \begin{cases} (y,\tau) \mapsto y = \pi(y), \\ ((x,t),\tau) \mapsto \hbar(x, t\tau), \end{cases}$$

where $\hbar : X \times I \to \mathrm{Cyl}(f)$ is the canonical map as usual. Check that this has the desired properties: For $\tau = 1$, we get $\mathrm{id}_{\mathrm{Cyl}(f)}$. For $\tau = 0$, we get

$$\begin{tikzcd} \mathrm{Cyl}(f) \arrow[r, "q"] & Y \arrow[r, "\pi"] & \mathrm{Cyl}(f) . \\ X \times I \arrow[u] \arrow[r] & X \arrow[u, "f"] \arrow[r] & X \times I \arrow[u, "\hbar"] \end{tikzcd}$$

$$(x,\tau) \xmapsto[\text{(def. of } q)]{} x \xmapsto[\text{(def. of Cyl}(f))]{} (x,0)$$

□

Remark 1.4.13. This factorization is functorial.

Application: $X \xrightarrow{f} Y$ yields $\pi_n(X \xrightarrow{f} Y) =: \pi_n(\mathrm{Cyl}(f), X)$, the *homotopy group of* $f : X \to Y$. For these groups, we obtain a long exact sequence because $\pi_n(\mathrm{Cyl}(f)) \cong \pi_n(Y)$.

This will sometimes also be used to define relative homotopy groups $\pi_n(Y, X, x)$, where X is not a subspace of Y.

1.5 Higher connectivity

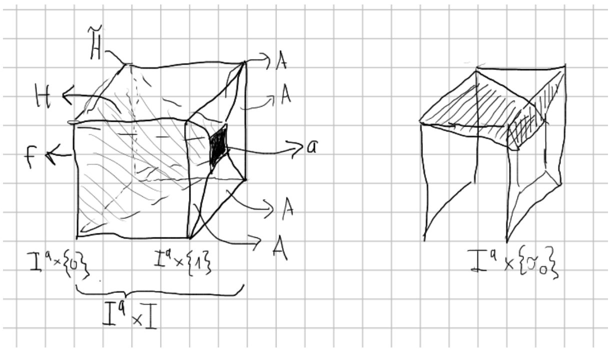
Definition 1.5.1. A map $f : A \to X$ is *n-connected* $(-1 \le n \le \infty)$ if

0. $X \neq \emptyset$.
1. $f_* : \pi_0 A \twoheadrightarrow \pi_0 X$ is surjective if $n \ge 0$.
2. $\pi_i(X, A, a) = 0$ for all $a \in A$, $1 \le i \le n$.

Proposition 1.5.2. *Let (X, A) be a pair in* Top*. The following are equivalent:*

1. *Each map $(I^q, \partial I^q) \to (X, A)$ is homotopic to a constant map, $1 \le q \le n$, and $\pi_0 A \twoheadrightarrow \pi_0 X$ is surjective.*
2. *Each map $(I^q, \partial I^q) \to (X, A)$ is homotopic* $\mathrm{rel}\, \partial I^q$ *to a map into A, $0 \le q \le n$.*
3. $\pi_0 A \twoheadrightarrow \pi_0 X$ *is surjective,* $\pi_q(A, a) \xrightarrow{\cong} \pi_q(X, a)$ *is an isomorphism for all $a \in A$, $1 \le q \le n-1$, and* $\pi_n(A, a) \twoheadrightarrow \pi_n(X, a)$ *is surjective.*
4. *The inclusion $A \hookrightarrow X$ is n-connected.*

Proof. – 1. $\Rightarrow$ 2.: $q = 0$ can be checked directly. Let $q \geq 1$, let $f : (I^q, \partial I^q) \to (X, A)$. By assumption there exists $H : I^q \times I \to X$ such that $H|_{I^q \times \{0\}} = f$, $H|_{I^q \times \{1\}} = a$ for $a \in A$, $H|_{\partial I^q \times I} \subset A$.[6] We define $\tilde{H} : I^q \times I$ by the following picture:



Thus on the inner "truncated pyramid" (resp. "deformed cube"), we define $\tilde{H}$ to be H (up to reparametrization). For a given time slice $\tau_0 \in I$, we define $\tilde{H}$ on the upper shaded trapezium in this time slice by the identification with the other shaded trapezium (as indicated in the sketch on the right-hand side). This goes to A. Similarly we define it on the left, right, and lower trapezia, and analogously in higher dimensions. Explicit formulas can be given and continuity can be verified.

– 2. $\Rightarrow$ 4. Let $q > 0$ and $f : (I^q, \partial I^q, J^{q-1}) \to (X, A, a)$. We know that there is $g : (I^q, \partial I^q, J^{q-1}) \to (A, A, a)$ such that $f \sim g$. Thus $[f]$ comes from $\pi_q(A, A, a) \to \pi_q(X, A, a)$. However $\pi_q(A, A, a)$ is trivial due to the long exact sequence for homotopy groups (theorem 1.1.5) applied to the pair (A, A).

– 3. $\Leftrightarrow$ 4. by the long exact sequence of homotopy groups.

– 4. $\Rightarrow$ 1. Let $f : (I^q, \partial I^q) \to (X, A)$ be a map of pairs. Note that $J^{q-1} \hookrightarrow \partial I^q \hookrightarrow I^q$ are cofibrations. We have $f|_{J^{q-1}} \sim a$ for some a, since $J^{q-1} \simeq *$. Extend the corresponding homotopy using the cofibration property (twice) to get a homotopy of pairs (using suggestive notation)

$$\begin{aligned} &H : (I^q, \partial I^q) \times I \to (X, A), \\ &H|_{I^q \times \{0\}} = f, \\ &H|_{I^q \times \{1\}} =: g, \end{aligned}$$

where $g : (I^q, \partial I^q, J^{q-1}) \to (X, A, a)$ is a map of triples. By 4., $g \sim a$ (constant map) as a map of triples. Thus $f \sim g \sim a$.

$\square$

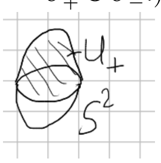
Theorem 1.5.3 (Connectivity theorem)**.** *Let X, Y be topological spaces, $X_0, X_1 \subset X$ be such that $X_0^\circ \cup X_1^\circ = X$, $Y_0, Y_1 \subset Y$ be such that $Y_0^\circ \cup Y_1^\circ = Y$, and $f(X_i) \subset Y_i$ for $i = 0, 1$. Assume furthermore that $f|_{X_i}$, $i = 0, 1$, is n-connected and $f|_{X_0 \cap X_1}$ is $(n-1)$-connected for some $n \geq 1$.*

Then f is n-connected.

(Proof later.)

Corollary 1.5.4. *S^n is n-connected.*

Proof. By induction on n. We decompose $S^n = U_+ \cup U_-$, such that $U_+ \cap U_- \simeq S^{n-1}$, $U_+ \simeq *$, $U_- \simeq *$. (We choose U_+ and U_- slightly larger than the upper and lower hemisphere such that $S^n = U_+^\circ \cup U_-^\circ$.)



[6] Here we denote the constant map to a also by a.

By theorem 1.5.3 applied to

$$\begin{tikzcd} X_0 = * \arrow[d, hook] & Y_0 = U_+ \arrow[d, hook] \\ X = * \arrow[r, "f"] & S^n \\ X_1 = * \arrow[u, hook] & Y_1 = U_- \arrow[u, hook] \end{tikzcd},$$

we obtain the claim. □

Remark 1.5.5. Let $i : A \hookrightarrow B$ be a cofibration. Consider the diagram, where the solid arrows commute:

$$\begin{tikzcd} A \arrow[r, "\phi|_A"] \arrow[d, hook, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "\phi"'] \arrow[ur, dashed, "\psi"] & Y \end{tikzcd}.$$

Are we able to find a map ψ (as indicated) such that $\psi \circ i = \phi|_A$ and $f \circ \psi \sim \phi \operatorname{rel} A$?[7]

For example, consider

$$\begin{tikzcd} \partial I^q \arrow[r] \arrow[d, hook] & X \arrow[d, hook, "f"] \\ I^q \arrow[r] \arrow[ur, dashed, "\psi"] & Y \end{tikzcd},$$

where $f : X \hookrightarrow Y$ is an inclusion map which is q-connected. Then by 2. of proposition 1.5.2, we can find such a ψ.

If we can solve this problem for

$$\begin{tikzcd} A \arrow[r, "\tilde{\phi}|_A = \phi|_A"] \arrow[d, hook, "i"'] & X \arrow[d, hook, "j"] \\ B \arrow[r, "\tilde{\phi}"'] \arrow[ur, dashed, "\tilde{\psi}"] & \mathrm{Cyl}(f) \end{tikzcd},$$

then we can solve the original problem (this is called the "mapping cylinder trick", cf. proposition 1.4.12 and finds also application in other cases): We extend our original diagram to

$$\begin{tikzcd} A \arrow[r, "\phi|_A"] \arrow[d, hook, "i"'] & X \arrow[d, hook, "j"] \\ B \arrow[r, dashed, "\pi\circ\phi"] \arrow[dr, "\phi"'] & \mathrm{Cyl}(f) \arrow[d, bend left, "q"] \\ & Y \arrow[u, bend left, "\pi"] \end{tikzcd}.$$

The square does not commute. However we have (since $q \circ j = f$)

$$\pi \circ \phi \circ i = \pi \circ f \circ \phi|_A = \underbrace{\pi \circ q}_{\sim \mathrm{id} \operatorname{rel} Y} \circ j \circ \phi|_A \sim j \circ \phi|_A \operatorname{rel} A.$$

Thus, since i is a cofibration, there exists

$$\begin{aligned} &H : B \times I \to \mathrm{Cyl}(f), \\ &H|_{B\times\{0\}} = \pi \circ \phi, \\ &H|_{B\times\{1\}} = \tilde{\phi}, \end{aligned}$$

[7]Note that ψ does not need to make the diagram commute, only "up to homotopy".

where $\tilde{\phi}$ makes this square commutative, i.e. $\tilde{\phi} \circ i = j \circ \phi|_A$.[8]

Now suppose that we have solved the problem of finding $\tilde{\psi}$ in the mapping cylinder push-out diagram above. I.e. $\tilde{\psi} \circ i = \phi|_A$, and $j \circ \tilde{\psi} \sim \tilde{\phi} \operatorname{rel} A$. Furthermore $\tilde{\phi} \sim \pi \circ \phi$ by the homotopy H. We conclude

$$f \circ \tilde{\psi} = q \circ j \circ \tilde{\psi} \sim q \circ \tilde{\phi} \sim \underbrace{q \circ \pi}_{\mathrm{id}_Y} \circ \phi = \phi \operatorname{rel} A.$$

We have solved the original problem if we simply choose $\psi = \tilde{\psi}$.

Definition 1.5.6. A *subdivision* of I^n of width $\frac{1}{N}$ is the representation $I^n = \bigcup_{k_1,\ldots,k_n \in \{0,\ldots,N-1\}} \underbrace{\prod_{j=1}^{n} [\frac{k_j}{N}, \frac{k_j+1}{N}]}_{n-\text{dim. cube}}$.

A *cubical complex* $B \subset I^n$ is a union of cubes $\underbrace{\prod_{j=1}^{n} [\frac{k_j}{N}, \frac{k'_j}{N}]}_{\text{“elementary cubes”}}$, where $0 \leq k_j \leq N$ and $k'_j \in \{k_j, k_{j+1}\}$ (with $k'_j = k_j$ if $k_j = N$; the cube can also be “flat” in dimension n). The *k'th skeleton* of B is defined as $B(k) := \bigcup \leq k-$ dim. elementary cubes in B.

A *subcomplex* $A \subseteq B$ is a subset of the union of cubes of B.

Lemma 1.5.7 (Strictification lemma)**.** *Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, phantom, "H"] & Y \end{tikzcd}$$

in Top *be homotopy commutative, i.e.* $p \circ f \sim^H g \circ i$.

If i is a cofibration, then there exists $g' : B \to Y$ such that $g' \circ i = p \circ f$, $g \sim g'$.

If p is a fibration, then there exists $f' : A \to X$ such that $p \circ f' = g \circ i$, $f' \sim f$.

Proof. We only show the first claim, i.e. let i be a cofibration. We can extend H to

$$\begin{aligned} &\tilde{H} : B \times I \to Y, \\ &\tilde{H}_0 = g, \\ &\tilde{H}_1 =: g', \end{aligned}$$

(where g' is defined by $\tilde{H}_1$) such that $\tilde{H} \circ (i \times \mathrm{id}_I) = H$. Thus $g' \circ i = H_1|_A = p \circ f$. □

Lemma 1.5.8. *Let $f : X \to Y$ be n-connected, and (B, A) be a cubical pair of dimension less than n (i.e. B is a cubical complex of dimension $\leq n$ and $A \subseteq B$ is a subcomplex). Then every commutative square indicated by the solid arrows*

$$\begin{tikzcd} A \arrow[r, "\phi|_A"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "\phi"'] \arrow[ur, dashed, "\psi"] & Y \end{tikzcd},$$

where i denotes the inclusion, admits ψ with

$$\begin{aligned} &\psi \circ i = \phi|_A, \\ &f \circ \psi \sim \phi \operatorname{rel} A. \end{aligned}$$

Proof. 1. By a mapping cylinder trick (cf. proposition 1.4.12), we can reduce the problem to the case that f is an inclusion.

[8] $\tilde{\phi}$ is in general not unique.

2. Proof by induction on the relative dimension $\dim(B,A) =: m$. This shall be defined as the highest dimension which occurs in B but not in A. We prove that there exists $\psi_m : A \cup B(m) \to X$ such that $\psi_m|_A = \phi|_A$ and $f \circ \psi_m \sim \phi|_{A\cup B(m-1)} \operatorname{rel} A \cup B(m-1)$. By induction assumption we have the diagram (which commutes "up to homotopy")

$$\begin{tikzcd} A \arrow[r, "\phi|_A"] \arrow[d, hook] & X \arrow[d, "f"] \\ A \cup B(m-1) \arrow[ur, "\psi_{m-1}"] \arrow[r, "\phi|_{A\cup B(m-1)}"] \arrow[d] & Y \\ B \arrow[ur, "\phi"'] & \end{tikzcd}.$$

Now consider

$$\begin{tikzcd} \coprod \partial I^{m-1} \arrow[r] \arrow[d, hook] & A \cup B(m-1) \arrow[r, "\psi_{m-1}"] \arrow[d] & X \arrow[d, "f"] \\ \coprod I^m \arrow[r] \arrow[urr, dashed, "\tilde\psi"] & A \cup B(m) \arrow[ul, phantom, "\ulcorner"] \arrow[r, "\tilde\phi"'] \arrow[ur, dashed, "\psi_m"'] & Y \end{tikzcd}.$$

The map $\tilde\phi$ is obtained by the strictification lemma 1.5.7; we have $\tilde\phi \sim \phi \operatorname{rel} A \cup B(m-1)$ and the right square is commutative (the left one is so trivially). We get the map $\tilde\psi$ and a homotopy from proposition 1.5.2, 2. (i.e. the upper left triangle is commutative, the lower right one up to homotopy). The left square is a push-out diagram (similar to the construction of CW-complexes), yielding the map ψ_m with $\psi_m|_A = \psi_{m-1}|_A = \phi|_A$. Furthermore $A \cup B(m-1) \to A \cup B(m)$ is a cofibration by proposition 1.4.10; therefore we get a suitable homotopy $f \circ \psi_m \sim \tilde\phi \operatorname{rel} A \cup B(m-1)$. This completes the induction step, since by glueing two homotopies we obtain then $f \circ \psi_m \sim \phi \operatorname{rel} A \cup B(m-1)$.

In the end, we get $\psi := \psi_n : A \cup B(n) = B \to X$ with $\psi|_A = \phi|_A$, $f \circ \phi \sim \phi \operatorname{rel} A$.

□

Now we can turn on to prove theorem 1.5.3.

Proof of theorem 1.5.3. 1. Reduce to the case of a pair (Y,X) and the inclusion map. This follows almost by definition of n-connectivity of a map, using the mapping cylinder construction (cf. proposition 1.4.12). We note the elementary statements that $\mathrm{Cyl}(f|_{X_i}) \subset \mathrm{Cyl}(f)$ for $i = 0,1$ may be regarded as a subspace and $\mathrm{Cyl}(f) = \mathrm{Cyl}(f|_{X_0})^\circ \cup \mathrm{Cyl}(f|_{X_1})^\circ$, $\mathrm{Cyl}(f|_{X_0}) \cap \mathrm{Cyl}(f|_{X_1}) = \mathrm{Cyl}(f|_{X_0 \cap X_1})$. In the following we abbreviate $X_{01} := X_0 \cap X_1$, $Y_{01} := Y_0 \cap Y_1$. We have commutative diagrams

$$\begin{tikzcd} X \arrow[r, hook] & \mathrm{Cyl}(f) \arrow[r, "\simeq"] & Y \\ X_i \arrow[u, hook] \arrow[r, hook] & \mathrm{Cyl}(f|_{X_i}) \arrow[u, hook] \arrow[r, "\simeq"'] & Y_i \arrow[u, hook] \end{tikzcd}$$

for $i = 0, 1, 01$ (and similar ones for $X_{01} \hookrightarrow X_0$, $X_{01} \hookrightarrow X_1$). Thus we can indeed reduce to this case of an inclusion.

2. Now consider $f : (I^n, \partial I^n) \to (Y, X)$. (We abuse the letter f here; the map $f : X \to Y$ from the statement of the claim is by step 1. just the inclusion $j : X \hookrightarrow Y$ any more. Furthermore for $q < n$, the construction can be done very similarly.) We want to find a homotopy $\operatorname{rel} \partial I^n$ to a map into X (because of 1.5.2, 2.). To this end, let us construct a suitable decomposition $I^n = K_0 \cup K_1$ into cubical complexes such that $f(K_i) \subseteq Y_i$ for $i = 0,1$ and $f(K_i \cap \partial I^n) \subseteq X_i$. To construct these, consider $A_i := f^{-1}(Y \backslash Y_i^\circ) \cup f^{-1}(X \backslash X_i^\circ)$. This is closed in I^n. We have $A_0 \cap A_1 = \emptyset$, thus $(I^n \backslash A_0) \cup (I^n \backslash A_1)$ is an open cover of I^n. By the Lebesgue lemma, there

exists therefore a subdivision of I^n into cubes W such that for all W, we have $W \subset I^n \backslash A_0$ or $W \subset I^n \backslash A_1$ (or both). Now define

$$K_i := \bigcup_{W \text{ such that } \substack{f(W) \subset Y_i^\circ, \\ f(W \cap \partial I^n) \subset X_i^\circ}} W.$$

3. Let $K_{01} := K_0 \cap K_1$, and for $i = 0, 1, 01$, denote $K_i^\bullet := (n-1) -$ skeleton of K_i. At first, we "compress f on $K_{01}^\bullet$". Consider the diagram (where j_{01} denotes the inclusion)

$$\begin{tikzcd} \partial I^n \cap K_{01} \arrow[r, "f"] \arrow[d, hook] & X_{01} \arrow[d, hook, "j_{01}"] \\ K_{01}^\bullet \arrow[r, "f"'] \arrow[ur, dashed, "g_{01}"] & Y_{01} \end{tikzcd} .$$

[9] Note that $K_{01}^\bullet$ is a $(n-1)$-dimensional cubical complex, and $j_{01} : X_{01} \hookrightarrow Y_{01}$ is $(n-1)$-connected by assumption. By the construction of K_0, K_1, indeed $f|_{K_{01}^\bullet} : K_{01}^\bullet \to Y_{01}$. Thus the solid arrows form a well-defined commutative square. By the lemma before, there exists $g_{01} : K_{01}^\bullet \to X_{01}$ as indicated, such that the upper left triangle commutes, i.e. $g_{01}|_{\partial I^n \cap K_{01}} = f$, and the lower right triangle is "homotopy commutative", i.e. $j_{01} \circ g_{01} \sim^{h_{01}} f \operatorname{rel} \partial I^n \cap K_{01}$. (We denote h_{01} the homotopy.)

4. Now we extend g_{01} to a map $g_0 : K_0 \cap (\partial I^n \cup K_1^\bullet) = (K_0 \cap \partial I^n) \cup K_{01}^\bullet \to X_0$ by

$$\begin{aligned} g_0|_{K_0 \cap \partial I^n} &= f, \\ g_0|_{K_{01}^\bullet} &= g_{01}. \end{aligned}$$

[10] Furthermore we glue h_{01} together with a constant homotopy to get a homotopy H_0 such that $j_0 \circ g_0 \sim^{H_0} f \operatorname{rel} \partial I^n \cap K_0$. (Here $j_0 : X_0 \to Y_0$ denotes the inclusion. Recall that $h_{01}|_{\partial I^n \cap K_{01}}$ is constant.)

5. $K_0 \cap (\partial I^n \cup K_1^\bullet) \hookrightarrow K_0$ is a cofibration (it is a relative CW-complex). Therefore we are able to extend H_0 to a homotopy $\Psi_0 : K_0 \times I \to Y_0$ with $\Psi_0|_{K_0 \times \{0\}} = f$. Denote $\Psi_0|_{K_0 \times \{1\}} =: F_0$. This yields the solid arrows in

$$\begin{tikzcd} K_0 \cap (\partial I^n \cup K_1^\bullet) \arrow[r, "g_0"] \arrow[d, hook] & X_0 \arrow[d, "j_0"] \\ K_0 \arrow[r, "F_0"'] \arrow[ur, dashed, "h_0"] & Y_0. \end{tikzcd}$$

The square is commutative. Now note that j_0 is by assumption n-connected. Thus by the lemma before there exists h_0 as indicated, making the upper left triangle commutative and the lower right one homotopy commutative. To be precise,

$$\begin{aligned} &h_0|_{K_0 \cap (\partial I^n \cup K_1^\bullet)} = g_0, \\ &j_0 \circ h_0 \sim F_0 \operatorname{rel} K_0 \cap (\partial I^n \cup K_1^\bullet). \end{aligned} .$$

Then $(j_0 \circ h_0)|_{K_0 \cap (\partial I^n \cup K_1^\bullet)} \sim f \operatorname{rel} \partial I^n \cap K_0$ by construction of F_0 (and by gluing together two homotopies).

6. Performing analogous constructions as in steps 4. and 5. with the roles of K_0 and K_1 swapped, one gets $g_1 : K_1 \cap (\partial I^n \cup K_0^\bullet) \to X_1$, $h_1 : K_1 \to X_1$. One verifies that these return the same on $K_{01}^\bullet$ (namely g_{01}). Thus they glue together to a map $h : K_0^\bullet \cup K_1^\bullet \to X$, and

[9] We have $\partial I^n \cap K_{01} = \partial I^n \cap K_{01}^\bullet$.

[10] These glue together to a continuous map by construction of g_{01}.

similarly the homotopies glue together.[11] Then

$$\begin{aligned} h &\sim f|_{K_0^\bullet \cup K_1^\bullet} \operatorname{rel} \partial I^n, \\ h|_{\partial I^n} &= f. \end{aligned}$$

7. Finally, we extend h to I^n; i.e. we need to "fill in the cubes of the subdivision". Consider any cube W of the subdivision of I^n, without loss of generality $W \subset K_0$. Consider

$$\begin{tikzcd} \partial I^n \arrow[r, "\cong"] \arrow[d, hook] & \partial W \subset \arrow[d, hook] \arrow[rr, bend left, "h_0"] & K_0^\bullet \arrow[r] & X_0 \arrow[d, hook, "j_0"] \\ I^n \arrow[r, "\cong"] & W \arrow[rru, dashed, "\tilde{h}_W"] \arrow[rr, "j_0 \circ h_0"'] & & Y_0 \end{tikzcd}$$

and recall that j_0 is n-connected. Clearly the solid arrows form a commutative diagram, thus by the lemma before, there exists $\tilde{h}_W : W \to X_0$ as indicated such that

$$\begin{aligned} \tilde{h}_W|_{\partial W} &= h_0|_{\partial W} = h|_{\partial W}, \\ j_0 \circ \tilde{h}_W &\sim j_0 \circ h_0 \operatorname{rel} \partial W. \end{aligned}$$

This we do for all such cubes W (for $W \subset K_{01}$, we make the arbitrary choice to define $\tilde{h}_W$ using h_0). Glueing everything together, we get a map $\tilde{h} : I^n \to X$. This map satisfies then $\tilde{h} \sim f \operatorname{rel} \partial I^n$, by at first applying the homotopies $j_0 \circ \tilde{h}_W \sim j_0 \circ h_0$ resp. $j_1 \circ \tilde{h}_W \sim j_1 \circ h_1$ for all cubes W, and then applying the homotopies $j_0 \circ h_0 \sim F_0 \sim f$ (resp. $j_1 \circ h_1 \sim F_1 \sim f$, where F_1 is defined analogously to F_0).[12] This proves the n-connectivity of the map $X \to Y$. □

Corollary 1.5.9 ("Easy excision")**.** *Let $Y \in$ Top, $U, V \subseteq Y$ an open cover (i.e. $Y = U^\circ \cup V^\circ$) with $W := U \cap V \neq \emptyset$. If (V, W) id n-connected, then (Y, U) is n-connected.*

Proof. The situation is described by the (pushout) diagram of inclusions

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & U \arrow[d, hook] \\ V \arrow[r, hook] & Y \end{tikzcd}.$$

We apply the previous theorem to

$$\begin{aligned} X &= U \longrightarrow Y \\ X_0 &= W \xrightarrow{n-\text{conn.}} Y_0 = V \\ X_1 &= U \xrightarrow{\infty-\text{conn.}} Y_1 = U \\ X_{01} &= W \xrightarrow{\infty-\text{conn.}} Y_{01} = W. \end{aligned}$$

13 □

Exercise 1.5.10. Let $i : A \to X$ be a cofibration, $f : A \to A'$ be a homotopy equivalence.

[11] They do in general not glue together on $K_0 \cup K_1 = I^n$, thus we need to consider the $(n-1)$-skeleta at first.

[12] One verifies that these homotopies glue together on the intersections of the cubes W (regardless of the choice involved for $W \subset K_{01}$) and that the homotopy is relative to ∂I^n. Here it is used that the homotopies $j_0 \circ h_0 \sim F_0$ and $j_1 \circ h_1 \sim F_1$ are in particular relative to $K_{01}^\bullet$ and that Ψ_0 and Ψ_1 are extensions of h_{01}, where Ψ_1 is defined analogously to Ψ_0.

[13] Note that this requires $W \neq \emptyset$.

Consider the pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] \arrow[dr, phantom, "\ulcorner"] & A' \arrow[d, "i'"] \\ X \arrow[r, "f'"'] & X' \end{tikzcd}.$$

Then f' is a homotopy equivalence.

Proposition 1.5.11. *Consider the pushout square*

$$\begin{tikzcd} \coprod S^n \arrow[r, "\phi"] \arrow[d, "i"'] \arrow[dr, phantom, "\ulcorner"] & A \arrow[d, "f"] \\ \coprod D^{n+1} \arrow[r, "\Phi"'] & X \end{tikzcd}.$$

Then f is n-connected.

Proof. Consider the factorization

$$\begin{tikzcd} \coprod S^n \arrow[r] \arrow[d, "i"'] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Cyl}(\phi) \arrow[r] \arrow[d, "\tilde{i}"] \arrow[dr, phantom, "\ulcorner"] & A \arrow[d, "f"] \\ \coprod D^{n+1} \arrow[r] \arrow[rr, bend right, "\Phi"'] & Z \arrow[r] & X \end{tikzcd}.$$

Here we have constructed the pushout Z and got the map $Z \to X$ from its universal property. The diagram is commutative. Now it is a basic exercise in category theory to show that, since the big rectangle and the left square are pushouts, so must be the right square (as indicated). Furthermore, by proposition 1.4.10, $\tilde{i}$ is a cofibration (since i is one). Thus by the exercise before, $Z \to X$ is a homotopy equivalence. Therefore, f is n-connected if and only if $\tilde{i}$ is.

We apply easy excision, corollary 1.5.9. Let $U := A \cup_{\mathrm{Cyl}(\phi)} \coprod (S^n \times [0, \frac{1}{2}]) \sim A \sim \mathrm{Cyl}(\phi) \subset Z$. Let $V := \coprod D^{n+1} \cup (S^n \times [\frac{1}{4}, 1]) \sim \coprod D^{n+1} \sim \coprod *$, where $*$ denotes the one-point space. We have $U \cap V \sim \coprod S^n$. Now $(\coprod D^{n+1}, \coprod S^n)$ is n-connected (e.g. by the long exact sequence of homotopy groups). Then, by easy excision, $\tilde{i}$ is n-connected. $\square$

Corollary 1.5.12. *If (X, A) is a relative CW-complex, then $(X, X^{(n)})$ is n-connected.*

Proof. Finite-dimensional case: Apply inductively the proposition before.

In infinite-dimensional case, use that by a compactness argument: $\pi_m X \cong \mathrm{colim}_n\, \pi_m(X^{(n)})$. $\square$

1.6 CW-complexes (recollection)

Definition 1.6.1 (CW-complex)**.** – Let $(X, Y) \in \mathrm{Top}^2$. A *relative CW-structure on* (X, Y) is a sequence of subspaces of X:

$$Y := X^{(-1)} \subset \cdots \subset X^{(n)} \subset \cdots \subset X$$

such that

1. $X \cong \mathrm{colim}_n\, X^{(n)}$ (with the colimit topology).
2. For all $n \geq 0$, $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching n-cells: Precisely, $X^{(n)} \backslash X^{(n-1)} \cong \bigcup_{\alpha \in I_n} D^{n\circ}$ for some index set I_n, and for all $\alpha \in I_n$ there exist

$$\begin{array}{ll} \Phi_n^\alpha : D^n \to X^{(n)} & \text{(characteristic maps),} \\ \phi_n^\alpha : \partial D^n \to X^{(n-1)} & \text{(attaching maps),} \end{array}$$

such that we have a pushout diagram

$$\begin{tikzcd} \coprod_{\alpha\in I_n} \partial D^n \arrow[r, "\coprod_{\alpha\in I_n}\phi_n^\alpha"] \arrow[d, hook] & X^{(n-1)} \arrow[d, hook] \arrow[dl, phantom, "\ulcorner"] \\ \coprod_{\alpha\in I_n} D^n \arrow[r, "\coprod_{\alpha\in I_n}\Phi_n^\alpha"'] & X^{(n)} \end{tikzcd} .$$

- A *CW-structure* on $X \in \mathrm{Top}$ is a relative CW-structure on $(X, \emptyset)$.
- A *CW-complex* is $X \in \mathrm{Top}$ together with a CW-structure.
- A CW-complex X has *dimension* n if $X = X^{(n)}$ and $X \neq X^{(n-1)}$.

Example 1.6.2. $\mathbb{C}P^n$ is a CW-complex with one cell in each dimension $2i$, for $0 \leq i \leq n$.
$\mathbb{R}P^n$ is a CW-complex with one cell in each dimension i, for $0 \leq i \leq n$.

Definition 1.6.3. A *subcomplex* Y of a CW-complex X is $Y \subseteq X$ such that for all cells $e \subseteq X$ with $e \cap Y \neq \emptyset$, already $\overline{e} \subseteq Y$.

Remark 1.6.4. If Y is a subcomplex of X, then Y is itself a CW-complex with $Y^{(n)} = Y \cap X^{(n)}$.

Proposition 1.6.5 (Closure finiteness (i.e. each cell intersects only finitely many cells))**.** *Let X be a CW-complex, $K \subseteq X$ compact. Then*

1. *K is contained in some finite subcomplex of X, in particular*
2. *there exists n such that $K \subseteq X^{(n)}$.*

Proof.

- *Claim 1:* Let S be a set of points which lie in different cells of X. Then $S \subseteq X$ is closed.

  Proof of claim 1: By definition of the weak (i.e. colimit) topology, $S \subseteq X$ is closed if and only if $S \cap X^{(n)}$ is closed for all $n \in \mathbb{N}$. Now argue by induction on n: We have for all α (in the index set for the attached n-cells):

  $$(\Phi_n^\alpha)^{-1}(S) = \underbrace{(\phi_n^\alpha)^{-1}(S)}_{\subset \partial D^n,\text{ closed by induction}} \cup \underbrace{\{\text{preimage of at most one point in the cell } D^n_\alpha\}}_{\subset D^{n\circ},\text{closed}}$$

  This yields the induction step by definition of the pushout topology, thus S is closed. This proves Claim 1.

- *Observation:* In fact we even have, iterating the previous argument, that for all $S' \subseteq S$, $S' \subseteq X$ is closed. Thus S carries the discrete topology. Now, for $K \subseteq X$ compact, let S be a set of representatives of cells intersecting K (i.e. one point in each cell). Then $S \subseteq K$ is a closed subset of a compact set, thus it itself is compact. On the other hand, S carries the discrete topology. Thus S is finite, i.e. K intersects only finitely many cells.

- *Claim 2:* Every cell in X is contained in a finite subcomplex.

  Proof of claim 2: Let $e := \Phi_n^\alpha(D^{n\circ})$ be a n-cell. By the previous part, $\Phi_n^\alpha(\partial D^n) \subseteq X$ is contained in a finite subcomplex Y of X, since it is compact. Then $\overline{e} \subseteq Y \cup e$ is contained in a finite subcomplex.

The two claims and the observation together prove the proposition. □

Corollary 1.6.6. *Let X be a CW-complex, K be a compact topological space. Then* $[K, X] = \mathrm{colim}_n [K, X^{(n)}]$*. The same holds in* Top_*.

Proof. There is a canonical map $B : \operatorname{colim}_n [K, X^{(n)}] \to [K, X]$. Let us show that it is bijective.

Surjective: For all $f : K \to X$, there exists a factorization (i.e. a commutative diagram)

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[dr] & X \\ & X^{(n)} \arrow[u, hook] \end{tikzcd}$$

for some n.

Injective: Let $f_0, f_1 : K \to X^{(n)}$ be homotopic in X via $H : K \times I \to X$. Since $K \times I$ is compact, H factors through some $X^{(m)}$, so that $[f_0] = [f_1] \in [K, X^{(m)}]$. Then they are also equal in the colimit. □

Definition 1.6.7. A map $f : X \to Y$ of CW-complexes is *cellular*, if $f(X^{(n)}) \subseteq Y^{(n)}$ for all $n \in \mathbb{N}$.

Theorem 1.6.8 (Cellular approximation)**.** *Let X, Y be CW-complexes and $B \subseteq X$ a subcomplex. If $f : X \to Y$ is a map such that $f|_B$ is cellular, then there exists a cellular map $g : X \to Y$ with $f \sim g \operatorname{rel} B$.*

Proof. We prove: For all $n \geq 0$, there exists $H^n : X \times I \to Y$ such that

1. $H^0|_{X \times \{0\}} = f$, $H^{n-1}|_{X \times \{1\}} = H^n|_{X \times \{0\}}$ for $n \geq 1$.
2. $H^n|_{X^{(i)} \times \{1\}} \subseteq Y^{(i)}$ for $i \leq n$.
3. H^n is constant on $X^{(n-1)} \cup B$.

Then we can prove existence of $H : X \times I \to Y$, $f \sim^H g$ as follows: Define H by

$$H(x,t) := \begin{cases} H^i(x, 2^{i+1}(t - 1 + 2^{-i})), & 1 - 2^{-i} \leq t \leq 1 - 2^{-i+1}, \\ H^i(x, 1), & x \in X^{(i)}, t = 1. \end{cases}$$

This can be done because the "new" homotopies H^j for $j \geq i$ change nothing on $X^{(i)} \cup B$. Then H is continuous on $X^{(i)} \times I$ for all i. Using $X \times I \cong \operatorname{colim}_i X^{(i)} \times I$ (since X is a CW-complex), H is continuous.

Proof of the existence of H^n: By induction on n. We assume that there exists H^{n-1}, f_{n-1} such that $H^{n-1}|_{X \times \{1\}} = f_{n-1}$ and $f_{n-1}(X^{(i)}) \subseteq Y^{(i)}$ for $i \leq n-1$. Consider the commutative diagram

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, hook] & X^{(n-1)} \arrow[r, "f_{n-1}"] \arrow[d] & Y^{(n-1)} \arrow[d] \\ D^n \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X^{(n)} \arrow[r, "f_{n-1}"'] & Y \end{tikzcd} .$$

Since $(Y, Y^{(n)})$ is n-connected, the composition $(D^n, S^{n-1}) \longrightarrow (X^{(n)}, X^{(n-1)}) \xrightarrow{f_{n-1}} (Y, Y^{(n-1)})$ is homotopic to a map $g : (D^n, S^{n-1}) \to (Y^{(n)}, Y^{(n-1)})$ rel S^{n-1}.

We glue g and the homotopy (together with maps and homotopies defined similarly for all other cells in $X^{(n)}$) to get $H^n|_{X^{(n)} \times I} : X^{(n)} \times I \to Y$. Since $X^{(n)} \hookrightarrow X$ is a cofibration, we can extend H^n to $X \times I$. Defining $f_n := H^n|_{X \times \{1\}}$ completes the induction step, since $f_n(X^{(n)}) \subseteq Y^{(n)}$ by construction. □

Corollary 1.6.9. *Let $f_0 \sim f_1$ be cellular maps from a CW-complex X to a CW-complex Y. Then there exists $H : X \times I \to Y$ such that $H(X^{(n)} \times I) \subseteq Y^{(n+1)}$, $H_0 = f_0$, $H_1 = f_1$, i.e.* a cellular homotopy *between f_0 and f_1.*

The proof consists of noting that $X \times I$ has a CW-structure as well, and then applying the previous theorem.

1.7 Whitehead's theorem

Proposition 1.7.1. *Let (Y, B) be an n-connected pair, (X, A) be a relative CW-complex,* $\operatorname{rel}\dim(X, A) \leq n \leq \infty$*. Then any map $f : (X, A) \to (Y, B)$ is homotopic to a map into B relative to A: In the commutative square (formed by the solid arrows)*

$$\begin{tikzcd} A \arrow[r, "f|_A"] \arrow[d, hook] & B \arrow[d, hook, "i"] \\ X \arrow[r, "f"'] \arrow[ur, dashed] & Y \end{tikzcd},$$

there exists a dashed map such that upper triangle is commutative and the lower one is homotopy commutative $\operatorname{rel} A$.

Furthermore, if $\operatorname{rel}\dim(X, A) < n$*, then the homotopy class of the map $X \to B$ is unique relative to A.*

Proof. Observation: Consider for $q \leq n$ the diagram

$$\begin{tikzcd} \bigsqcup \partial I^q \arrow[r] \arrow[d] & A \arrow[r, "f|_A"] \arrow[d] & B \arrow[d, "i"] \\ \bigsqcup I^q \arrow[r, "\bigsqcup \Phi_q := \Phi"'] \arrow[urr, dashed, "g"] & X \arrow[r, "f"'] \arrow[ur, dashed, "G"'] & Y \end{tikzcd}.$$

- There exists $g : \bigsqcup I^q \to B$ such that $i \circ g \sim f \circ \Phi \operatorname{rel} \bigsqcup \partial I^q$ via a homotopy $H : (\bigsqcup I^q) \times I \to Y$.
- Define $G : X \to B$ to be equal to $f|_A$ on A and to be equal to g on $\bigsqcup \partial I^q$ (a map out of a pushout).
- There exists $h : X \times I \to Y$ such that $G \sim^h f \operatorname{rel} A$: It suffices to note that $H : (\bigsqcup I^q) \times I \to Y$ agrees with the constant homotopy $A \times I \to Y$, $(a, t) \mapsto f|_A(a)$ on $(\bigsqcup \partial I^q) \times I$. Thus H and this constant homotopy glue together and define h.

Now on to the actual proof. We construct the needed homotopy by induction on the skeletal filtration.

Induction step: For $q \leq n$, suppose that we have found $G_q : X^{(q)} \to B$, $h_q : X^{(q)} \times I \to Y$ such that $G_q = f|_A$ and $i \circ G_q \sim^{h_q} f|_{X^{(q)}} \operatorname{rel} A$, i.e. we have a diagram

$$\begin{tikzcd} A \arrow[r, "f|_A"] \arrow[d] & B \arrow[d, "i"] \\ X^{(q)} \arrow[r, "f"'] \arrow[ur, dashed, "G_q"] & Y \end{tikzcd}$$

such that the upper triangle is commutative and the lower one up to the homotopy h_q. Since $X^{(q)} \hookrightarrow X^{(q+1)}$ is a cofibration, we can extend h_q to

$$\begin{aligned} &H_q : X^{(q+1)} \times I \to Y, \\ &H_q|_{X^{(q+1)} \times \{0\}} = f|_{X^{(q+1)}}. \end{aligned}$$

By the observation made before with $X^{(q)}$ instead of A and $G_q (= H_{q+1}|_{X^{(q)} \times \{1\}})$ instead of $f|_A$, there exist

$$\begin{aligned} &G_{q+1} : X^{(q+1)} \to B, \\ &i \circ G_{q+1} \sim H_q|_{X^{(q+1)} \times \{1\}} \operatorname{rel} X^{(q)}, \end{aligned}$$

i.e. we have a diagram

$$\begin{tikzcd} X^{(q)} \arrow[r, "G_q"] \arrow[d, hook] & B \arrow[d, "i"] \\ X^{(q+1)} \arrow[r, "H_q|_{X^{(q+1)}\times\{0\}}"'] \arrow[ur, dashed, "G_{q+1}"] & Y \end{tikzcd} .$$

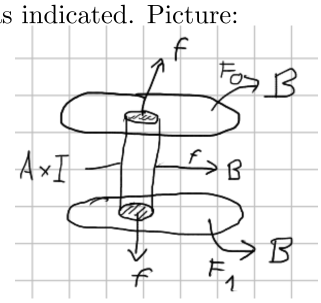
This yields the desired maps and homotopies by induction.[14] For $n = \infty$, glue all homotopies using $X \cong \operatorname{colim}_q X^{(q)}$. This proves the first part of the claim.

For the uniqueness in case $\operatorname{rel}\dim(X, A) < n$, let $F_0, F_1 : X \to B$ such that $F_j|_A = f|_A$, $i \circ F_j \sim^{H_j} f \operatorname{rel} A$, $j = 0, 1$.[15] Consider

$$\begin{tikzcd} X \times \partial I \cup_{A\times\partial I} A \times I \arrow[rr, bend left, "(F_0,F_1)"] \arrow[rr] \arrow[d, hook] \arrow[dr, bend right, "\mathrm{proj}"'] & & B \arrow[d, hook] \\ & A \arrow[ur, "f"] & \\ X \times I \arrow[rr, "\text{“}H_0 - H_1\text{”}"'] \arrow[urr, dashed, "h"] & & Y \end{tikzcd} .$$

on $X \times [0, \frac{1}{2}] : H_0$,
on $X \times [\frac{1}{2}, 1] : \overline{H_1}$.

[16]Here the map $X \times \partial I \cup_{A\times\partial I} A \times I \to B$ is defined by gluing the maps defined on the subspaces as indicated. Picture:



Since $(X \times I, X \times \partial I \cup_{A\times\partial I} A \times I)$ is a relative CW-complex of relative dimension equal to $\operatorname{rel}\dim(X, A) + 1 \leq n$, there exists (as indicated)

$$h : X \times I \to B,$$
$$h|_{X\times\{0\}} = F_0,$$
$$h|_{X\times\{1\}} = F_1$$

by the first part. This proves the uniqueness. □

Corollary 1.7.2. *Suppose that X is a CW-complex, $g : B \to Y$ is n-connected, $n \geq 0$.*
If $\dim(X) < n$, then $[X, B] \xrightarrow{\sim} [X, Y]$ is an isomorphism (of groupoids).
If $\dim(X) = n$, then $[X, B] \twoheadrightarrow [X, Y]$ is surjective.

Proof. $(X, \emptyset)$ is a relative CW-complex. Apply the proposition before. □

Definition 1.7.3. A map $f : Y \to Z$ is a *weak equivalence* if $f_* : \pi_0 Y \to \pi_0 Z$ and $f_* : \pi_i(Y, y) \to \pi_i(Z, f(y))$ for $i \geq 0$ are isomorphisms of sets resp. groups.

[14]Note also $H_q|_{X^{(q+1)}\times\{1\}} \sim f|_{X^{(q+1)}} \operatorname{rel} A$.

[15]I.e. they are homotopic through Y, but we want to find a homotopy inside B.

[16]$\overline{H_1}$ is the inverse parametrization of H_1; strictly speaking we also need to speed up both H_0 and $\overline{H_1}$ by a factor of 2.

(This is the same as ∞-connected.)

Theorem 1.7.4 (Whitehead's theorem)**.** *Let $f : Y \to Z$ be a map of CW-complexes.*

1. *f is a homotopy equivalence if and only if f is a weak equivalence.*
2. *If $\dim(Y) \leq k$, $\dim(Z) \leq k$, $f_* : \pi_q(Y, y) \to \pi_y(Z, f(y))$ is an isomorphism for $q \leq k$, then f is a homotopy equivalence.*

Remark 1.7.5. Note that for Y and Z to be homotopy equivalent, it is not sufficient that they merely have (abstractly) isomorphic homotopy groups. There needs to be a map $f : Y \to Z$ which induces these isomorphisms.

Proof of theorem 1.7.4.

1. "only if": Clear.

   "if": Let f be ∞-connected. Therefore $f_* : [X, Y] \xrightarrow{\sim} [X, Z]$ is an isomorphism for all CW-complexes X (by the corollary before). By the Yoneda lemma in the category hCW, f is an isomorphism in hCW $\subset$ hTop. So f is a homotopy equivalence.[17]

2. If f is k-connected, then $f_*[Z, Y] \twoheadrightarrow [Z, Z]$, $g \mapsto f \circ g$ is surjective. In particular there exists $[g] \in [Z, Y]$ with $f_*([g]) = [f \circ g] = [\mathrm{id}]$. Thus we can find a right inverse g to f in hCW.

   On the other hand, we have isomorphisms $g_* : \pi_i(Z, z) \xrightarrow{\sim} \pi_i(Y, g(z))$ for $i \leq k$, since f_* is an isomorphism and $f_* \circ g_* : \pi_i(Z, z) \to \pi_i(Z, f(g(z)))$ is an isomorphism.[18] Applying the same reasoning as before, $g_* : [Y, Z] \to [Y, Y]$ is surjective, thus g admits a right inverse h in hCW. Then in hCW, $[g \circ f] = [g \circ f \circ g \circ h] = [g \circ h] = [\mathrm{id}]$. Thus g is also a left inverse to f in hCW, i.e. f is an isomorphism in hCW.

$\square$

Corollary 1.7.6. *A CW-complex X is contractible if and only if $\pi_0 X \cong \{*\}$, $\pi_i(X, x) = 0$ for $i \geq 1, x \in X$.*

There is nothing left to prove.

Example 1.7.7. S^∞ is contractible. It is a CW-complex since $S^n \hookrightarrow S^{n+1}$ defines a CW-structure $(S^\infty)^{(n)} \hookrightarrow (S^\infty)^{(n+1)}$. Thus

$$\pi_n(S^\infty) \cong \pi_n((S^\infty)^{(n+1)}) = \pi_n(S^{n+1}) = 0$$

(the first isomorphism by cellular approximation). Then apply the corollary before.

1.8 CW-approximation

Theorem 1.8.1. *Let A, Y be topological spaces and $f : A \to Y$ be k-connected, $k \geq -1$. Then for each $n > k$, $n \leq \infty$ (where by definition $\infty < \infty$), there exists a space X and an injective map $A \hookrightarrow X$ such that (X, A) is a relative CW-complex with cells in dimension $k+1, k+2, \ldots, n$ and there is an n-connected map $F : X \to Y$ such that*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & Y \\ X \arrow[ur, "F"] & \end{tikzcd}$$

commutes.

If A is a CW-complex, then $A \subset X$ is a subcomplex.

[17] Here hCW denotes the homotopy category of CW-complexes, i.e. the full subcategory of hTop generated by CW-complexes. By cellular approximation, this is the same as the homotopy category of CW-complexes with only cellular maps and cellular homotopies.

[18] By construction $f \circ g \circ \phi$ is homotopic to ϕ for all $\phi \in \pi_i(Z, z)$. Thus $f(g(z))$ and z must lie in the same path-connected component and then it is well known (see also chapter 2.5) that the change of basepoint from z to $f(g(z))$ yields an isomorphism of homotopy groups.

Proof. Special cases $n = 0$, $n = 1$: Exercise.

Assume $n \geq 2$. We argue by induction (assuming the base case given). It suffices to treat $n = k + 1$, i.e. we need to find F in the diagram

$$\begin{tikzcd} A \arrow[r, "k-\text{conn.}", "f"'] \arrow[d, hook] & Y \, . \\ X^{(k+1)} \arrow[ur, "F", "(k+1)-\text{conn.}"'] & \end{tikzcd}$$

If $n = k + 2$, we would just start from

$$\begin{tikzcd} X^{(k+1)} \arrow[r, "(k+1)-\text{conn.}", "F"'] \arrow[d, hook] & Y \, , \\ X^{(k+2)} \arrow[ur, "F'", "(k+2)-\text{conn.}"'] & \end{tikzcd}$$

and so on. (If $n = \infty$, we need to glue all these maps together using the colimit topology.) So assume $n = k + 1$.

We reduce to the case of the mapping cylinder: It suffices to find F in the diagram

$$\begin{tikzcd} A \arrow[r, hook, "k-\text{conn}"] \arrow[d] & \mathrm{Cyl}(f) \arrow[r, "q"] & Y \, , \\ X \arrow[ur, "F"'] & & \end{tikzcd}$$

where q is a homotopy equivalence (cf. proposition 1.4.12). Thus we can assume that (Y, A) is a pair, i.e. $f : A \hookrightarrow Y$ is an inclusion. (Then by the above diagram we can solve the problem for arbitrary Y.)

Choose a point a in any path-connected component of Y and fix a set of generators of $\pi_n(Y, A, a)$:

$$\{\Phi_j : (D^n, S^{n-1}, *) \to (Y, A, a)\}_{j \in J}.$$

[19] Consider then

$$\begin{tikzcd} \bigsqcup S^{n-1} \arrow[r, "\bigsqcup \Phi_j|_{S^{n-1}}"] \arrow[d, hook] & A \arrow[r, hook, "f"] \arrow[d] & Y \, . \\ \bigsqcup D^n \arrow[r, "\bigsqcup X_j"'] \arrow[rru, bend right, "\bigsqcup \Phi_j"'] \arrow[ur, phantom, "\ulcorner"] & X \arrow[ur, dashed, "F"'] & \end{tikzcd}$$

Here the right square is defined as the pushout, which yields the map F. $(F, \mathrm{id}) : (X, A) \to (Y, A)$ induces a commutative diagram of long exact sequences

$$\begin{tikzcd} \pi_n(A,a) \arrow[r] \arrow[d, "\mathrm{id}"] & \pi_n(X,a) \arrow[r] \arrow[d, "(1)"'] & \pi_n(X,A,a) \arrow[r] \arrow[d, "(2)"] & \pi_{n-1}(A,a) \arrow[r] \arrow[d, "\mathrm{id}"'] \arrow[dr, "f_*"] & \pi_{n-1}(X,a) \arrow[r] \arrow[d, "(3)"] & 0 \, . \\ \pi_n(A,a) \arrow[r] & \pi_n(Y,a) \arrow[r] & \pi_n(Y,A,a) \arrow[r] & \pi_{n-1}(A,a) \arrow[r] & \pi_{n-1}(Y,a) \arrow[r] & 0 \end{tikzcd}$$

(2) is surjective (since $\{X_j : (D^n, S^{n-1}, *) \to (X, A, a)\}_{j \in J} \xrightarrow{F_*} \{\Phi_j : (D^n, S^{n-1}, *) \to (Y, A, a)\}_{j \in J}$ by construction of F). Furthermore we have $\pi_{n-1}(X, A, a) = 0$ since we only attach n-dimensional cells, and $\pi_{n-1}(Y, A, a) = 0$ since f is n-connected. Thus by exactness and a diagram chase, (3) is injective.

On the other hand, (3) is surjective, as the surjection $f_* : \pi_{n-1}(A, a) \twoheadrightarrow \pi_{n-1}(Y, a)$ factors

[19] This is possible since $\pi_0(A) \xrightarrow{\sim} \pi_0(Y)$ is bijective.

through (3). Thus (3)($= F_*$) is an isomorphism.

By another diagram chase, (1)($= F_*$) is surjective. Thus by proposition 1.5.2, 3., F is n-connected.[20] □

Corollary 1.8.2. *Let Y be a topological space. Then there exists a CW-complex X and a weak equivalence $F : X \to Y$.*

Proof. Apply the preceding theorem to $A = \emptyset$. □

Corollary 1.8.3. *Let Y be a CW-complex with $\pi_i(Y) = 0$ for $0 \leq i \leq k$, $k \in \mathbb{N}$. Then Y is homotopy equivalent to a CW-complex X with $X^{(k)} = *$.*

Proof. Consider

$$\begin{tikzcd} * \arrow[r, "f"] \arrow[d] & Y . \\ X \arrow[ur, "F"'] & \end{tikzcd}$$

f is k-connected by assumption. By CW-approximation (the preceding theorem), F exists, where we choose $n = \infty$. Then F is a homotopy equivalence by Whitehead's theorem 1.7.4. □

1.9 Excision for homotopy groups and applications

Motivation:

Example 1.9.1. Let $i : A \hookrightarrow X$ be an inclusion. Often (e.g. if A is a NDR, neighbourhood deformation retract) this induces an isomorphism

$$H_*(X, A) \xrightarrow{\sim} H_*(X/A)$$

on homology (in all degrees, indicated by the subscript). This is not true for

$$\pi_*(X, A) \to \pi_*(X/A, *).$$

Counterexample: $(X, A) = (D^2, S^1)$, thus $X/A \cong S^2$: From the long exact sequence we have

$$\ldots \longrightarrow \pi_1(D^2) = 0 \longrightarrow \pi_1(S^1) \cong \mathbb{Z} \longrightarrow \pi_1(D^2, S^1) \cong \mathbb{Z} \longrightarrow \pi_0(S^1) \longrightarrow \pi_0(D^2) .$$

Thus $\pi_1(D^2, S^1) \to \pi_1(S^2) = 0$ is not injective. On the other hand, for $i \geq 2$, $\pi_i(D^2, S^1) = 0$. Thus e.g. $\pi_2(D^2, S^1) \to \pi_2(S^2)$ is not surjective.

Excision states that $\pi_i(X, A) \cong \pi_i(X/A)$ in a certain range of i.

Theorem 1.9.2*. (Excision for arbitrary topological spaces) *Let Y be a topological space, $Y = Y_0 \cup Y_1$ with Y_0, Y_1 open and $Y_{01} := Y_0 \cap Y_1 \neq \emptyset$. If (Y, Y_{01}) is p-connected, (Y_0, Y_{01}) is q-connected for $p, q \geq 0$, then $\pi_i(Y_0, Y_{01}) \to \pi_i(Y, Y_1)$ is*

$$\begin{cases} \text{an isomorphism for } i \leq p + q - 1, \\ \text{surjective for } i = p + q. \end{cases}$$

Diagram:

$$\begin{tikzcd} Y_{01} \arrow[r, hook, "p\text{-conn.}"] \arrow[d, hook, "q\text{-conn.}"'] & Y_1 . \arrow[d, hook] \\ Y_0 \arrow[r, hook] & Y \end{tikzcd}$$

[20] In all lower degrees, the relative homotopy groups $\pi_k(X, A, a)$ are zero, thus $\pi_k(X, a) \cong \pi_k(A, a)$ for $k < n - 1$. Then there F_* is an isomorphism because f_* is one.

Theorem 1.9.3*. (Excision for CW-complexes) *Let X be a CW-complex, $X = A \cup B$ with A, B CW-complexes such that the inclusions are cellular maps, and $C := A \cap B \neq \emptyset$. If (A, C) is p-connected, (B, C) is q-connected, $p, q \geq 0$, then $\pi_i(A, C) \to \pi_i(X, B)$ is*

$$\begin{cases} \text{an isomorphism for } i \leq p + q - 1, \\ \text{surjective for } i = p + q. \end{cases}$$

Excision for CW-complexes implies excision for arbitrary topological spaces: Consider

$$\begin{tikzcd}
C \arrow[rr, hook, "p-\text{conn.}"] \arrow[dr, "\text{weak eq.}"] \arrow[dd, hook, "q-\text{conn.}"'] & & A \arrow[dr, "\text{weak eq.}"] & \\
& Y_{01} \arrow[rr, hook] \arrow[dd, hook] & & Y_1 \\
B \arrow[dr, "\text{weak eq.}"'] & & & \\
& Y_0 & &
\end{tikzcd} .$$

The weak equivalences exist by CW-approximation (theorem 1.8.1 and its first corollary). We have an induced map

$$X := A \cup_C B \to Y = Y_0 \cup_{Y_{01}} Y_1$$

(where C is non-empty because it is weakly equivalent to a non-empty space). This is a weak equivalence by the higher connectivity theorem 1.5.3.[21] Then excision for CW-complexes is recognized to imply excision for arbitrary topological spaces.

Furthermore, excision for arbitrary topological spaces implies excision for CW-complexes (exercise). However, the actual proofs of the theorems (actually theorem 1.9.3) are postponed to section 2.7.[22]

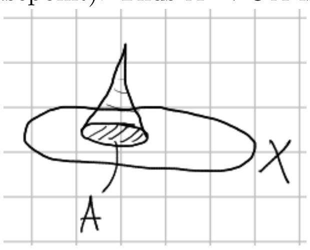
Corollary 1.9.4*. (Quotient theorem) *Let (X, A) be a p-connected CW-pair, where A is q-connected, $p, q \geq 0$. Then $\pi_i(X, A) \to \pi_i(X/A)$ (induced by the canonical projection) is*

$$\begin{cases} \text{an isomorphism for } i \leq p + q, \\ \text{surjective for } i = p + q + 1. \end{cases}$$

Proof. Consider the cone $CA := (A \times I)/(A \times \{1\})$; this is contractible. Now consider the pushout

$$\begin{tikzcd}
A \arrow[r, hook, "p-\text{conn.}"] \arrow[d, hook, "(q+1)-\text{conn.}"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d] \\
CA \arrow[r] & X \cup_A CA
\end{tikzcd} .$$

By the long exact sequence, we have isomorphisms $\pi_i(CA, A) \cong \pi_{i-1}(A)$ for $i \geq 1$ (for any basepoint). Thus $A \hookrightarrow CA$ is $q + 1$-connected since A is q-connected.



[21] The theorem must be applied to suitable open neighbourhoods of A and B, which exist since relative CW-complexes are neighbourhood deformation retracts.

[22] For this reason, they have been marked with an asterisk.

$(X \cup_A CA, CA)$ is a CW-pair, and in $X \cup_A CA$ we have $X \cap CA = A$. By the excision theorem for CW-complexes, $\pi_i(X,A) \to \pi_i(X \cup_A CA, CA)$ is

$$\begin{cases} \text{an isomorphism for } i \le p+q, \\ \text{surjective for } i = p+q+1. \end{cases}$$

Since $CA \simeq *$, $\pi_i(X \cup_A CA, CA) \cong \pi_i(X \cup_A CA)$. Now observe that $\pi_i(X \cup CA) \cong \pi_i(X/A)$: Consider the pushout

$$\begin{tikzcd} CA \arrow[r, "\text{cofibration}"] \arrow[d, "\text{homotopy eq.}"'] \arrow[dr, phantom, "\ulcorner" very near end] & X \cup_A CA \arrow[d] \\ * \arrow[r] & X/A \end{tikzcd} .$$

By an exercise in the section on higher connectivity, the right map is a homotopy equivalence.[23] □

Corollary 1.9.5*. (Freudenthal's suspension theorem) *Let X be an n-connected pointed CW-complex. Then there are natural morphisms*

$$\pi_i(X) \to \pi_{i+1}(\Sigma X),$$

where ΣX is the (reduced or unreduced) suspension. These are

$$\begin{cases} \text{isomorphisms for } i \le 2n, \\ \text{surjective for } i = 2n+1. \end{cases}$$

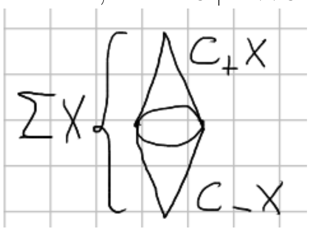
In particular,

$$\pi_i(S^{n+1}) \cong \pi_{i+1}(S^{n+2})$$

for $i \le 2n$. Therefore there is a chain of isomorphisms[24]

$$\mathbb{Z} \cong \pi_2(S^2) \xrightarrow{\cong} \pi_3(S^3) \xrightarrow{\cong} \pi_4(S^4) \xrightarrow{\cong} \dots \xrightarrow{\cong} \pi_n(S^n) \xrightarrow{\cong} \dots .$$

Proof. (For CW-complexes, the reduced and unreduced suspension are homotopy equivalent. Thus it suffices to consider the unreduced one.) We decompose $\Sigma X = C_+X \cup C_-X$ into the upper and lower cone, where $C_+X \cap C_-X = X$.



As in the proof of the quotient theorem 1.9.4, $(C_\pm X, X)$ are recognized to be $(n+1)$-connected because of the long exact sequence of homotopy groups. We have a commutative diagram

$$\begin{tikzcd} \pi_{i+1}(C_+X, X) \arrow[r] \arrow[d, "\cong"'] & \pi_{i+1}(\Sigma X, C_-X) \\ \pi_i(X) \arrow[r, dashed] & \pi_{i+1}(\Sigma X) \arrow[u, "\cong"'] \end{tikzcd} .$$

The left map is an isomorphism because of the long exact sequence for (C_+X, X), the right one because of the long exact sequence for $(\Sigma X, C_-X)$. Because of the excision theorem, the upper

[23]By functoriality, the resulting isomorphism is indeed the one induced by the canonical projection. We may glue the two pushout diagrams together; then the resulting large diagram is again a pushout and the resulting composed map $X \to X \cup_A CA \to X/A$ is exactly the canonical projection.

[24]It starts at $\pi_2(S^2) \cong \mathbb{Z}$ because of the Hopf fibration.

map is an isomorphism for $i+1 < n+1+n+1$, i.e. $i \leq 2n$, and surjective for $i = 2n+1$. The dashed arrow is defined as the composition; this is a natural morphism. □

Corollary 1.9.6*. *The degree map*

$$\deg : \mathbb{Z} \cdot [\mathrm{id}_{S^n}] \cong \pi_n(S^n) \to H_n(S^n) \cong \mathbb{Z}$$

is an isomorphism for all $n \in \mathbb{N}$.

Proof. We have $\deg([\mathrm{id}_{S^n}]) = 1$, so deg sends the generator of $\pi_n(S^n)$ to the generator of $H_n(S^n)$. □

Remark 1.9.7. Let X, Y be finite CW-complexes. Using Freudenthal's suspension theorem one can show that

$$[X,Y]^{\mathrm{st}} := \mathrm{colim}_n [\Sigma^n X, \Sigma^n Y] \cong [\Sigma^k X, \Sigma^k Y] \qquad \text{for } k \gg 0.$$

This leads to the subject of *stable homotopy theory.* In particular, one can define the stable homotopy groups

$$\pi_i^{\mathrm{st}}(X) := \mathrm{colim}_k\, \pi_{i+k}(\Sigma^k X).$$

Theorem 1.9.8*. (Hurewicz' theorem) *Let X be an $(n-1)$-connected topological space, $n \geq 2$. Then*

$$\widetilde{H}_i(X) = 0 \qquad \text{for } i \leq n-1$$

and the Hurewicz homomorphism

$$h : \pi_n(X) \xrightarrow{\sim} H_n(X)$$

is an isomorphism.

If (X, A) is $(n-1)$-connected and A is simply path-connected, then

$$H_i(X, A) = 0 \qquad \text{for } i \leq n-1$$

and

$$h : \pi_n(X, A) \xrightarrow{\sim} H_n(X, A)$$

is an isomorphism.

Remark 1.9.9.

1. Recall the definition of the Hurewicz homomorphism:

$$\begin{array}{ll} h_n : \pi_n(X,x) = [(S^n,*),(X,x)] & \to H_n(X), \\ (f : S^n \to X) & \mapsto f_*([S^n]), \end{array}$$

   where $[S^n] \in H_n(S^n) \cong \mathbb{Z} \cdot [S^n]$ is the generator (before also called $[\mathrm{id}_{S^n}]$. Similarly we define $h_n : \pi_n(X, A, x) \to H_n(X, A)$.

2. If X is not simply connected, but path-connected, we have at least a surjection

$$\pi_1(X) \twoheadrightarrow H_1(X) \cong \pi_1(X)/[\pi_1(X), \pi_1(X)]$$

   of the fundamental group onto its abelianization (without proof here).

Proof of theorem 1.9.8.

1. At first, let us only consider CW-complexes. Then we can reduce the relative statement to the absolute one for CW-pairs: consider an $(n-1)$-connected CW-pair (X, A), where A is 1-connected. By the quotient theorem 1.9.4, $\pi_i(X,A) \cong \pi_i(X/A)$ for $i \leq n$. Furthermore $H_i(X, A) \cong \widetilde{H}_i(X/A)$, because A is a neighbourhood deformation retract in X. $Y := X/A$ is an $(n-1)$-connected CW-complex.

2. Thus in the following, after the reduction step, we can assume to have a $(n-1)$-connected CW-complex given. By CW-approximation (theorem 1.8.1 resp. one of its corollaries), we can assume that $Y^{(n+1)} = \{*\}$. Thus

$$Y^{(n)} = \bigvee_{\alpha \in A} S^n$$

is a wedge sum, where A is an index set. $Y^{(n+1)}$ is obtained from $Y^{(n)}$ by attaching $(n+1)$-cells e_β for $\beta \in B$ and some index set B.

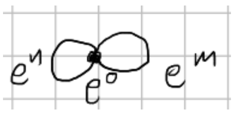
Claim:

$$\begin{tikzcd}
\pi_n(\bigvee_{\alpha\in A} S^n) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \bigoplus_{\alpha\in A} \mathbb{Z}[i_\alpha : S^n \hookrightarrow \bigvee_{\alpha\in A} S^n] \arrow[d, "\cong"] \\
\operatorname{colim}_{A'\subseteq A \text{ finite}} \pi_n(\bigvee_{\alpha\in A'} S^n) \arrow[r, "\cong"'] & \operatorname{colim}_{A'\subseteq A \text{ finite}} \bigoplus_{\alpha\in A'} \mathbb{Z}[i_\alpha]
\end{tikzcd}$$

is a commutative diagram of isomorphisms. Here the left map is an isomorphism by a compactness argument, the right isomorphism is purely algebraic. The actual claim is that the upper map is an isomorphism, but by commutativity of the diagram, it suffices to consider finite $A' \subseteq A$, since this will then prove that the lower map is an isomorphism.

Proof of the claim: $\bigvee_{\alpha\in A'} S^n \hookrightarrow \prod_{\alpha\in A'}$ is a CW-pair that is $(2n-1)$-connected, since $\prod_{\alpha\in A'} S^n$ is obtained from $\bigvee_{\alpha\in A'} S^n$ by attaching $2n, 3n, 4n, \ldots, |A'|n$-cells. For example, if $|A'| = 2$, we have $S^n \vee S^m \hookrightarrow S^n \times S^m$, where $S^n \times S^m$ consists of

$$\left\{\begin{array}{ll} \left.\begin{array}{c} e^0 \times e^0 \\ e^n \times e^0 \\ e^0 \times e^m \end{array}\right\} & S^n \vee S^m \\ e^n \times e^m & (n+m) - \text{cell} \end{array}\right.$$



Thus

$$\pi_n(\bigvee_{\alpha\in A'} S^n) \xrightarrow{\cong} \pi_n(\prod_{\alpha\in A'} S^n) \xrightarrow{\cong} \bigoplus_{\alpha\in A'} \pi_n(S^n) \xrightarrow{\cong} \bigoplus_{\alpha\in A'} \mathbb{Z}\,,$$

where the second isomorphism exists since the product is finite, $n > 1$ (thus one gets the direct sum instead of some product) and by Algebraic Topology 1. This proves the claim.

Furthermore $(Y, Y^{(n+1)})$ is $(n+1)$-connected (see a corollary to the "easy excision" theorem 1.5.9). Thus $\pi_n(Y) \cong \pi_n(Y^{(n+1)})$. Also, by cellular homology $H_n(Y) \cong H_n(Y^{(n+1)})$. Therefore we have

$$\begin{tikzcd}
\pi_{n+1}(Y^{(n+1)}, Y^{(n)}) \arrow[r] \arrow[d, "\cong", "(1)"'] & \pi_n(Y^{(n)}) \arrow[r] \arrow[d, "\cong", "(4)"'] & \pi_n(Y^{(n+1)}) \arrow[r] \arrow[ddd, "h-\text{Hurewicz homomorphism}"] & \pi_n(Y^{(n+1)}, Y^{(n)}) = 0 \quad (6)\,. \\
\pi_{n+1}(\underbrace{Y^{(n+1)}/Y^{(n)}}_{\bigvee_{\beta\in B} S^{n+1}}) \arrow[d, "\cong", "(3)"'] & \bigoplus_{\alpha\in A} \mathbb{Z}\cdot[i_\alpha] & & \\
\bigoplus_{\beta\in B} \mathbb{Z}e_\beta & & & \\
H_{n+1}(Y^{(n+1)}, Y^{(n)}) \arrow[r] \arrow[u, "\cong", "(2)"'] & H_n(Y^{(n)}) \arrow[r] \arrow[uu, "\cong", "(5)"'] & H_n(Y^{(n+1)}) \arrow[r] & H_n(Y^{(n+1)}, Y^{(n)}) = 0 \quad (7)
\end{tikzcd}$$

The upper and lower row are the long exact sequences of homotopy resp. homology groups.

(1): This is an isomorphism due to the quotient theorem; note $2n - 1 \geq n + 1$ if $n \geq 2$.

(2): This is an isomorphism since $(Y^{(n+1)}, Y^{(n)})$ is a good pair, cf. excision on homology.

(3): By the claim above, with B instead of A.

(4): By the claim above.

(5): By cellular homology.

(6): Since $(Y^{(n+1)}, Y^{(n)})$ is n-connected, this is zero.

(7): $H_n(Y^{(n+1)}, Y^{(n)}) = 0$ since it is isomorphic to $\widetilde{H}_n(Y^{(n+1)}/Y^{(n)})$, and we attach $(n+1)$-cells. (Use e.g. cellular homology.)

Now verify commutativity. Then h is an isomorphism by taking quotients of isomorphic groups (or the five lemma).

3. The case of arbitrary topological spaces may be reduced to the case of CW-complexes, by virtue of the following proposition.

□

Proposition 1.9.10. *Let $f : X \to Y$ be a weak equivalence. Then for all coefficient modules $A \in \mathrm{Ab}$*

$$f_* : H_*(X, A) \xrightarrow{\sim} H_*(Y, A),$$
$$f^* : H^*(Y, A) \xrightarrow{\sim} H^*(X, A)$$

are isomorphisms between the homology and cohomology of X and Y.

Proof. 1. We start with the proof for homology and with the "usual" coefficients in $\mathbb{Z}$. We can reduce to the inclusion using the mapping cylinder construction (proposition 1.4.12):

$$\begin{tikzcd} X \arrow[r, hook] \arrow[rr, bend right, "f"'] & \mathrm{Cyl}(f) \arrow[r, "\text{h. eq.}"] & Y \end{tikzcd}$$

(where h. eq. denotes a homotopy equivalence). Then $X \hookrightarrow \mathrm{Cyl}(f)$ is a weak equivalence as well, and due to $\pi_i(\mathrm{Cyl}(f)) \cong \pi_i(Y)$, $H_i(\mathrm{Cyl}(f)) \cong H_i(Y)$ it suffices to prove the claim for the inclusion.

2. Consider the diagram of long exact sequences

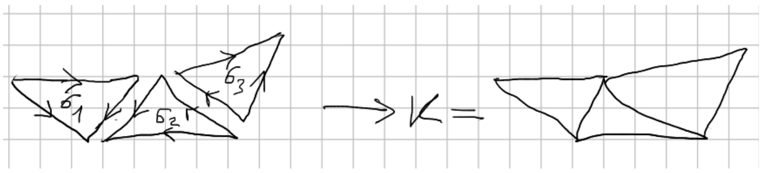
$$\begin{tikzcd} \dots \arrow[r] & \pi_i(X) \arrow[r] \arrow[d, "h"] & \pi_i(Y) \arrow[r] \arrow[d, "h"] & \pi_i(Y, X) \arrow[r] \arrow[d, "h"] & \pi_{i-1}(X) \arrow[r] \arrow[d, "h"] & \dots, \\ \dots \arrow[r] & H_i(X) \arrow[r] & H_i(Y) \arrow[r] & H_i(Y, X) \arrow[r] & H_{i-1}(X) \arrow[r] & \dots \end{tikzcd}$$

where all vertical maps are the (relative resp. absolute) Hurewicz homomorphism. If we can prove that $H_i(Y, X) = 0$ for all $i \in \mathbb{N}$, then we obtain the desired isomorphisms $H_i(X) \cong H_i(Y)$. It suffices to show that if (Y, X) is n-connected, then $H_i(Y, X) = 0$ for $i \leq n$.

3. Thus assume (Y, X) to be n-connected. Let $[\alpha] \in H_j(Y, X)$ be a homology class. This can be "lifted" to $\alpha = \sum_{i \in I} n_i \sigma_i$, where I is an index set $n_i \in A$ for $i \in I$, $\sigma_i : \Delta^j \to Y$ are simplices, such that ∂A is a chain in X. Define a CW-complex

$$K := \bigsqcup_{i \in I} \Delta^j / \left(\begin{array}{c} \text{glue } (j-1)\text{-dimensional faces of } \Delta^j\text{'s that are} \\ \text{sent to } Y \text{ identically under different } \sigma_i \end{array} \right).$$

Example for $i = 2$, in Y:



Define $\sigma : K \to Y$ such that

$$\begin{tikzcd} K \arrow[r, "\sigma"] & Y \\ \Delta^j \arrow[u, hook] \arrow[ur, "\sigma_i"'] & \end{tikzcd}$$

commutes. Furthermore let $L \subseteq K$ be the subcomplex of $(i-1)$-dimensional faces of Δ^i that appear non-trivially in $\partial\alpha$. Now consider

$$\begin{tikzcd} L \arrow[r, "\sigma|_L"] \arrow[d, hook] & X \arrow[d, hook] \\ K \arrow[r, "\sigma"'] & Y \end{tikzcd} .$$

Since $X \hookrightarrow Y$ is by assumption n-connected, there exists $g : K \to X$ such that $g \sim \sigma \operatorname{rel} L$ (cf. proposition 1.7.1). There exists $\alpha' \in C_i^{\text{sing}}(K)$ with $\sigma \circ \alpha' = \alpha$ (by construction). Thus

$$[\alpha] = \sigma_*([\alpha']) = (\text{Image of } g_*([\alpha']) \text{ along } H_i(X, X) \to H_i(Y, X)) = 0,$$

since $H_i(X, X) = 0$. This proves $H_i(Y, X) = 0$ and therefore the claim for homology with $\mathbb{Z}$-coefficients.

4. Now, since $f_* : H_*(X) \xrightarrow{\sim} H_*(Y)$ is an isomorphism by the first part, also

$$\begin{aligned} f_* &: H_*(X, A) \xrightarrow{\sim} H_*(Y, A), \\ f^* &: H^*(Y, A) \xrightarrow{\sim} H^*(X, A) \end{aligned}$$

are isomorphisms for all $A \in \text{Ab}$ by the universal coefficient theorem.

$\square$

2 Homotopy pullbacks and homotopy pushouts

Literature:

- M. Mather, *Pullbacks in homotopy theory*
- M. Mather, *Hurewicz theorem for pairs and squares*
- P. Fantham, I. James, M. Mather, *On the reduced product construction*
- J. May, *Weak equivalences and quasifibrations*
- J. Milnor, *On spaces having the homotopy types of CW-complexes*
- P. Fantham, F. Moore, *Groupoid enriched categories and homotopy theory*
- J. Strom, *Modern classical homotopy theory*
- B. Munson, I. Volić, *Cubical Homotopy Theory*

2.1 "Problems" with categorical constructions in $\mathrm{Top}_{(*)}$ and $\mathrm{hTop}_{(*)}$

This section is meant to motivate that for a proper treatment of homotopies, a different categorical setting than the one previously used is needed.

- Top as a category "knows nothing" about homotopies. Consider the two pushout squares in Top

  $$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\ulcorner"] & * \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r] \arrow[d] & CX \arrow[d] \\ CX \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \Sigma X \end{tikzcd} .$$

  There is a levelwise homotopy equivalence

  $$\begin{tikzcd} X \arrow[rr, "\mathrm{id}_X"'] \arrow[dr] \arrow[dd] & & X \arrow[dr] \arrow[dd] & \\ & * \arrow[rr] & & CX \\ * \arrow[rr] & & CX & \end{tikzcd} ,$$

  but $\Sigma X \to *$ is not a homotopy equivalence. Thus the pushout in Top is incompatible with homotopy equivalences.

- To find a "dual counterexample" for pullbacks instead of pushouts, for a topological space X, define the *space of paths in* X $\Pi X := \mathrm{map}(I, X) = X^I$. As a set, this is $\{f \mid f : I \to X \text{ is a map}\}$. Now define the space $\Pi_x X$ by the pullback diagram

  $$\begin{tikzcd} \Pi_x X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi X \arrow[d, "\mathrm{ev}_0"] \\ * \arrow[r, "x"'] & X \end{tikzcd} ,$$

where $x \in X$ and the constant map to x is again denoted x. Then $\Pi_x X$ is the set of paths starting at x; it is also called the *fiber* over x. $\Pi_x X$ is contractible: Consider

$$\begin{array}{l} H : \Pi_x X \times I \to \Pi_x X, \\ (\gamma, t) \mapsto (\tau \mapsto \gamma_t(\tau) := \gamma(t\tau)) \end{array} .$$

We have $H_1 = \mathrm{id}_{\Pi_x X}$, and $H_0 = x : \Pi_x X \to \Pi_x X$ (i.e. every path gets mapped to the constant path x). Now compare the two pullback diagrams

$$\begin{tikzcd} * \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "x"] \\ * \arrow[r, "x"'] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Omega_x X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_x X \arrow[d, "\mathrm{ev}_1"] \\ * \arrow[r, "x"'] & X \end{tikzcd} .$$

Here $\Omega_x X$ is the space of loops in X based at x. By the above, we have a levelwise homotopy equivalence

$$\begin{tikzcd} & * \arrow[d] & \Pi_x X \arrow[l] \arrow[d] \\ & X & X \arrow[l, "\mathrm{id}_X"'] \\ * \arrow[ur] & * \arrow[l] \arrow[ur] & \end{tikzcd} .$$

But it can be shown[1] that $\pi_0(\Omega_x X) \cong \pi_1(X, x)$. Thus $\Omega_x X$ is not contractible if X is not simply connected, and therefore also the pullback in Top is incompatible with homotopy equivalences.

– However, it is also not favorable to simply work in the homotopy categories hTop resp. hTop_* instead. hTop_* does not have some simple pushouts: Consider the situation in hTop_*

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, "z^2"'] & * \arrow[d] \arrow[ddr] & \\ S^1 \arrow[r] \arrow[drr, "\gamma"'] & P & \\ & & Z \end{tikzcd} .$$

Here z^2 denotes the pointed homotopy class of the map of degree 2 (viewing S^1 as a subset of $\mathbb{C}$), and the arrows are assumed to commute in hTop_* (i.e. up to pointed homotopy). Suppose that there existed P such that P were a pushout in hTop_*. We construct a contradiction by considering three different spaces Z and analyzing the conditions on the maps $P \to Z$.

There is a fiber bundle $\mathbb{R}P^3 \cong SO(3) \longrightarrow SO(3)/SO(2) \cong S^2$, with fiber $S^1 \cong SO(2)$. ($SO(3)$ acts on S^2, and $SO(2)$ can be regarded as e.g. the stabilizer of the unit vector $(0,0,1)^T$.) Because it is a fibration, for any $T \in \mathrm{Top}_*$, we have an exact sequence of pointed sets

$$[T, S^1]_* \longrightarrow [T, \mathbb{R}P^3]_* \longrightarrow [T, S^2]_* .$$

[2]

Now note $[S^1, Z] = \pi_1(Z)$. Then an examination of the (assumed) pushout diagram above

[1]See later in a more general setting, or explicitly as an exercise.

[2]Here $[T, Y]_*$ denotes the set of pointed maps from T to Y, modulo homotopy. The statement about the exact sequence means just that every map $T \to \mathbb{R}P^3$ whose composition with the projection is nullhomotopic in S^2 is already homotopic to a map from T to S^1. In order to prove this, one just needs to lift the homotopy to $\mathbb{R}P^3$, which is possible since $\mathbb{R}P^3 \to S^2$ is a fibration.

shows that

$$[P, Z]_* = \{\gamma \in \pi_1(Z) \,|\, \gamma^2 = e\}$$

(where e is the neutral element). Then the above exact sequence reads

$$0 = [P, S^1]_* \longrightarrow [P, \mathbb{R}P^3]_* \cong \mathbb{Z}/2 \longrightarrow [P, S^2]_* = 0\ .$$

This is a contradiction (it clearly is not exact). Thus P cannot exist.

The idea to remedy these shortcomings is to include homotopies in the structure of the category (and develop language for this). This will be done in the following sections.

Future example of reasoning: Consider e.g. the data encoded in a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d, "z_0"] \\ * \arrow[r, "z_1"'] \arrow[ur, phantom, "\overset{H}{\Leftarrow}"] & Z \end{tikzcd}\ .$$

Here (and often in the following) this means that the diagram only commutes up to homotopy, i.e. there is

$$\begin{aligned} H &: X \times I \to Z \\ H_0(X) &= \{z_0\}, \\ H_1(X) &= \{z_1\}. \end{aligned}$$

This is equivalent to a map $\Sigma X \to Z$. Thus we will define ΣX as the homotopy pushout of the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \\ * & \end{tikzcd}\ .$$

Dually (or at least similarly), the data of the diagram

$$\begin{tikzcd} W \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & & * \arrow[d, "x"] \\ & * \arrow[r, "x"'] \arrow[ur, phantom, "\overset{H}{\Leftarrow}"] & X \end{tikzcd}$$

is equivalent to a map $W \to \Omega_x X$. Thus we define $\Omega_x X$ as the homotopy pullback of the diagram

$$\begin{tikzcd} & * \arrow[d, "x"] \\ * \arrow[r, "x"'] & X \end{tikzcd}\ .$$

Problem: composition (i.e. concatenation) of homotopies is not associative.
Solution: It is associative if we treat homotopies up to homotopies between homotopies.

2.2 Elements of 2-category theory

Definition 2.2.1. A 2-*category* $\mathcal{C}$ consists of the following data:

– A set/class of objects $\mathrm{Ob}\,\mathcal{C}$.

- For all $x, y \in \mathrm{Ob}\,\mathcal{C}$, a small category $\mathrm{HOM}_{\mathcal{C}}(x,y)$,[3] such that:
  - Objects of $\mathrm{HOM}_{\mathcal{C}}(x,y)$ are 1-*morphisms*

$$x \xrightarrow{f} y\ .$$

  - Morphisms of $\mathrm{HOM}_{\mathcal{C}}(x,y)$ are 2-*morphisms*

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow H"] & y \end{tikzcd}\ , \qquad H : f \to g$$

- For all $x, y, z \in \mathrm{Ob}\,\mathcal{C}$ composition functors

$$\underbrace{\mathrm{HOM}_{\mathcal{C}}(y,z) \times \mathrm{HOM}_{\mathcal{C}}(x,y)}_{\substack{\text{product of categories}\\ \text{objects: pairs of objects}\\ \text{morphisms: pairs of morphisms}}} \xrightarrow{\circ_{x,y,z}} \mathrm{HOM}_{\mathcal{C}}(x,z)\ .$$

- For $x \in \mathrm{Ob}_{\mathcal{C}}$ an identity morphism $\mathrm{id}_x \in \mathrm{Ob}\,\mathrm{HOM}_{\mathcal{C}}(x,x)$.

This satisfies the following requirements:

- Associativity of compositions: For all $x, y, z, t \in \mathrm{Ob}\,\mathcal{C}$,

$$\begin{tikzcd} \mathrm{HOM}_{\mathcal{C}}(z,t) \times \mathrm{HOM}_{\mathcal{C}}(y,z) \times \mathrm{HOM}_{\mathcal{C}}(x,y) \arrow[r, "\mathrm{id}\times\circ_{x,y,z}"] \arrow[d, "\circ_{y,z,t}\times\mathrm{id}"'] & \mathrm{HOM}_{\mathcal{C}}(z,t) \times \mathrm{HOM}_{\mathcal{C}}(x,z) \arrow[d, "\circ_{x,z,t}"] \\ \mathrm{HOM}_{\mathcal{C}}(y,t) \times \mathrm{HOM}_{\mathcal{C}}(x,y) \arrow[r, "\circ_{x,y,t}"'] & \mathrm{HOM}_{\mathcal{C}}(x,t) \end{tikzcd}$$

  commutes.

- For all $x, y \in \mathrm{Ob}\,\mathcal{C}$,

$$\begin{tikzcd} \mathrm{HOM}_{\mathcal{C}}(x,y) \times \mathrm{HOM}_{\mathcal{C}}(x,x) \arrow[r, "\circ_{x,x,y}"] & \mathrm{HOM}_{\mathcal{C}}(x,y) \\ \mathrm{HOM}_{\mathcal{C}}(x,y) \arrow[u, "\mathrm{id}\times\mathrm{id}_x"] \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

  commutes.[4] Furthermore

$$\begin{tikzcd} \mathrm{HOM}_{\mathcal{C}}(y,y) \times \mathrm{HOM}_{\mathcal{C}}(x,y) \arrow[r, "\circ_{x,y,y}"] & \mathrm{HOM}_{\mathcal{C}}(x,y) \\ \mathrm{HOM}_{\mathcal{C}}(x,y) \arrow[u, "\mathrm{id}_y\times\mathrm{id}"] \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

  commutes.

Definition 2.2.2. A *groupoid* is a small (1-)category, where all morphisms are isomorphisms.

[3] A small category is one where for all pairs of objects, the class of morphisms between them is a set.
[4] Here id is a functor, while id_x is an object in $\mathrm{HOM}_{\mathcal{C}}(x,x)$; thus the "product functor" in the left arrow is constant in one component.

Example 2.2.3. 1. Let G be a group. Then there is a category with one object $\bullet$ and $\mathrm{Hom}(\bullet,\bullet) = G$. This is a groupoid.[5]

2. Let X be a topological space. Then we define the *fundamental groupoid* of X to be the category $\Pi_1 X$ with

$$\begin{array}{ll} \mathrm{Ob}\,\Pi_1 X & := \text{points in } X \\ \forall x, y \in X : \mathrm{Hom}_{\Pi_1 X}(x,y) & := \{\gamma : I \to X \mid \gamma(0) = x, \gamma(1) = y\}/(\text{homotopies of paths}) \end{array}.$$

E.g. for $x \in X$, $\mathrm{Hom}_{\Pi_1 X}(x,x) = \pi_1(X,x)$.

Remark 2.2.4. If X is path-connected, then

$$\left(\begin{tikzcd} \bullet \arrow[loop left, "\pi_1(X{,}x)"] \end{tikzcd} \right) \hookrightarrow \Pi_1 X$$

is an inclusion of categories. Here we denote by $\begin{tikzcd} \bullet \arrow[loop left, "\pi_1(X{,}x)"] \end{tikzcd}$ the category defined by the group $\pi_1(X,x)$ (as in the first example above).

Definition 2.2.5. A *(2,1)-category* $\mathcal{C}$ is a 2-category where for all $x, y \in \mathrm{Ob}\,\mathcal{C}$, $\mathrm{HOM}_{\mathcal{C}}(x,y)$ is a groupoid.

(I.e. in a $(2,1)$-category, every "double arrow" or "homotopy" has an inverse.)

Example 2.2.6. 1. Cat, the category of categories, is a 2-category. Here for $\mathcal{C}, \mathcal{D} \in \mathrm{Ob}\,\mathrm{Cat}$, $\mathrm{HOM}_{\mathrm{Cat}}(\mathcal{C},\mathcal{D})$ consists of

$$\begin{array}{ll} \mathrm{Ob}\,\mathrm{HOM}_{\mathrm{Cat}}(\mathcal{C},\mathcal{D}) & = \text{functors from } \mathcal{C} \text{ to } \mathcal{D} \\ \forall F, G \in \mathrm{Ob}\,\mathrm{HOM}_{\mathrm{Cat}}(\mathcal{C},\mathcal{D}) : \mathrm{Mor}_{\mathrm{HOM}_{\mathrm{Cat}}(\mathcal{C},\mathcal{D})}(F,G) & = \text{natural transformations of functors from } F \text{ to } G. \end{array}$$

In diagrams

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D} \end{tikzcd}.$$

2. The category of groupoids $\mathrm{Gpd} \subset \mathrm{Cat}$ (this is an inclusion of categories) is a $(2,1)$-category. I.e. if $\mathcal{C}, \mathcal{D}$ are two groupoids, and we have a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D} \end{tikzcd},$$

then η is invertible. Verify this: Let $c_1, c_2 \in \mathrm{Ob}\,\mathcal{C}$. By definition of the natural transformation η, we have morphisms $\eta_{c_1} \in \mathrm{Hom}_{\mathcal{D}}(Fc_1, Gc_1)$, $\eta_{c_2} \in \mathrm{Hom}_{\mathcal{D}}(Fc_2, Gc_2)$ such that

$$\begin{tikzcd} Fc_1 \arrow[r, "\eta_{c_1}"] \arrow[d, "F\phi"'] & Gc_1 \arrow[d, "G\phi"] \\ Fc_2 \arrow[r, "\eta_{c_2}"'] & Gc_2 \end{tikzcd}$$

commutes in $\mathcal{D}$ for all $\phi \in \mathrm{Hom}_{\mathcal{C}}(c_1, c_2)$. I.e.

$$\eta_{c_2} \circ F\phi = G\phi \circ \eta_{c_1}.$$

Applying the inverses (which exist since $\mathcal{D}$ is a groupoid) we obtain

$$F\phi \circ \eta_{c_1}^{-1} = \eta_{c_2}^{-1} \circ G\phi.$$

[5] We denote homomorphism sets in 1-categories by lowercase Hom, while the homomorphism categories in 2-categories are denotes with uppercase HOM. Sometimes we use again subscripts to specify the category, sometimes not if it should be clear from the context.

Then $(\eta_c^{-1})_{c\in\mathcal{C}}$ is a natural transformation $G \Rightarrow F$ which is inverse to η.[6]

Remark 2.2.7. We have different compositions of morphisms between 2-categories:

- Composition of 1-morphisms:

$$\begin{tikzcd} \underset{x}{\bullet} \arrow[r, "f"] & \underset{y}{\bullet} \arrow[r, "g"] & \underset{z}{\bullet} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \underset{x}{\bullet} \arrow[r, "g\circ f"] & \underset{z}{\bullet} \end{tikzcd}.$$

- Composition of 2-morphisms:

$$\begin{tikzcd} \underset{x}{\bullet} \arrow[r, bend left=50, "f"] \arrow[r, "g" description] \arrow[r, bend right=50, "k"'] \arrow[r, phantom, shift left=3, "\Downarrow\eta"] \arrow[r, phantom, shift right=3, "\Downarrow\xi"] & \underset{y}{\bullet} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \underset{x}{\bullet} \arrow[r, bend left, "f"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\xi\circ\eta"] & \underset{y}{\bullet} \end{tikzcd}$$

- Composition of 1- and 2-morphisms:

$$\begin{tikzcd} \underset{x}{\bullet} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\eta"] & \underset{y}{\bullet} \arrow[r, "k"] & \underset{z}{\bullet} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \underset{x}{\bullet} \arrow[r, bend left, "k\circ f"] \arrow[r, bend right, "k\circ g"'] \arrow[r, phantom, "\Downarrow k\circ\eta"] & \underset{z}{\bullet} \end{tikzcd}.$$

To define $k \circ \eta$ precisely, consider the category $[1]$ consisting of two objects and a single morphism,

$$\underset{0}{\bullet} \longrightarrow \underset{1}{\bullet}.$$

A functor out of $[1]$ can be regarded as a way to specify a morphism.

Thus we can define $k \circ \eta$ by the commutative diagram

$$\begin{tikzcd} \mathrm{HOM}_{\mathcal{C}}(y,z) \times \mathrm{HOM}_{\mathcal{C}}(x,y) \arrow[r, "\circ_{x,y,z}"] & \mathrm{HOM}_{\mathcal{C}}(x,z) \\ {[1]} \arrow[u, "(f\xrightarrow{\eta}g,\, k\xrightarrow{\mathrm{id}}k)"] \arrow[ur, "(k\circ f\Rightarrow k\circ g)=:k\circ\eta"'] & \end{tikzcd},$$

where the arrow $k \circ \eta$ is defined as the composition of functors as indicated.[7]

(Analogously one can compose at first a 1-morphism with then a 2-morphism.)

We generically use the symbol "$\circ$" for all of these compositions.

Further examples of 2-categories:

Example 2.2.8. 3. Let $\mathcal{A}$ be an abelian category, e.g. $R-$Mod, where R is a ring (and $R-$Mod are the modules over it). Then there is a $(2,1)$-category $\mathrm{Ch}(\mathcal{A})$ of chain complexes over $\mathcal{A}$:

- Let $C_\bullet, D_\bullet \in \mathrm{Ch}(\mathcal{A})$. Then

$$\begin{aligned} &\mathrm{Ob}\,\mathrm{HOM}_{\mathrm{Ch}(\mathcal{A})}(C_\bullet, D_\bullet) \\ &\quad = \{(f_n : C_n \to D_n)_{n\in\mathbb{Z}} \mid d^D \circ f_n = f_{n-1} \circ d^C\}, \\ &\forall (f_n), (g_n) \in \mathrm{Ob}\,\mathrm{HOM}_{\mathrm{Ch}(\mathcal{A})} : \mathrm{Mor}_{\mathrm{HOM}_{\mathrm{Ch}(\mathcal{A})}(C_\bullet, D_\bullet)}((f_n),(g_n)) \\ &\quad := \{(h_n : C_n \to D_{n+1})_{n\in\mathbb{Z}} \mid d_n^D \circ h_n + h_{n-1} \circ d^C = f_n - g_n\} \end{aligned}.$$

[6] Just paste the two resulting commutative squares to each other.

[7] Commutativity of the diagram is used to define $k \circ \eta$, not a statement to be proven. Note that $(f \xrightarrow{\eta} g, k \xrightarrow{\mathrm{id}} k)$ denotes the images of the single morphism, not of the objects. The object 0 gets mapped onto (f,k), while the object 1 gets mapped onto (g,k).

In words, 1-morphisms are given by chain maps and 2-morphisms by chain homotopies. Diagram:

$$\begin{tikzcd} C_{n+1} \arrow[r, "d^C"] \arrow[d] & C_n \arrow[r, "d^C"] \arrow[d] \arrow[dl] & C_{n-1} \arrow[d] \arrow[dl] \\ D_{n+1} \arrow[r, "d^D"'] & D_n \arrow[r, "d^D"'] & D_{n-1} \end{tikzcd} .$$

– The composition of 2-morphisms is

$$f \xRightarrow{H} g \xRightarrow{L} k\ ; \qquad (H+L)_n = H_n + L_n.$$

– The 2-morphism $(h_n = 0)_{n\in\mathbb{Z}}$ is $\operatorname{id} f$.

– The 2-morphism $(-h_n)_{n\in\mathbb{Z}} =$ "$-h$" is the inverse 2-morphism to h.

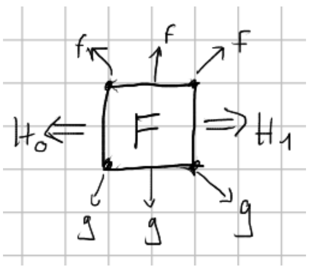
4. Main example: Top, Top$_*$. For $X, Y \in \operatorname{Ob}\mathrm{Top}_{(*)}$, we define $\mathrm{HOM}_{\mathrm{Top}_{(*)}}(X,Y)$ by

$$\begin{aligned} &\operatorname{Ob}\mathrm{HOM}_{\mathrm{Top}_{(*)}}(X,Y) \\ &\qquad = \{f : X \to Y \text{ (pointed) continuous maps}\}, \\ &\forall f, g \in \operatorname{Ob}\mathrm{HOM}_{\mathrm{Top}_{(*)}} : \operatorname{Mor}_{\mathrm{HOM}_{\mathrm{Top}_{(*)}}(X,Y)}(f,g) \\ &\qquad := \{H : X \times I \to Y \text{ (pointed) homotopy} \mid H_0 = f, H_1 = g\}/\sim, \end{aligned}$$

where $H_0 \sim H_1$ if there exists

$$\begin{aligned} &F : (X \times I) \times I \to Y, \\ &F|_{X\times I\times\{0\}} = H_0, \\ &F|_{X\times I\times\{1\}} = H_1, \\ &\forall x \in X, t \in I : F(x,0,t) = f(x), \quad F(x,1,t) = g(x). \end{aligned}$$

(In Top$_*$, F is also required to fix the basepoint.)[8]



In the following, we will often omit subscripts on the various categories and sets if they should be clear.

Lemma 2.2.9. *Let $X, Y \in \operatorname{Ob}\mathrm{Top}$. Then concatenation of homotopies make* $\mathrm{HOM}(X,Y)$ *a groupoid.*

Proof. We need to consider the situation

$$\begin{tikzcd} X \arrow[r, bend left=50, "f"{name=F}] \arrow[r, "g"{name=G, description}] \arrow[r, bend right=50, "k"'{name=K}] & Y \arrow[from=F, to=G, Rightarrow, shorten=2pt, "H_1"] \arrow[from=G, to=K, Rightarrow, shorten=2pt, "H_2"] \end{tikzcd} ,$$

where

$$(H_2 \circ H_1)(t) = \begin{cases} H_1(2t), & t \in [0, \frac{1}{2}], \\ H_2(2t-1), & t \in [\frac{1}{2}, 1]. \end{cases}$$

[8] F is the "homotopy of homotopies" mentioned earlier.

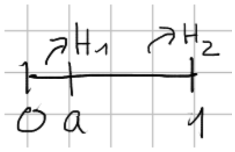
We need to check: If $H_1 \sim^F H_1'$, then there is $\widetilde{F}$ such that $H_2 \circ H_1 \sim^{\widetilde{F}} H_2 \circ H_1'$. To this end define

$$\widetilde{F}(x,t,\tau) := \begin{cases} F(x,2t,\tau), & t \in [0,\frac{1}{2}], \\ H_2(x,2t-1), & t \in [\frac{1}{2},1]. \end{cases}$$

Similarly, if $H_2 \sim H_2'$, then $H_2' \circ H_1 \sim H_2 \circ H_1$. Thus the composition is well-defined.

Furthermore the composition $\circ$ is associative: For $0 < a < 1$, define

$$(H_2 \circ_a H_1)(x,t) := \begin{cases} H_1(x, \frac{1}{a}t), & t \in [0,a], \\ H_2(x, \frac{1}{1-a}(t-a)), & t \in [a,1]. \end{cases}$$



Claim: For all $a, b \in (0,1)$, $H_2 \circ_a H_1 \sim H_1 \circ_b H_2$.

Proof of the claim: Define $F : (X \times I) \times I \to Y$ by

$$F(x,t,\tau) := \begin{cases} H_1(x, \frac{1}{a+(b-a)\tau}t), & t \in [0, a+(b-a)\tau], \\ H_2(x, \frac{1}{1-(a+(b-a)\tau)}(t-(a+(b-a)\tau))), & t \in [a+(b-a)\tau, 1]. \end{cases}$$

Then $F(\cdot,t,\tau) = H_2 \circ_{a+(b-a)\tau} H_1$, yielding the desired "homotopy of homotopies".

Using the claim, one can verify that $\circ$ is associative.

(Finally, note that every homotopy has an inverse, given by inverse parametrization. Thus $\mathrm{HOM}(X,Y)$ is indeed a groupoid.) $\square$

Remark 2.2.10. Let X, Y be topological spaces. If Y^X exists, then there is an isomorphism of categories $\mathrm{HOM}(X,Y) \cong \Pi_1(Y^X)$: For $T \in \mathrm{Ob}\,\mathrm{Top}$, we have $\mathrm{Hom}(T, Y^X) = \mathrm{Hom}(X \times T, Y)$ (where Hom denotes 1-morphisms in Top). By definition $\Pi_1(Y^X)$ is made into a category by choosing here

$$\left\{ \begin{array}{lll} T = * : & \mathrm{Hom}(*, Y^X) = \mathrm{Hom}(X,Y), & \text{objects of } \Pi_1(Y^X), \\ T = I : & \mathrm{Hom}(I, Y^X) = \mathrm{Hom}(X \times I, Y), & \left.\begin{array}{l} \\ \\ \end{array}\right\} \rightsquigarrow \begin{array}{l} \text{morphisms of } \Pi_1(Y^X) \\ \text{(homotopies modulo homotopies of homotopies)} \end{array} \\ T = I \times I : & \mathrm{Hom}(I \times I, Y^X) = \mathrm{Hom}(X \times I \times I, Y), & \end{array} \right.$$

Thus $\Pi_1(Y^X)$ has "the same" objects and morphisms as $\mathrm{HOM}(X,Y)$.

On the other hand, this can be used to define $(2,1)$-category structures on $\mathrm{Top}, \mathrm{Top}_*, \mathrm{CW}, \mathrm{CW}_*$ (which coincides by the above observation in the case of Top with the one defined above).

Remark 2.2.11. Every 1-category is a $(2,1)$-category trivially: In a 1-category $\mathcal{C}$, for $x, y \in \mathrm{Ob}\,\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(x,y) \in \mathrm{Ob}\,\mathrm{Sets}$. Then use the inclusion

$$\begin{array}{lll} \mathrm{Sets} & \hookrightarrow & \mathrm{Gpd}, \\ x & \mapsto & \begin{cases} \mathrm{Ob} : x \\ \mathrm{Mor} : \text{only identities} \end{cases} \end{array}.$$

However, in this chapter/generally, we only consider the "non-trivial" $(2,1)$-category structures on $\mathrm{Top}, \mathrm{Top}_*, \dots$.

Definition 2.2.12. Let $\mathcal{G}$ be a groupoid. Then we define

$$\pi_0\,\mathcal{G} := \mathrm{Ob}(\mathcal{G})/\text{isomorphism}.$$

Example 2.2.13. 1. If X is a topological space,

$$\pi_0\,\Pi_1 X \cong \pi_0(X),$$

where $\pi_0(X)$ on the right-hand side denotes the set of path-connected components of X as defined earlier.

2. If $\mathcal{G}_1, \mathcal{G}_2$ are groupoids, there is an isomorphism (i.e. a bijection)

$$\pi_0(\mathcal{G}_1 \times \mathcal{G}_2) \cong \pi_0\, \mathcal{G}_1 \times \pi_0\, \mathcal{G}_1.$$

(Here $\mathcal{G}_1 \times \mathcal{G}_2$ is the explicit "product category" used also earlier.)

In fact, $\pi_0 : \mathrm{Gpd} \to \mathrm{Sets}$ is a functor.

Definition 2.2.14. Let $\mathcal{C}$ be a $(2,1)$-category. Then define the *homotopy category* $\mathrm{h}\mathcal{C}$ *of* $\mathcal{C}$ by

$$\begin{array}{ll} \mathrm{Ob}\,\mathrm{h}\mathcal{C} & := \mathrm{Ob}\,\mathcal{C}, \\ \forall x, y \in \mathrm{Ob}\,\mathcal{C} : \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(x,y) & := \pi_0 \mathrm{HOM}_{\mathcal{C}}(x,y) =: [x,y]. \end{array}$$

This is a 1-category, where the composition is defined using the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(y,z) \times \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(x,y) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(x,z) \arrow[dd, equal] \\ \pi_0\, \mathrm{HOM}_{\mathcal{C}}(y,z) \times \pi_0\, \mathrm{HOM}_{\mathcal{C}}(x,y) & \\ \pi_0(\mathrm{HOM}_{\mathcal{C}}(y,z) \times \mathrm{HOM}_{\mathcal{C}}(x,y)) \arrow[r] \arrow[u, "\cong"] & \pi_0\, \mathrm{HOM}(x,z) \end{tikzcd} ,$$

where the lower map is induced by the composition functor (which is a morphism in Gpd).

Example 2.2.15. – The usual homotopy categories hTop, hTop_* are homotopy categories in the sense defined above.

– One can define a homotopy category $\mathrm{hCh}(\mathcal{A})$ of chain complexes over an abelian category $\mathcal{A}$.

Definition 2.2.16. A 1-morphism $f : x \to y$ in a $(2,1)$-category $\mathrm{h}\mathcal{C}$ is called an *equivalence*, if one of the following equivalent condition holds:

1. $[f]$ is an isomorphism in $\mathrm{h}\mathcal{C}$.
2. There exists a 1-morphism $g : y \to x$ and 2-morphisms $\xi : g \circ f \Rightarrow \mathrm{id}_x$, $\eta : f \circ g \Rightarrow \mathrm{id}_y$.

Example 2.2.17. – In Top, an equivalence is a homotopy equivalence.

– In Gpd, an equivalence is an equivalence of categories.

Definition 2.2.18 (imprecise)**.** A diagram in a 2-category $\mathcal{C}$ is a collection of:

– objects,

– 1-morphisms between some of the objects,

– 2-morphisms between some of the morphisms.

A diagram is 2-*commutative*, if whenever there are different compositions of 1-morphisms between the same two objects x, y of the diagram, then 2-morphisms between these 1-morphisms form a commutative diagram in $\mathrm{HOM}_{\mathcal{C}}(x,y)$.

(Usually the conditions to verify will be clear from the context.)

Example 2.2.19. – The most trivial 2-commutative diagram is a 2-commutative triangle:

$$\begin{tikzcd} \overset{\bullet}{x} \arrow[rr, "w"] \arrow[dr, "u"'] & \arrow[d, phantom, "\Uparrow \xi"] & \overset{\bullet}{z} \\ & \overset{\bullet}{y} \arrow[ur, "v"'] & \end{tikzcd} .$$

There is no condition on 2-morphisms.

– A 2-commutative 3-simplex is defined by the data:
  – objects $0, 1, 2, 3$,
  – 1-morphisms $01, 02, 03, 12, 13, 23$,
  – 2-morphisms $012, 013, 023, 123$.

  These are fit into the following diagram:

$$\begin{tikzcd}
\underset{0}{\bullet} \arrow[rr, "02"] \arrow[drrr, "03"] \arrow[ddr, "01"'] & & \underset{2}{\bullet} \arrow[dr, "23"] & \\
& & & \underset{3}{\bullet} \\
& \underset{1}{\bullet} \arrow[uur, "12"'] \arrow[urr, "13"'] & &
\end{tikzcd}$$

  Above, the 2-morphisms have not been drawn. The 012-face looks actually like

$$\begin{tikzcd}
\underset{0}{\bullet} \arrow[rr, "02"] \arrow[dr, "01"'] & \arrow[d, phantom, "\Uparrow 012"] & \underset{2}{\bullet}, \\
& \underset{1}{\bullet} \arrow[ur, "12"'] &
\end{tikzcd}$$

  and similarly the other ones. The 2-commutativity condition means here that in $\mathrm{HOM}(0,3)$ (a 1-category), the diagram

$$\begin{tikzcd}
03 & 23 \circ 02 \arrow[l, Rightarrow, "023"'] \\
13 \circ 01 \arrow[u, Rightarrow, "013"] & 23 \circ 12 \circ 01 \arrow[l, Rightarrow, "123 \circ 01"] \arrow[u, Rightarrow, "23 \circ 012"']
\end{tikzcd}$$

  commutes.

2.3 2-pullbacks and 2-pushouts

In the following, we will often use the terms “2-morphisms” and “homotopies” interchangeably. We work in an abstract $(2,1)$-category $\mathcal{C}$, often without specifying this.

Definition 2.3.1. A *weak* (resp. *strict*) 2-*pullback* of a diagram

$$\begin{tikzcd}
& B \arrow[d, "g"'] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

is the data (X, u, v, H) such that there is a 2-commutative square

$$\begin{tikzcd}
X \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\Swarrow H"] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

satisfying the following two conditions:

– “Existence”: For any (T, u', v', H') such that there is a 2-commutative diagram

$$\begin{tikzcd} T \arrow[r, "v'"] \arrow[d, "u'"'] \arrow[dr, phantom, "H'"] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd},$$

there exist

$$\alpha : T \to X,$$
$$h_u : u \circ \alpha \Rightarrow u',$$
$$h_v : v \circ \alpha \Rightarrow v'$$

such that

$$\begin{tikzcd} T \arrow[dr, dashed, "\alpha"] \arrow[drr, bend left, "v'"] \arrow[ddr, bend right, "u'"'] & & \\ & X \arrow[r, "v"] \arrow[d, "u"] \arrow[dr, phantom, "H"] \arrow[ur, phantom, "h_v"] \arrow[l, phantom, "h_u"] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd},$$

is 2-commutative when adding H' to it, i.e. in $\mathrm{HOM}(T, C)$, the diagram

$$\begin{tikzcd} f \circ u \circ \alpha \arrow[r, Rightarrow, "f \circ h_u"] & f \circ u' \\ g \circ v \circ \alpha \arrow[u, Rightarrow, "H \circ \alpha"] \arrow[r, Rightarrow, "g \circ h_v"'] & g \circ v' \arrow[u, Rightarrow, "H'"'] \end{tikzcd}$$

commutes.

– “Uniqueness”: Suppose that there is (T, u', v', H') as above and (α, h_u, h_v), $(\tilde{\alpha}, \tilde{h}_u, \tilde{h}_v)$ both yield 2-commutative diagrams as above, then there exists $\xi : \alpha \Rightarrow \tilde{\alpha}$ such that the diagrams

$$\begin{tikzcd} T \arrow[rr, bend left, "\alpha"] \arrow[rr, bend right, "\tilde{\alpha}"'] \arrow[rr, phantom, "\Downarrow\xi"] \arrow[dr, "u'"'] & & X \arrow[dl, "u"] \\ & A & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T \arrow[rr, bend left, "\alpha"] \arrow[rr, bend right, "\tilde{\alpha}"'] \arrow[rr, phantom, "\Downarrow\xi"] \arrow[dr, "v'"'] & & X \arrow[dl, "v"] \\ & B & \end{tikzcd}$$

are 2-commutative (when adding the other relevant homotopies to them). I.e.

in $\mathrm{HOM}(T, A)$,

$$\begin{tikzcd} u' & & u \circ \alpha \arrow[ll, Rightarrow, "h_u"'] \arrow[dl, Rightarrow, "u \circ \xi"] \\ & u \circ \tilde{\alpha} \arrow[ul, Rightarrow, "\tilde{h}_u"] & \end{tikzcd}$$

commutes.

in $\mathrm{HOM}(T, B)$,

$$\begin{tikzcd} v' & & v \circ \alpha \arrow[ll, Rightarrow, "h_v"'] \arrow[dl, Rightarrow, "v \circ \xi"] \\ & v \circ \tilde{\alpha} \arrow[ul, Rightarrow, "\tilde{h}_v"] & \end{tikzcd}.$$

commutes.

If (X, u, v, H) is a strict 2-pullback, ξ is unique.

Dually, one defines *weak/strict* 2-*pushouts*.

Remark 2.3.2. – A strict 2-pullback is a weak 2-pullback (and similarly with pushouts), but we will see that in Top/Top$_*$, only weak 2-pullbacks/2-pushouts exist.

– Note the difference to a simple pullback in the homotopy category: Here also the explicit 2-morphisms are required/provided.

Lemma 2.3.3 (Uniqueness of 2-pullbacks I). *Assume that*

$$\begin{tikzcd} X \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "H"] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X' \arrow[r, "v'"] \arrow[d, "u'"'] \arrow[dr, phantom, "H'"] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

are two weak 2-pullbacks. Then there exist equivalences $\alpha : X' \to X$, $\beta : X \to X'$ that fit into a 2-commutative diagram (where not all homotopies have been drawn)

$$\begin{tikzcd} X \arrow[dr, dashed, "\beta"] \arrow[drrr, bend left, "v"] \arrow[ddrr, bend right, "u"'] & & & \\ & X' \arrow[dr, dashed, "\alpha"] \arrow[drr, bend left, "v'"] \arrow[dr, bend right=60, "u'"'] \arrow[u, phantom, "h'_v"] \arrow[d, phantom, "h'_u"] & & \\ & & X \arrow[r, "v"] \arrow[d, "u"] \arrow[dr, phantom, "H"] \arrow[u, phantom, "h_v"] \arrow[l, phantom, "h_u"] & B \arrow[d, "g"] \\ & & A \arrow[r, "f"'] & C \end{tikzcd} .$$

In particular, weak 2-pullbacks are unique up to equivalence.

Proof. By the "existence" parts for X resp. X', we get $\alpha, \beta, h_u, h_v, h'_u, h'_v$ such that

$$\begin{tikzcd} f \circ u \circ \alpha \arrow[r, Rightarrow, "f \circ h_u"] & f \circ u' \\ g \circ v \circ \alpha \arrow[r, Rightarrow, "g \circ h_v"'] \arrow[u, Rightarrow, "H \circ \alpha"] \arrow[ur, phantom, "(1)"] & g \circ v' \arrow[u, Rightarrow, "H'"'] \end{tikzcd}$$

and similar other diagrams commute.

Define $H_u : \ u \circ \alpha \circ \beta \xRightarrow{h_u \circ \beta} u' \circ \beta \xRightarrow{h'_u} u$, i.e.

$$\begin{tikzcd} X \arrow[dr, dashed, "\beta \circ \alpha"] \arrow[ddr, bend right, "u"'] & \\ & X \arrow[d, "u"] \arrow[dl, phantom, "H_u"] \\ & A \end{tikzcd} .$$

Similarly define H_v. We check that the black arrows in

$$\begin{tikzcd} g \circ v \circ \alpha \circ \beta \arrow[r, Rightarrow, "g \circ h_v \circ \beta"] \arrow[d, Rightarrow, "H \circ \alpha \circ \beta"'] \arrow[dr, phantom, "(1) \circ \beta"] & g \circ v' \circ \beta \arrow[r, Rightarrow, "g \circ h'_v"] \arrow[d, Rightarrow, "H' \circ \beta"] \arrow[dr, phantom, "(2)"] & g \circ v \arrow[d, Rightarrow, "H"] \\ f \circ u \circ \alpha \circ \beta \arrow[r, Rightarrow, "f \circ h_u \circ \beta"'] & f \circ u' \circ \beta \arrow[r, Rightarrow, "f \circ h'_u"'] & f \circ u \end{tikzcd}$$

commute: We insert the red arrow and observe (as indicated) that the left square is just $(1) \circ \beta$, which commutes since (1) commutes. Furthermore also (2) commutes, since it is the 2-commutativity condition of the data (β, h'_u, h'_v).

Therefore, the diagram

$$\begin{tikzcd}
X \arrow[dr, "\beta\circ\alpha"] \arrow[dr, bend right, "\mathrm{id}_X"'] \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "u"'] & & \\
& X \arrow[r, "v"] \arrow[d, "u"] \arrow[dr, phantom, "H"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd},$$

is made 2-commutative by the data $(\beta \circ \alpha, H_u, H_v)$ (as indicated, but the homotopies have been omitted in the diagram). On the other hand (as also indicated) it is made 2-commutative by the data $(\mathrm{id}_X, \mathrm{id}_u, \mathrm{id}_v)$.[9] Thus by "uniqueness", there exists $\beta \circ \alpha \Rightarrow \mathrm{id}_X$.

Similarly, one gets $\alpha \circ \beta \Rightarrow \mathrm{id}_{X'}$; thus α and β are equivalences which are inverse to each other. $\square$

Definition 2.3.4. In a $(2,1)$-category $\mathcal{C}$, $* \in \mathrm{Ob}\,\mathcal{C}$ is an *initial/final object* if for all $T \in \mathrm{Ob}\,\mathcal{C}$, $\mathrm{HOM}_{\mathcal{C}}(*, T)$ resp. $\mathrm{HOM}_{\mathcal{C}}(T, *)$ are equivalent to a trivial groupoid (with object 0 and morphism $0 \to 0$).

$*$ is a *zero object*, if it is both an initial and a final object.

Proposition 2.3.5. *Let $*$ be a zero object in a $(2,1)$-category $\mathcal{C}$.*[10]

Assume that

$$\begin{tikzcd}
F \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\swarrow"] & E \arrow[d, "p"] \\
* \arrow[r] & X
\end{tikzcd}$$

is a weak 2-pullback diagram. Then for all $T \in \mathrm{Ob}\,\mathcal{C}$, the sequence

$$[T, F] \xrightarrow{i\circ} [T, E] \xrightarrow{p\circ} [T, X]$$

is exact. Dually, if

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\swarrow"] & B \arrow[d, "p"] \\
* \arrow[r] & Y
\end{tikzcd}$$

is a weak 2-pushout square, then for all $T \in \mathrm{Ob}, \mathcal{C}$,

$$[Y, T] \xrightarrow{\circ p} [B, T] \xrightarrow{\circ i} [A, T]$$

is exact.

Proof. We only prove the statement for 2-pullbacks.

Let $[k] \in [T, E]$ such that $[p \circ k] = [*]$, where $*$ denotes the zero morphism which factors through $*$. Thus there exists $\xi : p \circ k \Rightarrow (T \to * \to X)$. Fitting these into a diagram we obtain the black part of

$$\begin{tikzcd}
T \arrow[dr, "g"] \arrow[drr, bend left, "k"] \arrow[ddr, bend right] & & \\
& F \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\xi"] & E \arrow[d, "p"] \\
& * \arrow[r] & X
\end{tikzcd},$$

[9] Here $\mathrm{id}_u, \mathrm{id}_v$ are morphisms in $\mathrm{HOM}(X, A)$ resp. $\mathrm{HOM}(X, B)$.

[10] Then for all $X, Y \in \mathrm{Ob}\,\mathcal{C}$, $[X, Y]$ contains the equivalence class of the "zero morphism" which factors through $*$. Thus we can speak about exactness.

which is 2-commutative. Since the original diagram was a 2-pullback diagram, we obtain g as indicated in red, such that the resulting diagram still is 2-commutative. Thus in particular there is $i \circ g \Rightarrow k$, i.e. $[i \circ g] = [k]$.

This shows one of the necessary inclusion to prove exactness. The other one follows directly from 2-commutativity of the weak 2-pullback diagram. □

Proposition 2.3.6. *Given a 2-commutative diagram*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] \arrow[dr, phantom, "(1)\ \Swarrow H_1"] & B \arrow[r, "\beta"] \arrow[d, "g"] \arrow[dr, phantom, "(2)\ \Swarrow H_2"] & E \arrow[d, "k"] \\ C \arrow[r, "\gamma"'] & D \arrow[r, "\delta"'] & F \end{tikzcd},$$

we define $H: \ k \circ \beta \circ \alpha \xRightarrow{H_2 \circ \alpha} \delta \circ g \circ \alpha \xRightarrow{\delta \circ H_1} \delta \circ \gamma \circ f$ *and*

$$\begin{tikzcd} A \arrow[r, "\beta\circ\alpha"] \arrow[d, "f"'] \arrow[dr, phantom, "(1{+}2)\ \Swarrow H"] & E \arrow[d, "k"] \\ C \arrow[r, "\delta\circ\gamma"'] & F \end{tikzcd}.$$

Then the following are equivalent:

– *The squares (1) (ABCD) and (2) (BEDF) are weak 2-pullbacks.*

– *The squares (2) (BEDF) and (1 + 2) (AECF) are weak 2-pullbacks.*

Proof (sketch). "⇒": "Existence": Given $(T, v, u, h : k \circ v \Rightarrow \delta \circ \gamma \circ u)$, we consider

$$\begin{tikzcd} T \arrow[drr, bend left, "v"] \arrow[dr, dashed, "w"] \arrow[ddr, bend right, "\gamma\circ u"'] & & \\ & B \arrow[r, "\beta"] \arrow[d, "g"] \arrow[dr, phantom, "H_2\ \Swarrow"] & E \arrow[d, "k"] \\ & D \arrow[r, "\delta"'] & F \end{tikzcd}.$$

We are given the black part and obtain the red part $(w, h_\beta : \beta \circ w \Rightarrow v, h_g : g \circ w \Rightarrow \gamma \circ u)$ because (2) is a weak 2-pullback. By definition then

$$\begin{tikzcd} \delta\circ\gamma\circ u & k\circ v \arrow[l, Leftarrow, "h"'] \\ \delta\circ g\circ w \arrow[u, Rightarrow, "\delta\circ h_g"] & k\circ\beta\circ w \arrow[l, Leftarrow, "H_2\circ w"] \arrow[u, Rightarrow, "k\circ h_\beta"'] \end{tikzcd}$$

commutes. Now by the existence part for (1), applied for the data (w, u, h_g), we obtain $(\chi, h_\alpha : \alpha \circ \chi \Rightarrow w, h_f : f \circ \chi \Rightarrow u)$. We need to check that this gives a 2-commutative diagram for

$T \xrightarrow{\chi} (1+2)$, i.e. the black arrows in

$$\begin{tikzcd}
\delta\circ\gamma\circ f\circ\chi \arrow[rr, Rightarrow, "\delta\circ\gamma\circ h_f"] & & \delta\circ\gamma\circ u \\
\delta\circ g\circ\alpha\circ\chi \arrow[u, Rightarrow, "\delta\circ H_1\circ\chi"] \arrow[r, Rightarrow, "\delta\circ g\circ h_\alpha"] \arrow[ur, phantom, "\delta\circ TABCD"] & \delta\circ g\circ w \arrow[ur, Rightarrow, "\delta\circ h_g"] \arrow[r, phantom, "TBEDF"] & k\circ v \arrow[u, Rightarrow, "h"'] \\
k\circ\beta\circ\alpha\circ\chi \arrow[u, Rightarrow, "H_2\circ\alpha\circ\chi"] \arrow[rr, Rightarrow, "k\circ\beta\circ h_\alpha"'] \arrow[ur, phantom, "(*)"] & & k\circ\beta\circ w \arrow[u, Rightarrow, "k\circ h_\beta"'] \arrow[ul, Rightarrow, "H_2\circ w"]
\end{tikzcd}$$

commute. (Then "existence" is proven by the data $(\chi, h_f, (k\circ h_\beta)\circ(k\circ\beta\circ h_\alpha))$.)

As indicated, we insert the red arrows and verify commutativity. The "square" indicated with "$TBEDF$" commutes because it is the 2-commutativity condition of this diagram (constructed above). The "square" indicated with "$\delta\circ TABCD$" commutes because it is the 2-commutativity condition of $TABCD$, composed with δ. Thus it remains to show that $(*)$ commutes. To this end consider the piece of the diagram defining it:

$$\begin{tikzcd}
T \arrow[dr, "\chi"'] \arrow[drr, bend left, "w"] \arrow[drr, phantom, "\Nearrow h_\alpha"] & & & \\
& A \arrow[r, "\alpha"'] & B \arrow[r, "\beta"'] \arrow[d, "g"'] \arrow[dr, phantom, "\Swarrow H_2"] & E \arrow[d, "k"] \\
& & D \arrow[r, "\delta"'] & F
\end{tikzcd}
.$$

Let again $[1] := 0 \longrightarrow 1$ be the category with 2 objects and one non-trivial morphism. In the product category $[1]\times[1]$, there is a commutative square

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (1,0) \arrow[d] \\
(0,1) \arrow[r] & (1,1)
\end{tikzcd}
.$$

Actually, $[1]\times[1]$ is "the" commutative square in the following sense: A functor out of $[1]\times[1]$ into another category yields a commutative square in this category. Thus in order to show that $(*)$ commutes, we want to find a functor H out of $[1]\times[1]$ to $\mathrm{HOM}(T,F)$ which takes the right values on objects and morphisms (i.e. it maps $(0,0)$ onto $k\circ\beta\circ\alpha\circ\chi$, etc.).

Now observe that we can define H just as the composition

$$\begin{tikzcd}
\mathrm{HOM}(B,F)\times\mathrm{HOM}(T,B) \arrow[rr, "\circ_{T,B,F}"] & & \mathrm{HOM}(T,F) \\
{[1]\times[1]} \arrow[u, "\alpha\circ\chi \xRightarrow{h_\alpha} w", "k\circ\beta \xRightarrow{H_2} \delta\circ g"'] \arrow[urr, bend right, "H"'] & &
\end{tikzcd}
.$$

(I.e. the diagram commutes by definition.) The left part of the diagram is understood as a functor $G : [1]\times[1] \to \mathrm{HOM}(B,F)\times\mathrm{HOM}(T,B)$.[11]

The composed functor $[1]\times[1] \xrightarrow{G} \mathrm{HOM}(B,F)\times\mathrm{HOM}(T,B) \xrightarrow{\circ_{T,B,F}} \mathrm{HOM}(T,F)$ (i.e. H)

[11] To be precise, G shall be defined on objects by

$$\begin{aligned}
G((0,0)) &= (\alpha\circ\chi, k\circ\beta),\\
G((0,1)) &= (\alpha\circ\chi, \delta\circ g),\\
G((1,0)) &= (w, k\circ\beta),\\
G((1,1)) &= (w, \delta\circ g)
\end{aligned}$$

corresponds now to precisely the diagram (∗):

$$\begin{tikzcd}
k \circ \beta \circ \alpha \circ \chi \arrow[r, Rightarrow, "k \circ \beta \circ h_\alpha"] \arrow[d, Rightarrow, "H_2 \circ \alpha \circ \chi"'] \arrow[dr, phantom, "(*)"] & k \circ \beta \circ w \arrow[d, Rightarrow, "H_2 \circ w"] \\
\delta \circ g \circ \alpha \circ \chi \arrow[r, Rightarrow, "\delta \circ g \circ h_\alpha"'] & \delta \circ g \circ w
\end{tikzcd} .$$

Thus (∗) commutes, finishing the proof of "existence".

"Uniqueness": Assume that we have given (T, u, v, h) and $(\chi, h_f, h_{\beta\circ\alpha})$, $(\tilde{\chi}, \tilde{h}_f, \tilde{h}_{\beta\circ\alpha})$ both make the diagram below 2-commutative:

$$\begin{tikzcd}
T \arrow[dr, bend right, "\chi"'] \arrow[dr, bend left, "\tilde{\chi}"] \arrow[drrr, bend left, "v"] \arrow[ddr, bend right, "u"'] & & & \\
& A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[r, "\beta"] \arrow[d, "g"'] & E \arrow[d, "k"] \\
& C \arrow[r, "\gamma"'] & D \arrow[r, "\delta"'] & F
\end{tikzcd}$$

[12] Then also the following diagram is 2-commutative by definition:

$$\begin{tikzcd}
T \arrow[dr, bend right, "\alpha\circ\chi"'] \arrow[dr, bend left, "\alpha\circ\tilde{\chi}"] \arrow[drr, bend left, "\alpha\circ v"] \arrow[ddr, bend right, "\gamma\circ u"'] & & \\
& B \arrow[r, "\beta"] \arrow[d, "g"'] & E \arrow[d, "k"] \\
& D \arrow[r, "\delta"'] & F
\end{tikzcd} .$$

By the uniqueness part for (2), we obtain a 2-morphism $\alpha \circ \chi \overset{K}{\Longrightarrow} \alpha \circ \tilde{\chi}$. These fit into a diagram

$$\begin{tikzcd}
T \arrow[drr, bend left=60, "\alpha\circ\chi"] \arrow[drr, bend left, "\alpha\circ\tilde{\chi}"'] \arrow[dr, bend left, "\chi"] \arrow[dr, bend right, "\tilde{\chi}"'] \arrow[ddr, bend right, "u"'] \arrow[rr, phantom, "\Downarrow K"] & & \\
& A \arrow[r, "\alpha"'] \arrow[d, "f"'] \arrow[ur, phantom, "\Uparrow K"] & B \arrow[d, "g"] \\
& C \arrow[r, "\gamma"'] & D
\end{tikzcd} .$$

As indicated in red, here we can view K as a 2-morphism in two different "two-cells". We want to

---

and on morphisms (denoting the single morphism in [1] simply by "→") by

$$\begin{aligned}
G((\mathrm{id}_0, \to)) &= (\mathrm{id}_{\alpha\circ\chi}, H_2), \\
G((\mathrm{id}_1, \to)) &= (\mathrm{id}_w, H_2), \\
G((\to, \mathrm{id}_0)) &= (h_\alpha, \mathrm{id}_{k\circ\beta}), \\
G((\to, \mathrm{id}_1)) &= (h_\alpha, \mathrm{id}_{\delta\circ g})
\end{aligned} .$$

(All other images of morphisms are either identities or can be obtained by composing these basic ones.) Then G is by definition a functor.

[12] The homotopies are omitted to avoid clutter.

use apply the uniqueness part for (1) to get a 2-morphism $\chi \Rightarrow \tilde{\chi}$. To this end, consider

$$\begin{tikzcd}
g\circ\alpha\circ\chi \arrow[rr, Rightarrow, "g\circ K"] \arrow[d, Rightarrow, "H_1\circ\chi"'] \arrow[dr, Rightarrow] & & g\circ\alpha\circ\tilde{\chi} \arrow[dl, Rightarrow] \arrow[d, Rightarrow, "H_1\circ\tilde{\chi}"] \\
\gamma\circ f\circ\chi \arrow[r, Rightarrow, "\gamma\circ h_u"'] & \gamma\circ u & \gamma\circ f\circ\tilde{\chi} \arrow[l, Rightarrow, "\gamma\circ\tilde{h}_u"]
\end{tikzcd} .$$

Here the two unnamed arrows are just defined as the respective compositions, such that the left and the right triangle trivially commute. The last triangle also commutes, since this is a condition on K (obtained from the "uniqueness" for (2)). Thus the black arrows form a commutative "square". On the other hand, also

$$\begin{tikzcd}
g\circ\alpha\circ\tilde{\chi} \arrow[rr, Rightarrow, "g\circ\mathrm{id}_{\alpha\circ\tilde{\chi}}"] \arrow[d, Rightarrow, "H_1\circ\tilde{\chi}"'] & & g\circ\alpha\circ\tilde{\chi} \arrow[dl, Rightarrow] \arrow[d, Rightarrow, "H_1\circ\tilde{\chi}"] \\
\gamma\circ f\circ\tilde{\chi} \arrow[r, Rightarrow, "\gamma\circ\tilde{h}_u"'] & \gamma\circ u & \gamma\circ f\circ\tilde{\chi} \arrow[l, Rightarrow, "\gamma\circ\tilde{h}_u"]
\end{tikzcd}$$

commutes (clearly). Thus by "uniqueness" for (1), we obtain a 2-morphism $\chi \Rightarrow \tilde{\chi}$. One verifies the necessary 2-commutativity conditions, proving "uniqueness" for $(1+2)$.

"$\Leftarrow$" (assuming that all weak 2-pullbacks exist): We construct a weak 2-pullback P of BCD and insert it into the given diagram, as indicated in red (omitting some homotopies):

$$\begin{tikzcd}
A \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "f"'] \arrow[dr, dashed, "\chi"] & & & \\
& P \arrow[r, "\gamma'"] \arrow[d, "g'"] \arrow[ur, phantom, "\Uparrow h_{\gamma'}"] \arrow[dl, phantom, "\Leftarrow h_{g'}"] & B \arrow[r, "\beta"] \arrow[d, "g"] \arrow[dl, phantom, "H_3"] & E \arrow[d, "k"] \arrow[dl, phantom, "H_2"] \\
& C \arrow[r, "\gamma"'] & D \arrow[r, "\delta"'] & F
\end{tikzcd} .$$

The map $\chi : A \to P$ exists since P is a weak 2-pullback.

By "$\Rightarrow$", $PECF$ is a weak 2-pullback. On the other hand, by assumption $AECF$ is a weak 2-pullback. Thus by lemma 2.3.3, χ is an equivalence.[13] Then also $ABCD$ is a weak 2-pullback square, which was to be shown.[14] □

Corollary 2.3.7. *Assume that $\mathcal{C}$ admits a zero object $*$ and all weak 2-pullbacks/2-pushouts. Then for $X \in \operatorname{Ob}\mathcal{C}$, there exists a weak 2-pullback*

$$\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] & * \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\
* \arrow[r] & X
\end{tikzcd} .$$

This gives a functor $\Omega : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{C}$ (exercise). Dually, there exists a weak 2-pushout

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & * \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\
* \arrow[r] & \Sigma X
\end{tikzcd} .$$

[13] Note that the map w in the proof of "$\Rightarrow$" may here again be taken as α, and then from lemma 2.3.3 it follows that it is indeed the map χ which is an equivalence and not some other map $A \to P$.

[14] The formal proof that $ABCD$ is a weak 2-pullback square if $PBCD$ is one and w is an equivalence can be done using proposition 2.3.13 later.

that gives a functor $\Sigma : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{C}$. *Consider then the weak* 2*-pullback diagram*

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & E \arrow[d, "p"] \\ * \arrow[r] & X \end{tikzcd}$$

and the weak 2*-pushout diagram*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & B \arrow[d, "p"] \\ * \arrow[r] & Y \end{tikzcd} \quad .$$

Then there are long exact sequences (the Puppe long exact sequences*) for every* $T \in \mathrm{Ob}\,\mathcal{C}$

$$\ldots \longrightarrow [T, \Omega F] \longrightarrow [T, \Omega E] \longrightarrow [T, \Omega X] \longrightarrow [T, F] \longrightarrow [T, E] \longrightarrow [T, X]$$

and

$$[A, T] \longleftarrow [B, T] \longleftarrow [Y, T] \longleftarrow [\Sigma A, T] \longleftarrow [\Sigma B, T] \longleftarrow [\Sigma Y, T] \longleftarrow \ldots .$$

Proof (Sketch). We only show the statement for 2-pullbacks, for 2-pushouts, the argument is dual. By proposition 2.3.6, we can construct

$$\begin{tikzcd} \Omega E \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & * \arrow[d] \\ * \arrow[r] & F \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & E \arrow[d] \\ & * \arrow[r] & X \end{tikzcd} \quad ,$$

where all squares resp. rectangles are 2-pullbacks.[15] This construction can be repeated.

Then one only needs to assemble the sequences from proposition 2.3.5 to obtain an exact sequence

$$\ldots \longrightarrow [T, \Omega F] \longrightarrow [T, \Omega E] \longrightarrow [T, \Omega X] \longrightarrow [T, F] \longrightarrow [T, E] \longrightarrow [T, X] \ ,$$

as claimed. □

Definition 2.3.8. Weak 2-pullbacks/pushouts in Top/Top$_*$/CW/CW$_*$ are called *homotopy pullbacks/homotopy pushouts.*

(Sometimes, sloppily, we will use these terms for general 2-pullbacks/pushouts in arbitrary categories as well.)

Theorem 2.3.9. *Let*

$$\begin{tikzcd} & C \arrow[d, "g"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

be a diagram in Top*, where* g *is a fibration. Then, denoting the ordinary pullback of this diagram*

[15] The lower square is the original 2-pullback; then one constructs the upper right square as a 2-pullback. Due to proposition 2.3.6, mirroring the rectangle used in this proposition, the big vertical rectangle is then again a 2-pullback. Due to uniqueness of 2-pullbacks, lemma 2.3.3, the space in the upper left corner is then indeed ΩX.
Afterwards one constructs the left square as a 2-pullback and obtains that also the horizontal rectangle is a 2-pullback, thus the space in its upper left corner is indeed ΩE.

by $A \times_B C$, the diagram

$$\begin{tikzcd} A \times_B C \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\text{(trivial)}\ \Swarrow"] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a homotopy pullback.[16]

Dually, if

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \\ C & \end{tikzcd}$$

is a diagram in Top*, where i is a cofibration, then, denoting the ordinary pushout of this diagram by $B \cup_A C$,*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] \arrow[dr, phantom, "\text{(trivial)}\ \Swarrow"] & B \arrow[d, "i'"] \\ C \arrow[r, "f'"'] & B \cup_A C \end{tikzcd}$$

is a homotopy pushout.

Sketch of proof. We only prove the statement for pullbacks. "Existence": Suppose we are given

$$\begin{tikzcd} T \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "u"'] & & \\ & & C \arrow[d, "g"] \\ & A \arrow[r, "f"'] \arrow[ur, phantom, "\Swarrow H"] & B \end{tikzcd} .$$

By the strictification lemma 1.5.7, there exists $v' : T \to C$, $v \overset{h_v}{\Longrightarrow} v'$ with $g \circ h_v = H$, and such that the corresponding diagram with v' instead of v is 1-commutative. By the universal property of the 1-pullback, we obtain a map $(u, v') : T \to A \times_B C$. Recalling that f' is just the projection onto C (by definition of the 1-pullback $A \times_B C := \{(a, c) \in A \times C \mid f(a) = g(c)\}$), then $f' \circ (u, v') = v'$, we can fit everything into a diagram

$$\begin{tikzcd} T \arrow[drr, bend left, "v"] \arrow[dr, dashed, "(u{,}v')"] \arrow[ddr, bend right, "u"'] & & \\ & A \times_B C \arrow[r, "f'"] \arrow[d, "g'"] \arrow[ur, phantom, "\Nearrow h_v^{-1}"] & C \arrow[d, "g"] \\ & A \arrow[r, "f"'] & B \end{tikzcd} ,$$

where h_v^{-1} denotes the inverse homotopy to h_v. Here the left triangle and the square are understood 1-commutative, i.e. with "zero homotopy" resp. "identity homotopy".[17] The irregular "square"

[16] Here the trivial homotopy is just a constant one.
[17] Use here that also g' is actually just a projection.

$TCAB$ also contains the homotopy H. Thus one needs to prove that in $\mathrm{HOM}(T, B)$, the diagram

$$\begin{tikzcd}
g \circ f' \circ (u, v') \arrow[r, Rightarrow, "g \circ h_v^{-1}"] \arrow[d, equal] & g \circ v' \arrow[d, Rightarrow, "H"] \\
f \circ g' \circ (u, v') \arrow[r, equal] & f \circ u
\end{tikzcd}$$

commutes. This is indeed true since $g \circ h_v = H$ by the strictification lemma (then also $g \circ h_v^{-1} = H^{-1}$).

"Uniqueness": Suppose we are given (T, u, v, H) and have $((u_1, v_1), u_1 \Rightarrow u, v_1 \Rightarrow v)$, $((u_2, v_2), u_2 \Rightarrow u, v_2 \Rightarrow v)$ both satisfying the necessary 2-commutativity conditions.[18]

$$\begin{tikzcd}
T \arrow[drr, bend left, "v"] \arrow[dr, bend left, "(u_1, v_1)"] \arrow[dr, bend right, "(u_2, v_2)"'] \arrow[ddr, bend right, "u"'] & & \\
& A \times_B C \arrow[r, "f'"] \arrow[d, "g'"'] & C \arrow[d, "g"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

We need to find a homotopy $(u_1, v_1) \Rightarrow (u_2, v_2)$, which is equivalent to finding two homotopies $u_1 \Rightarrow u_2$, $v_1 \Rightarrow v_2$. To this end, consider the commutative diagrams

$$\text{in } \mathrm{HOM}_{\mathrm{Top}}(T, A) \quad \begin{tikzcd}
u_1 \arrow[rr, Rightarrow] \arrow[dr, Rightarrow] & & u_2 \arrow[dl, Rightarrow] \\
& u &
\end{tikzcd} \qquad \Bigg| \qquad \text{in } \mathrm{HOM}_{\mathrm{Top}}(T, C) \quad \begin{tikzcd}
v_1 \arrow[rr, Rightarrow] \arrow[dr, Rightarrow] & & v_2 \arrow[dl, Rightarrow] \\
& v &
\end{tikzcd} .$$

Since we are working in Top, all 2-morphisms are invertible, thus the composition of one of the red arrows in the right diagram with the inverse of another one gives a 2-morphism $u_1 \Rightarrow u_2$ (as indicated by the black arrow). Similarly for $v_1 \Rightarrow v_2$. The necessary 2-commutativity conditions are just again given by the diagrams above and are therefore satisfied by construction. $\square$

Definition 2.3.10. A 2-*commutative cube* in the category $\mathcal{C}$ consists of the data

- Objects $A, B, C, D, A', B', C', D' \in \mathrm{Ob}\,\mathcal{C}$
- 1-morphisms $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta', f_A, f_B, f_C, f_D$
- 2-morphisms $H, H', F_1, F_2, F_3, F_4$

forming a diagram

$$\begin{tikzcd}
A \arrow[rr, "\alpha"] \arrow[dr, "\beta"] \arrow[dd, "f_A"'] & & B \arrow[dr, "\gamma"] \arrow[dd, "f_B"'] \arrow[d, phantom, "H"] & \\
& C \arrow[rr, "\delta"'] \arrow[dd, "f_C"] \arrow[dr, phantom, "F_1"] & & D \arrow[dd, "f_D"] \arrow[dl, phantom, "F_3"] \\
A' \arrow[rr, "\alpha'"'] \arrow[dr, "\beta'"'] \arrow[r, phantom, "F_2"] & & B' \arrow[dr, "\gamma'"'] \arrow[d, phantom, "F_4"] \arrow[dr, phantom, "H'"'] & \\
& C' \arrow[rr, "\delta'"'] & & D'
\end{tikzcd}$$

[18]Using that f', g' are projections, we already have abbreviated e.g. $f' \circ (u_1, v_1) = v_1$.

satisfying that in $\mathrm{HOM}_{\mathcal{C}}(A, D')$, the diagram

$$\begin{tikzcd}
& \delta' \circ f_C \circ \beta \arrow[dl, Rightarrow, "\delta' \circ F_2"'] & f_D \circ \delta \circ \beta \arrow[l, Rightarrow, "F_4 \circ \beta"'] & \\
\delta' \circ \beta' \circ f_A & & & f_D \circ \gamma \circ \alpha \arrow[ul, Rightarrow, "f_D \circ H"'] \arrow[dl, Rightarrow, "F_3 \circ \alpha"] \\
& \gamma' \circ \alpha' \circ f_A \arrow[ul, Rightarrow, "H' \circ f_A"] & \gamma' \circ f_B \circ \alpha \arrow[l, Rightarrow, "\gamma' \circ F_1"] &
\end{tikzcd}$$

commutes.

Often we will not specify all homotopies, but they are understood implicitly.

Definition 2.3.11. A 2-commutative square

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "\beta"'] \arrow[dr, phantom, "H \Swarrow"] & B \arrow[d, "\gamma"] \\
C \arrow[r, "\delta"'] & D
\end{tikzcd}$$

is *equivalent* to

$$\begin{tikzcd}
A' \arrow[r, "\alpha'"] \arrow[d, "\beta'"'] \arrow[dr, phantom, "H' \Swarrow"] & B' \arrow[d, "\gamma'"] \\
C' \arrow[r, "\delta'"'] & D'
\end{tikzcd}$$

if there exists a 2-commutative cube as in definition 2.3.10 before, where f_A, f_B, f_C, f_D are equivalences.

Lemma 2.3.12. *Equivalence of 2-commutative squares as in definition 2.3.11 is an equivalence relation.*

Proof. Reflexivity: Clear.

Transitivity: We may "compose" 2-commutative cubes, "glueing" them onto each other.

Symmetry:[19] We are given a 2-commutative cube for which we use the same notation as in definition 2.3.10 and furthermore 1-morphisms $g_A : A' \to A, \dots$ and 2-morphisms $h_A : g_A \circ f_A \Rightarrow \mathrm{id}_A, \dots, h_{A'} : f_A \circ g_A \Rightarrow \mathrm{id}_{A'}, \dots$. We proceed in three steps.

1. First we define the 2-morphisms of the "inverse cube", which shall be denoted G_1, G_2, G_3, G_4. The construction is described at the example of G_2, for the other ones it is analogous. We

[19]Note that if $\mathcal{A}$ is a 1-category, I is an "index category" like e.g. $[1] \times [1]$ considered before, $\mathcal{D}_1, \mathcal{D}_2 : I \to \mathcal{A}$ are "diagram functors" and $\eta : \mathcal{D}_1 \Rightarrow \mathcal{D}_2$ is a natural transformation such that $\eta(i)$ is an isomorphism for all $i \in \operatorname{Ob} I$, then η has an inverse η^{-1} defined by $\eta^{-1}(i) := \eta(i)^{-1}$.

$$\begin{tikzcd}
I \arrow[r, bend left, "\mathcal{D}_1"] \arrow[r, bend right, "\mathcal{D}_2"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathcal{A}
\end{tikzcd}$$

But this is no such obvious construction in a $(2,1)$-category. Thus the construction of an "inverse cube" requires some work.

are in the situation

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, "\beta"] \arrow[dd, "f_A"'] & & B \arrow[dr] \arrow[dd] & \\
& C \arrow[rr] \arrow[dd, "f_C"] & & D \arrow[dd] \\
A' \arrow[rr] \arrow[dr, "\beta'"'] \arrow[dd, "g_A"'] \arrow[r, phantom, "\Leftarrow F_2"] & & B' \arrow[dr] \arrow[dd] & \\
& C' \arrow[rr] \arrow[dd, "g_C"] & & D' \arrow[dd] \\
A \arrow[rr] \arrow[dr, "\beta"'] \arrow[r, phantom, "\Leftarrow G_2"] & & B \arrow[dr] & \\
& C \arrow[rr] & & D
\end{tikzcd} .$$

The intention is to find G_2 such that the resulting lower cube is again 2-commutative. In hindsight view of the next two steps, however, we want to find G_2 such that the homotopy $K_2 : g_C \circ f_C \circ \beta \xRightarrow{g_C \circ F_2} g_C \circ \beta' \circ f_A \xRightarrow{G_2 \circ f_A} \beta \circ g_A \circ f_A$ in the left face of the composition of the two cubes takes on a particularly simple form:

$$\begin{tikzcd}
A \arrow[r, "\beta"] \arrow[d, "g_A \circ f_A"'] \arrow[dr, phantom, "\Swarrow K_2"] & C \arrow[d, "g_C \circ f_C"] \\
A \arrow[r, "\beta"'] & C
\end{tikzcd} .$$

We want to achieve $K_2 = (\beta \circ h_A)^{-1} \circ (h_C \circ \beta)$, i.e. the square

$$\begin{tikzcd}
g_C \circ f_C \circ \beta \arrow[r, Rightarrow, "g_C \circ F_2"] \arrow[d, Rightarrow, "h_C \circ \beta"'] \arrow[dr, Rightarrow, "K_2"] & g_C \circ \beta' \circ f_A \arrow[d, Rightarrow, "G_2 \circ f_A"] \\
\beta & \beta \circ g_A \circ f_A \arrow[l, Rightarrow, "\beta \circ h_A"]
\end{tikzcd}$$

shall commute.[20] Reformulating this, we need to find G_2 such that $\mathrm{Hom}(g_C \circ \beta' \circ f_A, \beta \circ g_A \circ f_A) \ni H_2 := (\beta \circ h_A)^{-1} \circ (h_C \circ \beta) \circ (g_C \circ F_2)^{-1}$ satisfies $H_2 = G_2 \circ f_A$.

This is achieved by defining G_2 as the 2-morphism which makes the diagram below commutative (i.e. as the composition of the other arrows resp. their inverses):

$$\begin{tikzcd}
g_C \circ \beta' \arrow[d, Rightarrow, "G_2"'] & g_C \circ \beta' \circ f_A \circ g_A \arrow[l, Rightarrow, "g_C \circ \beta' \circ h_{A'}"'] & g_C \circ \beta' \circ f_A \circ g_A \circ f_A \circ g_A \arrow[l, Rightarrow, "g_C \circ \beta' \circ f_A \circ h_A \circ g_A"'] \arrow[r, Rightarrow, "g_C \circ \beta' \circ h_{A'} \circ f_A \circ g_A"] & g_C \circ \beta' \circ f_A \circ g_A \arrow[d, Rightarrow, "H_2 \circ g_A"] \\
\beta \circ g_A & \beta \circ g_A \circ f_A \circ g_A \arrow[l, Rightarrow, "\beta \circ g_A \circ h_{A'}"] & \beta \circ g_A \circ f_A \circ g_A \circ f_A \circ g_A \arrow[l, Rightarrow, "\beta \circ g_A \circ f_A \circ h_A \circ g_A"] \arrow[r, Rightarrow, "\beta \circ g_A \circ h_{A'} \circ f_A \circ g_A"'] & \beta \circ g_A \circ f_A \circ g_A
\end{tikzcd}$$

Claim: G_2 as defined above has the desired property $H_2 = G_2 \circ f_A$ (and hence $K_2 = (\beta \circ h_A)^{-1} \circ (h_C \circ \beta)$).

Proof of claim: The composition of all arrows but the right three ones (when taking the necessary inverses) is of the form $H_2' \circ g_A$ with $H_2' \in \mathrm{Hom}(g_C \circ \beta' \circ f_A, \beta \circ g_A \circ f_A)$. Note that in $\mathrm{HOM}(A, C)$ we have $H_2' \circ g_A \circ f_A = (\beta \circ g_A \circ f_A \circ h_A)^{-1} \circ H_2' \circ (g_C \circ \beta' \circ f_A \circ h_A)$ due

[20] The upper triangle commutes always by definition of K_2 and the lower one does so if and only if the square does.

to the commutative diagram resulting out of

$$
\begin{tikzcd}
A \arrow[r, bend left, "\mathrm{id}_A"] \arrow[r, bend right, "g_A \circ f_A"'] \arrow[r, phantom, "\Downarrow h_A"] & A \arrow[r, bend left, "g_C \circ \beta' \circ f_A"] \arrow[r, bend right, "\beta \circ g_A \circ f_A"'] \arrow[r, phantom, "\Downarrow H_2'"] & C \, .
\end{tikzcd}
$$

(Compare the proof of proposition 2.3.6.) Thus upon composing the above diagram with f_A, we can rewrite it also as

$$
\begin{tikzcd}[column sep=small]
g_C \circ \beta' \circ f_A \arrow[d, Rightarrow, "G_2 \circ f_A"'] & g_C \circ \beta' \circ f_A \circ g_A \circ f_A \arrow[l, Rightarrow, "g_C \circ \beta' \circ h_{A'} \circ f_A"'] \arrow[r, Rightarrow, "g_C \circ \beta' \circ f_A \circ h_A"] & g_C \circ \beta' \circ f_A & g_C \circ \beta' \circ f_A \circ g_A \circ f_A \arrow[l, Rightarrow, "g_C \circ \beta' \circ f_A \circ h_A"'] \arrow[r, Rightarrow, "g_C \circ \beta' \circ h_{A'} \circ f_A"] & g_C \circ \beta' \circ f_A \arrow[d, Rightarrow, "H_2"] \, . \\
\beta \circ g_A \circ f_A & \beta \circ g_A \circ f_A \circ g_A \circ f_A \arrow[l, Rightarrow, "\beta \circ g_A \circ h_{A'} \circ f_A"] \arrow[r, Rightarrow, "\beta \circ g_A \circ f_A \circ h_A"'] & \beta \circ g_A \circ f_A & \beta \circ g_A \circ f_A \circ g_A \circ f_A \arrow[l, Rightarrow, "\beta \circ g_A \circ f_A \circ h_A"] \arrow[r, Rightarrow, "\beta \circ g_A \circ h_{A'} \circ f_A"'] & \beta \circ g_A \circ f_A
\end{tikzcd}
$$

In the top and the bottom row, all arrows "cancel each other"; these are identity morphisms. This proves the claim.

2. We need to show that the lower cube is 2-commutative, i.e. in $\mathrm{HOM}(A', D)$,

$$
\begin{tikzcd}
 & \delta \circ g_C \circ \beta' \arrow[dl, Rightarrow, "\delta \circ G_2"'] & g_D \circ \delta' \circ \beta' \arrow[l, Rightarrow, "G_4 \circ \beta'"'] & \\
\delta \circ \beta \circ g_A & & & g_D \circ \gamma' \circ \alpha' \arrow[ul, Rightarrow, "g_D \circ H'"'] \arrow[dl, Rightarrow, "G_3 \circ \alpha'"] \\
 & \gamma \circ \alpha \circ g_A \arrow[ul, Rightarrow, "H \circ g_A"] & \gamma \circ g_B \circ \alpha' \arrow[l, Rightarrow, "\gamma \circ G_1"] &
\end{tikzcd}
$$

commutes.

Observation: In a $(2,1)$-category, 2-morphisms yield natural transformations between functors between HOM-categories: Consider

$$
\begin{tikzcd}
\underset{X}{\bullet} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow \eta"] & \underset{Y}{\bullet} \, .
\end{tikzcd}
$$

For any object T, we obtain a natural transformation between functors, that we still call η,

$$
\begin{tikzcd}
\mathrm{HOM}(Y, T) \arrow[r, bend left, "\circ f"] \arrow[r, bend right, "\circ g"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathrm{HOM}(X, T) \, .
\end{tikzcd}
$$

In particular, an equivalence $f_A : A \to A'$ induces equivalences of categories

$$
\mathrm{HOM}(A', T) \xrightarrow{\cong} \mathrm{HOM}(A, T) \, ,
$$
$$
\mathrm{HOM}(T, A) \xrightarrow{\cong} \mathrm{HOM}(T, A') \, .
$$

[21]

Because of this observation, the lower cube is commutative if and only if its composition with f_A is commutative. On the other hand, we can construct a diagram relating the 2-

[21] An equivalence of categories is what its name suggests: An equivalence in the 2-category Cat. Alternatively it may be proven to be exhibited by a functor which is fully faithful and essentially surjective. Such a functor admits an inverse functor. *Caveat:* The definition of "inverse functor" does not imply that the composition of a functor which its inverse functor is the identity, but only up to natural equivalence.

commutativity conditions of the lower cube with f_A to the 2-commutativity condition of the composed cube: Consider

$$\begin{tikzcd}
 & & \underline{g_D \circ f_D \circ \delta \circ \beta} \arrow[dl, Rightarrow, "g_D\circ F_4\circ\beta"'] \arrow[ddll, Rightarrow, bend right, "\underline{K_4\circ\beta}"'] & \underline{g_D \circ f_D \circ \gamma \circ \alpha} \arrow[l, Rightarrow, "\underline{g_D \circ f_D \circ H}"'] \arrow[dr, Rightarrow, "g_D\circ F_3\circ\alpha"] \arrow[ddrr, Rightarrow, bend left, "\underline{K_3\circ\alpha}"] & & \\
 & g_D \circ \delta' \circ f_C \circ \beta \arrow[dr, Rightarrow, "g_D\circ\delta'\circ F_2"] & g_D\circ(\text{upper cube}) & & g_D \circ \gamma' \circ f_B \circ \alpha \arrow[dl, Rightarrow, "g_D\circ\gamma'\circ F_1"'] & \\
\underline{\delta \circ g_C \circ f_C \circ \beta} \arrow[ddrr, Rightarrow, bend right, "\underline{\delta\circ K_2}"'] & & g_D \circ \delta' \circ \beta' \circ f_A \arrow[dl, Rightarrow, "G_4\circ\beta'\circ f_A"] & g_D \circ \gamma' \circ \alpha' \circ f_A \arrow[l, Rightarrow, "g_D \circ H' \circ f_A"'] \arrow[dr, Rightarrow, "G_3\circ\alpha'\circ f_A"] & & \underline{\gamma \circ g_B \circ f_B \circ \alpha} \arrow[ddll, Rightarrow, bend left, "\underline{\gamma\circ K_1}"] \\
 & \delta \circ g_C \circ \beta' \circ f_A \arrow[dr, Rightarrow, "\delta\circ G_2\circ f_A"] & \text{lower cube}\circ f_A & & \gamma \circ g_B \circ \alpha' \circ f_A \arrow[dl, Rightarrow, "\gamma\circ G_1\circ f_A"'] & \\
 & & \underline{\delta \circ \beta \circ g_A \circ f_A} & \underline{\gamma \circ \alpha \circ g_A \circ f_A} \arrow[l, Rightarrow, "\underline{H \circ g_A \circ f_A}"'] & &
\end{tikzcd}$$

Here the lower central hexagon is, as indicated, the 2-commutativity condition of the lower cube composed with f_A (which we want to prove), whereas the upper central hexagon is g_D composed with the 2-commutativity condition of the upper cube (which we know). The underlined 1- and 2-morphisms form the 2-commutativity condition for the composed cube.

The left and right irregular hexagons always commute: For example, in the left one one can insert the red arrows depicted below:

$$\begin{tikzcd}
 & & g_D \circ f_D \circ \delta \circ \beta \ . \arrow[dl, Rightarrow, "g_D\circ F_4\circ\beta"] \arrow[ddll, Rightarrow, bend right, "K_4\circ\beta"'] \\
 & g_D \circ \delta' \circ f_C \circ \beta \arrow[dl, Rightarrow, "G_4\circ f_C\circ\beta"] \arrow[dr, Rightarrow, "g_D\circ\delta'\circ F_2"] & \\
\delta \circ g_C \circ f_C \circ \beta \arrow[dr, Rightarrow, "\delta\circ g_C\circ F_2"] \arrow[ddrr, Rightarrow, bend right, "\delta\circ K_2"'] & & g_D \circ \delta' \circ \beta' \circ f_A \arrow[dl, Rightarrow, "G_4\circ\beta'\circ f_A"] \\
 & \delta \circ g_C \circ \beta' \circ f_A \arrow[dr, Rightarrow, "\delta\circ G_2\circ f_A"] & \\
 & & \delta \circ \beta \circ g_A \circ f_A
\end{tikzcd}$$

Then the two triangles commute by definition of K_2 resp. K_4, and the square commutes because of a similar argument to the one made in the proof of proposition 2.3.6 (for the square called $(*)$ there).

We conclude that the lower cube is 2-commutative if and only if the composed cube is 2-commutative.[22]

3. Finally, we verify that the composed cube is 2-commutative. We need to check that in

[22] *Note:* The diagram also shows that the composition of two 2-commutative cubes is always 2-commutative. This is hardly surprising, but useful when examining larger diagrams.

$\mathrm{HOM}(A, D)$, the black arrows in

$$\begin{tikzcd}
& \delta \circ g_C \circ f_C \circ \beta \arrow[dl, Rightarrow, "\delta\circ K_2"'] \arrow[d, Rightarrow, "\delta\circ h_C\circ\beta"] & g_D \circ f_D \circ \delta \circ \beta \arrow[l, Rightarrow, "K_4\circ\beta"'] \arrow[dl, Rightarrow, "h_D\circ\delta\circ\beta"] \arrow[d, phantom, "(*)"] & \\
\delta \circ \beta \circ g_A \circ f_A \arrow[r, Rightarrow, "\delta\circ\beta\circ h_A"'] & \delta \circ \beta \arrow[d, phantom, "(*)"] & \gamma \circ \alpha \arrow[l, Rightarrow, "H"] & g_D \circ f_D \circ \gamma \circ \alpha \arrow[l, Rightarrow, "h_D\circ\gamma\circ\alpha"'] \arrow[ul, Rightarrow, "g_D\circ f_D\circ H"'] \arrow[dl, Rightarrow, "K_3\circ\alpha"] \\
& \gamma \circ \alpha \circ g_A \circ f_A \arrow[ul, Rightarrow, "H\circ g_A\circ f_A"] \arrow[ur, Rightarrow, "\gamma\circ\alpha\circ h_A"'] & \gamma \circ g_B \circ f_B \circ \alpha \arrow[l, Rightarrow, "\gamma\circ K_1"] \arrow[u, Rightarrow, "\gamma\circ h_B\circ\alpha"'] &
\end{tikzcd}$$

commute. We insert the red objects and morphisms and verify commutativity:

The triangles all commute by the construction of the G_i in step 1 (as proven there). The lower "square" marked by $(*)$ commutes because of

$$\begin{tikzcd}
\mathrm{HOM}(A, D) \times \mathrm{HOM}(A, A) \arrow[rr, "\circ_{A,A,D}"] & & \mathrm{HOM}(A, D)\ . \\
{[1] \times [1]} \arrow[u, "\delta\circ\beta \xRightarrow{H} \gamma\circ\alpha", "g_A\circ f_A \xRightarrow{h_A} \mathrm{id}_A"'] \arrow[urr, bend right] & &
\end{tikzcd}$$

(Compare the proof of proposition 2.3.6.) For the upper "square" marked by $(*)$, there is an analogous diagram.

This concludes the proof of the lemma.

□

Proposition 2.3.13. *If a 2-commutative square is equivalent to a weak 2-pullback/weak 2-pushout, then it is also such.*

Sketch of proof. Only a brief argument for 2-pullbacks shall be given here. Let $ABCD$ be a 2-commutative square which is equivalent to a weak 2-pullback $A'B'C'D'$. Let (T, u, v, H) be the necessary data for a 2-pullback map into $ABCD$, which shall be shown to exist. This is used in the following composition of 2-commutative cubes (the other homotopies are omitted for brevity):

$$\begin{tikzcd}
T \arrow[drrr, bend left, "v"] \arrow[ddrr, bend right, "u"'] \arrow[dddr, dashed] \arrow[dddrrr] \arrow[ddddrr] \arrow[dddddr, red, bend right] & & & & \\
& A \arrow[rr] \arrow[dr] \arrow[dd] & & B \arrow[dr] \arrow[dd] \arrow[d, phantom, "H"] & \\
& & C \arrow[rr] \arrow[dd] & & D \arrow[dd] \\
& A' \arrow[rr] \arrow[dr] \arrow[dd] & & B' \arrow[dr] \arrow[dd] & \\
& & C' \arrow[rr] \arrow[dd] & & D' \arrow[dd] \\
& A \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\
& & C \arrow[rr] & & D
\end{tikzcd} .$$

The arrows $T \to B', T \to C'$ are defined as the compositions; then the dashed arrow exists because

$A'B'C'D'$ is a weak 2-pullback. The red arrow is again defined as the composition and shows "existence" for $ABCD$.

For "uniqueness", the proof is similar if the following lemma is used. □

Lemma 2.3.14. *Let $f : A \to A'$ be an equivalence. Let $g, h : A' \to X$ be 1-morphisms and $H : g \circ f \Rightarrow h \circ f$ be a 2-morphism. Then there exists a 2-morphism $G : g \Rightarrow h$ with $H = G \circ f$.*

Proof. This was already shown in the proof of lemma 2.3.12 in step 1. for "symmetry". The replacements in the notation to be made correspond to identifying $f_A = f$, $g_C \circ \beta' = g$, $\beta \circ g_A = h$, $H_2 = H$ and $G_2 = G$. □

2.4 Homotopy pullbacks and pushouts in Top/Top$_*$

We begin with the dual of proposition 1.4.12 (a "mapping cocylinder construction").

Proposition 2.4.1. *Let $f : X \to Y$ be a map of topological spaces. Then there exists a factorization $X \xrightarrow{s} W(f) \xrightarrow{p} Y$ of f such that s is a homotopy equivalence and p is a fibration.*

For the proof, we need:

Lemma 2.4.2. *Let Y^I be the path space of Y and $\mathrm{ev}_0, \mathrm{ev}_1 : Y^I \to Y$ denote the evaluations at 0 resp. 1. Then $(\mathrm{ev}_0, \mathrm{ev}_1) : Y^I \to Y \times Y$ is a fibration.*

Proof. Suppose we are given a space T and maps $H : T \to Y^I$, $(H_1, H_2) : T \times I \to Y \times Y$ such that

$$H|_{T\times\{0\}} = H_1|_{T\times\{0\}}, \qquad H|_{T\times\{1\}} = H_2|_{T\times\{0\}}.$$

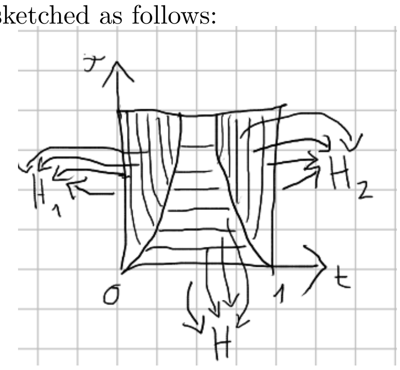
[23] Thus the solid arrows in

$$\begin{tikzcd} T \arrow[r, "H"] \arrow[d, "\mathrm{id}\times\{0\}"'] & Y^I \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ T \times I \arrow[r, "(H_1,H_2)"'] & Y \times Y \end{tikzcd}$$

commute, and we want to find a lift $\tilde{H}$ as indicated by the dashed lines. This can be interpreted[24] as a map $\tilde{H} : T \times I \times I \to Y$ such that

$$\begin{array}{l} \tilde{H}|_{T\times I\times\{0\}} = H, \\ \tilde{H}|_{T\times\{0\}\times I} = H_1, \\ \tilde{H}|_{T\times\{1\}\times I} = H_2 \end{array}.$$

We denote the coordinate in the first interval by t and in the second one by τ. Then $\tilde{H}$ can be sketched as follows:



□

[23] H may also be regarded as a map $T \times I \to Y$ by the exponential property of the mapping space, and (H_1, H_2) can always be written in such a product form.

[24] One uses again the exponential property of Y^I.

Proof of Proposition 2.4.1. Define $W(f), s, p$ via the 1-pullback diagram

$$\begin{tikzcd} W(f) \arrow[r] \arrow[d, "{(q,p)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y^I \arrow[d, "{(\mathrm{ev}_0,\mathrm{ev}_1)}"] \\ X \times Y \arrow[r, "f\times\mathrm{id}"'] & Y \times Y \end{tikzcd} .$$

Because of the lemma before, $(\mathrm{ev}_0, \mathrm{ev}_1)$ is a fibration. Now fibrations are stable under pullbacks (this is the dual of proposition 1.4.10), thus also (q, p) is a fibration. Furthermore the projection $\mathrm{pr}_Y : X \times Y \to Y$ onto Y is a fibration and fibrations are stable under composition. Therefore $p = \mathrm{pr}_Y \circ (q, p) : W(f) \to Y$ is a fibration.

By definition of the 1-pullback, $W(f) = \{(x, y, \gamma) \in X \times Y \times Y^I \mid (f(x), y) = (\gamma(0), \gamma(1))\}$. This is homeomorphic to $X \times Y^I \supset \{(x, \gamma) \in X \times Y^I \mid \gamma(0) = f(x)\}$ and in the following we will view $W(f)$ as a subset of $X \times Y^I$.

For the maps, we have by commutativity of the pullback diagram: $q(x, \gamma) = x$, $p(x, \gamma) = \gamma(1)$. Therefore if we define $s : X \to W(f)$, $x \mapsto (x, \overline{f(x)})$, where $\overline{f(x)}$ denotes the constant path to $f(x)$, we indeed obtain $p \circ s = f$.

Furthermore $q \circ s = \mathrm{id}_X$, and $s \circ q : W(f) \to W(f)$, $(x, \gamma) \mapsto (x, \overline{f(x)}) = (x, \overline{\gamma(0)})$. This is homotopic to $\mathrm{id}_{W(f)}$ by $H : W(f) \times I \to W(f)$,

$$(x, \gamma, \tau) \mapsto (x, \gamma_\tau), \qquad \text{where } \gamma_\tau(t) := \gamma(t\tau).$$

□

Theorem 2.4.3. *Homotopy pullbacks and pushouts exist in the categories* Top *and* Top$_*$.

Proof (for Top*).* Consider at first homotopy pullbacks. Given

$$\begin{tikzcd} & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd} ,$$

consider the cube

$$\begin{tikzcd} E_{f,g} \arrow[rr, "v"] \arrow[dr] \arrow[dd, "u"'] & & C \arrow[dr, "s"] \arrow[dd, "g"'] \arrow[dr, phantom, "\Swarrow H_2"', near end] & \\ & E_{f,g} \arrow[rr, "\pi"'] \arrow[dd, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "\Swarrow H_1", near end] & & W(g) \arrow[dd, "p"] \\ A \arrow[rr, "f"', near start] \arrow[dr] & & B \arrow[dr] & \\ & A \arrow[rr, "f"'] & & B \end{tikzcd}$$

where $W(g)$ is the mapping cocylinder defined in proposition 2.4.1, the maps p, s were also defined there, the unspecified maps are identities and all faces with no explicitly depicted homotopies have a trivial one. The space $E_{f,g}$ is defined as the 1-pullback of the front face, i.e.

$$A \times W(g) \supset E_{f,g} = \{(a, c, \gamma) \in A \times C \times B^I \mid \gamma(0) = g(c), \gamma(1) = f(a)\}.$$

The maps u and π are projections; furthermore we define $v(a, c, \gamma) = c$. The two non-trivial homotopies are defined as

$$H_1 : g \circ v \Rightarrow \gamma \circ u, \qquad H_1 : E_{f,g} \times I \to B, \qquad (\alpha, c, \gamma, \tau) \mapsto \gamma(\tau),$$

$$H_2 : s \circ v \Rightarrow \pi, \qquad H_1 : E_{f,g} \times I \to W(g), \qquad (\alpha, c, \gamma, \tau) \mapsto (c, \gamma_\tau), \qquad \text{where } \gamma_\tau(t) = \gamma(t\tau).$$

(One computes $H_1(a, c, \gamma, 0) = \gamma(0) = g(c) = (g \circ v)(a, c, \gamma)$ and $H_1(a, c, \gamma, 1) = f(a) = (f \circ u)(a, c, \gamma)$, $H_2(a, c, \gamma, 0) = (c, \gamma(0)) = (c, g(c)) = (s \circ v)(a, c, \gamma)$, $H_2(a, c, \gamma, 1) = (c, \gamma) = \pi(a, c, \gamma)$.)

The cube is 2-commutative, because $p \circ H_2 = H_1$ (equality of maps): $\gamma(\tau) = \gamma(\tau)$. Thus the square

$$\begin{tikzcd} E_{f,g} \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\swarrow H_1"] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is equivalent to the homotopy pullback square

$$\begin{tikzcd} E_{f,g} \arrow[r, "\pi"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & W(g) \arrow[d, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd} .$$

(The latter one is a homotopy pullback square since p is a fibration, cf. theorem 2.3.9.) Thus by proposition 2.3.13, also the former one is a homotopy pullback, as desired.

Dually, one shows that for a diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd} ,$$

the square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\swarrow H_1"] & B \arrow[d, "u"] \\ C \arrow[r, "v"'] & Q_{f,g} \end{tikzcd}$$

is a homotopy pushout, where $Q_{f,g}$ is defined by the 1-pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner", very near end] & \mathrm{Cyl}(f) \arrow[d] \\ C \arrow[r] & Q_{f,g} \end{tikzcd} .$$

Explicitly,

$$Q_{f,g} = B \sqcup C \sqcup A \times I \Big/ \left(\begin{array}{c} (a,0) \sim f(a) \\ (a,1) \sim g(a) \end{array} \right)$$

and

$$H_1 : u \circ f \Rightarrow v \circ g, \qquad H_1 : A \times I \to Q_{f,g}, \qquad (a, \tau) \mapsto [a, \tau].$$

□

Definition 2.4.4. The homotopy pullback and pushout squares constructed in theorem 2.4.3 are called the *standard homotopy pullback* and *standard homotopy pushout* of their respective input diagrams.[25]

Sometimes we will also use the notation $A \times_B^h C := E_{f,g}$, $B \cup_A^h C := Q_{f,g}$.

[25]Sometimes, the standard homotopy pushout is also called the double mapping cylinder.

Example 2.4.5. Examples of homotopy pushouts:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \arrow[dl, phantom, "\swarrow"] \\ * \arrow[r] & \mathrm{Cone}(f) \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[d] \\ * \arrow[r] & B/A \end{tikzcd}$$

where i is a cofibration

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \arrow[dl, phantom, "\swarrow"] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & * \arrow[d] \arrow[dl, phantom, "\swarrow"] \\ * \arrow[r] & S^{n+1} \end{tikzcd}$$

Example of a homotopy pullback:

$$\begin{tikzcd} \Omega_x X \arrow[r] \arrow[d] & * \arrow[d] \arrow[dl, phantom, "\swarrow"] \\ * \arrow[r] & X \end{tikzcd}$$

, where $\Omega_x X$ is the loop space based at $x \in X$.

Remark 2.4.6 (Notes on the pointed case)**.** The *standard homotopy pushout in* Top$_*$ is defined as

$$\begin{tikzcd} (A, a_0) \arrow[r, "f"] \arrow[d] & (B, b_0) \arrow[d] \arrow[dl, phantom, "\swarrow h"] \\ (C, c_0) \arrow[r] & (Q'_{f,g}, *) \end{tikzcd} ,$$

where

$$(Q'_{f,g}, *) := \left(B \sqcup C \sqcup A \times I \Big/ \begin{pmatrix} (a,0) \sim f(a), \\ (a,1) \sim g(a), \\ (a_0, t_1) \sim (a_0, t_2) \end{pmatrix}, \underbrace{[a_0, t]}_{=[f(a_0)]=[b_0]=[g(a_0)]=[c_0]} \right)$$

and $h : A \times I \to Q'_{f,g}$ is the canonical map. Then $h(a_0, t) = [a_0, t] = *$ equals the distinguished point for all t, i.e. h is a pointed homotopy.

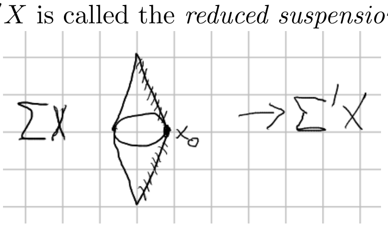
Example:

$$\begin{tikzcd} (X, x_0) \arrow[r] \arrow[d] & * \arrow[d] \arrow[dl, phantom, "\swarrow"] \\ * \arrow[r] & (\Sigma' X, *) \end{tikzcd} ,$$

where

$$(\Sigma' X, *) = \left(X \times I \Big/ \begin{pmatrix} (x,0) \sim (x',0), \\ (x,1) \sim (x',1), \\ (x_0, t_1) \sim (x_0, t_2) \end{pmatrix}, [x_0, t] \right).$$

$\Sigma' X$ is called the *reduced suspension* of X.



The *standard homotopy pullback in* Top$_*$ is defined as

$$\begin{tikzcd} (E_{f,g}, *) \arrow[r] \arrow[d] & (C, c_0) \arrow[d] \arrow[dl, phantom, "\swarrow H"] \\ (A, a_0) \arrow[r] & (B, b_0) \end{tikzcd} ,$$

where

$$(E_{f,g},*) := \left(\{(a,c,\gamma) \in A \times C \times B^I \mid \gamma(0) = g(c), \gamma(1) = f(a)\}, (a_0, c_0, \overline{b_0})\right).$$

Thus the underlying topological space of the 2-pullback in Top$_*$ is the same as the 2-pullback in Top. For the pushout, this is not the case.

Reason for this: The forgetful functor $\mathrm{Top}_* \to \mathrm{Top}$, $(X,x) \mapsto X$ has a left adjoint

$$\mathrm{Top} \to \mathrm{Top}_*, \qquad X \mapsto X_+ := (X \sqcup *, *).$$

[26] This means thast we have isomorphisms of categories for all $X \in \mathrm{Ob}\,\mathrm{Top}$, $(Y,y) \in \mathrm{Ob}\,\mathrm{Top}_*$

$$\mathrm{HOM}_{\mathrm{Top}_*}(X_+, (Y,y)) \xrightarrow{\sim} \mathrm{HOM}_{\mathrm{Top}}(X,Y).$$

This follows since

$$\mathrm{Ob}\,\mathrm{HOM}_{\mathrm{Top}_*}(X_+, (Y,y)) = \{f : X \to Y \mid f(*) = y\} \cong \{f : X \to Y\} = \mathrm{Ob}\,\mathrm{HOM}_{\mathrm{Top}}(X,Y)$$

and for $f, g \in \mathrm{Ob}\,\mathrm{HOM}_{\mathrm{Top}_*}(X_+, (Y,y))$,

$$\mathrm{Mor}_{\mathrm{HOM}_{\mathrm{Top}_*}}(f,g) = \{h : \underbrace{X_+ \times I}_{X \times I \sqcup * \times I} \to Y \mid h(*,t) = y, h_0 = f, h_1 = g\}/\sim$$

$$\cong \{h : X \times I \to Y \mid h_0 = f, h_1 = g\}/\sim = \mathrm{Mor}_{\mathrm{Top}}(f,g)$$

(where "$\sim$" denotes higher pointed resp. unpointed homotopies). Then the reason for the above observation follows from the next lemma.

Lemma 2.4.7. *The forgetful functor* $\mathrm{Top}_* \to \mathrm{Top}$ *preserves homotopy pullbacks.*

Proof. Given a homotopy pullback in Top$_*$ [27]

$$\begin{tikzcd}
(Y,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (C,c_0) \arrow[d, "g"] \arrow[dl, phantom, "\overset{H}{\Leftarrow}"] \\
(A,a_0) \arrow[r, "f"'] & (B,b_0)
\end{tikzcd} ,$$

we consider

$$\begin{tikzcd}
T \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "u"'] & & \\
& Y \arrow[r] \arrow[d] & C \arrow[d, "g"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

with $h : g \circ v \Rightarrow f \circ u$ in $\mathrm{HOM}_{\mathrm{Top}}(T,B)$. By adjointness we have $\mathrm{HOM}_{\mathrm{Top}}(T,C) \cong \mathrm{HOM}_{\mathrm{Top}_*}(T_+, (C,c_0))$, $\mathrm{HOM}_{\mathrm{Top}}(T,B) \cong \mathrm{HOM}_{\mathrm{Top}_*}(T_+, (B,b_0))$, $\mathrm{HOM}_{\mathrm{Top}}(T,A) \cong \mathrm{HOM}_{\mathrm{Top}_*}(T_+, (A,a_0))$, thus there are $v_+ : T_+ \to (C,c_0)$, $u_+ : T_+ \to (A,a_0)$ and $h_+ : g \circ v_+ \Rightarrow f \circ u_+$. Since the square we started out

[26]These are functors between $(2,1)$-categories; we do not consider such ones in their full generality here.
[27]We also indicate it by the symbol "$\lrcorner$" as the 1-pullback if there should not arise confusion.

with was a 2-pullback in Top_*, we get by its universal property the map α in

$$\begin{tikzcd}
T_+ \arrow[dr, dashed, "\alpha"] \arrow[drr, bend left, "v_+"] \arrow[ddr, bend right, "u_+"'] & & \\
& (Y, y_0) \arrow[r] \arrow[d] & (C, c_0) \arrow[d, "g"] \\
& (A, a_0) \arrow[r, "f"'] \arrow[ur, phantom, "\overset{H}{\Leftarrow}"] & (B, b_0)
\end{tikzcd} .$$

Applying the forgetful functor to α (and the homotopies), we obtain a map $T \to Y$ (and the required homotopies) to make the necessary diagram in Top 2-commutative. This shows "existence" for $YCAB$; "uniqueness" is shown similarly. $\square$

Remark 2.4.8. The forgetful functor $\mathrm{Top}_* \to \mathrm{Top}$ does not necessarily preserve homotopy pushouts.[28]

Theorem 2.4.9*. (Milnor) *Homotopy pullbacks of CW-complexes can be chosen to be CW-complexes. In particular, $\Omega_x X$ has the homotopy type of a CW-complex if $(X, x) \in \mathrm{Ob}\,\mathrm{CW}_*$.*

(Without proof here, hence marked with an asterisk.)

Corollary 2.4.10*. *Let*

$$\begin{tikzcd}
(F, f) \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (E, e) \arrow[d, "g"] \\
* \arrow[r] & (X, x)
\end{tikzcd}$$

be a 1*-pullback in* CW_**, where g is a Serre fibration and X, E are connected. Then it is a homotopy pullback square in* Top *and* Top_**.*[29]

In particular, by the Puppe long exact sequence, corollary 2.3.7, for all $T \in$ Ob Top *equipped with any basepoint, the sequence*

$$\ldots \longrightarrow [T, \Omega_x X]_* \longrightarrow [T, F]_* \longrightarrow [T, E]_* \longrightarrow [T, X]_*$$

is exact, where we denote the equivalence classes of pointed maps by $[\cdot, \cdot]_*$.[30]

(Proof later.)

Proposition 2.4.11. *There is an adjunction $\Sigma' \dashv \Omega$ of functors* $\mathrm{hTop}_* \to \mathrm{hTop}_*$*.*[31]

Proof (sketch), in an arbitrary $(2,1)$-category with zero object $*$ and weak 2-pullbacks, weak 2-pushouts. [32]

We need to construct a (natural) isomorphism $[\Sigma' X, Y] \cong [X, \Omega Y]$ for objects X, Y in our

---

[28] This also holds for general colimits: If one considers e.g. an "unrelated diagram", then the homotopy colimit is the same as the ordinary colimit since one cannot consider any homotopy. In Top_*, the resulting colimit is the wedge sum, but in Top, it is the disjoint union.

[29] F is necessarily connected as well. By definition of the pullback, F is the fiber at the point x. In other words, the statement of the corollary is that the fiber (over some basepoint) of a Serre fibration is the same as its homotopy fiber (over this basepoint).

[30] To be precise, we should write e.g. $[T, F]_* = [(T, t_0), (F, f_0)]_*$ for $t_0 \in T$.

[31] The symbol "$\dashv$" denotes the adjunction and can safely be ignored in the following.

[32] It could also be proven not hard "by hands" in Top_*.

category. Given $f : X \to Y$, consider

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "p"'] & * \arrow[d, "i"] \arrow[dl, phantom, "\Swarrow H_X"] & \\ * \arrow[r, "i"'] & \Sigma' X \arrow[dr, "f"] & \\ & & Y \end{tikzcd} ,$$

thus by the existence part of the homotopy pullback ΩY we get the map $\tilde{f}$ in the diagram below.

$$\begin{tikzcd} X \arrow[dr, dashed, "\tilde{f}"] \arrow[drr, bend left, "p"] \arrow[ddr, bend right, "p"'] & & \\ & \Omega Y \arrow[r] \arrow[d] \arrow[dr, phantom, "f\circ H_X \ \Swarrow"] & * \arrow[d] \\ & * \arrow[r] & Y \end{tikzcd}$$

[33]

In order that the assignment $f \mapsto \tilde{f}$ descends to a well-defined map $[\Sigma' X, Y] \to [X, \Omega Y]$, we need to show that if there is a 2-morphism $h : f \Rightarrow f'$ and $\tilde{f}, \tilde{f}'$ are defined as above for f, f', there is also a 2-morphism $\tilde{f} \Rightarrow \tilde{f}'$.

Claim: If such an h exists, then $f \circ H_X = f' \circ H_X$.

Proof of claim: Consider

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \arrow[dl, phantom, "\Swarrow H_X"] & \\ * \arrow[r] & \Sigma' X \arrow[dr, bend left, "f"] \arrow[dr, bend right, "f'"'] \arrow[dr, phantom, "\Swarrow h"] & \\ & & Y \end{tikzcd} .$$

By the argument made already at some times before, we obtain from the composition functor a commutative diagram:

$$\begin{aligned} &\mathrm{HOM}(\Sigma' X, Y) \times \mathrm{HOM}(X, \Sigma' X) \to \mathrm{HOM}(X, Y), \\ &(\ f \xrightarrow{h} f'\ ) \in \mathrm{HOM}(\Sigma' X, Y), \\ &(\ i \circ p \xrightarrow{H_X} i \circ p\ ) \in \mathrm{HOM}(X, \Sigma' X) \end{aligned}$$

yield commutativity of

$$\begin{tikzcd} f \circ i \circ p \arrow[r, Rightarrow, "h\circ i\circ p"] \arrow[d, Rightarrow, "f\circ H_X"'] & f' \circ i \circ p \arrow[d, Rightarrow, "f'\circ H_X"] \\ f \circ i \circ p \arrow[r, Rightarrow, "h\circ i\circ p"'] & f' \circ i \circ p \end{tikzcd} .$$

Since $*$ is the zero object, $\mathrm{HOM}(*, Y)$ is the trivial groupoid, having a single object and as only morphism the identity. In particular, the maps $f \circ i$ and $f' \circ i$ must coincide; they shall be called i'. Also the morphism $h \circ i$ must then be equal to $\mathrm{id}_{i'}$.[34] Inserting this into the above diagram we

[33] Here $f \circ H_x$ is understood as the homotopy in the large, distorted "square", not the small one.
[34] Slogan: "Maps into/out of $*$ do not have any homotopies."

obtain

$$\begin{tikzcd} i' \circ p \arrow[r, Rightarrow, "\mathrm{id}_{i'} \circ p = \mathrm{id}_{i' \circ p}"] \arrow[d, Rightarrow, "f \circ H_X"'] & i' \circ p \arrow[d, Rightarrow, "f' \circ H_X"] \\ i' \circ p \arrow[r, Rightarrow, "\mathrm{id}_{i'} \circ p = \mathrm{id}_{i' \circ p}"'] & f' \circ i \circ p \end{tikzcd} .$$

This shows the claim.

By "uniqueness" for the 2-pullback ΩY and the claim, we get a 2-morphism $\tilde{f} \Rightarrow \tilde{f}'$.[35] This shows that there is a well-defined map $[\Sigma' X, Y] \to [X, \Omega Y]$.

Similarly, one gets the map $[X, \Omega Y] \to [\Sigma' X, Y]$ and verifies that the respective compositions are the identity. Also one verifies that this is functorial in X and Y. □

Corollary 2.4.12. *For a homotopy pullback in* Top$_*$ *(using some unspecified basepoints)*

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[d] \arrow[dl, phantom, "\swArrow"] \\ * \arrow[r] & B \end{tikzcd} ,$$

one can rewrite the Puppe long exact sequence 2.3.7 as

$$\ldots \longrightarrow [\Sigma' T, E]_* \longrightarrow [\Sigma' T, B]_* \longrightarrow [T, F]_* \longrightarrow [T, E]_* \longrightarrow [T, B]_*$$

for $T \in \mathrm{Top}_*$.

In particular, for $T = (S^n, *)$*, one gets a long exact sequence of homotopy groups*[36]

$$\ldots \longrightarrow \pi_{n+1}(E) \longrightarrow \pi_{n+1}(B) \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \longrightarrow \ldots .$$

The dual result holds for homotopy pushouts.[37]

This follows from the previous proposition and corollary 2.4.10.

Proof of corollary 2.4.10. Consider

$$\begin{tikzcd} (F,f) \arrow[dr, dashed, "\phi"] \arrow[drr, bend left, "i"] \arrow[ddr, bend right] & & \\ & (Y,y) \arrow[r] \arrow[d] & (E,e) \arrow[d, "g"] \arrow[dl, phantom, "\swArrow"] \\ & * \arrow[r] & (X,x) \end{tikzcd} ,$$

where the inner square is a homotopy pullback.[38] Then consider the following long exact sequences of homotopy groups:

$$\begin{tikzcd} \ldots \arrow[r] & \pi_{n+1}(E) \arrow[r] \arrow[d, "\mathrm{id}"] & \pi_{n+1}(X) \arrow[r] \arrow[d, "\mathrm{id}"] & \pi_n(F) \arrow[r] \arrow[d, "\phi_*"] & \pi_n(E) \arrow[r] \arrow[d, "\mathrm{id}"] & \pi_n(X) \arrow[r] \arrow[d, "\mathrm{id}"] & \ldots . \\ \ldots \arrow[r] & \pi_{n+1}(E) \arrow[r] & \pi_{n+1}(X) \arrow[r] & \pi_n(Y) \arrow[r] & \pi_n(E) \arrow[r] & \pi_n(X) \arrow[r] & \ldots \end{tikzcd}$$

[35] By the claim, both $\tilde{f}, \tilde{f}'$ satisfy, together with their respective homotopies, the necessary 2-commutativity conditions for $(X \to *, X \to *, K)$, where $K := f \circ H_X = f' \circ H_X$.

[36] The basepoints shall be omitted for brevity.

The Puppe long exact sequence thus generalizes the long exact sequence for Serre fibrations 1.2.7 resp. the "usual" long exact sequence for homotopy groups. Sometimes we will thus call also this sequence a long exact sequence for homotopy groups.

[37] But there is no statement about homotopy groups here.

[38] The map ϕ always exists since the outer four arrows even commute.

The upper one is the long exact sequence of a Serre fibration obtained in Corollary 1.2.7, the lower one comes from the Puppe long exact sequence, cf. Corollary 2.4.12 before. By the five-lemma, ϕ_* is an isomorphism. Since this holds for all n and by theorem 2.4.9, both spaces are homotopy equivalent to CW-complexes, by Whitehead's theorem 1.7.4, ϕ is a homotopy equivalence. This proves the corollary. □

2.5 Pointed versus unpointed

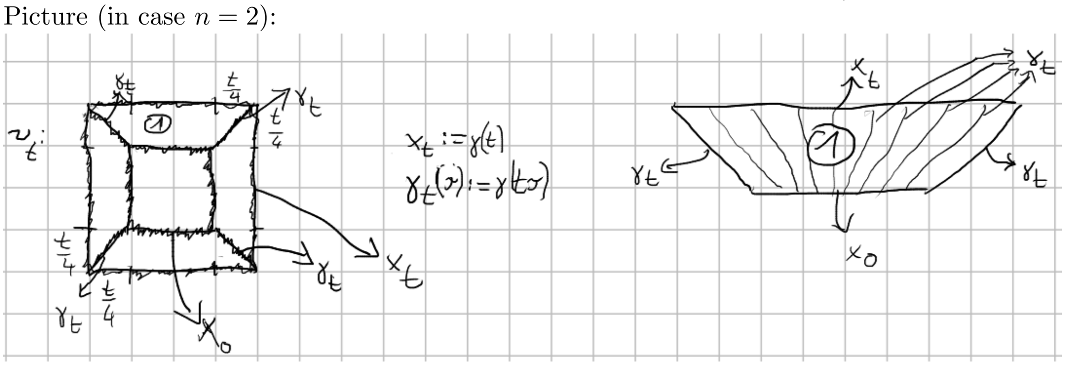
Lemma 2.5.1. *There is a natural functor* $\Pi_1(X) \to \mathrm{Grp}$ *for all* $X \in \mathrm{Top}, n \geq 1$ *that sends*

$$\begin{aligned}&\mathrm{Ob}\,\Pi_1(X) \ni x \mapsto \pi_n(X,x) \in \mathrm{Ob}\,\mathrm{Grp},\\ &\forall x_0, x_1 \in \mathrm{Ob}\,\Pi_1(X), (\gamma : x_0 \to x_1) \mapsto (\gamma_\sharp : \pi_n(X,x_0) \to \pi_n(X,x_1)).\end{aligned}$$

Proof. One can give a concrete description of $\gamma_\sharp$ for $\gamma \in \mathrm{Hom}(x_0,x_1)$ (i.e. γ is a path from x_0 to x_1):

Given $\nu_0 : (I^n, \partial I^n) \to (X, x_0)$, i.e. $[\nu_0] \in \pi_n(X,x_0)$, consider the homotopy $\partial I^n \times I \xrightarrow{\mathrm{pr}_2} I \xrightarrow{\gamma} X$. This can be extended to $\nu : I^n \times I \to X$ starting with ν_0, using that $\partial I^n \hookrightarrow I^n$ is a cofibration (cf. example 1.4.9). At time 1, we obtain $\nu_1 : (I^n, \partial I^n) \to (X, x_1)$ and define $\gamma_\sharp([\nu_0]) = [\nu_1]$.

Picture (in case $n = 2$):



[39] □

Lemma 2.5.2. *Let* $\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\eta"] & Y \end{tikzcd}$ *be a diagram in* Top. *Let* $x_0 \in X$ *and denote by* η_{x_0} *the restriction of* $\eta : X \times I \to Y$ *to* $\{x_0\} \times I$ *(this is a path in* Y*). Then for* $n \geq 1$*, there is a commutative diagram*

$$\begin{tikzcd} \pi_n(X,x_0) \arrow[rr, "f_*"] \arrow[dr, "g_*"'] & & \pi_n(Y, f(x_0)) \arrow[dl, "\cong", "(\eta_{x_0})_\sharp"'] \\ & \pi_n(Y, g(x_0)) & \end{tikzcd} .$$

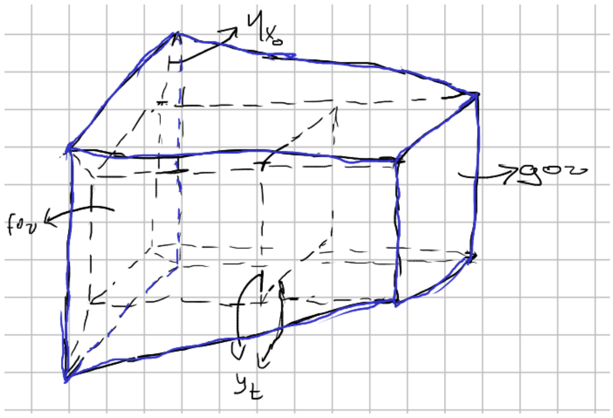
Proof (Sketch). Given $\nu : (I^n, \partial I^n) \to (X, x_0)$, we need to construct a homotopy between maps of pairs $(\eta_{x_0})_\sharp(f \circ \nu) \Rightarrow g \circ \nu$. This will establish their equality in $\pi_n(Y, g(x_0))$. Denote $y_0 := f(x_0), y_1 := g(x_0), y_t := \eta_{x_0}(t)$. For $n = 2$, the situation looks like follows:

---

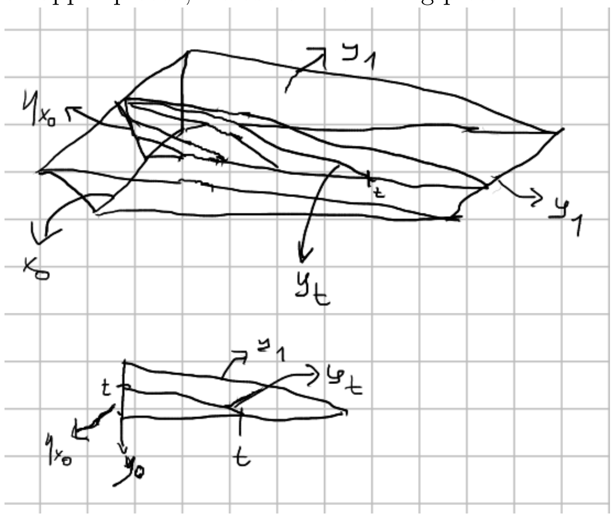
[39] If $f : Y \to X$ is a map, then it induces a functor (i.e. a map between groupoids) $f_* : \Pi_1(Y) \to \Pi_1(X)$. That the constructed functor is natural means that the resulting triangle

$$\begin{tikzcd} \Pi_1(Y) \arrow[rr, "f_*"] \arrow[dr] & & \Pi_1(X) \arrow[dl] \\ & \mathrm{Grp} & \end{tikzcd}$$

commutes. The proof of this is left as an exercise.



$\eta \circ \nu$ is defined on the interior, black cube, and we want to extend it to the blue (deformed) cube, such that for all t it sends ∂I^n to y_1. Thus one needs to "fill in" the remaining parts. For the "upper piece", we use the following picture:



I.e. we define the map to be constant along the lines in the vertical slices of the "upper piece" (one of them is sketched above). This we do for all such slices. □

Corollary 2.5.3. *Let*

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "H_x \Swarrow"] & E \arrow[d, "p"] \\ * \arrow[r, "x"'] & X \end{tikzcd}$$

be a homotopy pullback in Top_**, where the constant map to* $x \in X$ *is also denoted* x*. Then for all points* $f \in F$ *(not necessarily the distinguished one), there exists a long exact sequence*

$$\begin{tikzcd} \dots \arrow[r] & \pi_n(F,f) \arrow[r, "i_*"] & \pi_n(E,i(f)) \arrow[rr] \arrow[dr] & & \pi_n(X,x) \arrow[r, "\partial"] & \pi_{n-1}(F,f) \arrow[r] & \dots, \\ & & & \pi_n(X,p(i(f))) \arrow[ur, "\cong", "(H_f)_\sharp"'] & & & \end{tikzcd}$$

where H_f *denotes the restriction of* H *to* f*.*

Idea of proof. One attempts to find a 2-commutative cube that gives an equivalence from the given homotopy pullback to the standard homotopy pullback defined in theorem 2.4.3. This can be "pointed" by any point in the (homotopy) fiber, since it is a 1-commutative square. Then one carefully needs to compute how all 2-commutative faces of the cube induce maps on homotopy groups. □

Question: For X, Y (pointed) topological spaces, how do $[X,Y]_*$ and $[X,Y]$ differ? (Answer later, in section 4.5.)

2.6 On the importance of the homotopy fiber

Definition 2.6.1. For a map $p : E \to X$ and $x \in X$, the space F that fits into the homotopy pullback square

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ * \arrow[r, "x"'] \arrow[ur, phantom, "\Swarrow H"] & X \end{tikzcd}$$

is called the *homotopy fiber of p over X*. Here we denote the constant map to x again by x. Sometimes we will also use the symbol E_x instead of F.[40]

Lemma 2.6.2. *Let (X, A, a) be a pointed pair in* Top *and let*

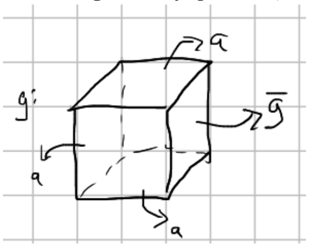
$$\begin{tikzcd} (F,f) \arrow[r] \arrow[d] & (A,a) \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\Swarrow H"] & (X,a) \end{tikzcd}$$

be a homotopy pullback in Top_**. Then there is an isomorphism $\pi_{k+1}(X,A,a) \xrightarrow{\sim} \pi_k(F,f)$ for all $k \geq 0$.*

Proof (Sketch). Given $[g] \in \pi_{k+1}(X,A,a)$, i.e. $g : (I^{k+1}, \partial I^{k+1}, J^k) \to (X,A,a)$ is a map of triples, we get a map $\overline{g} : \partial I^{k+1}/J^k \to A$ and a homotopy h that form a diagram

$$\begin{tikzcd} (\partial I^{k+1}/J^k, *) \arrow[r, "\overline{g}"] \arrow[d] & (A,a) \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\Swarrow h"] & (X,a) \end{tikzcd} .$$

Here h is given by g itself, if $\partial I^k \times \{t\}$ is collapsed to a point for all $t \in I$:



h is a homotopy between maps of pointed spaces. Because (F,f) is a homotopy pullback, we therefore get a map $(S^k, *) \cong (\partial I^{k+1}/J^k, *) \to (F,f)$. This defines the map $\pi_{k+1}(X,A,x) \to \pi_k(F,f)$.

There is a map between long exact sequences

$$\begin{tikzcd} \dots \arrow[r] & \pi_{k+1}(A,a) \arrow[r] \arrow[d, "\mathrm{id}"] & \pi_{k+1}(X,a) \arrow[r] \arrow[d, "\mathrm{id}"] & \pi_{k+1}(X,A,a) \arrow[r] \arrow[d] & \pi_k(A,a) \arrow[r] \arrow[d, "\mathrm{id}"] & \pi_k(X,a) \arrow[r] \arrow[d, "\mathrm{id}"] & \dots , \\ \dots \arrow[r] & \pi_{k+1}(A,a) \arrow[r] & \pi_{k+1}(X,a) \arrow[r] & \pi_k(F,f) \arrow[r] & \pi_k(A,a) \arrow[r] & \pi_k(X,a) \arrow[r] & \dots \end{tikzcd}$$

where the upper sequence is the one obtained in theorem 1.1.5, while the lower one is the Puppe long exact sequence from corollary 2.4.12. One verifies commutativity of this diagram (thus it

[40] *Caution:* Sometimes we will also use E_x for the "usual" fiber of a fiber bundle, not the homotopy fiber.

indeed is a map between sequences). Then the five-lemma yields the claim. □

Definition 2.6.3 (Redefinition)**.** A map $p : E \to X$ is *n-connected*, $n \geq 0$, if all homotopy fibers of p are $(n-1)$-connected.

Lemma 2.6.4. *The old definition of n-connectedness 1.5.1 and the above redefinition are equivalent.*

Proof.
- $n = 0$: Definition 1.5.1 required that $p_* : \pi_0 E \twoheadrightarrow \pi_0 X$ was surjective.

  The new definition requires that for all $x \in X$, the homotopy fiber E_x defined by the 2-pullback

$$\begin{tikzcd} E_x \arrow[r] \arrow[d] & E \arrow[d] \\ * \arrow[r, "x"'] \arrow[ur, phantom, "\Swarrow"] & X \end{tikzcd}$$

  is (-1)-connected, i.e. $E_x \neq \emptyset$. This is satisfied if and only if $\pi_0 E \twoheadrightarrow \pi_0 X$ is surjective.

- $n > 0$: Definition 1.5.1 required that $\pi_0 E \twoheadrightarrow \pi_0 X$ was surjective and $\pi_i(X, E, e) = 0$ for all $e \in E$, $1 \leq i \leq n$. By Proposition 1.5.2 and Corollary 2.5.3, this is equivalent to $\pi_i(E_x, f) = 0$ for all $x \in X$, $f \in E_x$, $1 \leq i \leq n-1$, i.e. E_x is $(n-1)$-connected.

□

Definition 2.6.5. A 2-commutative square in Top

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "p"'] & A \arrow[d, "f"] \\ B \arrow[r, "j"'] \arrow[ur, phantom, "\Swarrow H"] & C \end{tikzcd}$$

is called *n-weakly homotopy cartesian* if the map $X \to X \times^h_C B$ is n-connected. Here $A \times^h_C B$ denotes the (resp. a) homotopy pullback of ABC in the diagram above.[41]

Proposition 2.6.6. *The square*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "p"'] & A \arrow[d, "f"] \\ B \arrow[r, "j"'] \arrow[ur, phantom, "\Swarrow H"] & C \end{tikzcd}$$

is n-weakly homotopy cartesian if and only if for all $b \in B$, the map

$$X \times^h_B \{b\} =: X_b \to A_{j(b)} := A \times^h_C \{j(b)\}$$

is n-connected.

Proof. Denote $W := A \times^h_C B$. For $b \in B$, consider

$$\begin{tikzcd} W_b \arrow[r] \arrow[d] & W \arrow[r, "j'"] \arrow[d, "f'"] & A \arrow[d, "f"] \\ * \arrow[r, "b"'] \arrow[ur, phantom, "\Swarrow"] & B \arrow[r, "j"'] \arrow[ur, phantom, "\Swarrow"] & C \end{tikzcd},$$

where the constant map to b is again denoted b and the left square is defined as the homotopy pullback (i.e. W_b is the homotopy fiber of f' at b). By proposition 2.3.6, the big rectangle is then

[41] The map exists because $A \times^h_C B$ is a homotopy pullback.

again a homotopy pullback. Therefore both W_b and $A_{j(b)}$ are the homotopy pullbacks of

$$\begin{tikzcd} & A \arrow[d, "f"] \\ * \arrow[r, "j\circ b"'] & C \end{tikzcd} .$$

Since homotopy pullbacks are unique up to homotopy equivalence by lemma 2.3.3, $W_b \simeq A_{j(b)}$.[42] Therefore, it suffices to prove the claim (both directions) with W_b instead of $A_{j(b)}$.

Now consider in Top_* (however without specifying the basepoints; the pullbacks are anyways always the same due to lemma 2.4.7)

$$\begin{tikzcd} X_b \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & W_b \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & A_{j(b)} \arrow[d] \\ X \arrow[r, dashed] \arrow[d, "p"] \arrow[dr, phantom, "\Downarrow"] & W \arrow[r, "j'"'] \arrow[d, "f'"] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, "f"] \\ B \arrow[r, equal] & B \arrow[r, "j"'] & C \end{tikzcd} .$$

The map $X \to W$ is obtained since W is a homotopy pullback (as already stated); the right lower square is the pullback square. The map $X_b \to W_b$ is obtained since W_b is again a homotopy pullback. The left part of the diagram yields therefore a morphism of Puppe long exact sequences (cf. corollary 2.4.12)

$$\begin{tikzcd}[column sep=small] \dots \arrow[r] & \pi_{k+1}(B) \arrow[r] \arrow[d, equal] & \pi_k(X_b) \arrow[r] \arrow[d] & \pi_k(X) \arrow[r] \arrow[d] & \pi_k(B) \arrow[r] \arrow[d, equal] & \dots \arrow[r] & \pi_1(B) \arrow[r] \arrow[d, equal] & \pi_0(X_b) \arrow[r] \arrow[d] & \pi_0(X) \arrow[r] \arrow[d] & \pi_0(B) \arrow[d, equal] \\ \dots \arrow[r] & \pi_{k+1}(B) \arrow[r] & \pi_k(W_b) \arrow[r] & \pi_k(W) \arrow[r] & \pi_k(B) \arrow[r] & \dots \arrow[r] & \pi_1(B) \arrow[r] & \pi_0(W_b) \arrow[r] & \pi_0(W) \arrow[r] & \pi_0(B) \end{tikzcd} .$$

Now

- For $n = 0$: "If": Let $[w] \in \pi_0(W)$ (connected component of $w \in W$). Then $f(w) = b$ for some $b \in B$. By assumption we have a surjection $\pi_0(X_b) \twoheadrightarrow \pi_0(W_b)$, thus there is a preimage $[x]$ of $[w] \in \pi_0(W_b)$.[43] By homotopy commutativity of the square X_bW_bXW constructed above, the image of the map $\pi_0(X) \to \pi_0(W)$ contains $[w]$, i.e. $\pi_0(X) \twoheadrightarrow \pi_0(W)$ is surjective.

  "only if": By assumption $\pi_0(X) \twoheadrightarrow \pi_0(W)$ is surjective. From the morphism of long exact sequences considered above, we obtain that also $\pi_0(X_b) \twoheadrightarrow \pi_0(W_b)$ is surjective for all $b \in B$.
- For $n > 0$: Use the five-lemma in the above diagram of exact sequences to get the claim: $X \to W$ is n-connected if and only if $X_b \to W_b$ is n-connected for all $b \in B$.

This proves the proposition. □

2.7 Statement of Blakers-Massey theorem

Theorem 2.7.1*. (Blakers-Massey) *Let*

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, "f'"] \\ B \arrow[r, "g'"'] & B \end{tikzcd}$$

[42] Recall that homotopy equivalence is obtained by fitting the homotopy pullback $A_{j(b)}$ in the horizontal rectangle in the diagram above.

[43] Note here that W_b is defined as the standard homotopy pullback, in which W embeds.

be a homotopy pushout in Top. *Assume that f is m-connected, g is n-connected, $n, m \geq 0$. Then this square is a $(n+m-1)$-weakly homotopy pullback.*

(Proof later, in section 2.11.)

Remark 2.7.2. E.g. in the gategory $\mathrm{Ch}(\mathcal{A})$ of chain complexes over an abelian category $\mathcal{A}$, or in the stable homotopy category $\mathcal{D}(\mathcal{A})$, homotopy pullback squares are the same as homotopy pushout squares.

Corollary 2.7.3*. (Excision for CW-complexes, theorem 1.9.3) *Proof.* Recall the situation in theorem 1.9.3:

$$\begin{tikzcd} C \arrow[r, hook, "g"] \arrow[d, hook, "f"'] \arrow[dr, phantom, "(*)"] & A \arrow[d, hook, "v"] \\ B \arrow[r, hook, "u"'] & X \end{tikzcd},$$

where C, A, B are subcomplexes of a CW-complex X. We need to show: If f is m-connected, g is n-connected, then the map $\pi_i(B, C, c) \to \pi_i(X, A, c)$, for $c \in C$, is

$$\begin{cases} \text{an isomorphism for } i \leq n+m-1, \\ \text{surjective for } i = n+m. \end{cases}$$

Now since $(*)$ above is also a homotopy pushout square (since f is a cofibration, cf. 1.4.11, and by theorem 2.3.9), the Blakers-Massey theorem yields that it is also a $(n+m-1)$-weakly homotopy pullback. By proposition 2.6.6, this implies that $C_{f(c)} \to A_{v(g(c))}$ is $(n+m-1)$-connected. By proposition 1.5.2, this means that $\pi_i(C_{f(c)}) \to \pi_i(A_{v(g(c))})$ is

$$\begin{cases} \text{an isomorphism for } i \leq n+m-2, \\ \text{surjective for } i = n+m-1. \end{cases}$$

Since by lemma 2.6.2, we have $\pi_i(C_{f(c)}) \cong \pi_{i+1}(B, C, c)$ and $\pi_i(A_{v(g(c))}) \cong \pi_{i+1}(X, A, g(c))$ (and $g(c) = c$), we obtain the excision theorem. $\square$

Example 2.7.4*. (Freudenthal's suspension theorem (again, cf. corollary 1.9.5)) Let X be n-connected, $n \geq 0$. Consider the homotopy pushout

$$\begin{tikzcd} X \arrow[r, "(n+1)-\text{conn.}"] \arrow[d, "(n+1)-\text{conn.}"'] & * \arrow[d] \\ * \arrow[r] & B \end{tikzcd}.$$

The Blakers-Massey theorem yields that $X \to \Omega\Sigma X$ is $(2n+1)$-connected.[44] Thus the map $\pi_i(X) \to \pi_i(\Omega\Sigma X) \cong \pi_{i+1}(\Omega\Sigma X)$ (the latter isomorphism by the adjunction 2.4.11 and $\Sigma' S^i = S^{i+1}$) is

$$\begin{cases} \text{an isomorphism for } i \leq 2n, \\ \text{surjective for } i = 2n+1. \end{cases}$$

This is Freudenthal's suspension theorem.

2.8 Quasi-fibrations

Note: Many results will be stated in $\mathrm{Top}/\mathrm{Top}_*$, but proven only in $\mathrm{CW}/\mathrm{CW}_*$.

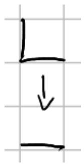
Definition 2.8.1. A map $p : E \to B$ is a *quasi-fibration*, if it is surjective and for all $b \in B$, the map

$$p^{-1}(b) \to E \times_B^h \{b\}$$

is a weak equivalence.

[44] This is a reformulation of the square being $(2n+1)$-weakly homotopy cartesian.

Example 2.8.2 (Examples of quasi-fibrations)**.**

- Fibrations
- Serre fibrations of CW-complexes[45]
- 
[46]

Remark 2.8.3 (Warning)**.** Quasi-fibrations are not stable under pullbacks (contrarily to e.g. fibrations or Serre fibrations).

Definition 2.8.4. Let $p : E \to B$ be a map. A subspace $A \subset B$ is called *distinguished* for p if the induced map $p^{-1}(A) \to A$ is a quasi-fibration.

(We will not need this definition but in the next few sections.)

Lemma 2.8.5. *Let $p : E \to B$ be a map and $A \subset B$ be distinguished for p. Then the following are equivalent:*

1. *$p : (E, p^{-1}(\{a\})) \to (B, a)$ is a weak equivalence for all $a \in A$ (i.e. $p_* : \pi_i(E, p^{-1}(\{a\}), a') \xrightarrow{\sim} \pi_i(B, a)$ is an isomorphism for all $a' \in p^{-1}(\{a\})$ and all i)*
2. *$(E, p^{-1}(A)) \to (B, A)$ is a weak equivalence.*

Proof (for CW-complexes). For CW-complexes, a weak equivalence is the same as a homotopy equivalence by Whitehead's theorem 1.7.4. Then consider

$$\begin{tikzcd}
p^{-1}(\{a\}) \arrow[r] \arrow[d] \arrow[dr, phantom, "(*)"] & E \arrow[d, "p"] \\
* \arrow[r, "a"'] & B
\end{tikzcd} .$$

Mirroring this and constructing the homotopy fibers of the map $p^{-1}(\{a\}) \to E$, one obtains (i.e. the left square is a homotopy pullback)

$$\begin{tikzcd}
F \arrow[r] \arrow[d] \arrow[dr, phantom, "\swArrow"] & p^{-1}(\{a\}) \arrow[r] \arrow[d] & * \arrow[d, "a"] \\
* \arrow[r] & E \arrow[r, "p"'] & B
\end{tikzcd} .$$

Then 1. holds if and only if the large rectangle is also a homotopy pullback, i.e. the homotopy fibers of $p^{-1}(\{a\}) \to E$ and $* \to B$ are equivalent. ("If" follows from lemma 2.6.2; "only if" follows from the same lemma and Whitehead's theorem.)

We thus have a chain of logical equivalences:

$$1. \overset{\text{(see above)}}{\Longleftrightarrow} \begin{array}{c}\text{the homotopy fibers of the horizontal} \\ \text{maps in } (*) \text{ are equivalent for all } a \in A\end{array} \overset{\text{proposition 2.6.6}}{\Longleftrightarrow} \begin{array}{c}(*) \text{ is a homotopy} \\ \text{pullback for all } a \in A\end{array} .$$

[45] Recall that $E \times^h_B \{b\}$ is defined by a homotopy pullback diagram. The "usual" fiber $p^{-1}(b)$ is defined as the 1-pullback of the same input diagram. By theorem 2.3.9 and corollary 2.4.10, respectively, the 1-pullback is here a 2-pullback as well, thus the two coincide up to homotopy equivalence.

[46] This is a homotopy equivalence, thus the homotopy fibers are points.

By the same reasoning, 2. holds if and only if the square

$$\begin{tikzcd} p^{-1}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "(**)"] & E \arrow[d, "p"] \\ A \arrow[r, hook] & B \end{tikzcd} .$$

is a homotopy pullback. Now consider

$$\begin{tikzcd} p^{-1}(\{a\}) \arrow[r] \arrow[d] \arrow[dr, phantom, "(+)"] & p^{-1}(A) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ * \arrow[r, "a"'] & A \arrow[r, hook] & B \end{tikzcd} .$$

The square $(+)$ is a homotopy pullback, because A is distinguished for p (and by definition of a quasi-fibration). The right square is $(**)$, while the total rectangle is $(*)$. Thus by proposition 2.3.6, the claim follows: $(*)$ is a homotopy pullback if and only if $(**)$ is. □

Proposition 2.8.6 (Being a quasi-fibration is local on the base)**.** *Let $p : E \to B$ be a map, $B_0, B_1 \subseteq B$ with $B = B_0^\circ \cup B_1^\circ$. Let $B_{01} := B_0 \cap B_1$. If B_0, B_1, B_{01} are distinguished for p, then B is distinguished for p, i.e. p is a quasi-fibration.*

In the proof, we need:

Theorem 2.8.7*. (May (see literature list)) *Let $f : X \to Y$ be a map, $X_0, X_1 \subset X$, $Y_0, Y_1 \subset Y$, $X_{01} := X_0 \cap X_1$, $Y_{01} := Y_0 \cap Y_1$ such that $X = X_0^\circ \cup X_1^\circ$, $Y = Y_0^\circ \cup Y_1^\circ$, $f(X_i) \subset Y_i$ for $i = 0, 1$. If $(X_i, X_{01}) \to (Y_i, Y_{01})$ is a weak equivalence for $i = 0, 1$, then $(X, X_i) \to (Y, Y_i)$ is a weak equivalence for $i = 0, 1$.*

The proof is similar to the one of the higher connectivity theorem 1.5.3 and is omitted here.

Proof of proposition 2.8.6. Denote $E_i := p^{-1}(B_i)$, $i = 0, 1$, and $E_{01} := p^{-1}(B_{01})$. Consider the commutative diagram, where "q-fib." denotes a quasi-fibration, the horizontal maps are all inclusions and the vertical maps are restrictions of p:

$$\begin{tikzcd} E_{01} \arrow[r, hook] \arrow[d, "\text{q-fib.}"'] & E_i \arrow[r, hook] \arrow[d, "\text{q-fib.}"'] & E \arrow[d] \\ B_{01} \arrow[r] & B_i \arrow[r] & B \end{tikzcd} ,$$

for $i = 0, 1$. By lemma 2.8.5.2, $(E_i, E_{01}) \to (B_i, B_{01})$, is a weak equivalence for $i = 0, 1$: Assumption 1. in this lemma is satisfied since we have shown in its proof that it is equivalent to

$$\begin{tikzcd} p^{-1}(b) \arrow[r] \arrow[d] & E_i \arrow[d] \\ * \arrow[r, "b"'] & B_i \end{tikzcd}$$

being a homotopy pullback for all $b \in B$. However, it is ∞-weakly homotopy cartesian since E_i is a quasi-fibration, and since we work in CW, it is a homotopy pullback.

By May's theorem 2.8.7, $(E, E_i) \to (B, B_i)$ is a weak equivalence (for $i = 0, 1$). Reversing the logic from above, from lemma 2.8.5.1 we now obtain that $(E, p^{-1}(b)) \to (B, b)$ is a weak equivalence for all $b \in B_i$, $i = 0, 1$, thus for all $b \in B$. By the same reasoning as before this is equivalent to

$$\begin{tikzcd}
p^{-1}(b) \arrow[r] \arrow[d] & E \arrow[d] \\
* \arrow[r, "b"'] & B
\end{tikzcd}$$

being a homotopy pullback for all $b \in B$, i.e. $p^{-1}(b)$ is the homotopy fiber over b for all $b \in B$. This concludes the proof. ☐

2.9 Mather's cube theorems

Theorem 2.9.1 (Mather's first cube theorem)**.** *Let*

$$\begin{tikzcd}
A \arrow[rr, "\alpha"] \arrow[dr, "\beta"] \arrow[dd, "f_A"'] & & B \arrow[dr, "\gamma"] \arrow[dd, "f_B"' near start] \arrow[d, phantom, "H" near end] & \\
& C \arrow[rr, "\delta"' near start] \arrow[dd, "f_C" near start] \arrow[r, phantom, "F_1"] & & D \arrow[dd, "f_D"] \arrow[d, phantom, "F_3"] \\
A' \arrow[rr, "\alpha'"' near end] \arrow[dr, "\beta'"'] \arrow[r, phantom, "F_2"] & & B' \arrow[dr, "\gamma'"'] \arrow[r, phantom, "F_4"] & \\
& C' \arrow[rr, "\delta'"'] & \arrow[r, phantom, "H'"] & D'
\end{tikzcd}$$

be a 2-commutative cube as in definition 2.3.10. Assume that the top and the bottom face are homotopy pushouts.

- *Strict version: If the rear (containing F_1) and left (containing F_2) faces are homotopy pullbacks, then so are the front (containing F_4) and the right face (containing F_3).*
- *Weak version: If the rear (containing F_1) and left (containing F_2) faces are n-weakly homotopy cartesian, $n \in \mathbb{N}$, then so are the front (containing F_4) and the right face (containing F_3).*

Proof (Sketch). The proof of the strict version for CW-complexes is sketched. It suffices furthermore to prove that the front face is a homotopy pullback, for the right one the proof is analogous.

1. *Preparation step:* We may assume that f_A, f_B, f_C are fibrations, that A is the 1-pullback $A' \times_{B'} B$, and that F_1, F_2 are trivial. This is done using the following construction:
   - There exists an equivalence from the F_1-face to the standard homotopy pullback of $A'B'B$ (e.g. due to "Uniqueness of 2-pullbacks I", lemma 2.3.3). Composing this with the cube from the definition of the standard homotopy pullback in theorem 2.4.3 (resp. actually its inverse cube, which can be constructed as in lemma 2.3.12) yields a cube (omitting homotopies and names of maps for brevity)

$$\begin{tikzcd}
\overline{A} \arrow[rr] \arrow[dr, "\simeq"] \arrow[dd] & & \overline{B} \arrow[dr, "\simeq"] \arrow[dd] & \\
& C' \arrow[rr] \arrow[dd] & & D' \arrow[dd] \\
A \arrow[rr] \arrow[dr, "\simeq"] & & B \arrow[dr, "\simeq"] & \\
& A' \arrow[rr] & & C'
\end{tikzcd},$$

where the rear face is a (commutative) 1-pullback square.[47] Then this cube can be composed with the original one. The assumptions on the new, big cube are still satisfied: Consider e.g. its top face, which looks (after mirroring) like

$$\begin{tikzcd}
\overline{A} \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & A \arrow[r, "\beta"] \arrow[d, "\alpha"] \arrow[dr, phantom, "H \Nearrow"] & C \arrow[d, "\delta"] \\
\overline{B} \arrow[r, "\simeq"'] & B \arrow[r, "\gamma"'] & D
\end{tikzcd} .$$

The right square is a homotopy pushout by assumption, the left one also (exercise). Thus by the dual of proposition 2.3.6, the composed square $\overline{A}C\overline{B}D$ is a homotopy pushout, which is the condition for the top face of the composed cube.

Similar reasonings are applied to conclude that also the left and bottom faces of the composed cube still satisfy the conditions of the theorem.

Thus we may assume now (after renaming spaces and maps) that $A = A' \times_{B'} C$ and F_1 is trivial. Furthermore, f_B is a fibration because of the construction using theorem 2.4.3, and then f_A is a fibration because it is the pullback of a fibration (the dual of proposition 1.4.10).

– Now f_C is replaced by a fibration as follows: Consider a factorization of the map $f_C : C \to C'$ into a composition $C \xrightarrow[s_C]{\text{(homotopy eq.)}} \overline{C} \xrightarrow[\overline{f_C}]{\text{(fibration)}} C'$ (which can be obtained by the mapping cocylinder construction, proposition 2.4.1). Let p_C be the homotopy inverse to s_C. Consider then the cube

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, "\overline{\beta}:=s_C\circ\beta"] \arrow[dd] & & B \arrow[dr] \arrow[dd] & \\
& \overline{C} \arrow[rr, "\overline{\delta}:=\delta\circ p_C"] \arrow[dd, "\overline{f_C}"] & & D \arrow[dd] \\
A' \arrow[rr] \arrow[dr] & & B' \arrow[dr] & \\
& C' \arrow[rr] & & D'
\end{tikzcd} ,$$

where the maps $\overline{\beta}$, $\overline{\delta}$ and $\overline{f_C}$ and the space $\overline{C}$ have changed compared to the cube obtained from the first reduction step (and none else). In order that the new cube is still 2-commutative, also the homotopies H and F_4 in the top and front face need to be altered to $\overline{H}$ and $\overline{F_4}$.[48] We define $\overline{H} := (\delta \circ h_C^{-1} \circ \beta) \circ H$, where $h_C : p_C \circ s_C \Rightarrow \mathrm{id}_C$ is a homotopy, and $\overline{F_4}$ shall be such that $F_4 \circ (f_D \circ \delta \circ h_C) = \overline{F_4} \circ s_C$ (this exists by lemma 2.3.14).

The new cube is still 2-commutative, since $(\overline{F_4} \circ \overline{\beta}) \circ (f_D \circ \overline{H}) = (F_4 \circ \beta) \circ (f_D \circ H)$. Thus, removing the overlines from the notation again, we may assume that f_C is a fibration (since s_C is a homotopy equivalence, the top face is still a homotopy pushout, the right one is still a homotopy pullback, and the new front face is a homotopy pullback if and only if the old front face is one).

---

[47] $\overline{A}, \overline{B}$ are likely different from A, B, but the bottom face contains two times the same spaces.

[48] Note that F_2 can remain the same; although one space in the left face changed, the maps $A \to C'$ did not.

– To finalize the preparation step, we need to alter the left face. Currently it looks like

$$\begin{tikzcd}
A \arrow[r, "\beta"] \arrow[d, "f_A"'] & C \arrow[d, "f_C"] \arrow[dl, phantom, "F_2"] \\
A' \arrow[r, "\beta'"'] & C'
\end{tikzcd},$$

where f_C is a fibration. Now by the strictification lemma 1.5.7, there exists $\overline{\beta} : A \to C$ with $\beta' \circ f_A = f_C \circ \overline{\beta}$, $\overline{\beta} \sim \beta$. Then upon composing the given cube with the 2-commutative cube[49]

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, "\overline{\beta}"'] \arrow[dd] & & A \arrow[dr, "\beta"] \arrow[dd] & \\
& C \arrow[rr] \arrow[dd] & & C \arrow[dd] \\
A' \arrow[rr] \arrow[dr] & & A' \arrow[dr] & \\
& C' \arrow[rr] & & C'
\end{tikzcd},$$

we can alter the left face of the given cube to contain $\overline{\alpha}$ instead of α. Furthermore, since the maps to the right are all identities, the front, rear and bottom faces remain unchanged, and the top face remains a homotopy pushout. Thus we may assume that F_1 is trivial, which concludes the preparation step.

2. *Construction step:* We now investigate the cube (omitting homotopies for brevity, and the unnamed maps are canonical ones)

$$\begin{tikzcd}
A \arrow[rr, "\alpha"] \arrow[dr, "\beta"] \arrow[dd, "f_A"'] & & B \arrow[dr] \arrow[dd, "f_B"] & \\
& C \arrow[rr] \arrow[dd, "f_C"] & & C \cup^h_A B \arrow[dd, "q"] \\
A' \arrow[rr, "\alpha'"] \arrow[dr, "\gamma'"'] & & B' \arrow[dr] & \\
& C' \arrow[rr] & & C' \cup^h_{A'} B'
\end{tikzcd},$$

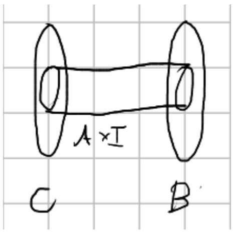
where $C \cup^h_A B$, $C' \cup^h_{A'} B'$ are standard homotopy pushouts. Furthermore we assume that the left and back faces have the properties obtained in the preparation step. The map $q : C \cup^h_A B \to C' \cup^h_{A'} B'$ is induced from the composed maps $B \to C' \cup^h_{A'} B'$ and $C \to C' \cup^h_{A'} B'$. We want to show that the right face

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "f_C"'] \arrow[dr, phantom, "(*)"] & C \cup^h_A B \arrow[d, "q"] \\
C' \arrow[r, "g'"'] & C' \cup^h_{A'} B'
\end{tikzcd}$$

is a homotopy pullback. Now recall from theorem 2.4.3 that $C \cup^h_A B = C \sqcup B \sqcup A \times$

[49] In this cube, all unnamed arrows are either identities or come from the left face of the original cube. Only the top and right face contain nontrivial homotopies; the 2-commutativity follows from the (proof of the) strictification lemma.

$$I \Big/ \left(\begin{array}{c} (a,0) \sim \alpha(a) \\ (\alpha,1) \sim \beta(a) \end{array} \right):$$



Analogously, $C' \cup^h_{A'} B'$ is defined. Then it is observed that the map q is defined as (viewing e.g. $A \times I$ as subspace of $C \cup_A B$)

$$q = \begin{cases} f_A \times \mathrm{id}_I, & \text{on } A \times I, \\ f_B, & \text{on } B, \\ f_C, & \text{on } C. \end{cases}$$

Claim: q is a quasi-fibration.

Proof of claim: Since $f_A \times \mathrm{id}, f_B, f_C$ are fibrations, they are quasi-fibrations. Since quasi-fibrations are local on the base (by proposition 2.8.6), q is a quasi-fibration.[50]

Now by proposition 2.6.6, $(*)$ is ∞-weakly homotopy cartesian (i.e. a homotopy pullback, in case of CW-complexes) if and only if for all $c' \in C'$, the map between the fibers $C_{c'} \to (C \cup^h_A B)_{g'(c')}$ is ∞-connected. But f_C is a fibration, and by the claim, q is a quasi-fibration. Thus the homotopy fibers coincide with the fibers (by theorem 2.3.9 resp. by definition of a quasi-fibration). These fibers are the same: $C_{c'} = f_C^{-1}(\{c'\}) = q^{-1}(g(c'))$, since g is an inclusion and q is defined to be f_C on C (which is the preimage of $g(C')$).

We conclude that in this particular cube, the front (and right) faces are indeed homotopy pullbacks.

3. Finally, we return to the original cube with arbitrary homotopy pushouts D, D' instead of the

[50] We apply proposition 2.8.6 to $B_0 := B' \sqcup A' \times [0, \frac{2}{3}]/((a',0) \sim \alpha'(a'))$, and $B_1 := C' \sqcup A' \times [\frac{1}{3}, 1])/((a',1) \sim \beta'(a'))$. These are homeomorphic to mapping cylinders.

Claim: The restrictions $q|_{q^{-1}(B_0)} : q^{-1}(B_0) \to B_0$ and $q|_{q^{-1}(B_1)} : q^{-1}(B_1) \to B_1$ are quasi-fibrations.

Proof of claim: For B_0, we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{incl}_{2/3}"] \arrow[d, "f_A"'] & q^{-1}(B_0) \arrow[r, "\mathrm{retraction}"] \arrow[d, "q|_{q^{-1}(B_0)}"] & B \arrow[d, "f_B"] \\ A' \arrow[r, "\mathrm{incl}_{2/3}"'] & B_0 \arrow[r, "\mathrm{retraction}"'] & B' \end{tikzcd} .$$

The big rectangle is a homotopy pullback by assumption. The right square is a homotopy pullback since both horizontal maps are homotopy equivalences. So by proposition 2.3.6, the left square is a homotopy pullback. By proposition 2.6.6, the homotopy fibers of all the vertical maps are weakly equivalent. As f_A is a (quasi-)fibration, its homotopy fiber is weakly equivalent to its actual fiber, analogously with f_B.

One concludes for $x' \in B' \subset B_0$ from the right square that its homotopy fiber is weakly equivalent to its "usual" fiber (since then, viewing B as a subspace of $q^{-1}(B_0)$, we have $q^{-1}(x') = f_B^{-1}(x')$, which is weakly equivalent to the corresponding homotopy fiber of f'_B, which is weakly equivalent to the corresponding homotopy fiber of q.) For $(x',t') \in A' \times (0, \frac{2}{3}] \subset B_0$, its "usual" fiber is $q^{-1}((x',t')) = f_A^{-1}(x')$, regardless of t'. One concludes then from the left square that this is weakly equivalent to the corresponding homotopy fiber.

Thus the homotopy fibers over any point $x' \in B_0$ are weakly equivalent to $q^{-1}(x')$. proving that the corresponding restriction of q is a quasi-fibration.

For B_1, the proof is analogous.

standard ones: To this end, consider (omitting most of the names of maps and homotopies)

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd] & & B \arrow[dr] \arrow[dd] \arrow[ddrr, bend left, "\gamma"] & & \\
& C \arrow[rr] \arrow[dd] \arrow[drrr, bend right=10, "\delta"'] & & C \cup^h_A B \arrow[dr, "\simeq"] \arrow[dd, "q"] & \\
A' \arrow[rr] \arrow[dr] & & B' \arrow[dr] \arrow[ddrr, bend left, "\gamma'"] & & D \arrow[dd, "f_D"] \\
& C' \arrow[rr] \arrow[drrr, bend right=10, "\delta'"'] & & C' \cup^h_{A'} B' \arrow[dr, "\simeq"] & \\
& & & & D'
\end{tikzcd} ,$$

where the homotopy equivalences $C \cup^h_A B \to D$, $C' \cup^h_{A'} B'$ have been obtained from uniqueness of weak 2-pushouts (the dual of proposition 2.3.3). This is furthermore 2-commutative, if the face $(C \cup^h_A B)D(C' \cup^h_{A'} B')$ is equipped with the homotopy obtained from "uniqueness" for $C \cup^h_A B$. Now one can insert in the prism $C(C \cup^h_A B)DC'(C' \cup^h_{A'} B')$ the maps id_C, id'_C to "blow it up" to a homotopy commutative cube whose rear face is

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "f_C"'] & C \cup^h_A B \arrow[d, "q"] \arrow[dl, phantom, "\Swarrow"] \\
C' \arrow[r] & C' \cup^h_{A'} B'
\end{tikzcd} ,$$

and whose front face is

$$\begin{tikzcd}
C \arrow[r, "\delta"] \arrow[d, "f_C"'] & D \arrow[d, "f_D"] \arrow[dl, phantom, "\Swarrow F_4"] \\
C' \arrow[r, "\delta'"'] & D'
\end{tikzcd}$$

and the maps between the rear and front face are all homotopy equivalences (or even identities). This therefore describes an equivalence between its front and rear face, and since the rear face has been proven to be a homotopy pullback in step 2., also its front face is one (cf. proposition 2.3.13). This proves that $CDC'D'$ (the front face of the original cube) is a homotopy pullback.

□

Remark 2.9.2. Where do such cubes as in theorem 2.9.1 come from? Example:

1. Start with a homotopy pushout square $A'B'C'D'$.
2. Find f_C, f_B such that A is a homotopy pullback for both squares

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\
A' \arrow[r] & B'
\end{tikzcd} , \qquad
\begin{tikzcd}
A \arrow[r] \arrow[d] & C \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\
A' \arrow[r] & C'
\end{tikzcd} .$$

3. Construct D as a homotopy pushout

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\
& C \arrow[rr] \arrow[urr, phantom, "\Rightarrow"] & & D
\end{tikzcd} .$$

4. Use the universal property of homotopy pushouts to get f_D (check that this yields a 2-commutative cube; exercise).

Theorem 2.9.3 (Mather's second cube theorem)**.** *Let*

$$\begin{tikzcd}
A \arrow[rr, "\alpha"] \arrow[dr, "\beta"] \arrow[dd, "f_A"'] & & B \arrow[dr, "\gamma"] \arrow[dd, "f_B"' near start] \arrow[d, phantom, "\Downarrow H" near end] & \\
& C \arrow[rr, "\delta"' near start, crossing over] \arrow[dd, "f_C"' near start] \arrow[r, phantom, "\Swarrow F_1"] & & D \arrow[dd, "f_D"] \arrow[dl, phantom, "\Swarrow F_3"] \\
A' \arrow[rr, "\alpha'" near end] \arrow[dr, "\beta'"'] \arrow[r, phantom, "\Swarrow F_2"] & & B' \arrow[dr, "\gamma'"'] \arrow[d, phantom, "\Swarrow F_4"] & \\
& C' \arrow[rr, "\delta'"'] & & D' \arrow[ul, phantom, "\Downarrow H'" near start]
\end{tikzcd}$$

be a 2-commutative cube as in definition 2.3.10. Assume that the vertical faces (containing F_1, F_2, F_3, F_4) are homotopy pullbacks and the bottom face is a homotopy pushout. Then the top face is a homotopy pushout.

In the proof, we need:

Lemma 2.9.4. *Let*

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & D \arrow[d] \\
C' \arrow[r, "u"'] \arrow[ur, phantom, "\Swarrow"] & E' \arrow[r, "v"'] \arrow[ur, phantom, "\Swarrow"] & D'
\end{tikzcd}$$

be a 2-commutative diagram in CW*. If $u_* : \pi_0 C' \twoheadrightarrow \pi_0 E'$ is surjective and the left square and the total rectangle are homotopy pullbacks, then the right square is also a homotopy pullback.*[51]

Proof. By proposition 2.6.6 and since we work in CW, it suffices to check that for all $e' \in E'$, the map between the homotopy fibers $E_{e'} \to D_{v(e')}$ is a homotopy equivalence. In fact, in view of the isomorphisms constructed in lemma 2.5.1, it suffices to look at representatives of $\pi_0(E')$. Thus, by assumption on u, we can assume $e' = u(c')$ for $c' \in C'$. We can thus form

$$\begin{tikzcd}
C_{c'} \arrow[rr, "(**)"] \arrow[dr, "(*)"'] & \arrow[d, phantom, "\Nwarrow"] & D_{v(e')} \\
& E_{e'} \arrow[ur] &
\end{tikzcd} .$$

(where the homotopy is obtained using that $E_{e'}$ is a homotopy pullback; the horizontal map "factors up to homotopy" through $E_{e'}$ because $e' = u(c')$). Since the left square is a homotopy pullback, $(*)$ is a homotopy equivalence, and since the total rectangle is a homotopy pullback; $(**)$ is one. Thus by the 3-out-of-2-property for isomorphisms (in hCW), $E_{e'} \to D_{v(e')}$ is a homotopy equivalence; hence the lemma. $\square$

Proof of theorem 2.9.3 (sketch, for CW-complexes and in the case that $\pi_0 C' \twoheadrightarrow \pi_0 D'$ is surjective).

[51] Compare proposition 2.3.6 and in particular the additional required assumptions.

We construct a new 2-commutative cube $ABC(C \cup^h_A B)A'B'C'(C' \cup^h_{A'} B')$ and consider

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd] & & B \arrow[dr] \arrow[dd] \arrow[ddrr, bend left] & & \\
& C \arrow[rr] \arrow[dd] \arrow[drrr, bend right] & & C \cup^h_A B \arrow[dd] \arrow[dr, dashed, red] & \\
A' \arrow[rr] \arrow[dr] & & B' \arrow[dr] \arrow[ddrr, bend left] & & D \arrow[dd] \\
& C' \arrow[rr] \arrow[drrr, bend right] & & C' \cup^h_{A'} B' \arrow[dr, dashed, red] & \\
& & & & D'
\end{tikzcd}.$$

By theorem 2.9.1, in particular the face

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & C \cup^h_A B \arrow[d] \\
C' \arrow[r] & C' \cup^h_{A'} B'
\end{tikzcd}$$

is a homotopy pullback. Then consider[52]

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & C \cup^h_A B \arrow[r] \arrow[d] & D \arrow[d] \\
C' \arrow[r] & C' \cup^h_{A'} B' \arrow[r, "\simeq"'] & D'
\end{tikzcd},$$

where the map $C' \cup^h_{A'} B'$ is a homotopy equivalence, because the lower face of the original cube was already a homotopy pushout. The left square is a homotopy pullback by theorem 2.9.1, see above; the total square is a homotopy pullback by assumption.[53] By the lemma before, hence, the right square

$$\begin{tikzcd}
C \cup^h_A B \arrow[r] \arrow[d] & D \arrow[d] \\
C' \cup^h_{A'} B' \arrow[r, "\simeq"'] & D'
\end{tikzcd}$$

is a homotopy pullback.

Then $C \cup^h_A B \to D$ is a homotopy equivalence (exercise, already seen before). Thus $ABCD$ is a homotopy pushout, proving the theorem. $\square$

Remark 2.9.5. In the situation of theorem 2.9.3, we can prove the case that $\pi_0(C') \to \pi_0(D')$ is not surjective, as follows: In this case, we can write $B' = \widetilde{B}' \sqcup \overline{B}'$ such that $A' \to B'$ factors through $\widetilde{B}'$ (i.e. $A \to B' = A \to \widetilde{B}' \hookrightarrow B'$) and up to homotopy equivalence, $D' = (\widetilde{B}' \cup^h_{A'} C') \sqcup \overline{B}'$. Then we can "decompose" the cube into several ones with connected spaces to prove the theorem.

[52] Homotopies are omitted.

[53] Note that the change to homotopic maps does not change the homotopy pullback property in the case considered here; one can can construct a 2-commutative cube yielding an equivalence from the original front face to the total square above.

2.10 James' reduced product and Freudenthal's suspension theorem

Definition 2.10.1. Let $(X, e) \in \operatorname{Ob}\mathrm{CW}_*$. The *James' reduced product* (or James' construction) of (X, e) is a CW-complex

$$J(X) := \operatorname{colim}_{k\geq 0} J^k(X),$$

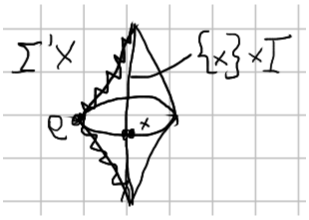
where for $k \geq 0$,

$$J^k(X) := \{(x_1, \dots, x_k) \in X^{\times k}\}\Big/(x_1, x_2, \dots, x_i, e, x_{i+1}, \dots, x_k) \sim (x_1, x_2, \dots, x_i, x_{i+1}, e, \dots, x_k).$$

Remark 2.10.2.
- We have $J^k(X) \subset J^{k+1}(X)$ for $k \geq 0$, if we embed it by $[x_1, \dots, x_k] \mapsto [x_1, \dots, x_k, e]$.
- We denote $[x_1, \dots, x_k] =: x_1 \cdot \cdots \cdot x_k$. These are "words in X, modulo e" (since arbitrary strings of e may by the above embeddings just be ignored).
- Thus $J(X)$ is the free associative monoid on X, viewed as a set; e is the neutral element.

Theorem 2.10.3 (James)**.** *Let (X, e) be a connected pointed compact CW-complex. Then there exists a homotopy equivalence $\lambda : J(X) \xrightarrow{\simeq} \Omega\Sigma' X$. Here $\lambda(x)$ is defined to be the image of $\{x\} \times I$ in $\Sigma' X$; this is a loop based at e.*

Remark 2.10.4.
- The theorem also holds in the noncompact case.
- In fact, λ is a map of H-spaces, i.e. monoid objects in the homotopy category. Picture:



Lemma 2.10.5. *For $n \in \mathbb{N}$, let $T : X \times J^n(X) \to J^{n+1}(X)$, $(x, x_1 \cdot \cdots \cdot x_n) \mapsto x \cdot x_1 \cdot x_n$. Then*

$$\begin{tikzcd} X \times J^n(X) \arrow[r, "T"] \arrow[d, "\mathrm{pr}_2"'] & J^{n+1}(X) \arrow[d] \\ J^n(X) \arrow[r] & * \end{tikzcd}$$

is a homotopy pushout in Top_*.[54]

Proof. By induction on n. For $n = 0$, we have

$$\begin{tikzcd} X \times \{e\} \arrow[r, "\mathrm{id}"] \arrow[d] & J^1(X) = X \arrow[d] \\ * \arrow[r] & * \end{tikzcd},$$

which is a homotopy pushout.

[54] Note that for any space Q, this means that there are only very few maps $J^n(X) \to Q$ and $J^{n+1}(X) \to Q$ that are compatible with T up to homotopy; they are constant (up to homotopy). In this sense, the statement is unexpected and strong.

Induction step $n \mapsto n+1$: Consider the 1-commutative diagram

$$\begin{tikzcd}
X \times J^n(X) \arrow[r, hook] \arrow[dd, "\mathrm{pr}_2"'] & X \triangleright J^{n+1}(X) \arrow[r, "\overline{T}"] \arrow[d, hook] \arrow[dr, phantom, "(1)"] & J^{n+1}(X) \arrow[d, hook] \\
\text{(2)} & X \times J^{n+1}(X) \arrow[r, "T"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "(3)"] & J^{n+2}(X) \arrow[d] \\
J^n(X) \arrow[r, hook] & J^{n+1}(X) \arrow[r] & *
\end{tikzcd},$$

where $X \triangleright J^{n+1}(X) := X \times J^n(X) \cup_{\{e\} \times J^n(X)} \{e\} \times J^{n+1}(X) \subset X \times J^{n+1}(X)$ is a 1-pushout and $\overline{T}$ is defined by

$$\overline{T} := \begin{cases} T, & \text{on } X \times J^n(X), \\ \mathrm{id}, & \text{on } \{e\} \times J^{n+1}(X). \end{cases}$$

(1) is a pushout square: Since X is compact, all spaces considered here are compact Hausdorff, so it suffices that it is a pushout for sets, and this can be done directly. Since $X \triangleright J^{n+1}(X) \hookrightarrow X \times J^{n+1}(X)$ is a cofibration, it is also a homotopy pushout square (by theorem 2.3.9).

(2) is a pushout square by the 3-out-of-2-property for 1-pushouts and since in

$$\begin{tikzcd}
\{e\} \times J^n(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \{e\} \times J^{n+1}(X) \arrow[d] \\
X \times J^n(X) \arrow[r] \arrow[d] & X \triangleright J^{n+1}(X) \arrow[d] \\
J^n(X) \arrow[r] & J^{n+1}(X)
\end{tikzcd}$$

the upper square and the total rectangle are 1-pushouts. Since $X \times J^n(X) \hookrightarrow X \triangleright J^{n+1}(X)$ is a cofibration, (2) is also a homotopy pushout.

By induction assumption, the total square in the diagram above (i.e. $(1+2+3)$) is a homotopy pushout. Since (2) is a homotopy pushout, by the 3-out-of-2-property (the dual of proposition 2.3.6), $(1+3)$ (i.e. $(X \triangleright J^{n+1}(X))J^{n+1}(X)J^{n+1}(X)*$) is a homotopy pushout. Since (1) is a homotopy pushout, then it follows that (3) is a homotopy pushout, and this was to be shown to conclude the induction step. □

Corollary 2.10.6. *Consider the maps $T : X \times J^n(X) \to J^{n+1}(X)$ as in the previous lemma; these glue together to a map $T : X \times J(X) \to J(X)$. Then*

$$\begin{tikzcd}
X \times J(X) \arrow[r, "T"] \arrow[d, "\mathrm{pr}_2"'] & J(X) \arrow[d] \\
J(X) \arrow[r] & *
\end{tikzcd}$$

is a homotopy pushout.

Proof. For $n \in \mathbb{N}$, let W^n be the standard homotopy pushout of

$$\begin{tikzcd}
X \times J^n(X) \arrow[r, "T"] \arrow[d, "\mathrm{incl.} \times \mathrm{id}"'] \arrow[dr, phantom, "\ulcorner", very near end] & J^{n+1}(X) \arrow[d] \\
C(X) \times J^n(X) \arrow[r] & W^n
\end{tikzcd},$$

where $C(X)$ is the cone of X. Since this is contractible, $C(X) \times J^n(X) \simeq J^n(X)$, but the map incl. $\times$ id is a cofibration. Therefore (by theorem 2.3.9), W^n may be chosen as the 1-pushout.

Let $W := \operatorname{colim}_n W^n$. Then

$$\begin{tikzcd}
X \times J(X) \arrow[r, "T"] \arrow[d, "\mathrm{incl.} \times \mathrm{id}"'] & J(X) \arrow[d] \\
C(X) \times J(X) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & W
\end{tikzcd}$$

is a 1-pushout square which is also a 2-pushout square by the same argument as before. Since $C(X)$ is contractible, W is the homotopy pushout of the diagram we want to consider.

For all $n \in \mathbb{N}$, $W^n \simeq *$ (by the previous lemma), in particular all its homotopy groups are trivial. Since it is known that for $k \in \mathbb{N}$, $\pi_k(W) = \operatorname{colim}_n \pi_k(W^n)$ (see Algebraic Topology I), W has trivial homotopy groups. By Whitehead's theorem 1.7.4, $W \simeq *$. $\square$

Proof of James' theorem 2.10.3. Consider the cube

$$\begin{tikzcd}
X \times J(X) \arrow[rr, "\mathrm{pr}_2"] \arrow[dr, "T"] \arrow[dd, "\mathrm{pr}_1"'] & & J(X) \arrow[dr] \arrow[dd] & \\
& J(X) \arrow[rr] \arrow[dd] & & * \arrow[dd] \\
X \arrow[rr] \arrow[dr] & & * \arrow[dr] & \\
& * \arrow[rr] & & \Sigma' X
\end{tikzcd}\quad .$$

Here the top and bottom face are homotopy pushouts (the top one by the previous corollary, the bottom one by definition), and the left and rear face are homotopy pullbacks. By Mather's first cube theorem (theorem 2.9.1), the front (or right) face

$$\begin{tikzcd}
J(X) \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & \Sigma'(X)
\end{tikzcd}$$

is a homotopy pullback. This implies $J(X) \simeq \Omega\Sigma' X$. $\square$

Corollary 2.10.7 (Freudenthal's suspension theorem (cf. corollary 1.9.5, but now independent of the excision and hence the Blakers-Massey theorem))**.** *If X is an n-connected CW-complex, then $X \to \Omega\Sigma' X$ is $(2n+1)$-connected.*[55]

Proof. Because of James' theorem (theorem 2.10.3), it suffices to prove that $X \to J(X)$ is $(2n+1)$-connected. We can assume that $X^{(n)} = \{*\}$ because of corollary 1.8.3. For $k \in \mathbb{N}$, consider the map $X \to J^k(X) = X^{\times k}/\sim$. The cells of $J^k(X)$ can be enumerated explicitly:

$$\left\{ \begin{array}{ll}
0 - \text{cell}: & \{e\} \\
(n+1) - \text{cells:} & \left. \begin{array}{l} X^{(n+1)} \times \{e\} \times \{e\} \times \cdots \times \{e\} \\ \{e\} \times X^{(n+1)} \times \{e\} \times \cdots \times \{e\} \\ \ldots \\ \{e\} \times \{e\} \times \cdots \times \{e\} \times X^{(n+1)} \end{array} \right\} \begin{array}{l} \text{these glue together} \\ \text{to } X^{(n+1)} \end{array} \\
(\text{higher dimensions}) & \ldots
\end{array} \right.$$

[55]The formulation from corollary 1.9.5 is obtained using the Σ'-Ω-adjunction, proposition 2.4.11.

The cells with the lowest possible dimensionality which differ from $X^{(n+1)}, X^{(n+2)}, \dots$ (i.e. from X) are of the form $X^{(n+1)} \times X^{(n+1)} \times \{e\} \times \cdots \times \{e\}$ (or analogously with other factors). Thus $J(X)$ is obtained from X by attaching cells with dimension greater than or equal to $2(n+1)$.

Now it only (up to some technical details) remains to note that if a space B is obtained from a space A by attaching an $(2n+2)$-cell, i.e. by a pushout diagram

$$\begin{tikzcd} S^{2n+1} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & A \arrow[d] \\ D^{2n+2} \arrow[r] & B \end{tikzcd},$$

then, since the inclusion $S^{2n+1} \hookrightarrow D^{2n+2}$ is $(2n+1)$-connected, also the map $B \to A$ is $(2n+1)$-connected. □

Remark 2.10.8.
- The absolute Hurewicz theorem follows from the Freudenthal suspension theorem (exercise).
- Both the absolute and the relative Hurewicz theorem can also be deduced from Serre spectral sequences.

Thus also the Hurewicz theorem has found a proof which does not depend on the Blakers-Massey theorem. We will use the former in the next paragraph to prove the latter.

2.11 Proof of Blakers-Massey theorem

We give the proof in CW and in the case $n, m \geq 2$, or $m = 0$, n arbitrary resp. $n = 0$, m arbitrary.

Theorem 2.11.1 (Blakers-Massey, theorem 2.7.1)**.** *Let*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow"] & B \arrow[d, "f'"] \\ C \arrow[r, "g'"'] & D \end{tikzcd}$$

be a homotopy pushout in Top *or* Top_**. Assume that f is m-connected, g is n-connected, $n, m \geq 0$. Then this square is a $(n+m-1)$-weakly homotopy pullback.*

Proof (in $\mathrm{CW}_{(*)}$ and in case $n, m \geq 2$, or $m = 0$, n arbitrary resp. $n = 0$, m arbitrary). Note: Since we work in CW, we can assume that $f : A \hookrightarrow X$ is a relative CW-complex with only cells of dimension greater than or equal to $(n+1)$ are attached. (Cf. theorem 1.8.1.) Then the square may be assumed to be an actual pushout (since f is then a cofibration) and thus D is obtained from B by attaching cells of dimension greater than or equal to $(n+1)$. By "easy excision", corollary 1.5.9 (resp. one of its corollaries), f' is n-connected.

Now:

- *Case $m = 0$:*[56] Let $P := C \times^h_D B$ be the (standard, or any other) homotopy pullback. Consider the diagram

$$\begin{tikzcd} A \arrow[dr, dashed, "w"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\ & P \arrow[r] \arrow[d, "\tilde{f}'"'] & B \arrow[d, "f'"] \\ & C \arrow[r, "g'"'] & D \end{tikzcd},$$

[56] $n = 0$ will not be proven; it is symmetric to this one.

(where w is obtained from the universal property of P, and homotopies have been omitted). Since f' is n-connected by the above observation, $\tilde{f}'$ is n-connected. (The homotopy fibers of $\tilde{f}'$ and f' are homotopy equivalent by proposition 2.6.6. Then by the new definition in section 6 of this part, $\tilde{f}'$ is n-connected.)

Now we consider

$$\begin{tikzcd} A \arrow[rr, "w"] \arrow[dr, "f"'] & & P \arrow[dl, "\tilde{f}'"] \\ & C & \end{tikzcd}$$

(which commutes up to homotopy). Applying the homotopy invariant functors π_i, $i \in \mathbb{N}$, we obtain

$$\begin{tikzcd} \pi_i(A) \arrow[rr, "w_*"] \arrow[dr, "f_*"'] & & \pi_i(P) \arrow[dl, "\tilde{f}'_*"] \\ & \pi_i(C) & \end{tikzcd} .$$

For $i \leq n-1$, both f_* and $\tilde{f}'_*$ are isomorphisms by proposition 1.5.2. Thus then $\pi_i(A) \cong \pi_i(P)$, i.e. w is $(n-1)$-connected, as claimed (since $m = 0$).[57]

- *Case* $m, n \geq 2$:
  - *Reduction step:* Choose $d \in D$, denote the constant map to d again by d and construct a 2-commutative cube consisting of the black objects and morphisms in

$$\begin{tikzcd} \tilde{A} \arrow[drrr, bend left] \arrow[dd] \arrow[ddrr, bend right] \arrow[dr, dashed, red] & & & & \\ & \color{red}{\tilde{P}} \arrow[rr, red] \arrow[dr, red] \arrow[dd, red, "w"] & & \tilde{B} \arrow[dr] \arrow[dd] & \\ A \arrow[drrr] \arrow[dr, dashed, red] \arrow[ddrr, bend right] & & \tilde{C} \arrow[rr] \arrow[dd] & & * \arrow[dd, "d"] \\ & \color{red}{P} \arrow[rr, red] \arrow[dr, red] & & B \arrow[dr] & \\ & & C \arrow[rr] & & D \end{tikzcd}$$

  such that all vertical faces are homotopy pullbacks. (Exercise.) By Mather's second cube theorem 2.9.3, the top face of this (black) cube is a homotopy pushout.

  Then, by their respective universal properties, we can fit in the homotopy pullbacks P of BCD and $\tilde{P}$ of $\tilde{B}\tilde{C}*$, indicated in red, such that the resulting diagram is 2-commutative.

  Claim:

$$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d] & \tilde{P} \arrow[d, "w"] \\ A \arrow[r] & P \end{tikzcd}$$

  is a homotopy pullback.

[57] Note that only a surjection for $i = n-1$ would be required. However we cannot obtain a surjection for $i = n$, thus this is the best possible result.

Proof of claim: Consider the top and right face of the "inner cube" ($\tilde{P}\tilde{B}\tilde{C} * PBCD$):

$$\begin{tikzcd} \tilde{P} \arrow[r] \arrow[d] & \tilde{B} \arrow[r] \arrow[d] & B \arrow[d] \\ \tilde{C} \arrow[r] & * \arrow[r, "d"'] & D \end{tikzcd}$$

Both squares are 2-pullbacks by construction, hence the total rectangle is a 2-pullback by proposition 2.3.6. It remains so if the two horizontal maps are replaced by the homotopic maps coming from the left and bottom instead of the top and right face.[58] Thus we may replace the map $\tilde{P} \to \tilde{B} \to B$, coming from the rear face, with the map $\tilde{P} \xrightarrow{w} P \to B$. Analogously we may replace the map $\tilde{C} \to * \xrightarrow{d} D$ with the map $\tilde{C} \to C \to D$. Therefore, in

$$\begin{tikzcd} \tilde{P} \arrow[r, "w"] \arrow[d] & P \arrow[r] \arrow[d] & B \arrow[d] \\ \tilde{C} \arrow[r] & C \arrow[r] & D \end{tikzcd}$$

the total rectangle is a homotopy pullback. Since the right square is a homotopy pullback by construction (this is the bottom face of the "inner cube"), so is the left, again by proposition 2.3.6.

Now consider

$$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d] & \tilde{P} \arrow[r] \arrow[d, "w"] & \tilde{C} \arrow[d] \\ A \arrow[r] & P \arrow[r] & C \end{tikzcd}$$

The total rectangle $\tilde{A}\tilde{C}AC$ is a homotopy pullback, since it is equivalent to the left face of the black cube (the top and bottom maps are replaced with homotopic ones similar to before).[59] Furthermore, the right square is also a homotopy pullback (see above). Thus by a final application of proposition 2.3.6, $\tilde{A}\tilde{P}AP$ is a homotopy pullback, proving the claim.

By proposition 2.6.6, for all $x \in \tilde{P}$, the homotopy fibers $\tilde{A}_x$ and $A_{w(x)}$ are homotopy equivalent. Thus, again by this proposition, it suffices to prove that the map $\tilde{A} \to \tilde{P}$ is $(n+m-1)$-connected, i.e. we prove the Blakers-Massey theorem for

$$\begin{tikzcd} \tilde{A} \arrow[r, "m-\text{conn.}"] \arrow[d, "n-\text{conn.}"'] & \tilde{B} \arrow[d, "n-\text{conn.}"] \\ \tilde{C} \arrow[r, "m-\text{conn.}"'] & * \end{tikzcd}$$

instead of a diagram with arbitrary D. We drop the tildes.[60]

– *Main step:* The map $B \to *$ is a fibration.[61] Hence by theorem 2.3.9, the 1-pullback is a 2-pullback here as well. However, the 1-pullback of $BC*$ is just $B \times C$.

[58] Exercise. This amounts to constructing a suitable 2-commutative cube which yields an equivalence to the original 2-pullback square. The cube can be obtained from $\tilde{P}\tilde{B}\tilde{C} * PBCD$ by inserting $\mathrm{id}_{\tilde{P}}, \mathrm{id}_B, \mathrm{id}_{\tilde{C}}, \mathrm{id}_D$.
[59] Note that the homotopy in $\tilde{A}\tilde{P}AP$ comes from "uniqueness" for $\tilde{P}$.
[60] We have seen the proof of the connectivities of the maps $\tilde{B} \to *$ and $\tilde{C} \to *$ in the case $m = 0$.
[61] Every homotopy may just be static.

Construct three 1-pushouts in the diagram

$$\begin{tikzcd}
A \arrow[dr, dashed, red] \arrow[drr, bend left] \arrow[ddr, bend right] & & & \\
& {\color{red}B \times C} \arrow[r, red] \arrow[d, red] & B \arrow[r] \arrow[d] & * \arrow[d] \\
& C \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] \arrow[ur, phantom, "\ulcorner"] & \Sigma B \arrow[d] \\
& * \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \Sigma C \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \Sigma C \vee \Sigma B
\end{tikzcd} .$$

(It is straightforward to verify what must be the spaces occurring there.) Since $ABC*$ is a homotopy pushout as well (by assumption), we get by successive usage of the 3-out-of-2-property (the dual of proposition 2.3.6) that $A * *(\Sigma C \vee \Sigma B)$ is a 2-pushout diagram resp. $\Sigma A \simeq \Sigma B \vee \Sigma C$.

We are interested in the connectivity of the map $A \to B \times C$.

Exercise: $\Sigma(B \times C) \simeq \Sigma B \vee \Sigma C \vee \Sigma(B \wedge C)$.[62] Furthermore, the map $\Sigma B \vee \Sigma C \simeq \Sigma A \to \Sigma(B \times C) \simeq \Sigma B \vee \Sigma C \vee \Sigma(B \wedge C)$ is just the inclusion.

Since $B \to *$, $C \to *$ are n resp. m-connected, B resp. C are $(n-1)$ resp. $(m-1)$-connected (cf. proposition 1.5.2).

Exercise: Then $B \wedge C$ is $(n+m-1)$-connected.[63]

It follows that $\Sigma(B \wedge C)$ is $(n+m)$-connected. Thus by CW-approximation (resp. corollary 1.8.3), we may assume that its $(n+m)$-skeleton is a point. Because of cellular homology, this means that the map $\Sigma A \to \Sigma(B \times C)$ induces isomorphisms on (reduced) homology up to degree $n+m$ (since the map $\Sigma B \vee \Sigma C \to \Sigma B \vee \Sigma C \vee \Sigma(B \wedge C)$ then clearly does).

I.e. $\widetilde{H}_{i+1}(\Sigma(B \times C), \Sigma A) = 0$ for $i+1 \leq n+m$. By the suspension isomorphism (see Algebraic Topology I), $\widetilde{H}_i(B \times C, A) = 0$ for $i \leq n+m-1$. Using the relative Hurewicz theorem 1.9.8 we conclude that $(B \times C, A)$ is $(n+m-1)$-connected. This proves the Blakers-Massey theorem.[64]

□

2.12 Seifert-van Kampen theorem revisited

Theorem 2.12.1 (Seifert-van Kampen)**.** *Let*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\
C \arrow[r] \arrow[ur, phantom, "\Leftarrow"] & D
\end{tikzcd}$$

[62]Hint: Prove at first $\Sigma'(B \times C) \simeq \Sigma' B \vee \Sigma' C \vee \Sigma'(B \wedge C)$. Then use $\Sigma X \simeq \Sigma' X$ for all CW-complexes X, because of the 1-pushout

$$\begin{tikzcd}
I \arrow[r, "\text{(cofibration)}"] \arrow[d, "\text{h. eq.}"'] & \Sigma X \arrow[d] \\
* \arrow[r] & \Sigma' X
\end{tikzcd}$$

and exercise 1.5.10.

[63]Hint: E.g. look at cells, using CW-approximation.

[64]Strictly speaking, we would have to care somewhat about the choice of d in the reduction step. When taking the homotopy fibers of $\tilde{A} \to \tilde{P}$, they depend on the connected component of d. Thus we would need to repeat this taking representatives for every element of $\pi_0(D)$.

be a homotopy pushout in Top*. Then*

$$\begin{tikzcd} \Pi_1 A \arrow[r, "\Pi_1 f"] \arrow[d, "\Pi_1 g"'] & \Pi_1 B \arrow[d] \arrow[dl, phantom, "\swArrow"] \\ \Pi_1 C \arrow[r] & \Pi_1 D \end{tikzcd}$$

is a weak (= strict) 2*-pushout in* Gpd*.*

Remark 2.12.2. – In the exercises, we will show that the group version of the Seifert-van Kampen theorem is a corollary of this statement. But, in fact, this is a more general result. (Note that the intersection does not need to be connected.)

– Example:

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & * \arrow[d] \arrow[dl, phantom, "\swArrow"] \\ * \arrow[r] & S^1 \end{tikzcd}$$

is a homotopy pushout in Top. One can verify by hand (checking the universal property) that

$$\begin{tikzcd} \bullet\ \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \arrow[dl, phantom, "\swArrow"] \\ \bullet \arrow[r] & \left(\bullet \circlearrowright \mathbb{Z}\right) \end{tikzcd}$$

is a 2-pushout in Gpd.

The reason for the Seifert-van Kampen theorem is that Π_1 has a right adjoint 2-functor B : Gpd $\to$ Top on the levels of 2-categories.

Definition 2.12.3 (informal)**.** A 2*-functor* $f : \mathcal{C} \to \mathcal{D}$ between $(2,1)$-categories $\mathcal{C}, \mathcal{D}$ is a collection of functors

$$(F_{X,Y} : \mathrm{HOM}_{\mathcal{C}}(X,Y) \to \mathrm{HOM}_{\mathcal{D}}(F(X), F(Y)))_{X,Y \in \mathrm{Ob}\,\mathcal{C}}$$

that respects the composition.[65]

Proposition 2.12.4. *There exists a (strict)* 2*-functor* $B : \mathrm{Gpd} \to \mathrm{Top}$ *such that*

1. *If G is a group and* $\bullet \circlearrowright G$ *denotes the groupoid with one object which has the automorphism group G, then* $B(\bullet \circlearrowright G)$ *has the homotopy type of the Eilenberg-Mac Lane space* $K(G,1)$ *(i.e. its fundamental group is isomorphic to G, and all higher homotopy groups are zero).*

2. *If $\mathcal{G}_i$ for i in some index set are groupoids, then* $B\left(\bigsqcup_i \mathcal{G}_i\right) = \bigsqcup_i B(\mathcal{G}_i)$.

3. *(Adjointness): For* $X \in \mathrm{Ob}\,\mathrm{Top}$, $\mathcal{G} \in \mathrm{Ob}\,\mathrm{Gpd}$,

$$\Pi_1 : \mathrm{HOM}_{\mathrm{Top}}(X, B\mathcal{G}) \xrightarrow{\sim} \mathrm{HOM}_{\mathrm{Gpd}}(\Pi_1 X, \Pi_1(B\mathcal{G})) \qquad \text{is an equivalence of groupoids.}$$

[66]

We will not prove properties 1. and 2., and the proof of property 3. is postponed.

Remark 2.12.5.

[65] There is a weak and a strict version: The corresponding diagrams may be required to commute up to natural equivalence or actually commute.

[66] A (1-)functor $F : \mathcal{C} \to \mathcal{D}$ is called an *equivalence*, if it is fully faithful and essentially surjective. Here *fully faithful* means that for all $X, Y \in \mathrm{Ob}\,\mathcal{C}$, the map $F : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is a bijection, and *essentially surjective* means that it is surjective on equivalence classes of objects (i.e. $\pi_0 F : \pi_0\,\mathcal{C} \twoheadrightarrow \pi_0\,\mathcal{D}$ is surjective).

The properties 1. and 2. of B as in proposition 2.12.4 imply that every $\mathcal{G} \in \operatorname{Ob}\mathrm{Gpd}$ is (non-canonically) equivalent to $\Pi_1 B\mathcal{G}$, and so $\mathrm{HOM}_{\mathrm{Gpd}}(\Pi_1 X, \Pi_1 B\mathcal{G}) \cong \mathrm{HOM}_{\mathrm{Gpd}}(\Pi_1 X, \mathcal{G})$ for all $X \in \operatorname{Ob}\mathrm{Top}$.

Sketch of proof of theorem 2.12.1. We need to consider the situation

$$
\begin{tikzcd}
\Pi_1 A \arrow[r, "\Pi_1 f"] \arrow[d, "\Pi_1 g"'] & \Pi_1 B \arrow[d] \arrow[ddr, bend left, "\beta"] & \\
\Pi_1 C \arrow[r] \arrow[drr, bend right, "\gamma"'] & \Pi_1 D \arrow[dr, phantom, "\Leftarrow \theta"] & \\
 & & T
\end{tikzcd} ,
$$

where T is a groupoid. Without loss of generality, we may assume that $T = \bullet \circlearrowright G$ is a connected groupoid with only one object. We denote such groupoids by $\mathbf{B}G$. By property 1. from proposition 2.12.4, there exists an isomorphism of groupoids $\phi : \mathbf{B}G \xrightarrow{\sim} \Pi_1 B(\mathbf{B}G)$. We abbreviate $BG := B(\mathbf{B}G)$. Now consider

$$
\begin{tikzcd}
\mathrm{HOM}_{\mathrm{Top}}(C, BG) \arrow[rr, "\Pi_1", "\sim"'] \arrow[dr] & & \mathrm{HOM}(\Pi_1 C, \Pi_1 BG) \arrow[dl, "\sim", "\circ\phi"'] \\
 & \mathrm{HOM}_{\mathrm{Top}}(\Pi_1 C, \mathbf{B}G) &
\end{tikzcd} ,
$$

where the unnamed arrow is defined as the composition. The top arrow is an equivalence because of property 3. from proposition 2.12.4 and the right one is an equivalence because of property 1. (see above for the construction of ϕ), thus the unnamed one is one as well. In particular it is essentially surjective. This means that we can "lift" γ to $c : C \to BG$ such that $\gamma \cong \phi \circ \Pi_1 c$ (are isomorphic). Similarly, one can lift β to $b : B \to BG$.

Using fullness of the equivalence above, one can "lift" furthermore θ to $t : b \circ f \Rightarrow c \circ g$. We then have

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \arrow[ddr, bend left, "b"] & \\
C \arrow[r] \arrow[drr, bend right, "c"'] & D \arrow[dr, dashed, "w"] \arrow[dr, phantom, "\Leftarrow t"'] & \\
 & & T
\end{tikzcd} .
$$

The map $w : D \to \mathbf{B}G$ is obtained using the universal property of D. Then one obtains a map $\Pi_1 D \xrightarrow{\Pi_1 w} \Pi_1 BG \xrightarrow{\phi} \mathbf{B}G$, together with the needed 2-morphisms. This shows "existence" for the diagram in Gpd. For the "uniqueness" part, one should use that $\mathrm{HOM}_{\mathrm{Top}}(C, \mathbf{B}G) \to \mathrm{HOM}_{\mathrm{Gpd}}(\Pi_1 C, \Pi_1 BG)$ is faithful. $\square$

Remark 2.12.6. "1-pushouts may always be reduced to construction with sets": In a category $\mathcal{C}$, the diagram

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}
$$

is a (1-)pushout if and only if for all $T \in \operatorname{Ob}\mathcal{C}$, we have an isomorphism $\mathrm{Hom}_{\mathcal{C}}(D, T) \cong \mathrm{Hom}_{\mathcal{C}}(C, T) \times_{\mathrm{Hom}_{\mathcal{C}}(A,T)} \mathrm{Hom}_{\mathcal{C}}(B, T)$, where the right-hand side is a pullback in Sets.

Exercise: In a $(2,1)$-category $\mathcal{C}$,

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\ C \arrow[r] & D \end{tikzcd}$$

is a strict 2-pushout, if and only if for all $T \in \mathrm{Ob}\,\mathcal{C}$,

$$\begin{tikzcd} \mathrm{HOM}_{\mathcal{C}}(D,T) \arrow[r] \arrow[d] & \mathrm{HOM}_{\mathcal{C}}(B,T) \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\ \mathrm{HOM}_{\mathcal{C}}(C,T) \arrow[r] & \mathrm{HOM}_{\mathcal{C}}(A,T) \end{tikzcd}$$

is a (weak=strict) 2-pullback in Gpd.

Definition 2.12.7. Let

$$\begin{tikzcd} & \mathcal{G}_2 \arrow[d, "F_2"] \\ \mathcal{G}_1 \arrow[r, "F_1"'] & \mathcal{G}_3 \end{tikzcd}$$

be a diagram in Gpd. Denote $\mathcal{G}_1 \times^h_{\mathcal{G}_3} \mathcal{G}_2$ the category[67] with

- Objects: Triples $(c_1, c_2, \eta : F_2(c_2) \xrightarrow{\sim} F_1(c_1))$ with $c_1 \in \mathrm{Ob}\,\mathcal{G}_1$, $c_2 \in \mathrm{Ob}\,\mathcal{G}_2$.
- Morphisms: For $(c_1, c_2, \eta), (c_1', c_2', \eta') \in \mathrm{Ob}(\mathcal{G}_1 \times^h_{\mathcal{G}_3} \mathcal{G}_2)$, morphisms between them are pairs $(\alpha_1 : c_1 \to c_1', \alpha_2 : c_2 \to c_2')$ such that

$$\begin{tikzcd} F_2(c_2) \arrow[r, "\eta", "\sim"'] \arrow[d, "F_2(\alpha_2)"'] & F_1(c_1) \arrow[d, "F_1(\alpha_1)"] \\ F_2(c_2') \arrow[r, "\sim", "\eta'"'] & F_1(c_1') \end{tikzcd}$$

commutes.

Proposition 2.12.8. *In the situation of the previous definition, the diagram*

$$\begin{tikzcd} \mathcal{G}_1 \times^h_{\mathcal{G}_3} \mathcal{G}_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathcal{G}_2 \arrow[d, "F_2"] \arrow[dl, phantom, "\Swarrow H"] \\ \mathcal{G}_1 \arrow[r, "F_1"'] & D \end{tikzcd}$$

is a strict 2-pullback, where $p_i(c_1, c_2, \eta) := c_i$ for $i = 1, 2$ and $H : F_2 \circ p_2 \Rightarrow F_1 \circ p_1$ is given by

$$H_{(c_1,c_2,\eta)} := \eta : F_2 \circ p_2(c_1, c_2, \eta) = F_2(c_2) \to F_1(c_1) = F_1 \circ p_1(c_1, c_2, \eta)$$

for $(c_1, c_2, \eta) \in \mathrm{Ob}(\mathcal{G}_1 \times^h_{\mathcal{G}_3} \mathcal{G}_2)$.

Remark 2.12.9. Similarly, one can describe all 2-limits in Gpd (and in Cat).[68]

[67] Remember: Gpd is a subcategory of Cat, i.e. its 1-morphisms are functors and its 2-morphisms are natural transformations.

[68] Remark: If one ever wants to consider some limit of categories, it should be a 2-limit, because categories almost always arise only up to equivalence, not to isomorphism. Thus a 1-limit would depend on choices.

Proof. Let $T \in \operatorname{Ob}\mathrm{Gpd}$ and let the black arrows in

$$\begin{tikzcd}
T \arrow[drr, bend left, "\Upsilon_1"] \arrow[dr, dashed, "\phi"] \arrow[ddr, bend right, "\Upsilon_2"'] & & \\
& \mathcal{G}_1 \times^h_{\mathcal{G}_3} \mathcal{G}_2 \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathcal{G}_2 \arrow[d, "F_2"] \\
& \mathcal{G}_1 \arrow[r, "F_1"'] \arrow[ur, phantom, "\swarrow h"] & C
\end{tikzcd}$$

form a diagram in Gpd. Then there exists $\phi : T \to \mathcal{G}_1 \times^h_{\mathcal{G}_3} \mathcal{G}_2$ defined by

$$\phi(t) = (\Upsilon_1(t), \Upsilon_2(t),\ F_2(\Upsilon_2(t)) \xrightarrow{h_t} F_1(\Upsilon_1(t))\ )$$

for $t \in \operatorname{Ob} T$.[69] One verifies that the diagram, including the red arrows, is 2-commutative if the two triangles have no homotopy at all. This shows "existence".

"Uniqueness": Given (ψ, ξ_1, ξ_2), $\xi_i : p_i \circ \psi \Rightarrow \Upsilon_i$ for $i = 1, 2$, there should exist a unique $\beta : \psi \to \phi$ making the diagram below 2-commutative:

$$\begin{tikzcd}
T \arrow[drr, bend left, "\Upsilon_2"] \arrow[dr, bend left=15, "\psi"] \arrow[dr, bend right=15, "\phi"'] \arrow[ddr, bend right, "\Upsilon_1"'] & & \\
& \mathcal{G}_1 \times^h_{\mathcal{G}_3} \mathcal{G}_2 \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathcal{G}_2 \arrow[d, "F_2"] \\
& \mathcal{G}_1 \arrow[r, "F_1"'] & \mathcal{G}_3
\end{tikzcd}$$

.

For $t \in \operatorname{Ob} T$, denote $\psi(t) = (\psi_1(t), \psi_2(t),\ F_2(\psi_2(t)) \xrightarrow{\eta_t} F_1(\psi_1(t))\ )$. The morphism $\beta(t) : (\psi_1(t), \psi_2(t), \eta_t) \to (\Upsilon_1(t), \Upsilon_2(t), h_t)$ can then be defined by $\beta(t) := ((\xi_1)_t : \psi_1(t) \to \Upsilon_1(t), (\xi_2)_t : \psi_2(t) \to \Upsilon_2(t))$. This is a morphism between triples because of the 2-commutativity of the diagram for (ψ, ξ_1, ξ_2) above. Unravelling all 2-commutativity conditions imposed yields also the uniqueness of β. $\square$

Example 2.12.10.

$$\begin{tikzcd}
{}_s \bullet \quad \bullet_t \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[ur, phantom, "\swarrow \eta"] & \left({}_0 \bullet \overset{\eta_s}{\underset{\eta_t}{\rightrightarrows}} \bullet {}_1 \right)
\end{tikzcd}$$

is a strict 2-pushout in Gpd. Here $\left({}_0 \bullet \overset{\eta_s}{\underset{\eta_t}{\rightrightarrows}} \bullet {}_1 \right)$ denotes the groupoid freely generated by η_s and η_t (i.e. there are no 2-morphisms).[70]

[69] For $t, t' \in \operatorname{Ob} T$, we define ϕ on morphisms $f : t \to t'$ by $\phi(f) := (\Upsilon_1(f), \Upsilon_2(f)) : (\Upsilon_1(t), \Upsilon_2(t), h_t) \to (\Upsilon_1(t'), \Upsilon_2(t'), h_{t'})$.

[70] Note that η_s, η_t are isomorphisms since we are in Gpd, but we mean "generated as a groupoid by η_s, η_t" and not "a group generated by $\eta_s \circ \eta_t^{-1}$" (which would also be a groupoid, with one object only).

Proof. Let $T \in \operatorname{Ob}\mathrm{Gpd}$ and define $\mathcal{X} := \mathrm{HOM}_{\mathrm{Gpd}}\big({}_0\,\bullet \underset{\eta_t}{\overset{\eta_s}{\rightrightarrows}} \bullet\,{}_1\,, T\big)$. Then consider

$$\begin{tikzcd}
\mathcal{X} \arrow[r] \arrow[d] & \mathrm{HOM}_{\mathrm{Gpd}}(\bullet, T) = T \arrow[d, "\Delta"] \\
T = \mathrm{HOM}_{\mathrm{Gpd}}(\bullet, T) \arrow[r, "\Delta"'] & \mathrm{HOM}_{\mathrm{Gpd}}(\bullet\quad\bullet, T) = T \times T
\end{tikzcd}.$$

Here Δ denotes the diagonal functor.[71]

Objects of $\mathcal{X}$ are triples $(t_0, t_1, \eta_s : t_0 \xrightarrow{\sim} t_1, \eta_t : t_0 \xrightarrow{\sim} t_1)$. For $(t_0, t_1, \eta_s, \eta_t), (t_0', t_1', \eta_s', \eta_t') \in \operatorname{Ob}\mathcal{X}$, a morphism from $(t_0, t_1, \eta_s, \eta_t)$ to $(t_0', t_1', \eta_s', \eta_t')$ is a pair $(\alpha_0 : t_0 \to t_0', \alpha_1 : t_1 \to t_1')$ such that the diagrams

$$\begin{tikzcd}
t_0 \arrow[r, "\alpha_0"] \arrow[d, "\eta_s"'] & t_0' \arrow[d, "\eta_s'"] \\
t_1 \arrow[r, "\alpha_1"'] & t_1'
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
t_0 \arrow[r, "\alpha_0"] \arrow[d, "\eta_t"'] & t_0' \arrow[d, "\eta_t'"] \\
t_1 \arrow[r, "\alpha_1"'] & t_1'
\end{tikzcd} \tag{$*$}$$

commute in T.

On the other hand, we can construct the 2-pullback in the diagram of HOM-categories above and obtain

$$\begin{tikzcd}
T \times^h_{T \times T} T \arrow[r] \arrow[d] & T \arrow[d, "\Delta"] \arrow[dl, phantom, "\Swarrow"] \\
T \arrow[r, "\Delta"'] & T \times T
\end{tikzcd}.$$

Note that for $t_0, t_1 \in \operatorname{Ob} T$, an isomorphism $(t_0, t_0) = \Delta(t_0) \xrightarrow{\sim} \Delta(t_1) = (t_1, t_1)$ consists just of a pair of isomorphisms $\eta_s, \eta_t : t_0 \xrightarrow{\sim} t_1$. Therefore, the objects of $T \times^h_{T\times T} T$ as in definition 2.12.7 consist of triples $(t_0, t_1, (\eta_s : t_0 \xrightarrow{\sim} t_1, \eta_t : t_0 \xrightarrow{\sim} t_1))$. Similarly, one verifies that the morphisms of $T \times^h_{T\times T} T$ are pairs $(\alpha_0 : t_0 \to t_0', \alpha_1 : t_1 \to t_1')$ that satisfy that the diagrams $(*)$ commute in T.

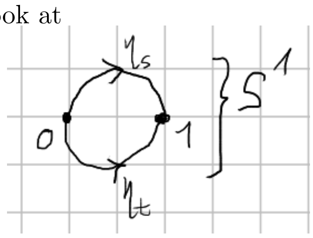
Therefore, we find an equivalence of categories $\phi : \mathcal{X} \to T \times^h_{T\times T} T = \mathrm{HOM}_{\mathrm{Gpd}}(\bullet, T) \times^h_{\mathrm{HOM}_{\mathrm{Gpd}}(\bullet\ \bullet, T)} \mathrm{HOM}_{\mathrm{Gpd}}(\bullet, T)$ defined on objects by $\phi(t_0, t_1, \eta_s, \eta_t) := (t_0, t_1, (\eta_s, \eta_t) : \Delta(t_0) \to \Delta(t_1))$.[72] $\square$

Remark 2.12.11.

There is an equivalence of categories (resp. groupoids) $\big({}_0\,\bullet \underset{\eta_t}{\overset{\eta_s}{\rightrightarrows}} \bullet\,{}_1\big) \approx \big(\bullet \circlearrowright \mathbb{Z}\big)$ (exercise).

Using the Seifert-van Kampen theorem and the previous example, one can use this to compute the fundamental group of the circle.

Geometric intuition for $\big({}_0\,\bullet \underset{\eta_t}{\overset{\eta_s}{\rightrightarrows}} \bullet\,{}_1\big) \approx \big(\bullet \circlearrowright \mathbb{Z}\big)$: Find $\big({}_0\,\bullet \underset{\eta_t}{\overset{\eta_s}{\rightrightarrows}} \bullet\,{}_1\big) \hookrightarrow \Pi_1(S^1)$, then look at



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[71] I.e. on objects $\Delta(t) = (t, t)$ and on morphisms $\Delta(f : t \to t') = (f, f) : (t, t) \to (t', t')$.

Exercise: Write down the reason for the "clear" equalities and why the maps $T \to T \times T$ must be Δ. In particular, $\mathrm{HOM}_{\mathrm{Gpd}}(\bullet\quad\bullet, T) = T \times T$ since there are no relations between the "points". Here $T \times T$ is the explicitly constructed product category already considered earlier, whose objects are pairs of objects and whose morphisms are pairs of morphisms.

[72] On morphisms, due to the notation used here, $\phi((\alpha_1, \alpha_2)) := (\alpha_1, \alpha_2)$.

To conclude this section, we sketch a proof of the adjointness of B and Π_1 in the proposition we started out with.

Sketch of proof of proposition 2.12.4, 3. We want to show: For $X \in \operatorname{Ob}\mathrm{Top}$, $\mathcal{G} \in \operatorname{Ob}\mathrm{Gpd}$,

$$\Pi_1 : \mathrm{HOM}_{\mathrm{Top}}(X, B\mathcal{G}) \xrightarrow{\sim} \mathrm{HOM}_{\mathrm{Gpd}}(\Pi_1 X, \Pi_1(B\mathcal{G})) \qquad \text{is an equivalence of groupoids.}$$

We assume properties 1. and 2. as given.

It suffices to treat the case that X is connected (since Π_1 maps the connected components of X onto disjoint groupoids). Furthermore we may assume by CW-approximation, theorem 1.8.1 and corollary 1.8.3 that X is a CW-complex with $X^{(0)} = \{*\}$. Also, because of property 2., we may assume that $\mathcal{G} = \mathbf{B}G$ for some group G, where $\mathbf{B}G := \begin{tikzcd} \bullet \arrow[loop right, "G"] \end{tikzcd}$ as defined earlier. We again abbreviate $B(\mathbf{B}G) =: BG$.

We need to show an equivalence of categories, i.e. we need to show that Π_1 is a fully faithful and essentially surjective functor.

– Essential surjectivity and fullness: Given $F_1, F_2 : \Pi_1 X \to \Pi_1 BG$, $\eta : F_1 \Rightarrow F_2$, i.e. a "two-cell"

$$\begin{tikzcd} \Pi_1 X \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\eta"] & \Pi_1 BG \end{tikzcd},$$

it suffices to find $f_1, f_2 : X \to BG$, $h : f_1 \Rightarrow f_2$ such that for the 0-cell[73] $x \in X$,

$$\begin{tikzcd} B\pi_1(X,x) \arrow[rr, bend left, "f_{1*}"] \arrow[rr, bend right, "f_{2*}"'] \arrow[rr, phantom, "\Downarrow h_*"] \arrow[dr, "x"'] & & \Pi_1 BG \\ & \Pi_1 X \arrow[ur, bend left, "F_1"] \arrow[ur, bend right, "F_2"'] \arrow[ur, phantom, "\Downarrow\eta"] & \end{tikzcd}$$

2-commutes (where $f_{i*} := \Pi_1(f_i)$ for $i = 1, 2$ and $h_* := \Pi_1(h)$).[74]

– Faithfulness: Given $f : X \to BG$, $H : f \Rightarrow f$ in $\mathrm{HOM}_{\mathrm{Top}}(X, BG)$ such that $\Pi_1 H = \mathrm{id}_{f_*}$, we need to show that already $H = \mathrm{id}_f$. I.e. if we look at a representative of H, there exists $F : X \times I \times I \to BG$, $H \sim^F \mathrm{id}_f$.

The items f_1, f_2, h, F thus required to prove essential surjectivity, fullness and faithfulness can all be constructed using the skeletal filtration on X, using that BG is $K(G,1)$, i.e. $\pi_1 BG = G$ and $\pi_i BG = 0$, $i > 1$. We will show it for f_1 only (for the other ones it is similar).

– $n = 0$. We have $X^{(0)} = \{x\}$. $F_1(x)$, where x is understood as an element of $\operatorname{Ob}\Pi_1 X$, is a point in BG (since an element of $\operatorname{Ob}\Pi_1 BG$). Let $f_1^{(0)} : X^{(0)} \to BG$, $f_1^{(0)}(x) := F_1(x)$.

– $n = 1$: We have $X^{(1)} = \bigvee_\alpha S^1$, where α is understood to index the 1-cells of X. Then the corresponding characteristic maps Φ_α can also be understood as automorphisms of x in $\Pi_1 X$ (i.e. elements of $\pi_1(X,x)$). Thus $F_1(\Phi_\alpha)$ is a loop in $(BG, F_1(x))$ up to homotopy (since an automorphism of $F_1(x)$ in $\Pi_1 BG$). We choose a representative γ_α of this loop and define $f_1^{(1)} : X^{(1)} \to BG$ such that for all α, $f_1^{(1)} \circ \Phi_\alpha = \gamma_\alpha$.[75]

[73] I.e. a 0-cell in the sense of CW-complexes, which has nothing straightforwardly to do with the 2-cells in the diagrams.

[74] More explicitly: We need to find g_1, g_2 with $g_{1*} \cong F_1, g_{2*} \cong F_2$, and we need to show that for any $\tilde{k} \in \mathrm{Hom}_{\mathrm{HOM}_{\mathrm{Gpd}}(\Pi_1 X, \Pi_1 BG)}(l_{1*}, l_{2*})$, where $l_1, l_2 \in \mathrm{HOM}_{\mathrm{Top}}(X, BG)$, there is $k \in \mathrm{Hom}_{\mathrm{HOM}_{\mathrm{Top}}(X,BG)}(l_1, l_2)$ with $\tilde{k} = k_*$. This general case follows from the diagram above.

[75] I.e. "we define $f_1^{(1)}$ on the 1-cells to be these representatives".

– $n = 2$: Consider the pushout square (which is also a homotopy pushout due to theorem 2.3.9)

$$\begin{tikzcd} \coprod_\beta S^1 \arrow[r, "\coprod_\beta \phi_\beta"] \arrow[d, hook] & X^{(1)} \arrow[d] \arrow[ddr, bend left, "f_1^{(1)}"] & \\ \coprod_\beta D^2 \arrow[r] \arrow[drr, bend right, "*"'] & X^{(2)} \arrow[dr, dashed, "f_1^{(2)}"] & \\ & & BG \end{tikzcd} ,$$

where β indexes the 2-cells and $*$ denotes the constant map (to $F_1(x)$). Since the relations given by the 2-cells are "respected" by F_1 (by definition, since it is a map $\Pi_1 X \to \Pi_1 BG$), $f_1^{(1)} \circ \phi_\beta$ is homotopic to a constant map for all β and we find the indicated homotopy. Then we can define $f_1^{(2)}$ as the dashed pushout map.

– For $n > 2$, proceed by induction. Induction step $n-1 \mapsto n$: Consider the (1- and 2-)pushout square

$$\begin{tikzcd} \coprod_\gamma S^{n-1} \arrow[r, "\coprod_\gamma \phi_\gamma^{(n-1)}"] \arrow[d, hook] & X^{(n-1)} \arrow[d] \arrow[ddr, bend left, "f_1^{(n-1)}"] & \\ \coprod_\gamma D^2 \arrow[r] \arrow[drr, bend right, "*"'] & X^{(n)} \arrow[dr, dashed, "f_1^{(n)}"] & \\ & & BG \end{tikzcd} .$$

We have $f_1^{(n-1)} \circ \phi_\gamma^{(n-1)} \sim *$ for all γ, because $\pi_i(BG) = 0$ for $i > 1$. Thus there is the homotopy $H^{(n-1)}$ as indicated and we construct $f_1^{(n)}$ using again that attaching cells makes a homotopy pushout square.

– Finally, use that $X = \operatorname{colim}_n X^{(n)}$ to define the map $f_1 : X \to BG$.

$\square$

It remains to define B with the properties 1. and 2. from 2.12.4 and to show that it is a functor. See the exercises.

3 Brown representability

Literature:

- A. Hatcher, *Algebraic Topology*

3.1 Brown representability

The goal of this part is to prove:

Theorem 3.1.1 (Brown). *Denote* $\mathrm{h}(\mathrm{CW}_*^\circ)^{\mathrm{op}}$ *be the homotopy category of the dual category of pointed, connected CW-complexes. Let* $h : \mathrm{h}(\mathrm{CW}_*^\circ)^{\mathrm{op}} \to \mathrm{Sets}_*$ *be a functor that satisfies:*

- Mayer-Vietoris property *(MV) or* excision property*: If*

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, hook] & A \arrow[d, hook] \\ B \arrow[r, hook] & X \end{tikzcd}$$

  is a diagram of CW-subcomplexes (in particular $X = A \cup B$, $C = A \cap B$), then, considering the diagram

$$\begin{tikzcd} h(X) \arrow[r] \arrow[d] & h(A) \arrow[d] \\ h(B) \arrow[r] & h(C) \end{tikzcd} ,$$

  the canonical map $h(X) \twoheadrightarrow h(B) \times_{h(C)} h(A)$ is surjective.[1]

- Additivity*: For any family $\{X_\alpha\}_\alpha$ of connected pointed CW-complexes (indexed by an arbitrary set), the inclusion maps $X_{\alpha'} \hookrightarrow \bigvee_\alpha X_\alpha$ (for all α') induce an isomorphism $h(\bigvee_\alpha X_\alpha) \xrightarrow{\sim} \prod_\alpha h(X_\alpha)$.*

Then there exists $K \in \mathrm{Ob}\,\mathrm{CW}_*^\circ$, which is unique up to homotopy equivalence, and a natural equivalence $[\cdot, K]_* \xrightarrow{\sim} h(\cdot)$.

Remark 3.1.2.
- One can replace in the Mayer-Vietoris property the diagram with any homotopy pushout square in CW_*°.[2]
- The connectedness assumption in the theorem is essential, but one can prove a Brown representability theorem for 2-functors $h : \mathrm{CW}^{\mathrm{op}} \to \mathrm{Gpd}$. Recall that for any category $\mathcal{C}$, $f : \mathrm{CW} \to \mathcal{C}$ factors over hCW if and only if it sends homotopic maps to equal maps.[3]

Lemma 3.1.3. *For $K \in \mathrm{CW}_*^\circ$, the functor $[\cdot, K]_* : (\mathrm{hCW}_*^\circ)^{\mathrm{op}} \to \mathrm{Sets}_*$ satisfies*

- *Additivity: If $X_\alpha \in \mathrm{Ob}\,\mathrm{CW}_*^\circ$, α in some index set, then the inclusions induce an isomorphism $[\bigvee_\alpha X_\alpha, K]_* \xrightarrow{\sim} \prod_\alpha [X_\alpha, K]_*$.*

[1] "$h(X)$ is a weak 1-pullback."

[2] This follows from CW-approximation (theorem 1.8.1), homotopy invariance of h and the strictification lemma 1.5.7 (a cellular version of it).

[3] So a statement about 2-categories is "stronger" than one about homotopy categories, but nevertheless we will only prove the statement given here.

– *Mayer-Vietoris property: If*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\overset{H}{\swarrow}"] & B \arrow[d, "g'"] \\ C \arrow[r, "f'"'] & D \end{tikzcd}$$

is a homotopy pushout square in CW_*°*, then the canonical map* $[D,K]_* \twoheadrightarrow [B,K]_* \times_{[A,K]_*} [C,K]_*$ *is surjective.*

Proof. – Additivity: Prove and use: $\bigvee_\alpha X_\alpha$ is the 1-coproduct in hCW_*° (exercise).

– Mayer-Vietoris property: Let $u : B \to K$, $v : C \to K$ such that $[u \circ f] = [v \circ g]$ in $[A,K]_*$. I.e. there is a homotopy $h : v \circ g \Rightarrow u \circ f$. Then, considering

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "g'"] \arrow[ddr, bend left, "u"] & \\ C \arrow[r, "f'"] \arrow[drr, bend right, "v"'] & D \arrow[dr, dashed, "\phi"] \arrow[dr, phantom, "\swarrow h"'] & \\ & & K \end{tikzcd},$$

we obtain the map $\phi : D \to K$ by the homotopy pushout property such that $[\phi \circ f'] = [v]$, $[\phi \circ g'] = [u]$. This is what we need: $([u],[v]) \in [B,K]_* \times_{[A,K]_*} [C,K]_*$ has the preimage $[\phi] \in [D,K]_*$.

$\square$

Remark 3.1.4. If we look at $\mathrm{HOM}(\cdot, K)$, then we get a 2-commutative square in Gpd:

$$\begin{tikzcd} \mathrm{HOM}(D,K) \arrow[r] \arrow[d] & \mathrm{HOM}(B,K) \arrow[d] \arrow[dl, phantom, "\swarrow"] \\ \mathrm{HOM}(C,K) \arrow[r] & \mathrm{HOM}(A,K) \end{tikzcd},$$

thus in particular there is a map $\mathrm{HOM}(D,K) \to \mathrm{HOM}(C,K) \times^h_{\mathrm{HOM}(A,K)} \mathrm{HOM}(B,K)$. One can check that this is essentially surjective (similar to above) and full, but it is not an equivalence.

This would be a necessary (and sufficient) condition for the Brown representability for 2-functors $\mathrm{CW}^{\mathrm{op}} \to \mathrm{Gpd}$.

Remark 3.1.5. *Idea of the proof of Brown representability:* Let us construct a pair (K,u), $K \in \mathrm{Ob}\,\mathrm{CW}_*^\circ$, $u \in h(K)$, such that for all $m \in \mathbb{N}$, we have an isomorphism $[S^m,K]_* \xrightarrow{\sim} h(S^m)$, $f \mapsto f^*(u)(=: h(f)(u))$.

By the Yoneda lemma, $u \in h(K)$ yields a natural transformation $\eta : [\cdot,K]_* \Rightarrow h$ that is defined by $[X,K]_* \ni f \mapsto f^*(u) \in h(X)$ (for $X \in \mathrm{CW}_*^\circ$). Now if $h = [\cdot,K']_*$ for $K' \in \mathrm{Ob}\,\mathrm{CW}_*^\circ$, we obtain in particular for S^m that $\eta : [S^m,K]_* \xrightarrow{\sim} [S^m,K']_*$ is a bijection. Let $\tilde{u}$ be any representative of $u \in h(K) = [K,K']_*$. Then $\eta = (\tilde{u}\circ)$; thus $\tilde{u}$ induces isomorphisms on all homotopy groups. By Whitehead's theorem 1.7.4, $\tilde{u}$ is a homotopy equivalence. Therefore K is uniquely determined (up to homotopy equivalence) by the requirements $[S^m,K]_* \xrightarrow{\sim} h(S^m)$ for $m \in \mathbb{N}$.

Furthermore, any $X \in \mathrm{Ob}\,\mathrm{CW}_*^\circ$ is obtained from spheres $\{S^n\}$ by homotopy pushouts

$$\begin{tikzcd} \bigvee S^n \arrow[r] \arrow[d] & X^{(n)} \arrow[d] \arrow[dl, phantom, "\swarrow"] \\ * \arrow[r] & X^{(n+1)} \end{tikzcd},$$

thus using the Mayer-Vietoris property (cf. also remark 3.1.2) and additivity, one can then show that if K, if it satisfies $[S^m, K]_* \cong h(S^m)$ for all $m \in \mathbb{N}$, it will also satisfy $[X, K]_* \cong h(X)$ for all $X \in \mathrm{CW}_*^\circ$.

Lemma 3.1.6. *Let h be as in theorem 3.1.1; let $Z \in \mathrm{CW}_*^\circ$, $z \in h(Z)$. Then there exists $K \in \mathrm{CW}_*^\circ$, $u \in h(K)$ and a map $f : Z \to K$ such that*

1. *$f^*(u) = z$ (where $f^* : h(K) \to h(Z)$).*
2. *$\Psi_u : [S^m, K]_* \xrightarrow{\sim} h(S^m), f \mapsto f^*(u)$ is an isomorphism for all $m \geq 1$.*

Proof. We define K by induction.

– *Base case:* Let $K_1 := Z \vee \bigvee_{\alpha \in h(S^1)} S^1_\alpha$.[4] Define $f_1 : Z \to K_1$ to be the canonical inclusion.

  By additivity, $h(K_1) \cong h(Z) \times_{\alpha \in h(Z)} h(S^1_\alpha)$. Define $h(K_1) \ni u_1 := (z, (\alpha)_\alpha)$ (for every α, α lies in $h(S^1_\alpha) = h(S^1)$). Then

  – We have $f_1^*(u_1) = z$.
  – We have a surjection $\Psi_{u_1} : [S^1, K_1]_* \twoheadrightarrow h(S^1)$, $g \mapsto g^*(u_1)$. Similar to above, Ψ_{u_1} is given by the Yoneda lemma; $u_1 \in h(K_1)$ yields a natural transformation $[\cdot, K_1]_* \Rightarrow h(\cdot)$. It is surjective by construction: For $\alpha \in h(S^1)$, the characteristic map $\Phi_\alpha : S^1 \to K_1$ satisfies $\Psi_{u_1}(\Phi_\alpha) = \Phi_\alpha^*(u_1) = \alpha$.

– *Induction step:* Suppose we have constructed $(K_n, u_n, f_n : Z \to K_n)$ satisfying that

  0. The diagram

$$\begin{tikzcd} K_{n-1} \arrow[rr, hook] & & K_n \\ & Z \arrow[ul, "f_{n-1}"] \arrow[ur, "f_n"'] & \end{tikzcd}$$

  commutes.

  1. The maps $\Psi_{u_n} : [S^m, K]_* \to h(S^m)$, $g_m \mapsto g_m^*(u_n)$ are isomorphisms for $m < n$ and surjective for $m = n$.
  2. $f_n^*(u_n) = z$.

  Let $U_n := \{g_\beta : S^n \to K_n\}$ be a maximal set of maps such that for all β, $g_\beta^*(u_n) = * \in h(S^n)$, and for $\beta \neq \beta'$, $g_\beta \not\simeq g_{\beta'}$.[5] Then the maps $g_\beta \in U_n$ generate the kernel of $\Psi_{u_n} : [S^n, K_n]_* \to h(S^n)$. Define a space C_n such that

$$\begin{tikzcd} \bigvee_{\beta \in U_n} S^n \arrow[r, "\bigvee_{\beta \in U_n} g_\beta"] \arrow[d] & K_n \arrow[d] \arrow[dl, phantom, "\swArrow"] \\ * \simeq \bigvee_{\beta \in U_n} D^n \arrow[r] & C_n \end{tikzcd} \qquad (*)$$

  is a homotopy pushout.[6]

  The Mayer-Vietoris property for $(*)$ yields a surjection $h(C_n) \twoheadrightarrow h(*) \times_{h(\bigvee_\beta S^n)} h(K_n)$. By additivity for an empty family, $h(*) = *$; thus this is equivalent to an exact sequence (in Sets_*)

$$h(C_n) \longrightarrow h(K_n) \longrightarrow h(\textstyle\bigvee_\beta S^n) .$$

  Now the image of u_n in $h(\bigvee_\beta S^n) = \prod_\beta h(S^n)$ is trivial (i.e. $*$) by choice of the g_β, thus by exactness we find $\overline{u_n} \in h(C_n)$ such that $\overline{u_n} \mapsto u_n \in h(K_n)$.

[4] In words, K_1 is obtained from Z by attaching 1-cells for each element of $h(S^1)$.

[5] I.e. choose representatives of the, generically uncountable, set $\{g_\beta : S^n \to K_n \mid g_\beta^*(u_n) = * \in h(S^n)\}$. With "$\not\simeq$", we denote that two maps are not homotopic.

[6] Intuitively speaking, we "kill" the g_β: We want no kernel.
The pushout above might be chosen as a 1-pushout, but does not need to.

Now let $K_{n+1} := C_n \vee \bigvee_{\alpha \in h(S^{n+1})} S^{n+1}_\alpha$, such that $h(K_{n+1}) = h(C_n) \times \prod_{\alpha \in h(S^{n+1})} h(S^{n+1})$, and then define $u_{n+1} := (\overline{u_n}, (\alpha)_\alpha)$. Furthermore, let f_{n+1} be the composition

$$\begin{tikzcd} Z \arrow[r, "f_n"] & K_n \arrow[rr, hook, "i_n"] \arrow[dr, hook, "j_n"'] & & K_{n+1} \\ & & C_n \arrow[ur, hook, "k_n"'] & \end{tikzcd},$$

(where the maps i_n, j_n, k_n are the inclusions as indicated).

We verify that the conditions 0., 1., 2. are satisfied for $(K_{n+1}, u_{n+1}, f_{n+1})$:

0. is satisfied by construction.

2. is also satisfied by construction and induction: $k_n^*(u_{n+1}) = \overline{u_n}$, $j_n^*(\overline{u_n}) = u_n$, $f_n^*(u_n) = z$.

Proof that 1. is satisfied: $j_n : K_n \hookrightarrow C_n$ is n-connected, and $k_n : C_n \to K_{n+1}$ is n-connected. Thus $i_n = k_n \circ j_n$ is n-connected. Now consider the diagram

$$\begin{tikzcd} {[S^m, K_n]_*} \arrow[rr, "i_n \circ"] \arrow[dr, "\Psi_{u_n}"'] & & {[S^m, K_{n+1}]_*} \arrow[dl, "\Psi_{u_{n+1}}"] \\ & h(S^m) & \end{tikzcd}.$$

This commutes since for $g \in [S^m, K_n]_*$, $\Psi_{u_n}(g) = g^*(u_n) = g^*(i_n^*(u_{n+1})) = \Psi_{u_{n+1}}(i_n \circ g)$. For $m < n$, $i_n\circ$ is a bijection because of the n-connectedness of (K_{n+1}, K_n), and Ψ_{u_n} is a bijection by induction assumption. Thus also $\Psi_{u_{n+1}}$ is a bijection. For $m = n$, both $i_n\circ$ and Ψ_{u_n} are surjections. In particular since Ψ_{u_n} is a surjection, also $\Psi_{u_{n+1}}$ is surjective. Furthermore it is injective since $i_n\circ$ is surjective and by construction, $\ker(\Psi_{u_n}) \subset \ker(i_n\circ)$. It follows that $\Psi_{u_{n+1}} : [S^n, K_{n+1}] \xrightarrow{\sim} h(S^n)$ is an isomorphism in Sets_*.

Finally, for $m = n+1$, as in the base case we obtain a surjection $[S^{n+1}, K_{n+1}]_* \twoheadrightarrow h(S^{n+1})$; the preimage of $\alpha \in h(S^{n+1})$ is given by the characteristic map Φ_α. This concludes the induction step.

– Now we have a commutative diagram

$$\begin{tikzcd} & Z \arrow[d, hook, "f_n"] \arrow[dr, "f_{n+1}"] \arrow[drr, "f_{n+2}"] & & & \\ \ldots \arrow[r, hook] & K_n \arrow[r, hook, "i_n"] & K_{n+1} \arrow[r, hook, "i_{n+1}"] & K_{n+2} \arrow[r, hook] & \ldots \end{tikzcd},$$

where all inclusions may be chosen as subcomplexes by using specific attaching pushouts. So we can define $K := \operatorname{colim}_n K_i$ and have a map $f : Z \to K$. We need to define $u \in h(K)$.

Claim: The diagram

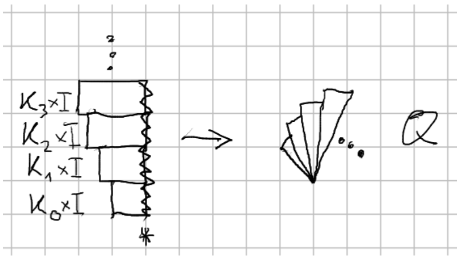
$$\begin{tikzcd} \bigvee_{i \in \mathbb{N}} K_i \arrow[r, "T_1"] \arrow[d, "T_2"'] & \bigvee_{i \in \mathbb{N}} K_{2i} \arrow[d] \\ \bigvee_{i \in \mathbb{N}} K_{2i+1} \arrow[r] & K \end{tikzcd}$$

is a homotopy pushout, where $K_0 := Z$, $T_1 := \mathrm{id}_Z \vee i_1 \vee \mathrm{id}_{K_2} \vee i_3 \vee \mathrm{id}_{K_4} \vee \ldots$, $T_2 := f_1 \vee \mathrm{id}_{K_1} \vee i_2 \vee \mathrm{id}_{K_3} \vee i_4 \vee \ldots$ and the other maps are given by the inclusion on each wedge summand.

Assuming the claim to be true, we define $u \in h(K)$ by the Mayer-Vietoris property such that if $l_n : K_n \to K$, $n \in \mathbb{N}$, are the inclusions, then $l_n^*(u) = u_n$ for all $n \in \mathbb{N}$ (with $l_0 := f : Z \to K$ and $u_0 := z$). This is possible since by additivity, $h(\bigvee_{i \in \mathbb{N}} K_i) = \prod_{i \in \mathbb{N}} h(K_i)$ and analogously for the other occurring wedge products.[7]

Proof of claim: Let Q be a homotopy pushout of $\left(\bigvee_{i\in\mathbb{N}} K_i\right)\left(\bigvee_{i\in\mathbb{N}} K_{2i}\right)\left(\bigvee_{i\in\mathbb{N}} K_{2i+1}\right)$. We choose the standard homotopy pushout as in theorem 2.4.3 (resp. the remarks on the pointed case). Using that $\left(\bigvee_{i\in\mathbb{N}} K_i \times I\right)/((*,t) \sim (*,t')) \cong \bigvee_{i\in\mathbb{N}}(K_i \times I/((*,t) \sim (*,t')))$ and that T_1, T_2 are surjective, Q admits an explicit description as a “mapping telescope” (with collapsed basepoint):

$$Q = \left(\coprod_{i\in\mathbb{N}} K_i \times I\right) \Bigg/ \left(\begin{array}{l} \forall j \in \mathbb{N}, x_j \in K_j : K_j \times I \ni (x_j, 1) \sim (i_j(x_j), 0) \in K_{j+1} \times I, \\ \forall j, j' \in \mathbb{N}, t, t' \in I : K_j \times I \ni (*, t) \sim (*, t') \in K_{j'} \times I \end{array}\right).$$



We get a canonical map $Q \to K$; it suffices to check that it is a weak equivalence. Now for $m \in \mathbb{N}$, $\pi_m(K) = \operatorname{colim}_i \pi_m(K_i) = \pi_m(K_n)$ for sufficiently large n (by cellular approximation, theorem 1.6.8, using that in the construction of K_{j+1} from K_j, only cells of dimension j and $j+1$ are attached). Thus it suffices to consider the maps $\pi_m(Q) \to \pi_m(K_n)$. Furthermore, again by cellular approximation, it suffices to consider $\pi_m(Q^{(m+1)}) \to \pi_m(K_n)$ (every map $f : S^m \to Q$ is homotopic to a map $f : S^m \to Q^{(m)}$, and if $f, f' : S^m \to Q^{(m)}$ are homotopic in Q, they are so in $Q^{(m+1)}$). Now

$$Q^{(m+1)} = \left(\coprod_{i\in\mathbb{N}} K_i^{(m)} \times I\right) \Bigg/ \left(\begin{array}{l} \forall j \in \mathbb{N}, x_j \in K_j : K_j^{(m)} \times I \ni (x_j, 1) \sim (i_j(x_j), 0) \in K_{j+1}^{(m)} \times I, \\ \forall j, j' \in \mathbb{N}, t, t' \in I : K_j^{(m)} \times I \ni (*, t) \sim (*, t') \in K_{j'}^{(m)} \times I \end{array}\right) =$$

$$= \left(K_n^{(m)} \times [n, \infty) \sqcup \coprod_{i \le n} K_i^{(m)} \times I\right) \Bigg/$$

$$\left(\begin{array}{l} \forall 0 \le j \le n, x_j \in K_j : K_j^{(m)} \times I \ni (x_j, 1) \sim (i_j(x_j), 0) \in K_{j+1}^{(m)} \times I, \\ \forall 0 \le j, j' \le n, t, t' \in I : K_j^{(m)} \times I \ni (*, t) \sim (*, t') \in K_{j'}^{(m)} \times I, \\ \forall 0 \le j \le n, t \in I, t' \in [n, \infty) : K_j^{(m)} \times I \ni (*, t) \sim (*, t') \in K_n^{(m)} \times [n, \infty) \end{array}\right)$$

(where the same n may be chosen). This is homotopy equivalent to $K_n^{(m)}$ (it retracts onto $K_n^{(m)} \times \{n\}$). Since $K_n^{(m)} \to K_n$ is m-connected, one obtains the claim and therefore the lemma.

□

Proof of Brown representability, theorem 3.1.1. We apply lemma 3.1.6 for $Z = *, z = * \in h(*)$ (recall $h(*) = *$ by additivity for an empty set). This yields (K, u) and the map $\Psi_u : [X, K]_* \to h(X), f \mapsto f^*(u)$ for all $X \in \operatorname{Ob} \mathrm{CW}_*^\circ$.

[7] One computes that $((u_0, u_2, u_4, \dots), (u_1, u_3, u_5, \dots)) \in \left(\prod_{i\in\mathbb{N}} h(K_{2i})\right) \times_{\prod_{i\in\mathbb{N}} h(K_i)} \left(\prod_{i\in\mathbb{N}} h(K_{2i+1})\right)$, because $T_1^*(u_0, u_2, u_4, \dots) = (u_0, i_1^*(u_2), u_2, i_3^*(u_4), u_4, \dots) = (i_0^*(u_1), u_1, i_2^*(u_3), u_3, i_4^*(u_5), \dots) = T_2^*(u_1, u_3, u_5, \dots)$. Since the map $h(K) \to \left(\prod_{i\in\mathbb{N}} h(K_{2i})\right) \times_{\prod_{i\in\mathbb{N}} h(K_i)} \left(\prod_{i\in\mathbb{N}} h(K_{2i+1})\right)$ is given by $((l_0^*, l_2^*, l_4^*, \dots), (l_1^*, l_3^*, l_5^*, \dots))$, the preimage $u \in h(K)$ of $((u_0, u_2, u_4, \dots), (u_1, u_3, u_5, \dots))$ obtained by the Mayer-Vietoris property will then indeed have the desired properties.

– Ψ_u is surjective: Let $\alpha \in h(X)$. Define a space Z' as the pushout

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & K \arrow[d, "j"] \\
X \arrow[r, "i"'] \arrow[ur, phantom, "\ulcorner"] & Z'
\end{tikzcd}$$

and let $z' := (\alpha, u) \in h(Z') = h(X) \times h(K)$ (the isomorphism by additivity). We apply again lemma 3.1.6, now for (Z', z'), and obtain (K', u'), $\Psi_{u'}$ and $f' : Z' \to K'$ (everything defined analogously as the unprimed variants). Thus we have the red arrows in

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & K \arrow[d, "j"] \arrow[ddr, bend left, "\theta"] & \\
X \arrow[r, "i"'] \arrow[ur, phantom, "\ulcorner"] \arrow[drr, bend right, "\chi"'] & Z' \arrow[dr, "f'"] & \\
 & & K'
\end{tikzcd}$$

(where χ, θ can be defined as the compositions; every map out of Z' yields such χ, θ). Then by construction $\chi^*(u') = i^*(f'^*(u')) = i^*((\alpha, u)) = \alpha$.

Now for $m \in \mathbb{N}$, $\theta\circ : [S^m, K]_* \to [S^m, K']_*$ is an isomorphism because

$$\begin{tikzcd}
{[S^m, K]_*} \arrow[rr, "\theta\circ"] \arrow[dr, "\Psi_u"'] & & {[S^m, K']_*} \arrow[dl, "\Psi_{u'}"] \\
 & h(S^m) &
\end{tikzcd}$$

commutes ($\Psi_u(g) = g^*(u) = g^*(j^*(\alpha, u)) = g^*(j^*(f'^*(u'))) = g^*(\theta^*(u')) = \Psi_{u'}(\theta \circ g)$ for $g \in [S^m, K]_*$) and both Ψ_u, $\Psi_{u'}$ are isomorphisms (for S^m). Thus θ is a weak equivalence and by Whitehead's theorem 1.7.4, θ is a homotopy equivalence.

Then let θ^{-1} be a homotopy inverse to θ and consider $\theta^{-1} \circ \chi : X \to K$. We compute $(\theta^{-1} \circ \chi)^*(u) = \chi^*({\theta^{-1}}^*(u)) = \chi^*(u') = \alpha$, because $\theta^*(u') = u$ by construction and the inverse to θ^* is ${\theta^{-1}}^*$. This proves surjectivity of Ψ_u.

– Ψ_u is injective: Let $f_0, f_1 : X \to K$ be pointed maps such that $f_0^*(u) = f_1^*(u)$. Define a space $\tilde{Z}$ as the pushout of the black arrows in

$$\begin{tikzcd}
X \vee X \arrow[r, "f_0 \vee f_1"] \arrow[d, hook, "\mathrm{incl}_0 \vee \mathrm{incl}_1"'] & K \arrow[d, "\tilde{j}"] \arrow[ddr, bend left, "\tilde{\theta}"] & \\
X \times I / * \times I \arrow[r, "\tilde{i}"'] \arrow[ur, phantom, "\ulcorner"] \arrow[drr, bend right, "\tilde{\chi}"'] & \tilde{Z} \arrow[dr, "\tilde{f}"] & \\
 & & \tilde{K}
\end{tikzcd}.$$

We apply lemma 3.1.6 to $(\tilde{Z}, \tilde{z})$, where $\tilde{z}$ is a preimage of $(u, f_0^*(u)) = (u, f_1^*(u)) \in h(X \times I / * \times I) \times_{h(X \vee X)} h(K) \cong h(X) \times_{h(X) \times h(X)} h(K)$.[8] This yields the space $\tilde{K}$ and the maps $\tilde{f}, \tilde{\chi}, \tilde{\theta}$. As in the proof of surjectivity, we find that $\tilde{\theta}$ is a homotopy equivalence. Let $\tilde{\theta}^{-1}$ be a homotopy inverse to $\tilde{\theta}$ and define $G := (\tilde{\theta}^{-1} \circ \tilde{\chi}) : X \times I / * \times I \to K$. Then G fits into a

[8] This exists by the Mayer-Vietoris property. We have $(u, f_0^*(u)) \in h(X \times I / * \times I) \times_{h(X) \times h(X)} h(K) = \{(v', u') \in h(X) \times h(K) \mid (\mathrm{incl}_0, \mathrm{incl}_1)^*(v') = (f_0, f_1)^*(u')\} = \{(v', u') \in h(X) \times h(K) \mid f_0^*(u') = f_1^*(u') = \mathrm{incl}_0^*(v')\}$ (the last equality by homotopy invariance of h).

2-commutative triangle

$$\begin{tikzcd}
X \vee X \arrow[rr, "f_0 \vee f_1"] \arrow[dr, "\mathrm{incl}_0 \vee \mathrm{incl}_1"'] & & K . \\
& X \times I / * \times I \arrow[ur, "G"'] \arrow[u, phantom, "\Uparrow"] &
\end{tikzcd}$$

[9] In the homotopy category hCW_*°, this means that (using also $X \times I/* \times I \sim X$)

$$\begin{tikzcd}
X \vee X \arrow[rr, "{[f_0 \vee f_1]}"] \arrow[dr, "{[\mathrm{id} \vee \mathrm{id}]}"'] & & K . \\
& X \arrow[ur, "{[G]}"'] &
\end{tikzcd}$$

commutes, and this means $[f_0] = [f_1] = [G]$.

$\square$

[9] The diagram above is 1-commutative, i.e. $\tilde{\theta} \circ (f_0 \vee f_1) = \tilde{\chi} \circ (\mathrm{incl}_0 \vee \mathrm{incl}_1)$. Then $(\tilde{\theta}^{-1} \circ \tilde{\chi}) \circ (\mathrm{incl}_0 \vee \mathrm{incl}_1) \sim f_0 \vee f_1$.

4 Principal G-bundles and vector bundles

Literature:

- T. tom Dieck, *Algebraic Topology*
- J. Milnor, J. Stasheff, *Lectures on Characteristic Classes*

4.1 Principal G-bundles

Idea: We know vector bundles (tangent bundles over manifolds, etc.). Now consider the same thing with groups as fibers instead of vector spaces.

Definition 4.1.1. Let G be a topological group. Assume furthermore $G \in \mathrm{CGHaus}$ (i.e. compactly generated, i.e. $S \subset G$ is open if and only if $S \cap K$ is open for all $K \subset G$ compact, and Hausdorff), and G has the homotopy type of a CW-complex.

A *principal G-bundle* over a topological space X is a topological space P ("total space"), together with a map $\pi : P \to X$ and a continuous action $a : G \times P \to P$, such that:

1. G "acts in the fibers/acts fiberwise":

$$\begin{tikzcd} G \times P \arrow[r, "a"] \arrow[d, "\pi \circ \mathrm{pr}_2"'] & P \arrow[d, "\pi"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

   commutes. In words: If $p \in \pi^{-1}(\{x\})$ for $x \in X$, then for all $g \in G$, also $a(g,p) =: g \cdot p \in \pi^{-1}(\{x\})$.

2. Local triviality: For all $x \in X$, there exists an open neighbourhood $U \subset X$ of x and a homeomorphism $\rho_U : G \times U \to \pi^{-1}(U)$ that is G-*equivariant* and the diagram

$$\begin{tikzcd} G \times U \arrow[rr, "\rho_U", "\cong"'] \arrow[dr, "\mathrm{pr}_2"'] & & \pi^{-1}(U) \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

   commutes. Here G-equivariance means that for all $g \in G$, $(h,u) \in G \times U$, we have $\rho_U(g \cdot (h,u)) = g \cdot \rho_U(h,u)$, where g acts on (h,u) by multiplication in the first factor: $g \cdot (h,u) := (gh, u)$.
   [1]

Remark 4.1.2. The "torsor condition" (sometimes additionally in the literature) is satisfied: The map $G \times P \to P \times_X P$, $(g,p) \mapsto (g \cdot p, p)$ is a homeomorphism.[2]

Example 4.1.3 (Examples of groups with the properties as above)**.**

- Finitely generated groups (with the discrete topology).
- $\mathrm{Gl}_n(\mathbb{R}), \mathrm{Gl}_n(\mathbb{C}), \dots$ - all the Lie groups.

[1]To be precise, this is a left principal G-bundle, where G acts from the left. A right principal G-bundle is defined analogously.

[2]Here $P \times_X P$ is the usual pullback.

Lemma 4.1.4. *Let $\pi : P \to E$ be a principal G-bundle.*

1. *π is a fiber bundle with fiber G.*
2. *The diagram*

$$\begin{tikzcd} P \arrow[r, equal] \arrow[d, "\pi"'] & P \arrow[d] \\ X & G\backslash P \arrow[l, "\cong"] \end{tikzcd}$$

   commutes and the lower map is a homeomorphism, where $G\backslash P$ is the quotient with respect to the group action.[3]

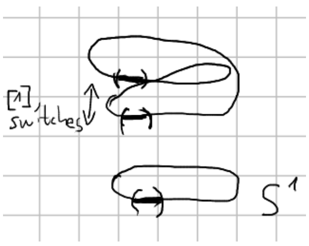
Proof. 1. Clear by definition (this is condition 2.).

2. It suffices to check it locally. Consider $U \subset X$ such that π is trivial over U; then use $G\backslash(G \times U) \cong U$.[4]

□

Example 4.1.5 (Examples of principal G-bundles)**.**

- The trivial (principal) G-bundle: $\mathrm{pr}_2 : G \times X \to X$.
- The double covering $S^1 \to S^1$:



  This is a prinicipal $\mathbb{Z}/2$-bundle.

- If G is discrete, we have a bijection

  $\{$principal G-bundles over $X\} \xrightarrow{\sim} \{$Galois (i.e. regular) coverings of X with deck transformation group $G\}$.

- $\mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{C}P^n, x \mapsto [x]$ is a principal $\mathbb{C}\backslash\{0\} =: \mathbb{C}^*$-bundle:[5] $[x] = [y]$ if there is $\lambda \in \mathbb{C}^*$ such that $x = \lambda y$.
- Similarly: $\mathbb{C}^\infty\backslash\{0\} = \mathrm{colim}_n\, \mathbb{C}^n\backslash\{0\} \to \mathrm{colim}_n\, \mathbb{C}P^n = \mathbb{C}P^\infty$ is a principal $\mathbb{C}^*$-bundle.

Proof (for $\mathbb{C}P^\infty$). The action $G \curvearrowright \mathbb{C}^\infty\backslash\{0\}$ is given by $\tau \cdot (z_0, z_1, \dots) = (\tau z_0, \tau z_1, \dots)$. This defines a fiberwise action.

Claim: This action is free and transitive on the fibers. (Exercise.)

To check: Local triviality. Thus we have to find a "trivializing cover". For $i \in \mathbb{N}$, let $U_i := \{(z_0 : z_1 : \dots) \mid z_i \neq 0\}$. The preimage by π is $\mathbb{C}^\infty\backslash\{z_i = 0\}$. We need a $\mathbb{C}^*$-equivariant homeomorphism ρ such that

$$\begin{tikzcd} \mathbb{C}^* \times U_i \arrow[rr, "\rho", "\cong"'] \arrow[dr, "\mathrm{pr}_2"'] & & \mathbb{C}^\infty\backslash\{z_i = 0\} \arrow[dl, "\pi"] \\ & U_i & \end{tikzcd}$$

commutes.

[3]The group action acts from the left, hence the symbol.
[4]Thus we have found a local homeomorphism, and bijectivity follows since G acts transitively on the fibers.
[5]$\mathbb{C}^*$ is a multiplicative group.

If it exists, ρ is uniquely determined by $\rho|_{\{1\}\times U_i}$, because of G-equivariance. Looking at

$$\begin{tikzcd} \{1\} \times U_i \arrow[rr] \arrow[dr, "\cong"'] & & \mathbb{C}^\infty \backslash \{z_i = 0\} \arrow[dl, "\pi"] \\ & U_i & \end{tikzcd} ,$$

it suffices to give a section of π over U_i.

Here, such a section is given by $(z_0 : z_1 : \cdots : z_i : \dots) \mapsto (\frac{z_0}{z_i}, \frac{z_1}{z_i}, \dots, \frac{z_i}{z_i}, \dots)$. (Check: This is well-defined and continuous.) Thus π is locally trivial. $\square$

Remark 4.1.6. Fact: If $H \subset G$ is a closed subgroups (not necessarily normal), where H, G are *Lie groups*[6], then $G \to H\backslash G$ is a principal H-bundle.[7]

Lemma 4.1.7. *Let $\pi : P \to X$ be a principal G-bundle and $f : Y \to X$ be a continuous map. Consider the pullback*

$$\begin{tikzcd} f^*P \arrow[r] \arrow[d, "f^*\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, "\pi"] \\ Y \arrow[r, "f"'] & X \end{tikzcd} .$$

Then $f^*\pi : f^*P \to Y$ is a principal G-bundle, where explicitly

$$f^*P = \{(y, p) \in Y \times P \,|\, f(y) = \pi(p)\} \subset Y \times P$$

$$G \curvearrowright f^*P \text{ by } g \cdot (y, p) := (y, g \cdot p).$$

Proof: Exercise (take the pullback of a trivializing cover).[8]

Definition 4.1.8. Let $\pi_1 : P_1 \to X$, $\pi_2 : P_2 \to X$ be two principal G-bundles. A *morphism* $\pi_1 \to \pi_2$ is a map $\rho : P_1 \to P_2$ that is G-equivariant and such that

$$\begin{tikzcd} P_1 \arrow[r, "\rho"] \arrow[d, "\pi_1"'] & P_2 \arrow[d, "\pi_2"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

commutes.

A *section* of $\pi : P \to X$ is a continuous map $s : X \to P$ such that $\pi \circ s = \mathrm{id}_X$.

Lemma 4.1.9. *Every morphism of principal G-bundles is an isomorphism.*

Proof. Let ρ be a morphism from a principal G-bundle $\pi_1 : P_1 \to X$ to a principal G-bundle $\pi_2 : P_2 \to X$. I.e.

$$\begin{tikzcd} P_1 \arrow[r, "\rho"] \arrow[d, "\pi_1"'] & P_2 \arrow[d, "\pi_2"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

commutes. To prove that ρ is a homeomorphism, look locally at ρ. Assume that U is a trivializing neighbourhood for both π_1 and π_2. We then have the following diagram, composing with the

[6] I.e. they are smooth manifolds such that multiplication and taking inverses are smooth maps.

[7] Note that $H\backslash G$ is not necessarily a group.

[8] Strictly speaking, $f^*\pi$ is the simple projection onto the first factor, but this is not a very useful description. We will also use the symbol f^* later for maps into pullbacks, not out of them; there should hopefully not arise confusion.

trivializing homeomorphisms:

$$\begin{tikzcd} G \times U \arrow[r, "\rho_U"] \arrow[d, "\mathrm{pr}_2"'] & G \times U \arrow[d, "\mathrm{pr}_2"] \\ U \arrow[r, equal] & U \end{tikzcd},$$

where ρ_U is a G-equivariant map such the diagram commutes. Explicitly, then by G-equivariance again we must have

$$\rho_U(g,u) = (g\overline{\rho}(u), u), \qquad \text{where } \overline{\rho} := \mathrm{pr}_1 \circ \rho_U|_{\{1\}\times U} : U \to G.$$

Then, an inverse is given by $(h,v) \mapsto (h\overline{\rho}(v)^{-1}, v)$.[9] □

Corollary 4.1.10. *Suppose that we have a commutative square*

$$\begin{tikzcd} \tilde{P} \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{\pi}"'] & P \arrow[d, "\pi"] \\ Y \arrow[r, "f"'] & X \end{tikzcd},$$

where $\pi, \tilde{\pi}$ are principal G-bundles and $\tilde{f}$ is G-equivariant. Then it is a pullback square; we have a homeomorphism such that

$$\begin{tikzcd} \tilde{P} \arrow[rr, "\cong"] \arrow[dr, "\tilde{P}"'] & & f^*P \arrow[dl] \\ & Y & \end{tikzcd}$$

commutes.

Proof. By the universal property of the pullback, we get a G-equivariant map $\tilde{P} \to f^*P$.[10] By the previous lemma, this map is an isomorphism. □

Corollary 4.1.11. *A principal G-bundle is trivial if and only if it has a section.*

Proof. Let $\pi : P \to X$ be a principal G-bundle that admits a section $s : X \to P$. We construct a G-equivariant map such that

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d, "\mathrm{pr}_2"'] & P \arrow[d, "\pi"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

commutes: It suffices to construct a map $X \cong \{1\} \times X \to P$, then extend by G-equivariance. This is given by s.[11] By the previous lemma, such a map is already an isomorphism, thus P is trivial.

(The converse direction: If we have an isomorphism $\rho : G \times X \xrightarrow{\sim} P$ such that

$$\begin{tikzcd} G \times X \arrow[r, "\rho", "\cong"'] \arrow[d, "\mathrm{pr}_2"'] & P \arrow[d, "\pi"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

commutes, then $s : X \to P$, $x \mapsto \rho(1,x)$ is a section.) □

Remark 4.1.12. A rank-1-vector bundle is trivial if and only if it admits a nowhere vanishing section. This statement is analogous to the one from the previous corollary, but with an additional condition.

[9] Thus ρ is a local homeomorphism, and since it is bijective, it is a homeomorphism.

[10] It is G-equivariant because $\tilde{f}$ is.

[11] Explicitly, "extending s by G-equivariance" means: Let $\rho : G \times X \to P$, $(g,x) \mapsto g \cdot s(x)$.

Our next goal will be to construct classifying spaces for principal G-bundles.

Definition 4.1.13. Let G be a topological group in CGHaus that has a CW-structure. Define the functor

$$
\begin{aligned}
&\mathrm{Bun}_G : \mathrm{CW}^{\mathrm{op}} \to \mathrm{Sets},\\
&\mathrm{Ob}\,\mathrm{CW} \ni X \mapsto \{\text{isomorphism classes of principal } G\text{-bundles } \pi : P \to X\},\\
&\forall X, Y \in \mathrm{Ob}\,\mathrm{CW} : \mathrm{Hom}_{\mathrm{CW}}(X, Y) \ni (f : X \to Y) \mapsto (\mathrm{Bun}_G(Y) \ni [P] \mapsto [f^*P] \in \mathrm{Bun}_G(X))\,.
\end{aligned}
$$

Pointed version: Define

$$
\begin{aligned}
&\mathrm{Bun}_G^* : \mathrm{CW}_*^{\mathrm{op}} \to \mathrm{Sets}_*,\\
&\mathrm{Ob}\,\mathrm{CW}_* \ni (X, x) \mapsto \{\text{isomorphism classes of pointed principal } G\text{-bundles } \pi : (P, p) \to (X, x)\},\\
&\forall (X, x), (Y, y) \in \mathrm{Ob}\,\mathrm{CW}_* :\\
&\quad \mathrm{Hom}_{\mathrm{CW}^*}((X, x), (Y, y)) \ni (f : (X, x) \to (Y, y)) \mapsto (\mathrm{Bun}_G^*((Y, y)) \ni [P] \mapsto [f^*P] \in \mathrm{Bun}_G((X, x)))\,.
\end{aligned}
$$

[12]

Remark 4.1.14. The natural map forgetting the basepoints $\mathrm{Bun}_G^*(X, x) \to \mathrm{Bun}_G(X)$ is surjective, but in general not injective.

The goal is to use Brown representability (theorem 3.1.1) to get a space BG such that $[X, BG]_* \cong \mathrm{Bun}_G^*(X)$ for all $X \in \mathrm{CW}_*^{\circ}$.[13] Then $[X, BG] \cong \mathrm{Bun}_G(X)$ as well (as we will see). BG will be called a *classifying space* for G.

We need to check: Bun_G is homotopy invariant, it satisfies the Mayer-Vietoris property and the additivity property.

4.2 Existence of classifying spaces

Lemma 4.2.1. *For all $n \in \mathbb{N}$, $\mathrm{Bun}_G(I^n) \cong \{*\}$ and $\mathrm{Bun}_G^*(I^n) \cong \{*\}$.*

Proof. Let $\pi : P \to I^n$ be a principal G-bundle. We need to show: π is "the" trivial bundle, i.e. (by corollary 4.1.11), we have to find a section $s : I^n \to P$. We do this by induction on n.

- *Base case $n = 0$:* trivial, any map $I^0 \to P$ is a section.
- *Induction step $n - 1 \to n$:* We write $P|_{I^{n-1}\times\{0\}} := i^*P = \pi^{-1}(I^{n-1} \times \{0\})$, where $i : I^{n-1} \times \{0\} \hookrightarrow I^n$ is the inclusion. By induction, this bundle (i.e. the map $i^*\pi$) has a section $s_{n-1} : I^{n-1} \times \{0\} \to P|_{I^{n-1}\times\{0\}}$. Consider the pullback diagram

$$
\begin{tikzcd}
P|_{I^{n-1}\times\{0\}} \arrow[r, "\rho"] \arrow[d, shift left, "i^*\pi"] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, "\pi"] \\
I^{n-1} \times \{0\} \arrow[r, hook, "i"'] \arrow[u, shift left, bend left, "s_{n-1}"] & I^n
\end{tikzcd}
\,.
$$

We need to find a section of π. Reordering the diagram, we obtain

$$
\begin{tikzcd}
I^{n-1} \times \{0\} \arrow[r, "\rho\circ s_{n-1}"] \arrow[d, hook, "i"'] & P \arrow[d, "\pi"] \\
I^n \arrow[r, "\mathrm{id}"'] \arrow[ur, dashed, "s_n"] & I^n
\end{tikzcd}
\,.
$$

The dashed map s_n exists, because π is a fiber bundle, in particular a Serre fibration. This proves the induction step and hence the lemma.

[12]Note that the definition implies $p \in \pi^{-1}(x)$ for $\pi \in \mathrm{Ob}\,\mathrm{Bun}_G^*((X, x))$.
The distinguished point in $\mathrm{Bun}_G^*((X, x))$ is the trivial bundle.

[13]When writing $X \in \mathrm{CW}_*^{\circ}$, it is understood with some basepoint, despite the notation.

$\square$

Lemma 4.2.2. *Let*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d, "\bar{j}"] \\ C \arrow[r, "\bar{i}"'] & X \end{tikzcd}$$

be a pushout square in Top. *Let $\pi : P \to X$ be a principal G-bundle over X. Then the canonical map*

$$P|_B \cup_{P|_A} P|_C \xrightarrow{\sim} P$$

is an isomorphism of G-bundles, where $P|_B := \bar{j}^* P$, $P|_C := \bar{i}^* P$, $P|_A := i^* P|_B = j^* P|_C$.

Moreover, if P_1, P_2 are principal G-bundles over X and there is an isomorphism $P_1 \xrightarrow{\sim} P_2$, then the induced square

$$\begin{tikzcd} P_1 \arrow[r, "\sim"] & P_2 \\ P_1|_B \cup_{P_1|_A} P_1|_C \arrow[u, "\sim"] \arrow[r, "\sim"'] & P_2|_B \cup_{P_2|_A} P_2|_C \arrow[u, "\sim"'] \end{tikzcd}$$

commutes.

Proof. Let $\tilde{P} := P|_B \cup_{P|_A} P|_C$. We need to show that this is a principal G-bundle and that the (canonical) map $\tilde{P} \to P$ is G-equivariant. Then lemma 4.1.9 yields that $\tilde{P} \xrightarrow{\sim} P$ is an isomorphism.

Remark: The map $\tilde{P} \to X$ is given as a pushout map:

$$\begin{tikzcd} P|_A \arrow[rr] \arrow[dr] \arrow[dd, "j^*(\bar{i}^*(\pi))"'] & & P|_C \arrow[dr] \arrow[dd, "\bar{i}^*(\pi)"'] & \\ & P|_B \arrow[rr] \arrow[dd, "\bar{j}^*(\pi)"'] & & \tilde{P} \arrow[dd, dashed] \\ A \arrow[rr, "j"] \arrow[dr, "i"'] & & C \arrow[dr, "\bar{i}"] & \\ & B \arrow[rr, "\bar{j}"'] & & X \end{tikzcd} .$$

Since $G \in$ CGHaus, the product "$G\times$" commutes with pushouts and the action $G \curvearrowright \tilde{P}$ (i.e. the multiplication map $G \times \tilde{P} \to \tilde{P}$) can be defined again as a pushout map. In particular it is continuous.[14]

We want to show: $\tilde{P}$ is locally trivial. Let $x \in X$. Because P is locally trivial, there exists a trivializing neighbourhood U of x for P. Denote $P_U := P|_U$; consider then $\tilde{P}|_U = P_U|_{B\cap U} \cup_{P_U|_{A\cap U}}$

[14] By definition it is fiberwise.

$G \times \cdot$ commutes with pushouts since it has a right adjoint $(\cdot)^G$ (the space of maps from G into $\cdot$, with the compact-open topology).

$P_U|_{C\cap U}$.[15] Since P_U is trivial, it suffices to show $\tilde{P}|_U \cong P_U$. Consider

$$\begin{tikzcd}
G\times(A\cap U) \arrow[rr, "\mathrm{id}_G\times j|_U"] \arrow[dr, "\mathrm{id}_G\times i|_U"'] \arrow[dd] & & G\times(C\cap U) \arrow[dr, "\mathrm{id}_G\times \bar{i}|_U"] \arrow[dd] & \\
& G\times(B\cap U) \arrow[rr, "\mathrm{id}_G\times \bar{j}|_U"'] \arrow[dd] & & P_U \cong G\times U \arrow[dd] \\
A\cap U \arrow[rr] \arrow[dr] & & C\cap U \arrow[dr] & \\
& B\cap U \arrow[rr] & & U
\end{tikzcd}.$$

Again, since $G\times$ commutes with pushouts, the pushout in the upper face is homeomorphic to $G\times U \cong P_U$ (since the pushout of $(A\cap U)(B\cap U)(C\cap U)$ is U itself). On the other hand, by definition of the pushout it is $P_U|_{B\cap U} \cup_{P_U|_{A\cap U}} P_U|_{C\cap U} = \tilde{P}|_U$.

Thus $\tilde{P}$ is locally trivial and hence a principal G-bundle. The map $\tilde{P}\to P$ is obtained from the universal property of the pushout and it is verified that it is G-equivariant. As stated, lemma 4.1.9 yields the first claim. The second one can directly be verified. □

Theorem 4.2.3 (Homotopy invariance, Part I)**.** *Let* $X\in \mathrm{Ob}\,\mathrm{CW}$ *or* $\mathrm{Ob}\,\mathrm{CW}_*$*. Denote* $\mathrm{pr}: X\times I \to X$ *the projection. Then* $\mathrm{Bun}_G(\mathrm{pr}) : \mathrm{Bun}_G(X) \xrightarrow{\sim} \mathrm{Bun}_G(X\times I)$ *and* $\mathrm{Bun}^*_G(\mathrm{pr}) : \mathrm{Bun}^*_G(X) \xrightarrow{\mathrm{Bun}} {}^*_G (X\times I)$ *are bijections.*

Proof. We only prove the unpointed case; the pointed one is similar. Let $i_0 : X \hookrightarrow X\times I, x\mapsto (x,0)$ be the inclusion. We show that $\mathrm{Bun}_G(i_0)$ is inverse to $\mathrm{Bun}_G(\mathrm{pr})$.

We have $\mathrm{pr}\circ i_0 = \mathrm{id}_X$, hence by functoriality $\mathrm{Bun}_G(i_0)\circ \mathrm{Bun}_G(\mathrm{pr}) = \mathrm{id}_{\mathrm{Bun}_G(X)}$. It remains to show: $\mathrm{Bun}_G(\mathrm{pr})\circ \mathrm{Bun}_G(i_0) = \mathrm{id}_{\mathrm{Bun}_G(X\times I)}$.

Let $\pi : P \to X\times I$ be a principal G-bundle. We need to show: $P \cong \mathrm{pr}^* i_0^* P$.[16] It suffices to construct a G-equivariant map $f : P \to P_0 := i_0^* P$ such that

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, "\pi"'] & P_0 \arrow[d, "\pi_0"] \\
X\times I \arrow[r, "\mathrm{pr}"'] & X
\end{tikzcd}$$

commutes, since then we get $P\cong \mathrm{pr}^* i_0^* P$ by corollary 4.1.10.

We construct f by induction over the "half-skeletal filtration" of $Y := X\times I$. Let $Y_0 := X\times\{0\}$ and for $n\in\mathbb{N}$, let $Y_n := (X\times\{0\})\cup(X^{(n-1)}\times I)$. So $Y_n\subset Y$ and $Y = \mathrm{colim}_n Y_n$.

- *Base case* $n=0$*:* Over $Y_0 = X\times\{0\}$, define f to be id_{P_0}.
- *Induction step* $n\mapsto n+1$*:* By induction assumption, f is already constructed over Y_n. We need to extend it to $P|_{Y_{n+1}}$.

  We need to understand how we attach cells to Y. Assume that we are given n-cells $(\Phi_j : I^n \to X^{(n)})_{j\in J}$ of X. This yields a pushout

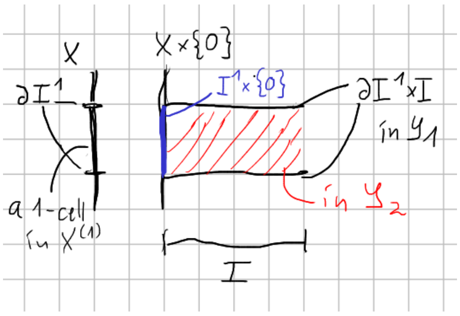
$$\begin{tikzcd}
\bigsqcup_{j\in J}(\partial I^n\times I \cup I^n\times\{0\}) \arrow[r] \arrow[d, hook] & Y_n \arrow[d] \\
\bigsqcup_{j\in J}(I^n\times I) \arrow[r] & Y_{n+1}\subset Y
\end{tikzcd}.$$

[15] Equality holds by definition of $\tilde{P}$ and some point-set topology. Alternatively it can actually be done using an analogy of Mather's second cube theorem for 1-pushouts and 1-pullbacks.

[16] Recall that Bun_G only considers isomorphism classes.

In the following, we will view the cubes $I^n \times I$ as subspaces of Y.

Sketch ($n = 1$):



Define $P_n := P|_{Y_n}$ and let $P_{n+1} = P_n \cup_{\coprod(\partial I^n \times I \cup I^n \times \{0\})} P|_{\coprod(I^n \times I)}$ be a pushout. We have a map $f_n : P_n \to P_0$ and want to extend it to $f_{n+1} : P_{n+1} \to P_0$. This we can do on every of the cubes individually: By lemma 4.2.1, $P|_{I^n \times I}$ is trivial. Then also $P|_{\partial I^n \times I \cup I^n \times \{0\}}$ is trivial (the restriction of a trivial bundle is trivial). So we need to find the dashed (G-equivariant) map ψ in the diagram

$$\begin{tikzcd} P|_{\partial I^n \times I \cup I^n \times \{0\}} \arrow[r] \arrow[d, hook] \arrow[dr, "\tilde{f}_n"] & P_n \arrow[d, "f_n"] \\ P|_{I^n \times I} \arrow[r, dashed, "\psi"'] & P_0 \end{tikzcd} ,$$

where the two bundles on the left are trivial ones and $\tilde{f}_n$ is defined as the composition. Using that the bundles are trivial, we want to find the dashed (G-equivariant) map ψ in

$$\begin{tikzcd} G \times (\partial I^n \times I \cup I^n \times \{0\}) \arrow[d, hook] \arrow[dr, "\tilde{f}_n"] & \\ G \times (I^n \times I) \arrow[r, dashed, "\psi"'] & P_0 \end{tikzcd} .$$

[17] Now, since π_0 is a fiber bundle (thus a Serre fibration), we can find the dashed lift in

$$\begin{tikzcd} \partial I^n \times I \cup I^n \times \{0\} \arrow[r, "\tilde{f}_n|_{\{e\} \times (\partial I^n \times I \cup I^n \times \{0\})}"] \arrow[d, hook] & P_0 \arrow[d, "\pi_0"] \\ I^n \times I \arrow[r, "\mathrm{pr}|_{I^n \times I}"'] \arrow[ur, dashed, "\psi|_{\{e\} \times I^n \times I}"] & X \end{tikzcd} .$$

(See lemma 1.2.8.) As indicated, the lift shall be $\psi|_{\{e\} \times I^n \times I} : \{e\} \times I^n \times I \to P_0$. We extend it by G-equivariance to $\psi : I^n \times I \to P_0$.[18] Doing this for all $j \in J$ (i.e. all attached cells), we can construct f_{n+1}.

– Finally, let $f := \operatorname{colim}_n f_n : P = \operatorname{colim}_n P_n \to P_0$. This is G-equivariant, hence by corollary 4.1.10 we obtain $P \cong \mathrm{pr}^* i_0^* P$.

Summarizing, we have shown that $\mathrm{pr}^* : \mathrm{Bun}_G(X) \to \mathrm{Bun}_G(X \times I)$ is an isomorphism with inverse i_0^*. □

Corollary 4.2.4. Bun_G *is a homotopy invariant functor, i.e.* Bun_G *factors through a functor* $\mathrm{Bun}_G : \mathrm{hCW}^{\mathrm{op}} \to \mathrm{Sets}$.[19] *The same holds for the pointed version: We have a functor* $\mathrm{Bun}_G^* : \mathrm{hCW}_*^{\mathrm{op}} \to \mathrm{Sets}_*$.

[17] Strictly speaking, we have redefined $\tilde{f}_n$ by composing with the trivializing homeomorphism.

[18] There is only one possible G-equivariant choice.

[19] We use the same symbol; there should not arise confusion.

Proof. Let $X, Y \in \mathrm{CW}$ and $f, g : X \to Y$ be homotopic through a homotopy H, i.e. $H : X \times I \to Y$ is such that $H \circ i_0 = f$, $H \circ i_1 = g$. We need to show: $\mathrm{Bun}_G(f) = \mathrm{Bun}_G(g)$. Now $\mathrm{Bun}_G(i_0)$ is an inverse to $\mathrm{Bun}_G(\mathrm{pr})$ by the previous theorem, and analogously to shown there for i_0, $\mathrm{Bun}_G(i_1)$ is also a left inverse (this is a purely formal argument). Thus $\mathrm{Bun}_G(i_1) = \mathrm{Bun}_G(i_1) \circ \mathrm{Bun}_G(\mathrm{pr}) \circ \mathrm{Bun}_G(i_0) = \mathrm{Bun}_G(i_0)$. Therefore

$$\mathrm{Bun}_G(f) = \mathrm{Bun}_G(i_0) \circ \mathrm{Bun}_G(H) = \mathrm{Bun}_G(i_1) \circ \mathrm{Bun}_G(H) = \mathrm{Bun}_G(g).$$

□

We continue with the proof of additivity for Bun_G.

Proposition 4.2.5. *Let $i : A \hookrightarrow X$ be a closed cofibration (e.g. a subcomplex of a CW-complex). Let $\pi : P \to X$ be a principal G-bundle over X and assume that there exists $s : A \to P$ such that*

$$\begin{tikzcd} & P \arrow[dr, "\pi"] & \\ A \arrow[ur, "s"] \arrow[rr, hook, "i"'] & & X \end{tikzcd}$$

commutes. Then there exists $U \subset X$ open with $A \subset U$ and $s_U : U \to P$, such that

$$\begin{tikzcd} & P \arrow[dr, "\pi"] & \\ U \arrow[ur, "s_U"] \arrow[rr, hook] & & X \end{tikzcd}$$

commutes and $s_U|_A = s$.

Proof. $i : A \hookrightarrow X$ is a neighbourhood deformation retract (NDR), i.e. there exists $h : X \times I \to X$ such that

$$\begin{aligned} & h_0 = \mathrm{id}_X, \\ & \forall a \in A, t \in I : h(a,t) = a, \\ & \exists U \subset X \text{ open}, A \subset U : h_1(U) = A. \end{aligned}$$

(Such an h can be found using the retraction $X \times I \to A \times I \cup X \times \{0\}$; this exists by a converse of corollary 1.4.7.) Denote $r := h_1|_U : U \to A$. We have a homotopy $h|_U : U \times I \to X$. Let $i_A : A \hookrightarrow U$, $i : A \hookrightarrow X$, $j : U \hookrightarrow X$ be the inclusions (thus $j \circ i_A = i$). We have $r \circ i_A = \mathrm{id}_A$, $h_0|_U = j$, $h_1|_U = i \circ r$ (when viewed as a map $U \to X$).

By homotopy invariance of Bun_G, $P_U := j^*P \cong r^*i^*P =: r^*P_A$. Equivalently (as follows from the proof of theorem 4.2.3), there is a pullback diagram

$$\begin{tikzcd} P_U \arrow[r, "\psi"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P_A \arrow[d] \\ U \arrow[r, "r"'] & A \end{tikzcd}$$

where ψ is G-equivariant. Define $s_U := r^*(s)$ using the universal property of pullbacks and the maps $\mathrm{id}_U, s \circ r$ using the following diagram:[20]

$$\begin{tikzcd}
U \arrow[dr, dashed, "s_U"] \arrow[drr, bend left, "s\circ r"] \arrow[ddr, bend right, "\mathrm{id}_U"'] & & \\
& P_U \arrow[r, "\psi"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P_A \arrow[d] \\
& U \arrow[r, "r"'] & A
\end{tikzcd} .$$

Verify that this extends s: Glueing another pullback diagram to the one above, we obtain the diagram

$$\begin{tikzcd}
A \arrow[dr, dashed, "i_A^*(s_U)"'] \arrow[drr, bend left, "s_U\circ i_A"] \arrow[ddr, bend right, "\mathrm{id}_A"'] & U \arrow[dr, dashed, "s_U"] \arrow[drr, bend left, "s\circ r"] \arrow[ddr, "\mathrm{id}_U"] & & \\
& P_A \arrow[r, "\phi"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P_U \arrow[r, "\psi"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P_A \arrow[d, bend left] \\
& A \arrow[r, "i_A"'] & U \arrow[r, "r"'] & A \arrow[u, dashed, "s"]
\end{tikzcd} .$$

Since $r \circ i_A = \mathrm{id}_A$, we have $P_A = \mathrm{id}_A^* P_A = i_A^* r^* P_A = i_A^* P_U$, thus the space in the left upper corner of the composed pullbacks is indeed P_A. Furthermore, $\psi \circ \phi = \mathrm{id}_{P_A}$.[21] Then

$$s_U|_A := i_A^*(s_U) = i_A^*(r^*(s)) \overset{(*)}{=} (r \circ i_A)^*(s) = \mathrm{id}_A^*(s) = s,$$

where the equality $(*)$ follows from functoriality of the pullback maps.[22] This was to be shown. $\square$

Corollary 4.2.6. *For any family $\{(X_\alpha, x_\alpha)\}_\alpha$, where $X_\alpha \in \operatorname{Ob}\mathrm{CW}_*$ for all α, the inclusions induce a canonical isomorphism* $\mathrm{Bun}_G^*(\bigvee_\alpha (X_\alpha, x_\alpha)) \xrightarrow{\sim} \prod_\alpha \mathrm{Bun}_G^*(X_\alpha, x_\alpha)$.

Proof. To be precise, the map exists because Bun_G is a functor.

- Surjectivity: For all α, let $\pi_\alpha : (P_\alpha, p_\alpha) \to (X_\alpha, x_\alpha)$ be principal G-bundles. We need to define a principal G-bundle $\pi : P \to \bigvee_\alpha (X_\alpha, x_\alpha)$. Let $P := (\coprod_\alpha P_\alpha)/(\forall \alpha, \beta \forall g \in G : g \cdot p_\alpha \sim g \cdot p_\beta)$ and π be defined as π_α for each α in the disjoint union. Then G acts on P in fibers of π freely and transitively.

  By the arguments as in corollary 4.1.11 or example 4.1.5, to prove local triviality it suffices to check that π locally on X has a section.[23]

  If $x \in X_\alpha \backslash \{x_\alpha\}$ for some α, then $X_\alpha \backslash \{x_\alpha\}$ is an open neighbourhood of x in $\bigvee_\alpha X_\alpha$, over which P equals $P_\alpha|_{X_\alpha \backslash \{x_\alpha\}}$. This admits a local section.

[20] Note that this is the definition of $r^*(s)$. Often in algebra r^* is also used for simple precomposition, i.e. $r^*(s) = s \circ r$, but not here. Both notions correspond to "pulling back".

[21] This uses a 2-out-of-3-property for 1-pullback squares and uniqueness of pullbacks, which yield a functoriality property.

[22] When writing $s_U|_A := i_A^*(s_U)$, we view P_A is a subspace of P_U. This is indeed the case, via ϕ. Similarly we did already when writing $s : A \to P_A$.

[23] If $\pi : Q \to Y$ is a map and $G \curvearrowright Q$ acts freely and transitively in fibers of q, then if $s : Y \to Q$ is a section of q we obtain a commutative diagram

$$\begin{tikzcd}
G \times Y \arrow[rr, "\phi"] \arrow[dr, "\mathrm{pr}"'] & & Q \arrow[dl, "\pi"] \\
& Y &
\end{tikzcd} ,$$

where $\phi(g, y) := g \cdot s(y)$ for $(g, y) \in G \times Y$. This is a bijection if G acts freely and transitively on the fibers.
Since $G \in \mathrm{CGHaus}$, it is a homeomorphism, thus Q is a (trivial, since we assumed the section globally) G-bundle. (The proof that ϕ even is a homeomorphism is not really trivial, but omitted. The main idea is the following: Also $Q, Y \in \mathrm{CGHaus}$, since they are CW-complexes. Then $G \times Y \in \mathrm{CGHaus}$. By the "compact-Hausdorff trick" from Algebraic Topology 1, for every compact subset $K \subset G \times Y$, $\phi|_K : K \to \phi(K)$ is a homeomorphism. Then one uses that $G \times Y$ is compactly generated to show that ϕ is a homeomorphism.)

The main part is to check that π admits sections in a neighbourhood of $* := [\{x_\alpha\}] \in \bigvee_\alpha x_\alpha$.[24] Because $\mathrm{Bun}_G^*(*) = \{*\}$, we have the lift s_α in

$$\begin{tikzcd} & P_\alpha \arrow[d, "\pi_\alpha"] \\ * \arrow[r, hook, "x_\alpha"'] \arrow[ur, dashed, "s_\alpha"] & X_\alpha \end{tikzcd} .$$

($s_\alpha(*) = p_\alpha$ by definition.) By the proposition before, for all α there exists $U_\alpha \subset X_\alpha$ open, $s_{U_\alpha} : U_\alpha \to P_\alpha$ with $s_{U_\alpha}(x_\alpha) = p_\alpha$. Then $s := \bigvee_\alpha s_{U_\alpha} : \bigvee_\alpha U_\alpha \to P$ is a section of P in a neighbourhood of $*$.

– Injectivity: Assume that $(P, p) \to (\bigvee_\alpha X_\alpha, *)$ is such that for all α, the map $(P_\alpha, p) \to (X_\alpha, x_\alpha)$ is trivial. By corollary 4.1.11, there hence exists a section $s_\alpha : X_\alpha \to P_\alpha$ (with $s_\alpha(x_\alpha) = p$). Then $\bigvee_\alpha s_\alpha =: s : \bigvee_\alpha X_\alpha \to P$ is a section of $(P, p) \to (\bigvee_\alpha X_\alpha, *)$. So again by corollary 4.1.11, P is trivial.

□

Corollary 4.2.7 (Mayer-Vietoris property for Bun_G^*)**.** *Let*

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, hook, "j"'] & B \arrow[d, hook] \\ C \arrow[r, hook] & X \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

be a pushout square, where i, j are CW-subcomplexes. Let P_A, P_B, P_C be principal G-bundles over A, B, C, respectively and assume that $j^*P_C \cong P_A \cong i^*P_B$.[25] *Then $P := P_C \cup_{P_A} P_B$ is a principal G-bundle over X. In particular we have a surjection* $\mathrm{Bun}_G^*(X) \twoheadrightarrow \mathrm{Bun}_G^*(C) \times_{\mathrm{Bun}_G^*(A)} \mathrm{Bun}_G^*(B)$.

Proof. As noted in the proof of lemma 4.2.2, the product $G\times$ commutes with pushouts. Thus we get an action of G on P. Furthermore this action is then free and transitive on the fibers of the (canonical) projection $\pi : P \to X$. Thus it remains to verify local triviality. By the argumentation from corollary 4.1.11 (resp. example 4.1.5), it suffices to find local sections of π.

– Let $x \in C\backslash A$ (or, analogously, $x \in B\backslash A$). Use then $P|_{C\backslash A} \cong P_C|_{C\backslash A}$ (resp. $P|_{B\backslash A} \cong P_B|_{B\backslash A}$), which has local sections.

– Let $x \in A$. There is an open neighbourhood $U_C \subset C$ of x trivializing P_C, which is equivalent to the existence of a section $s_C : U_C \to P_C$.

Fact: There exists $W \subset B$ open such that $W \cap A = U_C \cap A \hookrightarrow W$ is a closed cofibration.[26]

Then by proposition 4.2.5 we can extend j^*s_C to a section $s_B : U_B \to P_B$ for some $U_B \subset W$ open, where $W \cap A \subset U_B$ and hence $W \cap A = U_B \cap A = U_C \cap A$.[27] Thus we have: A section s_C of P_C over U_C, and a section s_B of P_B over U_B, that agree on $U_C \cap A = U_B \cap A$. These we can glue together to get a section of P over $U_C \cup U_B$, which is a neighbourhood of x.

□

Remark 4.2.8. The map in the previous corollary is not an isomorphism. For example, let $G = \mathbb{Z}$, then

$$* \neq \mathrm{Bun}_G^*(S^1) \twoheadrightarrow \mathrm{Bun}_G^*(I) \times_{\mathrm{Bun}_G^*(S^0)} \times \mathrm{Bun}_G^*(I) = *,$$

[24] All x_α are contained in the same equivalence class, hence the notation.

[25] This is equivalent, by corollary 4.1.10, to the existence of G-equivariant maps $P_A \to P_B$, $P_A \to P_C$.

[26] Note that we have relative closedness in W, not in B.

One can define W by induction on the cells of B, similarly to the construction of a neighbourhood of A retracting onto A.

[27] On one hand, $U_B \subset W$, thus $U_B \cap A \subset W \cap A$, but on the other hand $W \cap A \subset U_B$ and by definition $W \cap A \subset W$, thus $W \cap A \subset U_B \cap A$.

where the pullback on the right side must be trivial because already $\mathrm{Bun}_G^*(I) = *$ (by lemma 4.2.1).

Theorem 4.2.9. *There exists a space $BG \in \mathrm{CW}_*^\circ$ and a principal G-bundle $\pi : EG \to BG$ such that for all $X \in \mathrm{CW}_*^\circ$, the map*

$$[X, BG]_* \xrightarrow{\sim} \mathrm{Bun}_G^*(X), \qquad f \mapsto f^*EG$$

is an isomorphism in Sets_*.

Proof. We apply the Brown representability theorem (theorem 3.1.1) to Bun_G. The conditions are satisfied because of corollary 4.2.4, corollary 4.2.6 and corollary 4.2.7. □

Remark 4.2.10. Why do we care that Bun_G^* or Bun_G are representable?

- We get examples of principal G-bundles by constructing maps $X \to BG$.
- We can e.g. understand $\mathrm{Bun}_G^*(S^n)$ as $\pi_n(BG)$. We will find $\Omega BG \simeq G$ as topological spaces, so $\pi_n(BG) = \pi_{n-1}(G)$ for $n > 1$. This gives a classification of (pointed) principal G-bundles over S^n in terms of the homotopy groups of G.
- We can understand natural transformations $\mathrm{Bun}_G^*(\cdot) \to F(\cdot)$ for any functor $F : (\mathrm{CW}_*^\circ)^{\mathrm{op}} \to \mathrm{Sets}_*$ by the Yoneda lemma as $F(BG)$.

  Example: $F = H^n(\cdot, A)$ (cohomology with coefficients in A, e.g. $H^n(X, \mathbb{Q}) := \mathrm{Hom}_{\mathbb{Q}}(H_n(X, \mathbb{Q}), \mathbb{Q})$ - see later).

Upshot: It would be nice to have a geometric construction of BG.

Remark 4.2.11. The diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d] & EG \arrow[d] \\ * \arrow[r] & BG \end{tikzcd}$$

is a homotopy pullback (since $EG \to BG$ is a fiber bundle, hence a Serre fibration, and by corollary 2.4.10). We will see that $EG \simeq *$, thus $\Omega BG \simeq G$ (as stated before).

Assume that we have found a contractible CW-complex X with a free G-action, then $X \to G\backslash X$ is the desired universal bundle: We have a pullback diagram

$$\begin{tikzcd} * \simeq X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EG \simeq * \arrow[d] \\ G\backslash X \arrow[r] & BG \end{tikzcd}$$

(since BG is the classifying space; note that $X \to G\backslash X$ is a principal G-bundle and $G\backslash X$ is a CW-complex). By the long exact sequences for Serre fibrations (corollary 1.2.7) for

$$\begin{tikzcd} G \arrow[r] \arrow[d] & X \simeq * \arrow[d] \\ * \arrow[r] & G\backslash X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[r] \arrow[d] & EG \simeq * \arrow[d] \\ * \arrow[r] & BG \end{tikzcd}$$

we obtain that the map $G\backslash X \to BG$ induces isomorphisms on all homotopy groups, hence by Whitehead's theorem 1.7.4, it is a homotopy equivalence.

For $G = \mathrm{Gl}_k(\mathbb{C})$ or $G = \mathrm{Gl}_k(\mathbb{R})$, there is a principal G-bundle $V(k, n) \to \mathrm{Gr}(k, n)$, where $\mathrm{Gr}(k, n) = \{k$-dimensional subspaces in n-dimensional space (over $\mathbb{R}$ or $\mathbb{C}\}$ are Grassmannian manifolds. (Example: $\mathrm{Gr}(1, n) = \mathbb{R}P^n$ or $\mathbb{C}P^n$.)

There are inclusions $\mathrm{Gr}(k, n) \subset \mathrm{Gr}(k, n+1) \subset \ldots$. Letting $\mathrm{Gr}(k, \infty) := \mathrm{colim}_n \mathrm{Gr}(k, n)$, we obtain a bundle $V(k, \infty) \to \mathrm{Gr}(k, \infty)$. We will find $V(k, \infty) \simeq *$, thus $\mathrm{Gr}(k, \infty) = B\mathrm{Gl}_k(\mathbb{R})$ or $\mathrm{Gr}_{k,\infty} = B\mathrm{Gl}_k(\mathbb{C})$.

Example 4.2.12 (Principal $\mathbb{Z}$-bundles over S^1)**.** We will see $S^1 \simeq B\mathbb{Z}$, and $p : \mathbb{R} \to \mathbb{Z}$, $p(x) = e^{2\pi i x}$ is the universal principal $\mathbb{Z}$-bundle. I.e., by Brown representability (theorem 3.1.1), every pointed principal $\mathbb{Z}$-bundle is a pullback of p.

Therefore, $\mathrm{Bun}^*_{\mathbb{Z}}(S^1) = [S^1, B\mathbb{Z}]_* = [S^1, S^1]_* = \mathbb{Z}$. Explicitly: Let $f_n : S^1 \to S^1, z \mapsto z^n$ be a map of degree n (where S^1 is understood as a subset of $\mathbb{C}$). The principal $\mathbb{Z}$-bundle over S^1 corresponding to f_n is given by the pullback

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "q_n"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{R} \arrow[d, "p"] \\ S^1 \arrow[r, "f_n"'] & S^1 \end{tikzcd} .$$

The pullback of a covering of degree n is again a covering of degree n (exercise), thus $X_n \cong \mathbb{R} \times \mathbb{Z}/n$, where $\mathbb{Z}/n$ is the quotient group of $\mathbb{Z}$ with n elements (since $\mathbb{R}$ is contractible, all coverings are trivial).

Then, one computes that the map q_n is given by $q_n : \mathbb{R} \times \mathbb{Z}/n \to S^1$, $q_n(x, k) = e^{\frac{2\pi i}{n}(x-k)}$: In this case, the diagram

$$\begin{tikzcd} \mathbb{R} \times \mathbb{Z}/n \arrow[r, "\mathrm{pr}"] \arrow[d, "q_n"'] & \mathbb{R} \arrow[d, "p"] \\ S^1 \arrow[r, "f_n"'] & S^1 \end{tikzcd}$$

commutes, since

$$\begin{tikzcd} (x,k) \arrow[r, mapsto] \arrow[d, mapsto] & x \arrow[d, mapsto] \\ e^{\frac{2\pi i}{n}(x-k)} \arrow[r, mapsto] & e^{2\pi i(x-k)} = e^{2\pi i x} \end{tikzcd} .$$

The action of $\mathbb{Z}$ on $\mathbb{R}$: For $m \in \mathbb{Z}$, it is given by $x \mapsto x + m$.

The action of $\mathbb{Z}$ on $\mathbb{R} \times \mathbb{Z}/n$: For $m \in \mathbb{Z}$, $(x, k) \mapsto (x + m, k + m)$. This is free and transitive on fibers, thus $\mathbb{R} \times \mathbb{Z}/n$ is a principal $\mathbb{Z}$-bundle over S^1. Since the projection pr is $\mathbb{Z}$-equivariant (this follows directly from the definition), we obtain $\mathbb{R} \times \mathbb{Z}/n = f_n^*(\mathbb{R})$ by corollary 4.1.10.

Since $[S^1, B\mathbb{Z}]_* = \mathbb{Z}$ (see above), these are all pointed (or unpointed, see later) principal $\mathbb{Z}$-bundles over S^1 up to isomorphism.

4.3 Properties and construction of classifying spaces

Proposition 4.3.1 ("Clutching construction")**.** *Let $X \in \mathrm{Ob}\,\mathrm{CW}_*$. Then there exists a natural isomorphism of pointed sets $\mathrm{Bun}^*_G(\Sigma' X) \cong [X, G]_*$, where G is pointed by the unit element e.*

Proof. We construct maps in both directions.

– "$\to$": Let P be a principal G-bundle on $\Sigma' X$. Denote $C^+X := X \times I/(X \times \{1\} \cup \{x\} \times I)$, $C_-X := X \times I/(X \times \{0\} \cup \{x\} \times I)$, where x is the basepoint of X.[28] Then we have a pushout

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & C^+X \arrow[d] \\ C^-X \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \Sigma' X \end{tikzcd} .$$

Since $C^{\pm}X \sim *$ and Bun^*_G is homotopy invariant (corollary 4.2.4), $P|_{C^{\pm}X}$ is trivial. Choose

[28] For CW-complexes, these are homotopy equivalent to the "usual" cones. The reason has been stated in the proof of the Blakers-Massey theorem 2.7.1 in section II.11; use $I \simeq *$.

trivializations $G \times C^{\pm}X \xrightarrow{\sim} P|_{C^{\pm}X}$. Then consider the composition ψ defined by

$$\begin{tikzcd}
P|_X \arrow[r, equal] \arrow[dr, equal] & (P|_{C^-X})|_X & (G \times C^-X)|_X \arrow[l, "\rho_-|_{G\times X}"'] \arrow[r, equal] & G \times X \arrow[d, "\sim"', "\psi:=(\rho_+|_{G\times X})^{-1}\circ(\rho_-)|_{G\times X}"] \\
& (P|_{C^+X})|_X & (G \times C^+X)|_X \arrow[l, "\rho_+|_{G\times X}"] \arrow[r, equal] & G \times X
\end{tikzcd}.$$

Then ψ is a G-equivariant homeomorphism over X; the diagram

$$\begin{tikzcd}
G \times X \arrow[rr, "\psi"] \arrow[dr] & & G \times X \arrow[dl] \\
& X &
\end{tikzcd}$$

commutes. Since ψ is fiberwise, we can write $\psi(g,x) = (\tilde{\psi}(g,x), x)$ for $\tilde{\psi} : G \times X \to G$. Since ψ is G-equivariant, we furthermore have $\psi(g,x) = \psi(g \cdot (e,x)) = g \cdot \psi(e,x) = g \cdot (\tilde{\psi}(e,x), x)$, thus ψ is uniquely determined by $f : X \to G$, $x \mapsto \tilde{\psi}(e,x)$. In other words, from ψ we get a map $f : X \to G$.

The homotopy class of f does not depend on the choices of $\rho_\pm$: These are defined up to a G-equivariant fiberwise automorphism of $G \times C^{\pm}X$, i.e. (by the same reasoning as above for f) up to a map $\kappa_\pm : C^{\pm}X \to G$. Going through the construction, then f changes to $h : X \to G$, $h(x) = \kappa_-(x) \cdot f(x) \cdot \kappa_+(x)^{-1}$, where the multiplication and inverse are group operations. However, κ_+, κ_- are defined on the cones (and $\kappa_+(\cdot)^{-1}$ as well). Then $\kappa_-|_X, \kappa_+(\cdot)^{-1}|_X$ are homotopic to the constant map to e, where the homotopy is given by κ_- on C_-X and $\kappa_+(\cdot)^{-1}$ on $C_+(X)$. Hence $[f] = [h] \in [X, G]_*$.

Therefore we obtain a well-defined map $\mathrm{Bun}_G^*(\Sigma' X) \to [X, G]_*$.

– "$\leftarrow$": Let $f : X \to G$; we need to find a principal G-bundle over $\Sigma' X$. Consider the pushout

$$\begin{tikzcd}
P_X := G \times X \arrow[r, hook, "(g,x)\mapsto(f(x)g,\, i_X^+(x))"] \arrow[d, hook, "(g,x)\mapsto(g,\, i_X^-(x))"'] & G \times C_+X =: P_X^+ \arrow[d] \\
P_X^- := G \times C_-X \arrow[r] \arrow[ur, phantom, "\ulcorner"] & P(f)
\end{tikzcd},$$

where $i_X^\pm : X \to C^{\pm}X$ denotes the respective inclusions. Then $P(f)$ is a principal G-bundle over $\Sigma' X$ (essentially by the arguments used in the proof of the Mayer-Vietoris property for Bun_G^*, corollary 4.2.7).

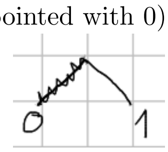
We verify that this only depends on the homotopy type of f: Let $f_0, f_1 : X \to G$ be homotopic maps via a homotopy $h : X \times I \to G$. This yields a bundle $P(h)$ over $\Sigma'(X \times I)$, and one verifies that with the inclusions $j_0, j_1 : \Sigma' X \to \Sigma'(X \times I)$ (into "beginning" and "end" of the interval), we have $j_{0,1}^* P(h) = P(f_{0,1})$. Since $j_0 \sim j_1$, homotopy invariance of Bun_G (corollary 4.2.4) yields $P(f_0) = P(f_1)$.

Therefore, we obtain a well-defined map $[X, G]_* \to \mathrm{Bun}_G^*(\Sigma' X)$.

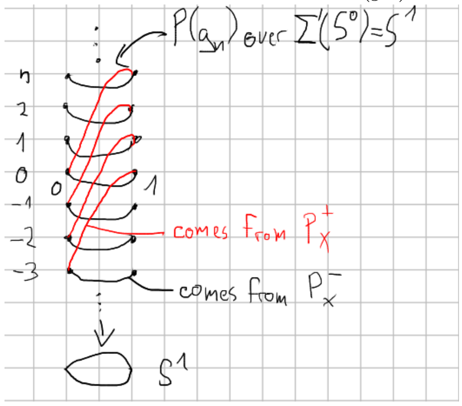
– One verifies that the maps are inverse to each other.

$\square$

Example 4.3.2. Let $G := \mathbb{Z}$, $g_n : \{0, 1\} =: S^0 \mapsto \mathbb{Z}$, $0 \mapsto 0$, $1 \mapsto n$ (where both S^0 and $\mathbb{Z}$ are pointed with 0). We have $C^{\pm}(S^0) = I = [0, 1]$ (up to homeomorphism).



Then we obtain the bundles $P(g_n)$ over $\Sigma' S^0 = S^1$; example $n = 3$:



(Compare example 4.2.12.)

The goal is now to understand classifying spaces concretely.

Remark 4.3.3. Let $p : EG \to BG$ be the universal bundle which exists by theorem 4.2.9. We now have a chain of isomorphisms of functors

$$[X, \Omega BG]_* \cong [\Sigma' X, BG]_* \cong \mathrm{Bun}^*_G(\Sigma' X) \cong [X, G]_*,$$

where the first isomorphism is due to the Σ'-Ω-adjunction (proposition 2.4.11), the second since BG is a classifying space (theorem 4.2.9), and the third due to the clutching construction just proved (proposition 4.3.1). By the Yoneda lemma, we get a homotopy equivalence (i.e. isomorphism in hCW°_*) $\psi : G \to \Omega BG$. ψ corresponds to the map $\Sigma' G \to BG$, which corresponds to the bundle $P(\mathrm{id}_G : G \to G)$ on $\Sigma' G$.

For $g \in G$, what is $\psi(g)$? Since ψ is a map to ΩBG, $\psi(g)$ corresponds to a map $S^1 \to BG$. This map corresponds to a principal G-bundle over S^1 that is obtained by clutching on S^0, i.e. the map

$$\begin{aligned} &S^0 \to G, \\ &0 \mapsto e, \\ &1 \mapsto g. \end{aligned}$$

[29]

On the other hand, we get a map $\phi : \Omega BG \to G$ as follows: The diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EG \arrow[d, "p"] \\ * \arrow[r] & G \end{tikzcd}$$

is a homotopy pullback, because p is a Serre fibration (see corollary 2.4.10). Therefore, we get a canonical map $\phi : \Omega BG \to G$ from the diagram

$$\begin{tikzcd} \Omega BG \arrow[rr] \arrow[dr, "\phi"] \arrow[ddr, bend right] & & * \arrow[d] \\ & G \arrow[r] \arrow[d] \arrow[ur, phantom, "\Swarrow"] & EG \arrow[d, "p"] \\ & * \arrow[r] & BG \end{tikzcd} .$$

[30] What is ϕ? By construction of ΩBG, the homotopy we used to obtain ϕ (i.e. the homotopy

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[29]This follows since the isomorphisms above are all natural in the space X. We use S^0 to find $\psi(g)$ for some specific $g \in G$.

[30]Note that this diagram shows, by the 2-out-of-3-property for homotopy pullbacks, proposition 2.3.6, that $(\Omega BG) * G(EG)$ is also a homotopy pullback.

$\Omega BG \to * \to EG \to BG \Rightarrow \Omega BG \to * \to BG$) is given by $h : \Omega BG \times I \to BG$, $h(\gamma, t) = \gamma(t)$.[31] Thus we are in the situation that we have the solid arrows in the diagram (where $*$ denotes the map factoring through $*$)

$$\begin{tikzcd} \Omega BG \arrow[r, "*"] \arrow[d, hook] & EG \arrow[d, "p"] \\ \Omega BG \times I \arrow[r, "h"'] \arrow[ur, dashed, "H"] & BG \end{tikzcd} .$$

Since $p : EG \to BG$ is a fibration, we find the lift H as indicated, such that $H_0 = *$. Then $\phi = H_1 : \Omega BG \to G = p^{-1}(*)$.[32]

If we are interested in a point of ΩBG (i.e. a loop $\gamma : S^1 \to BG$), then we need to consider

$$\begin{tikzcd} * \arrow[r, "*"] \arrow[d, hook] & EG \arrow[d, "p"] \\ I \arrow[r, "\tilde{\gamma}"'] \arrow[ur, dashed] \arrow[d] & BG \\ I/\{0,1\} = S^1 \arrow[ur, "\gamma"'] & \end{tikzcd}$$

(where $\tilde{\gamma}$ is defined as the composition).[33] Constructing such a lift is the same as choosing a section of $\tilde{\gamma}^* EG$ over I.[34]

Proposition 4.3.4. *In the situation of the previous remark, $\phi \circ \psi : G \to G$ is homotopic to the identity.*

Proof. We will show that we can even achieve $\phi \circ \psi = \mathrm{id}_G$ (when using some specific choices). Let $g \in G$. Then, as seen in the preceding remark, $\psi(g)$ corresponds to a principal G-bundle P_g over S^1 that is obtained by clutching on S^0,

$$\begin{aligned} &S^0 \to G, \\ &0 \mapsto e, \\ &1 \mapsto g. \end{aligned}$$

To get $\phi(\psi(g))$, we observed that we need a map $I \to EG$ such that

$$\begin{tikzcd} I \arrow[r, dashed] \arrow[d, "q"'] & EG \arrow[d, "p"] \\ S^1 \arrow[r, "\psi(g)"'] & BG \end{tikzcd}$$

commutes; equivalently (since P_g is a pullback of E_g by $\psi(g)$) we need to choose a map $\tilde{q} : I \to P_g$ such that

$$\begin{tikzcd} I \arrow[r, dashed, "\tilde{q}"] \arrow[dr, "q"'] & P_g \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EG \arrow[d, "p"] \\ & S^1 \arrow[r, "\psi(g)"'] & BG \end{tikzcd}$$

commutes. By construction of P_g we can choose $\tilde{q}$ such that $\tilde{q}(1) = g$, and then, by the definition of ϕ, we have $(\phi \circ \psi)(g) = \tilde{q}(1) = g$. $\square$

[31] Then $h_0 = h_1 = *$.
[32] This is just an explicit instance of why a pullback of a fibration is a homotopy pullback; essentially it is the strictification lemma 1.5.7.
[33] Recall also $h(\gamma, t) = \gamma(t)$.
[34] See also the next proposition.

Corollary 4.3.5. *$\phi : \Omega BG \xrightarrow{\sim} G$ is an homotopy equivalence, and EG is contractible.*

Proof. From the Yoneda lemma we already know that ψ is a homotopy equivalence (isomorphism in hCW_*°). We have shown that ϕ is a left inverse. Then, letting κ be a right inverse to ψ, we have $\phi = \phi \circ \psi \circ \kappa = \kappa$ (in hCW_*°), thus $\phi = \kappa$ is a homotopy equivalence (with homotopy inverse ψ).
Then consider the homotopy pullback diagram

$$\begin{tikzcd} \Omega BG \arrow[r] \arrow[d, "\phi"'] & * \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\ G \arrow[r] & EG \end{tikzcd} .$$

(This is a homotopy pullback by the 2-out-of-3-property; one glues the diagram $G(EG) * (BG)$ to it. See remark 4.3.3.) The Puppe long exact sequence for homotopy groups (corollary 2.4.12) reads

$$\ldots \longrightarrow \pi_i(\Omega BG) \xrightarrow{\cong} \pi_i(G) \longrightarrow \pi_i(EG) \longrightarrow \pi_{i-1}(\Omega BG) \xrightarrow{\cong} \pi_{i-1}(G) \longrightarrow \ldots,$$

and since the maps $\pi_i(\Omega BG) \to \pi_i(G)$ are all isomorphisms as ϕ is a homotopy equivalence, $\pi_i(EG) = 0$ for all i. Thus EG is contractible by Whitehead's theorem (theorem 1.7.4). □

Corollary 4.3.6. *Let E be a contractible CW-complex with a left G-action such that $E \to G\backslash E$ is a principal G-bundle over a CW-complex $G\backslash E$. Then, in the pullback diagram*

$$\begin{tikzcd} E \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EG \arrow[d] \\ G\backslash E \arrow[r, "\simeq"'] & BG \end{tikzcd} ,$$

the horizontal maps are homotopy equivalences.

Proof. Write down two long exact sequences

$$\begin{tikzcd} \ldots \arrow[r] & \pi_i(G) \arrow[r] \arrow[d, equal] & \pi_i(E) = 0 \arrow[r] \arrow[d, "\cong"] & \pi_i(G\backslash E) \arrow[r] \arrow[d] & \pi_{i-1}(G) \arrow[r] \arrow[d, equal] & \pi_{i-1}(E) = 0 \arrow[r] \arrow[d] & \ldots . \\ \ldots \arrow[r] & \pi_i(G) \arrow[r] & \pi_i(EG) = 0 \arrow[r] & \pi_i(BG) \arrow[r] & \pi_{i-1}(G) \arrow[r] & \pi_{i-1}(EG) = 0 \arrow[r] & \ldots \end{tikzcd}$$

By the five-lemma, $\pi_i(G\backslash E) \to \pi_i(BG)$ is an isomorphism for all i. Whitehead's theorem yields the claim. □

Result: We can take $E \to G\backslash E$ as $EG \to BG$!

Example 4.3.7. Let G be a discrete group. Then the universal principal G-bundle $EG \to BG$ is given by $\widetilde{K(G,1)} \to K(G,1)$, where $K(G,1)$ is an Eilenberg-Mac Lane space and $\widetilde{K(G,1)}$ is its universal covering.
In particular, the notation is consistent with the functor $B : \mathrm{Gpd} \to \mathrm{Top}$ used in the proof of the Seifert-van Kampen theorem 2.12.1.
We obtain, for example, $S^1 \simeq K(\mathbb{Z},1) \simeq B\mathbb{Z}$, and $\mathbb{R}P^\infty \simeq K(\mathbb{Z}/2,1) \simeq B(\mathbb{Z}/2)$.

4.4 Associated fiber bundle

Proposition 4.4.1. *Let $p : P \to X$ be a (left) principal G-bundle and F be a topological space with a continuous right action of G. Then $F \times P/((fg,p) \sim (f,gp)) =: F \times^G P \to X$ is a fiber bundle with fiber F (called* associated fiber bundle*).*

Proof. Let U be a trivializing open set for P. Then $P|_U \cong G \times U$ (G-equivariantly and over U), and $(F \times^G P)|_U = F \times^G (P|_U) \cong F \times^G (G \times U) \cong F \times U$, where the last homeomorphism is given by $(f, g, u) \mapsto (fg, u)$. $\square$

Example 4.4.2. – Consider a morphism $f : G_1 \to G_2$ of topological groups. We obtain $\mathrm{Bun}_{G_1}(\cdot) \to \mathrm{Bun}_{G_2}(\cdot)$: $P_1 \mapsto G_2 \times^{G_1} P_1$ (where G_1 acts on G_2 via the given morphism, i.e. $(g_2, g_1) \mapsto g_2 f(g_1)$). This is a natural transformation of functors.[35]

– Let Σ_n be the permutation group of n-elements. Then principal Σ_n-bundles are in bijection with coverings of degree n via $P \mapsto \{1, \ldots, n\} \times^{\Sigma_n} P$.

– Consider $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and for $\mathbb{K} = \mathbb{R}$, $G = O_n$ or $G = \mathrm{Gl}_n(\mathbb{R})$, for $\mathbb{K} = \mathbb{C}$, $G = U_n$ or $G = \mathrm{Gl}_n(\mathbb{C})$ (where O_n, U_n are the orthogonal and unitary groups). Then a principal G-bundle P yields an associated fiber bundle $\mathbb{K}^n \times^G P$: a topological vector bundle.[36] (In case $G = O_n$ resp. $G = U_n$, it is a vector bundle with a scalar product.)

4.5 Pointed versus unpointed (part 2)

Lemma 4.5.1. *Let $(X, x_0) \in \mathrm{CW}_*$. Then for any space Y where* $\mathrm{Map}(X, Y)$ *exists (e.g. CW-complexes), the evaluation* $\mathrm{ev}_{x_0} : \mathrm{Map}(X, Y) \to Y$, $f \mapsto f(x_0)$ *is a fibration.*

Proof. Suppose that we are given the solid arrows in the commutative diagram

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[d, hook, "{(\mathrm{id},0)}"'] & \mathrm{Map}(X,Y) \arrow[d, "\mathrm{ev}_{x_0}"] \\ T \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & Y \end{tikzcd} .$$

We need to find the indicated lift $\tilde{H} : T \times I \to \mathrm{Map}(X, Y)$, which is the same as a map (still called $\tilde{H}$) $\tilde{H} : T \times X \times I \to Y$. Now f corresponds (again abusing notation) to a map $f : T \times X \to Y$, and the above diagram means that then

$$\begin{tikzcd} T \arrow[r, hook, "{(\mathrm{id},0)}"] \arrow[d, hook, "{(\mathrm{id},x_0)}"] & T \times I \arrow[d, "H"] \\ T \times X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. Thus actually, we want to find a homotopy extension for $T \times \{x_0\} \hookrightarrow T \times X$, where we need to extend

$$\begin{cases} f : T \times X \times \{0\} \to Y, \\ H : T \times \{x_0\} \times I \to Y. \end{cases}$$

Thus the claim is true if $T \times \{x_0\} \hookrightarrow T \times X$ is a cofibration. Since $\{x_0\} \hookrightarrow X$ is a closed cofibration (X is well-pointed as a CW-complex), this is the case since closed cofibrations are stable under products (proposition 1.4.8). $\square$

Proposition 4.5.2. *Let $(X, x_0), (Y, y_0) \in \mathrm{Ob}\,\mathrm{CW}_*^\circ$. There exists a right action of $\pi_1(Y, y_0)$ on $[X, Y]_*$ and $[X, Y]_*/\pi_1(Y, y_0) \xrightarrow{\sim} [X, Y]$ is an isomorphism.*

Proof. Existence of the action: Exercise.

[35] $G_2 \times^{G_1} P_1$ admits a left G_2-action, because we multiply with $f(g_1)$ from the right.

[36] Later we will switch the roles of left and right actions such that G acts, as eventually expected, on $\mathbb{K}^n$ from the left, but $\mathbb{K}^n$ also allows for a right action.

Let $\mathrm{Map}_*(X,Y) := \{f : X \to Y \mid f(x_0) = y_0\}$. The diagram

$$\begin{tikzcd} \mathrm{Map}_*(X,Y) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(X,Y) \arrow[d, "\mathrm{ev}_{x_0}"] \\ * \arrow[r, "y_0"'] & Y \end{tikzcd}$$

is a homotopy pullback, since ev_{x_0} is a fibration by the previous lemma, and by theorem 2.3.9. Let $f \in \mathrm{Map}(X,Y)$ such that $f(x_0) = y_0$; choose this as basepoint in the mapping spaces. Then the long exact sequence of homotopy groups (resp. the Puppe sequence from corollary 2.4.12) reads

$$\begin{tikzcd} \ldots \arrow[r] & \pi_1(Y,y_0) \arrow[r, "\partial_f"] & \pi_0\mathrm{Map}_*(X,Y) = [X,Y]_* \arrow[r, "\text{forget}"] & \pi_0\mathrm{Map}(X,Y) = [X,Y] \arrow[r] & \pi_0 Y = * . \\ & \gamma \arrow[r, mapsto] & f \cdot \gamma = \partial_f(\gamma) & & \end{tikzcd}$$

(Here ∂_f depends on the choice of f; one verifies $\partial_f(\gamma) = f \cdot \gamma$. We have $\pi_0\mathrm{Map}(X,Y) = [X,Y]$, since a map $I \to \mathrm{Map}(X,Y)$ is the same as a homotopy $X \times I \to Y$; analogously for $\pi_0\mathrm{Map}_*(X,Y) = [X,Y]_*$. forget means "forget being a pointed map".)

By exactness, the kernel of forget (i.e., by our choice of basepoint, all maps g with $[g] = [f]$ in $[X,Y]$) is the image of ∂_f, i.e. $\pi_1(Y,y_0) \cdot [f]$. In other formulas, $\mathrm{forget}^{-1}([f]) = [f] \cdot \pi_1(Y,y_0)$. Since analogous sequences exist for all $f' \in \mathrm{Map}(X,Y)$ with $f'(x_0) = y_0$, we get $\mathrm{forget}^{-1}([f']) = [f'] \cdot \pi_1(Y,y_0)$ for all such maps.

However, for any arbitrarily chosen basepoint f, the above sequence shows that forget is surjective, i.e. every homotopy class $[h] \in [X,Y]$ contains a representative which is a pointed map. This yields the lemma: $[X,Y]_*$ surjects onto $[X,Y]$ with $[X,Y]_*/\pi_1(Y,y_0) \cong [X,Y]$. □

Corollary 4.5.3. *If Y is a simply connected CW-complex, then for every CW-complex X we have $[X,Y]_* \cong [X,Y]$.*

There is nothing left to prove.

Proposition 4.5.4. *Let $(X,x_0) \in \mathrm{Ob}\,\mathrm{CW}_*^\circ$. Then*

$$\begin{tikzcd} {[X,BG]_*/G} \arrow[r, "\sim"] \arrow[d, "\sim"'] & {[X,BG]} \arrow[d, "\sim"] \\ \mathrm{Bun}_G^*(X)/G \arrow[r, "\sim"] & \mathrm{Bun}_G(X) \end{tikzcd}$$

is a commutative diagram of bijections, where G acts on $\mathrm{Bun}_G^*(X)$ by $(P,p) \mapsto (P, g^{-1}p)$ (for $g \in G$), and on $[X,BG]_*$ via the action considered in the proposition before, using $\pi_1(BG) \cong \pi_0(G)$.[37]

Proof (sketch). We need to consider

$$\begin{tikzcd} {[X,BG]_*/G} \arrow[r, "(3)"] \arrow[d, "(1)"'] & {[X,BG]} \arrow[d, "(4)"] \\ \mathrm{Bun}_G^*(X)/G \arrow[r, "(2)"] & \mathrm{Bun}_G(X) \end{tikzcd} .$$

(3) is a bijection by the proposition before.

For (1), it suffices to check that the map $[X,BG]_* \to \mathrm{Bun}_G^*(X)$ is G-equivariant (then the bijection induces a bijection between quotients). The isomorphism $\pi_1(BG) \cong \pi_0(G)$ is obtained from the "clutching construction" map $\psi : G \to \Omega BG$ from remark 4.3.3. I.e. it is given by

[37] Thus the action actually only depends on the connected components of G.

$f \mapsto f \cdot \psi(g)$. Let $f : X \to BG$ with $f(x_0) = *$ (the basepoint of BG). Recalling the definition of ψ from remark 4.3.3, over x_0, the action of $\psi(g)$ just means multiplying by g^{-1}.[38]

For (2), surjectivity is clear (every bundle is a pointed one for some basepoint; the map is induced from the surjection $\mathrm{Bun}_G^*(X) \twoheadrightarrow \mathrm{Bun}_G(X)$). It also is injective: Let $(P_1, p_1), (P_2, p_2)$ be pointed principal G-bundles such that P_1, P_2 are (unpointedly) isomorphic, i.e. there exists an isomorphism $\phi : P_1 \to P_2$. Then $\phi(p_1) = g^{-1}p_2$ for some $g \in G$, since $\phi(p_1)$ lies in the same fiber as p_2 (the one over x_0). One concludes that $(P_1, p_1) \cong (P_2, g^{-1}p_2)$ and (P_2, p_2) lie in the same orbit of the action of G, i.e. they are the same in $\mathrm{Bun}_G^*(X)/G$.

Then (4) (i.e. $[X, BG] \to \mathrm{Bun}_G(X)$, $f \mapsto f^*EG$) must be a bijection as well, since the diagram is observed to commute. □

Corollary 4.5.5. *For all $X \in \mathrm{Ob}\,\mathrm{CW}$, the map $[X, BG] \to \mathrm{Bun}_G(X)$, $f \mapsto f^*EG$ is an isomorphism, where $EG \to BG$ is the universal bundle.*

Proof. We decompose $X = \coprod_\alpha X_\alpha$ into its connected components. Since the disjoint union is the coproduct in hCW, we have $[X, BG] \cong \prod_\alpha [X_\alpha, BG]$. On the other hand, also $\mathrm{Bun}_G(\coprod_\alpha X_\alpha) \cong \prod_\alpha \mathrm{Bun}_G(X_\alpha)$ (this can directly be verified). Now for every X_α, we can choose a basepoint $x_\alpha \in X_\alpha$; then by the previous proposition $[X_\alpha, BG] \to \mathrm{Bun}_G(X_\alpha)$ is an isomorphism. One obtains the claim: $[X, BG] \cong \prod_\alpha [X_\alpha, BG] \cong \prod_\alpha \mathrm{Bun}_G(X_\alpha) \cong \mathrm{Bun}_G(X)$. □

Thus BG does not only classify pointed bundles, but also unpointed ones.

4.6 Grassmann and Stiefel manifolds

In the following, let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (it would even work with $\mathbb{H}$).

Definition 4.6.1. The space

$$\mathrm{Gr}^{\mathbb{K}}(k, n) := \{V \subset \mathbb{K}^n \mid V \text{ is a vector subspace of dimension } k\}$$

is called the (real or complex) *Grassmannian manifold* of k-subspaces of n-dimensional space. The standard inner product of $\mathbb{K}^n$ shall be denoted $\langle \cdot, \cdot \rangle$; then

$$V^{\mathbb{K}}(k, n) := \{(v_1, \ldots, v_k) \mid \forall i : v_i \in \mathbb{K}^n, \forall i, j : \langle v_i, v_j \rangle = \delta_{ij}\}$$

is called the *Stiefel manifold* of orthonormal k-frames in n-dimensional space.[39] Then $V^{\mathbb{K}}(k, n) \subset (\mathbb{K}^{\times n})^{\times k}$, which shall define the topology on $V^{\mathbb{K}}(k, n)$. Furthermore, we have a surjective map of sets $\pi : V^{\mathbb{K}}(k, n) \to \mathrm{Gr}^{\mathbb{K}}(k, n)$, $(v_1, \ldots, v_k) \mapsto \mathrm{Span}^{\mathbb{K}}(v_1, \ldots, v_k)$. This shall define the topology on $\mathrm{Gr}^{\mathbb{K}}(k, n)$.

Lemma 4.6.2. *We have commutative diagrams, where the horizontal maps are homeomorphisms:*

$$\begin{tikzcd} O_n/O_{n-k} \arrow[r, "\alpha", "\sim"'] \arrow[d] & V^{\mathbb{R}}(k,n) \arrow[d, "\pi"] \\ O_n/(O_k \times O_{n-k}) \arrow[r, "\sim", "\beta"'] & \mathrm{Gr}^{\mathbb{R}}(k,n) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} U_n/U_{n-k} \arrow[r, "\alpha'", "\sim"'] \arrow[d] & V^{\mathbb{C}}(k,n) \arrow[d, "\pi"] \\ U_n/(U_k \times U_{n-k}) \arrow[r, "\sim", "\beta'"'] & \mathrm{Gr}^{\mathbb{C}}(k,n) \end{tikzcd} .$$

Proof. We only prove the real case; furthermore the proof that β is a homeomorphism is left to the reader.

O_n acts on $V^{\mathbb{R}}$ transitively by $A \cdot (v_1, \ldots, v_k) = (w_1, \ldots, w_k)$, where $A \cdot (\mathbf{v}_1, \ldots, \mathbf{v}_k) = (\mathbf{w}_1, \ldots, \mathbf{w}_k)$. (We denote column vectors with boldface and row vectors without boldface. The action of A on

[38] Note $f^*EG \cong (f \cdot \psi(g))^*EG$ as unpointed bundles, because $f \cdot \psi(g)$ is freely (i.e. non-basepoint preserving) homotopic to f, and Bun_G is homotopy invariant. Thus indeed only the basepoint changes when multiplying with $\psi(g)$.

One gets multiplication of the basepoint by g^{-1} instead of multiplication by g due to the "pulling back"-construction. Otherwise it would also not be a right action as required.

[39] The tuple $(v_1, \ldots, v_k)$ is called a *k-frame*.

a tuple of column vectors is given componentwise or equivalently may be seen as matrix multiplication.) This is indeed a well-defined action, since for an orthogonal matrix A, $A \cdot (v_1, \ldots, v_k)$ is still an orthonormal frame if $(v_1, \ldots, v_k)$ is one.

We define for $A \in O_n$: $\overline{\alpha}(A) := A \cdot (e_1, \ldots, e_k)$. Then $\overline{\alpha}$ is surjective: Given $(v_1, \ldots, v_k) \in V^{\mathbb{R}}(k, n)$, we can extend it to an orthonormal basis $(v_1, \ldots, v_k, w_{k+1}, \ldots, w_n)$ of $\mathbb{R}^n$. Then $A := (\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{w}_{k+1}, \ldots, \mathbf{w}_n)$ satisfies $\overline{\alpha}(A) = (v_1, \ldots, v_k)$.

Then $\overline{\alpha}$ descends to an isomorphism α from O_n modulo the stabilizer $\mathrm{St}(e_1, \ldots, e_k)$ to $V^{\mathbb{R}}(k, n)$. But this stabilizer is given by matrices A of the form

$$A = \begin{pmatrix} \mathbb{1}_{k \times k} & 0 \\ 0 & A' \end{pmatrix},$$

where $A' \in O_{n-k}$: The first k columns must have this particular shape because A stabilizes $(e_1, \ldots, e_k)$, and the right upper corner must contain the zero matrix due to $\langle Ae_j, e_i \rangle = \langle Ae_j, Ae_i \rangle = \langle e_j, e_i \rangle = 0$ for $j > k$, $i \leq k$. The right lower corner may contain any O_{n-k}-matrix (but it must be orthogonal in order that A is orthogonal). □

Lemma 4.6.3. *$\pi : V^{\mathbb{R}}(k, n) \to \mathrm{Gr}^{\mathbb{R}}(k, n)$ as defined before is a right principal O_k-bundle and $\mathrm{Gr}^{\mathbb{R}}(k, n)$ is locally homeomorphic to $\mathbb{R}^{k(n-k)}$.*

Proof. A matrix $A \in O_k$ acts on k-frames as follows: $(v_1, \ldots, v_k) \mapsto (w_1, \ldots, w_k)$, where $(\mathbf{v}_1, \ldots, \mathbf{v}_k) \cdot A = (\mathbf{w}_1, \ldots, \mathbf{w}_k)$. Let $W \subset \mathbb{R}^n$ be a point in $\mathrm{Gr}^{\mathbb{R}}(n, k)$. Choose $V \subset \mathbb{R}^n$ such that $W \oplus V = \mathbb{R}^n$ and consider

$$U := \{\tilde{W} \in \mathrm{Gr}^{\mathbb{R}}(k, n) \mid \tilde{W} \oplus V = \mathbb{R}^n\} = \{\tilde{W} \in \mathrm{Gr}^{\mathbb{R}}(k, n) \mid \tilde{W} \cap V = \{0\}\} \ni W.$$

Then, denoting $p : \mathbb{R}^n \to \mathbb{R}^n/V$ the canonical projection, we have

$$U \cong \{s : \mathbb{R}^n/V \to \mathbb{R}^n \text{ linear map} \mid p \circ s = \mathrm{id}_{\mathbb{R}^n/V}\}$$

(since $\tilde{W} \in \mathrm{Gr}^{\mathbb{R}}(k, n)$ lies in U if and only if $p|_{\tilde{W}} : \tilde{W} \to \mathbb{R}^n/V$ is an isomorphism; the bijection is given by $\tilde{W} \mapsto (p|_{\tilde{W}})^{-1}$, $\mathrm{im}(s) \leftarrow\!\shortmid s$). This is a vector space isomorphic to $\mathbb{R}^{k(n-k)}$. The resulting bijection $U \to \mathbb{R}^{k(n-k)}$ shall be called ρ_U.

- *Claim 1:* U is open in $\mathrm{Gr}^{\mathbb{R}}(k, n)$ and ρ_U is a homeomorphism.

  Proof of claim 1: U is open if and only if $\pi^{-1}(U)$ is open in $V^{\mathbb{R}}(k, n)$ (since $\mathrm{Gr}^{\mathbb{R}}(k, n)$ carries the quotient topology). Now

  $$\pi^{-1}(U) = \{(v_1, \ldots, v_k) \mid (v_1, \ldots, v_k) \text{ orthonormal family such that } \mathrm{Span}^{\mathbb{R}}(v_1, \ldots, v_k) \cap V = \{0\}\} =$$
  $$= V^{\mathbb{R}}(k, n) \cap \{(v_1, \ldots, v_k) \mid \mathrm{Span}^{\mathbb{R}}(v_1, \ldots, v_k) \cap V = \{0\}\}.$$

  Thus it suffices to show that $\{(v_1, \ldots, v_k) \mid \mathrm{Span}^{\mathbb{R}}(v_1, \ldots, v_k) \cap V = \{0\}\}$ is open in $(\mathbb{R}^n)^k$ (as $V^{\mathbb{R}}(k, n)$ carries the subspace topology). Choose $w_{k+1}, \ldots, w_n$ such that $V = \mathrm{Span}^{\mathbb{R}}(w_{k+1}, \ldots, w_n)$; then

  $$\{(v_1, \ldots, v_k) \mid \mathrm{Span}^{\mathbb{R}}(v_1, \ldots, v_k) \cap V = \{0\}\} = \{(v_1, \ldots, v_k) \mid \det(\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}_{k+1}, \ldots, \mathbf{w}_n) \neq 0\},$$

  which is indeed open as det is continuous. Furthermore ρ_U, ρ_U^{-1} are continuous (they are linear maps between vector spaces of finite dimension), proving claim 1.

  Claim 1 already shows that $\mathrm{Gr}^{\mathbb{R}}(k, n)$ is locally homeomorphic to $\mathbb{R}^{k(n-k)}$.

- *Claim 2:* π has a section over U.

  Proof of claim 2: Choosing a basis $w_1, \ldots, w_k$ of $\mathbb{R}^n/V$, we define $\sigma : U \to V^{\mathbb{R}}(k, n)$ using the homeomorphism from before as an identification and mapping

  $$\{s : \mathbb{R}^n/V \to \mathbb{R}^n \text{ linear map} \mid p \circ s = \mathrm{id}_{\mathbb{R}^n/V}\} \ni s \mapsto \text{Gram-Schmidt procedure for } (s(w_1), \ldots, s(w_k)).$$

  The Gram-Schmidt procedure is continuous, and $\pi \circ \sigma = \mathrm{id}_U$. Thus σ is the desired section.

By claim 2 and the reasoning from corollary 4.1.11 resp. example 4.1.5, π is trivial over U. The fiberwise action can be shown directly.

□

(An analogous lemma can be proven for the complex case.)

Theorem 4.6.4. *Let* $\mathrm{Gr}^{\mathbb{K}}(k,\infty) := \mathrm{colim}_n \mathrm{Gr}^{\mathbb{K}}(k,n)$, *and let* $V^{\mathbb{K}}(k,\infty) := \mathrm{colim}_n V^{\mathbb{K}}(k,n)$, *where the inclusions* $\mathrm{Gr}^{\mathbb{K}}(k,n) \hookrightarrow \mathrm{Gr}^{\mathbb{K}}(k,n+1)$ *and* $V^{\mathbb{K}}(k,n) \hookrightarrow V^{\mathbb{K}}(k,n+1)$ *are induced by* $\mathbb{K}^n \hookrightarrow \mathbb{K}^{n+1}$.

Then $V^{\mathbb{K}}(k,\infty) \to \mathrm{Gr}^{\mathbb{K}}(k,\infty)$ *is a universal principal* O_k*-bundle for* $\mathbb{K} = \mathbb{R}$, *and a universal principal* U_k*-bundle for* $\mathbb{K} = \mathbb{C}$. *I.e. the classifying spaces* $BO_k = \mathrm{Gr}^{\mathbb{R}}(k,\infty)$, $BU_k = \mathrm{Gr}^{\mathbb{C}}(k,\infty)$.

Example 4.6.5. For $k = 1$, we have $BO_1 = \mathrm{Gr}^{\mathbb{R}}(k,\infty) = \mathbb{R}P^\infty$ and $BU_1 = \mathrm{Gr}^{\mathbb{C}}(k,\infty) = \mathbb{C}P^\infty$.

For the proof, we need:

Proposition 4.6.6. *Let* $n < m$. *Then the inclusion* $O_n \hookrightarrow O_m$, $A \mapsto \begin{pmatrix} A & 0 \\ 0 & \mathbb{1}_{m-n} \end{pmatrix}$ *is* $(n-1)$*-connected, and the inclusion* $U_n \hookrightarrow U_m$, $A \mapsto \begin{pmatrix} A & 0 \\ 0 & \mathbb{1}_{m-n} \end{pmatrix}$ *is* $2n$*-connected.*

Proof. By induction, it suffices to prove it for $m = n+1$, since we have a chain of inclusions $O_n \hookrightarrow O_{n+1} \hookrightarrow \cdots \hookrightarrow O_m$ (similarly for the unitary groups).

O_{n+1} acts transitively on $S^n \subset \mathbb{R}^{n+1}$. Consider $v_0 := (0,\ldots,0,1) \in S^n$ and $\pi_n : O_{n+1} \to S^n$, $A \mapsto A \cdot v_0$.

Claim:* π_n is a principal $\mathrm{St}(v_0)$-bundle, where $\mathrm{St}(v_0)$ is the stabilizer of v_0.

(Without proof here.)

We have $\mathrm{St}(v_0) = \left\{ \begin{pmatrix} & & & 0 \\ & B & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix} \mid B \in O_n \right\} \cong O_n$. Since a fiber bundle is a Serre fibration, π_n is a Serre fibration with fiber O_n; we have the pullback diagram

$$\begin{tikzcd} O_n \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & O_{n+1} \arrow[d] \\ * \arrow[r] & S^n \end{tikzcd} .$$

The long exact sequence of a Serre fibration (corollary 1.2.7) reads

$$\ldots \longrightarrow \pi_{l+1}(S^n) \longrightarrow \pi_l(O_n) \xrightarrow{j_*} \pi_l(O_{n+1}) \longrightarrow \pi_l(S^n) \longrightarrow \ldots .$$

If $l < n-1$, $\pi_{l+1}(S^n) = 0$, and if $l < n$, $\pi_l(S^n) = 0$. By exactness, j_* is an isomorphism if $l < n-1$ and surjective if $l = n-1$. By proposition 1.5.2, j is $(n-1)$-connected.

Similarly, U_{n+1} acts transitively on $S^{2n+1} \subset \mathbb{C}^{n+1}$ with the stabilizer U_n. □

Corollary 4.6.7. *We have* $\pi_i(V^{\mathbb{R}}(k,n)) = 0$ *for* $i \leq n-k-1$ *and* $\pi_i(V^{\mathbb{C}}(k,n)) = 0$ *for* $i \leq 2n-2k$.

Proof. The construction of the homeomorphism α from lemma 4.6.2 yields a principal O_{n-k}-bundle over $V^{\mathbb{R}}(k,n)$:

$$\begin{tikzcd} O_{n-k} \arrow[r, "(n-k-1)-\text{conn.}"] \arrow[d] & O_n \arrow[d] \\ * \arrow[r] & V^{\mathbb{R}}(k,n) \cong O_n/O_{n-k} \end{tikzcd} .$$

The connectivity of the map $O_{n-k} \to O_n$ was shown in the previous lemma. The long exact sequence of homotopy groups for this Serre fibration yields

$$\dots \longrightarrow \pi_i(O_{n-k}) \twoheadrightarrow \pi_i(O_n) \longrightarrow \underbrace{\pi_i(V^{\mathbb{R}}(k,n))}_{=0} \longrightarrow \pi_{i-1}(O_{n-k}) \xrightarrow{\cong} \pi_{i-1}(O_k) \longrightarrow \dots$$

for $i \leq n-k-1$.[40]

Analogously one concludes for U_n, using the principal U_{n-k}-bundle

$$\begin{tikzcd} U_{n-k} \arrow[r, "2(n-k)-\text{conn.}"] \arrow[d] & U_n \arrow[d] \\ * \arrow[r] & V^{\mathbb{C}}(k,n) \cong U_n/U_{n-k} \end{tikzcd}.$$

□

Proof of theorem 4.6.4. 1. The map $V^{\mathbb{K}}(k,\infty) \to \mathrm{Gr}^{\mathbb{K}}(k,\infty)$ is a principal G-bundle, where $G = O_n$ if $\mathbb{K} = \mathbb{R}$ and $G = U_n$ if $\mathbb{K} = \mathbb{C}$: Let $W \in \mathrm{Gr}^{\mathbb{K}}(k,\infty)$, then $W \in \mathrm{Gr}^{\mathbb{K}}(k,n)$ for some $n \in \mathbb{N}$. As in the proof of lemma 4.6.3, one chooses V such that $\mathbb{K}^n = W \oplus V$. Then for all $m \geq n$, one defines an open neighbourhood $U_m \subset \mathrm{Gr}^{\mathbb{K}}(k,m)$ of W in $\mathrm{Gr}^{\mathbb{K}}(k,m)$, over which the map over which the map $V^{\mathbb{K}}(k,m) \to \mathrm{Gr}^{\mathbb{K}}(k,m)$ is a trivial fiber bundle. Using the same V for the definition of all U_m, one has $U_m \cap \mathrm{Gr}^{\mathbb{K}}(k,m') = U_{m'}$ for $m \geq m' \geq n$.

   By definition of the colimit topology and since the inclusions $\mathrm{Gr}^{\mathbb{K}}(k,m) \hookrightarrow \mathrm{Gr}^{\mathbb{K}}(k,n)$ are continuous for $m \leq n$, $U := \bigcup_{m=n}^{\infty} U_m$ is an open neighbourhood of W in $\mathrm{Gr}^{\mathbb{K}}(k,\infty)$. One verifies that over U, the map $V^{\mathbb{K}}(k,\infty) \to \mathrm{Gr}^{\mathbb{K}}(k,\infty)$ is a trivial fiber bundle. Thus $V^{\mathbb{K}}(k,\infty) \to \mathrm{Gr}^{\mathbb{K}}(k,\infty)$ is a principal G-bundle.

2. By corollary 4.3.6 (resp. a variant with a right G-action), it remains to show that $V^{\mathbb{K}}(k,\infty)$ is contractible. This follows by Whitehead's theorem 1.7.4, since by the corollary before, $\pi_i(V^{\mathbb{K}}(k,\infty)) = \mathrm{colim}_n\, \pi_i(V^{\mathbb{K}}(k,\infty)) = 0$ for all $i \in \mathbb{N}$.[41]

□

Remark 4.6.8. *Fact:* The inclusions $O_n \hookrightarrow \mathrm{Gl}_n(\mathbb{R})$ and $U_n \hookrightarrow \mathrm{Gl}_n(\mathbb{C})$ are homotopy equivalences. Therefore also $BO_n \cong B\mathrm{Gl}_n(\mathbb{R})$, $BU_n \cong B\mathrm{Gl}_n(\mathbb{C})$ (exercise).

Alternatively, one can consider the *non-compact Stiefel manifolds* $\mathcal{V}^{\mathbb{K}}(k,n) := \{(v_1,\dots,v_n) \,|\, \forall i : v_i \in \mathbb{K}^n, (v_1,\dots,v_n) \text{ are linearly independent}\}$. Then $\mathcal{V}^{\mathbb{K}}(k,n) \to \mathrm{Gr}^{\mathbb{K}}(k,n)$ is the universal $\mathrm{Gl}_k(\mathbb{K})$-bundle.

The advantage of working with $V^{\mathbb{K}}(k,n)$ is that these spaces are compact.[42]

4.7 Vector bundles: definition and properties

Definition 4.7.1. A *real vector bundle of rank n* over a topological space B is a space E, a map $p : E \to B$ and a vector space structure on $p^{-1}(b)$ for all $b \in B$, such that:

For all $b \in B$ there exists an open neighbourhood U of b in B and a homeomorphism ϕ_U :

[40] To be precise, we know the isomorphism and the surjection and conclude the "$= 0$" from exactness.

[41] The behaviour of π_i with respect to ascending unions is known from Algebraic topology 1.

[42] This can be proven using the compactness of O_n resp. U_n and lemma 4.6.2. The compactness of O_n resp. U_n in turn can be proven using the Heine-Borel theorem. In fact, the relations $A^T A = \mathbb{1}_{k\times k}$ define a closed subset of $\mathbb{R}^{k\times k}$. Furthermore O_n is also bounded, since for $i = 1,\dots,k$ and $A = (a_{ij})_{ij} \in O_n$ one has $a_{i1}^2 + \dots + a_{ik}^2 = (A^T A)_{ii} = 1$, thus $|a_{ij}| \leq 1$ for all $1 \leq i,j \leq n$. For U_n the proof is analogous.

$p^{-1}(U) \to U \times \mathbb{R}^n$ that is fiberwise $\mathbb{R}$-linear and such that the diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\phi_U", "\cong"'] \arrow[dr, "p"'] & & \mathbb{R}^n \times U \arrow[dl, "\mathrm{pr}_2"] \\ & U & \end{tikzcd}$$

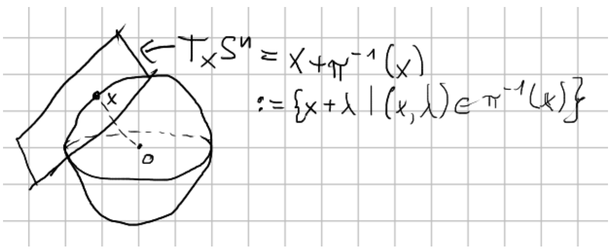
commutes.

A *complex vector bundle of rank* n shall be defined analogously.

Remark 4.7.2. If B is not connected, one can also consider vector bundles (v.b.) whose rank is different over each connected component of B.

Example 4.7.3.

- If V is a vector space, there is the trivial vector bundle $\mathrm{pr}_2 : V \times B \to B$.
- For any manifold M, we have the tangent bundle $TM \to M$. For example, for S^n, the tangent bundle has rank n: $TS^n = \{(x, \lambda) \,|\, x \in S^n, \lambda \in \mathbb{R}^{n+1}, \langle \lambda, x\rangle = 0\}$, $\pi : TS^n \to S^n$, $(x, \lambda) \mapsto x$. Picture (with abuse of notation):



- On the Grassmannians, there is the *tautological vector bundle*: For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, let $E^{\mathbb{K}}(k, n) := \{(x, v) \,|\, x \in \mathrm{Gr}^{\mathbb{K}}(k, n), v \in \mathbb{K}^n$ such that $v \in x\}$.[43] Then let $\pi : E^{\mathbb{K}}(k, n) \mapsto \mathrm{Gr}^{\mathbb{K}}(k, n)$, $\pi(x, v) = x$.

  Then "$\pi^{-1}(x) = x$", hence the name.

Definition 4.7.4. Let $p_V : V \to B, p_W : W \to B$ be vector bundles over a space B. A *morphism* from V to W is a continuous map ϕ such that:

- $p_W \circ \phi = p_V$, i.e. the diagram

$$\begin{tikzcd} V \arrow[rr, "\phi"] \arrow[dr, "p_V"'] & & W \arrow[dl, "p_W"] \\ & B & \end{tikzcd}$$

  commutes.

- ϕ is fiberwise linear, i.e. for all $b \in B$, the map $\phi|_{V_b} : V_b \to W_b$ is a linear map of vector spaces, where we denote $V_b := p_V^{-1}(b)$, $W_b := p_W^{-1}(b)$ the fibers of V resp. W over b.

Lemma 4.7.5. *Let $\pi : V \to B$ be a vector bundle and $\phi : B' \to B$ be a continuous map. Define the pullback ϕ^*V by the diagram*

$$\begin{tikzcd} \phi^*V := V' := V \times_B B' \arrow[r, "\tilde{\phi}"] \arrow[d] & V \arrow[d, "\pi"] \\ B' \arrow[r, "\phi"'] & B \end{tikzcd} .$$

[43] Recall that $x \in \mathrm{Gr}^{\mathbb{K}}(k, n)$ is a subspace of $\mathbb{K}^n$.

Then there exists a unique structure of a vector bundle on V' such that $\tilde{\phi}$ is linear on the fibers, i.e. for all $b' \in B'$, in the commutative diagram

$$\begin{tikzcd} V'_{b'} \arrow[r, "\tilde{\phi}|_{V'_{b'}}"] \arrow[d] & V_{\phi(b')} \arrow[d] \\ b' \arrow[r] & \phi(b') \end{tikzcd} ,$$

the map $\tilde{\phi}|_{V'_{b'}}$ is linear.

Proof.
- By definition of the pullback, we have $V'_{b'} = V_{\phi(b')}$ for all $b' \in B'$. So we can define the vector space structure on $V'_{b'}$ by inducing one from $V_{\phi(b')}$.
- To verify local triviality, let $b' \in B'$. Then there exists an open neighbourhood U of $\phi(b')$ in B such that $V \to B$ is trivial over U. Then $\phi^{-1}(U)$ is a trivializing neighbourhood of b': We can construct a cube

$$\begin{tikzcd} \mathbb{K}^n \times \phi^{-1}(U) \arrow[rr, "(\text{linear}){,}(1)"] \arrow[dr, "\rho"] \arrow[dd] & & \mathbb{K}^n \times U \arrow[dr, "(\text{linear}){,}(2)"] \arrow[dd] & \\ & V' \arrow[rr, "(\text{linear}){,}(3)"'] \arrow[dd] & & V \arrow[dd] \\ \phi^{-1}(U) \arrow[rr] \arrow[dr, hook] & & U \arrow[dr, hook] & \\ & B' \arrow[rr] & & B \end{tikzcd} .$$

Here the front face is the pullback we started out with. The right face is a pullback since V is trivial over U. The rear face can directly be verified to be a pullback. The map $\rho : \mathbb{K}^n \times \phi^{-1}(U) \to V'$ exists since V' is a pullback; it makes the diagram commute.

By repeated usage of the 2-out-of-3 property for 1-pullbacks and using commutativity of the diagram, the right face also is a pullback. Furthermore, as indicated, (1) is (fiberwise) linear (by definition of a pullback), (2) is linear because V is a vector bundle and (3) is linear by definition of the vector bundle structure on V' (using the pullback). Then also ρ is linear, since its composition with the linear map $V' \to V$ is again a linear map.[44]

□

Definition 4.7.6. $\phi^* V := V'$ as in the lemma before is called the *pullback of* V along ϕ.

Remark 4.7.7.
- This does *not* define a 1-functor

$$\begin{aligned} &\mathrm{Top} \to \mathrm{Sets}, \\ &\mathrm{Ob}\,\mathrm{Top} \ni X \mapsto \{\text{vector bundles on } X\} \in \mathrm{Sets} \\ &\mathrm{Hom}_{\mathrm{Top}}(X,Y) \ni f \mapsto (V \mapsto f^*V) \in \mathrm{Hom}_{\mathrm{Sets}}(\{\text{v.b. on} X\}, \{\text{v.b. on } Y\}) \end{aligned} ,$$

since for $f : X \to Y$, $g : Y \to Z$, $f^*(g^*V)$ is not equal to $(g \circ f)^*V$, but only canonically isomorphic.
- If B is a topological space, the category of vector bundles over B is additive ($\mathbb{R}$-linear), but *not* Abelian.[45]

[44] Use that the map $V'_{b'} \to V_{\phi(b')}$ between the fibers is an isomorphism for all $b' \in B'$. The composition of a map with a vector space isomorphism is linear if and only if the original map was linear as well.

[45] Example: The bundle $\mathrm{pr}_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has an endomorphism given by $(t,x) \mapsto (tx, x)$. This has no well-defined kernel (the fiberwise kernel is not a vector bundle).

Definition 4.7.8. Let $\pi : E \to B$ be a vector bundle of rank n over B. A subspace $E' \subset E$ is called a *subbundle of rank k* (of E) if there exist a trivializing open cover $\{U_\alpha\}_\alpha$ for E such that for all α, the left homeomorphism in the diagram

$$\begin{tikzcd} E' \cap \pi^{-1}(U_\alpha) \arrow[r, hook] \arrow[d, "\cong"'] & \pi^{-1}(U_\alpha) \arrow[d, "\cong"] \\ \mathbb{K}^k \times U \arrow[r, hook, "(\text{incl. of vector spaces})\times \mathrm{id}_U"'] & \mathbb{K}^n \times U \end{tikzcd} .$$

exists and the diagram commutes. In this case, $\pi' := \pi|_{E'} : E' \to B$ is a vector bundle of rank k over B.

Proposition 4.7.9*. *Let*

$$\begin{tikzcd} V \arrow[rr, "\phi"] \arrow[dr] & & W \arrow[dl] \\ & B & \end{tikzcd}$$

be a morphism of vector bundles $V \to B$, $W \to B$ over a topological space B with ranks $\mathrm{rk}(V)$, $\mathrm{rk}(W)$.[46] *Assume that there exists $m \in \mathbb{N}$ such that for all $b \in B$, the map between the fibers $\phi_b : V_b \to W_b$ has rank m. (I.e. ϕ is a morphism of constant rank.) Then*

1. $\ker(\phi) := \{v \in V \mid \phi(v) = 0\}$ *is a subbundle of V with rank* $\mathrm{rk}(V) - m$.
2. $\mathrm{im}(\phi) := \{w \in W \mid \exists v \in V : \phi(v) = w\}$ *is a subbundle of W with rank m.*
3. $\mathrm{coker}(\phi) := \bigsqcup_{b\in B} W_b/\mathrm{im}(\phi_b)$ *is a vector bundle with rank* $\mathrm{rk}(W) - m$, *where the disjoint union is meant as a set and* $\mathrm{coker}(\phi)$ *is equipped with the quotient topology of W.*

(No proof here.)

4.8 Linear algebra of vector bundles

Proposition 4.8.1. *Let F be a functor from (eventually a product of) the category of vector spaces to the category of vector spaces, e.g. $F = \bigoplus, \mathrm{Hom}(\cdot,\cdot), \bigotimes, \Lambda^k, \mathrm{Sym}^k, \ldots$. Let B be a topological space. Then F defines a functorial construction on vector bundles over B (still called F) such that the fibers over each $b \in B$ are isomorphic to F applied to the fibers of the input vector bundles. (E.g. if $F = \mathrm{Hom}(\cdot,\cdot)$ and $V \to B$, $W \to B$ are vector bundles over B, then $(\mathrm{Hom}(V,W))_b \cong \mathrm{Hom}(V_b, W_b)$ for all $b \in B$.)*

Proof (sketch). We prove the statement in the real case and for $\mathrm{Hom}(\cdot,\cdot)$. Let $p_W : V \to B$, $p_W : W \to B$ be vector bundles over B with ranks n, m, respectively. As a set, we define $\mathrm{Hom}(V,W) := \bigsqcup_{b\in B} \mathrm{Hom}(V_b, W_b)$ (where $\mathrm{Hom}(V_b, W_b)$ are linear homomorphisms between real vector spaces). The projection is the obvious one; we also write $\mathrm{Hom}(V,W)|_U$ for $U \subset B$ for the preimage of U under this projection (analogously for V, W). It remains to define the topology on $\mathrm{Hom}(V,W)$.

There exists an open cover $\{U_\alpha\}_\alpha$ of B that trivializes both V and W. Over U_α, we have commutative diagrams, where ρ_α, η_α are homeomorphisms:

$$\begin{tikzcd} V|_{U_\alpha} \arrow[rr, "\rho_\alpha", "\cong"'] \arrow[dr, "p_V"'] & & \mathbb{R}^n \times U_\alpha \arrow[dl, "\mathrm{pr}_2"] \\ & U_\alpha & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} W|_{U_\alpha} \arrow[rr, "\eta_\alpha", "\cong"'] \arrow[dr, "p_W"'] & & \mathbb{R}^m \times U_\alpha \arrow[dl, "\mathrm{pr}_2"] \\ & U_\alpha & \end{tikzcd} .$$

[46] I.e. the above diagram commutes.

In particular, for all $b \in U_\alpha$, we get isomorphisms $\rho_b : V_b \xrightarrow{\sim} \mathbb{R}^n$, $\eta_b : W_b \xrightarrow{\sim} \mathbb{R}^m$. We define

$$\begin{array}{rll} \psi_\alpha : \mathrm{Hom}(V,W)|_{U_\alpha} = & \bigsqcup_{b\in U_\alpha} \mathrm{Hom}(V_b, W_b) & \to U_\alpha \times \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \\ & \mathrm{Hom}(V_b, W_b) \ni \phi_b & \mapsto (b, \eta_b \circ \phi_b \circ \rho_b^{-1}) \end{array}.$$

The topology on $\mathrm{Hom}(V,W)$ shall be defined such that ψ_α is a homeomorphism and $\mathrm{Hom}(V,W)|_{U_\alpha}$ is open.[47] We need to check that for $U_{\alpha\beta} := U_\alpha \cap U_\beta$ (α, β some indices), the resulting topology on $\mathrm{Hom}(V,W)_{U_{\alpha\beta}}$ does not depend on the choice of α or β.

"Over $U_{\alpha\beta}$" (i.e. with appropriate restrictions), the composition $\psi_\alpha \circ \psi_\beta^{-1}$ has the following description: Let $\chi_{\alpha\beta} := \rho_\beta \circ \rho_\alpha^{-1}$, $\theta_{\alpha\beta} := \eta_\beta \circ \eta_\alpha^{-1}$. Note that these are isomorphisms "over $U_{\alpha\beta}$", i.e. e.g.

$$\begin{tikzcd} \mathbb{R}^n \times U_{\alpha\beta} \arrow[rr, "\chi_{\alpha\beta}", "\cong"'] \arrow[dr, "\mathrm{pr}_2"'] & & \mathbb{R}^n \times U_{\alpha\beta} \arrow[dl, "\mathrm{pr}_2"] \\ & U_{\alpha\beta} & \end{tikzcd}$$

commutes. Over $b \in B$, this means the application of an endomorphism $\chi_{\alpha\beta,b} \in \mathrm{Gl}_n(\mathbb{R})$ which depend continuously on b (this can be seen by expressing everything in terms of bases).[48] Similar observations are made for $W|_{U_{\alpha\beta}}$. Then

$$\begin{array}{rll} \psi_\alpha \circ \psi_\beta^{-1} : & \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \times U_{\alpha\beta} & \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \times U_{\alpha\beta}, \\ & (b, \kappa) & \mapsto (b, (\theta_{\alpha\beta,b} \circ \kappa \circ \chi_{\alpha\beta,b}^{-1})) \end{array}$$

is a homeomorphism. Thus $\psi_\alpha|_{U_{\alpha\beta}}$ is a homeomorphism if and only if $\psi_\beta|_{U_{\alpha\beta}}$ is a homeomorphism, i.e. the topology on $\mathrm{Hom}(V,W)_{U_{\alpha\beta}}$ is independent of the choice of α or β. This shows that the construction of $\mathrm{Hom}(V,W)$ is well-defined and therefore proves the proposition. □

Example 4.8.2. – The tautological complex line bundle on $\mathbb{C}P^n$ is usually denoted $\mathcal{O}(-1)$. Let then $\mathcal{O}(1) := \mathrm{Hom}(\mathcal{O}(-1), \underline{\mathbb{C}})$, where $\underline{\mathbb{C}} := \mathbb{C} \times \mathbb{C}P^n$ is the trivial line bundle. For $k \in \mathbb{Z}$, we then define

$$\mathcal{O}(k) := \begin{cases} \mathcal{O}(1)^{\otimes k}, & k > 0, \\ \underline{\mathbb{C}}, & k = 0, \\ \mathcal{O}(-1)^{\otimes(-k)}, & k < 0. \end{cases}$$

In fact, these are all complex line bundles on $\mathbb{C}P^n$ (see later).

– Over S^n, we have an isomorphism $\psi : TS^n \oplus \underline{\mathbb{R}} \to \underline{\mathbb{R}}^{n+1}$: Recall from example 4.7.3 that $TS^n = \{(x,\lambda) \mid x \in S^n, \lambda \in \mathbb{R}^{n+1}, \langle \lambda, x\rangle = 0\}$. Then let $\psi((\lambda, x), (t, x)) := (\lambda + t \cdot x, x)$. ψ is a morphism of vector bundles that is fiberwise an isomorphism. Thus ψ is an isomorphism.[49]

Exercise 4.8.3. We have an isomorphism $\phi : \mathbb{C}P^n = \mathrm{Gr}^{\mathbb{C}}(1, n+1) \xrightarrow{\sim} \mathrm{Gr}^{\mathbb{C}}(n, n+1)$ (sending each line to its orthogonal complement). Over $\mathbb{C}P^n$, by the previous example we have the tautological line bundle $\mathcal{O}(-1)$. On the other hand, denoting $\tau_n \to \mathrm{Gr}^{\mathbb{C}}(n, n+1)$ the tautological bundle on $\mathrm{Gr}^{\mathbb{C}}(n, n+1)$, we obtain a rank n vector bundle $\phi^*\tau_n$ on $\mathbb{C}P^n$. Then $\mathcal{O}(-1) \oplus \tau_n \cong \underline{\mathbb{C}}^{n+1}$.

4.9 Classifying spaces of vector bundles

Theorem 4.9.1. *Let $B \in \mathrm{Ob}\,\mathrm{CW}$. There exists natural isomorphisms of sets*

$$\begin{array}{l} \mathrm{Bun}_{\mathrm{Gl}_n(\mathbb{R})}(B) \xrightarrow{\sim} \{\text{real vector bundles or rank } m \text{ over } B\}/\text{isomorphism}, \\ \mathrm{Bun}_{\mathrm{Gl}_n(\mathbb{C})}(B) \xrightarrow{\sim} \{\text{complex vector bundles or rank } m \text{ over } B\}/\text{isomorphism}, \end{array}$$

where $\mathrm{Bun}_{\mathrm{Gl}_n(\mathbb{R})}$ *resp.* $\mathrm{Bun}_{\mathrm{Gl}_n(\mathbb{C})}$ *denote here the sets of right principal* $\mathrm{Gl}_n(\mathbb{R})$ *resp.* $\mathrm{Gl}_n(\mathbb{C})$*-bundles.*

[47] $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \cong \mathbb{R}^{nm}$ here carries the standard topology.

[48] Compare e.g. the definition of $\overline{\rho}$ in the proof of lemma 4.1.9; but here notational clutter is added since we have vector bundles and not only groups.

[49] One glues local homeomorphisms to obtain a homeomorphism.

Proof. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

– Given a principal $\mathrm{Gl}_n(\mathbb{K})$-bundle $P \to B$, the associated fiber bundle $P \times^{\mathrm{Gl}_n(\mathbb{K})} \mathbb{K}^n$ is a vector bundle of rank n over B. (Cf. example 4.4.2, resp. a variant switching the roles of left and right actions. The structure of vector spaces on the fibers is induced from $\mathbb{K}^n$.)

– On the other hand, given a vector bundle $V \to B$ of rank n over B, we define a set $E := \bigsqcup_{b\in B} \mathrm{Iso}(\mathbb{K}^n, V_b)$, where $\mathrm{Iso}(\mathbb{K}^n, V_b)$ denotes the set of linear isomorphisms from $\mathbb{K}^n$ to V_b. This has a right action of $\mathrm{Gl}_n(\mathbb{K})$ given by precomposition, i.e. by acting on $\mathbb{K}^n$. We define a topology on E similar to the one in proposition 4.8.1: If $U \subset B$ is trivializing V, i.e. there is a homeomorphism $\rho_U : V|_U \xrightarrow{\sim} \mathbb{K}^n \times U$, then define the topology on E by requiring that $E|_U$ is open and

$$\begin{array}{rll} \psi_U : E|_U = & \bigsqcup_{b\in U} \mathrm{Iso}(\mathbb{K}^n, V_b) & \to U \times \mathrm{Gl}_n(\mathbb{K}) \\ & \mathrm{Iso}(\mathbb{K}^n, V_b) \ni \phi_b & \mapsto (b, \rho_U \circ \phi_b) \end{array}$$

is a homeomorphism. Again this topology is independent of the choice of U.

– These constructions are inverse to each other: Given a principal G-bundle P, we obtain $P \times^{\mathrm{Gl}_n(\mathbb{K})} \mathbb{K}^n$ and then a principal G-bundle E with $E_b = \mathrm{Iso}(\mathbb{K}^n, P_b \times^{\mathrm{Gl}_n(\mathbb{K})} \mathbb{K}^n)$ for $b \in B$. There is a $\mathrm{Gl}_n(\mathbb{K})$-equivariant map $P \to E$ that looks on fibers as follows:

$$\begin{array}{ll} P_b & \to \mathrm{Iso}(\mathbb{K}^n, P_b \times^{\mathrm{Gl}_n(\mathbb{K})} \mathbb{K}^n), \\ p & \mapsto (\theta \mapsto (p, \theta)) \end{array} .$$

By lemma 4.1.9, it is an isomorphism, i.e. $[P] = [E]$ in $\mathrm{Bun}_{\mathrm{Gl}_n(\mathbb{K})}(B)$.

– If we start out with a vector bundle E, we obtain a principal G-bundle E as defined above and then the vector bundle $E \times^{\mathrm{Gl}_n(\mathbb{K})} \mathbb{K}^n$. There is a morphism of vector bundles $E \times^{\mathrm{Gl}_n(\mathbb{K})} \mathbb{K}^n \to V$ that looks on fibers as

$$\begin{array}{ll} \mathrm{Iso}(\mathbb{K}^n, V_b) \times^{\mathrm{Gl}_n(\mathbb{K})} \mathbb{K}^n & \to V_b, \\ (\phi, \theta) & \mapsto \phi(\theta) \end{array} .$$

One verifies that this is a fiberwise isomorphism, hence an isomorphism of vector bundles. Thus this composition yields, up to isomorphism, the same vector bundle again.

□

Corollary 4.9.2. *Let $B \in \mathrm{Ob}\,\mathrm{CW}$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then the sets $[B, \mathrm{Gr}^{\mathbb{K}}(n, \infty)]$, $\mathrm{Bun}_{\mathrm{Gl}_n(\mathbb{K})}$ and $\{\mathbb{K}\text{-vector bundles of rank } n \text{ over } B\}/\text{isomorphism}$ and $\mathrm{Bun}_{\tilde{G}}(B)$ are naturally isomorphic, where $\tilde{G} := O_n$ if $\mathbb{K} = \mathbb{R}$ and $\tilde{G} := U_n$ if $\mathbb{K} = \mathbb{C}$.*

Furthermore, the isomorphism $[B, \mathrm{Gr}^{\mathbb{K}}(n, \infty)] \xrightarrow{\sim} \{\mathbb{K}\text{-vector bundles of rank } n \text{ over } B\}/\text{isomorphism}$ is given by $f \mapsto f^*\tau_n$, where $f^*\tau_n$ is the pullback of the tautological bundle τ_n on $\mathrm{Gr}^{\mathbb{K}}(n, \infty)$.

The proof consists of assembling together the information from corollary 4.5.5, theorem 4.6.4, remark 4.6.8 and theorem 4.9.1.

In the following chapters, we assume familarity with the material from part 5.

4.10 Characteristic classes of vector bundles

Recall from corollary 4.9.2 that $BO_1 \simeq B\mathbb{Z}/2 \simeq \mathbb{R}P^\infty \simeq K(\mathbb{Z}/2, 1)$ classifies real line bundles, and $BU_1 \simeq \mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$ classifies complex line bundles.

If we want to construct natural transformations $\mathrm{Bun}_{O_k}(\cdot) \to H^m(\cdot, \mathbb{Z}/2)$, $\mathrm{Bun}_{U_k}(\cdot) \to H^m(\cdot, \mathbb{Z})$, we need to compute the cohomology of Grassmannians.

Theorem 4.10.1*. *We have*

$$\begin{array}{ll} H^*(\mathrm{Gr}^{\mathbb{R}}(k, \infty); \mathbb{Z}/2) \cong (\mathbb{Z}/2)[w_1, \ldots, w_k], & \text{where } \deg(w_i) = i \text{ for } i = 1, \ldots, k, \\ H^*(\mathrm{Gr}^{\mathbb{C}}(k, \infty); \mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_k], & \text{where } \deg(c_i) = 2i \text{ for } i = 1, \ldots, k. \end{array}$$

(No proof here.)

Definition 4.10.2. If X is a topological space, abusing notation we denote the natural transformations corresponding to the cohomology classes from the previous theorem

$$w_i : \{\text{real vector bundles on } X\}/(\text{isomorphism}) \Rightarrow H^i(X;\mathbb{Z}/2),$$
$$c_i : \{\text{complex vector bundles on } X\}/(\text{isomorphism}) \Rightarrow H^{2i}(X;\mathbb{Z}).$$

w_i is called the *i-th Stiefel-Whitney class*, and c_i is called the *i-th Chern class.*

[50]

Theorem 4.10.3*. *Characteristic classes satisfy the* Whitney sum formula*: If*

$$0 \longrightarrow V \longrightarrow W \longrightarrow U \longrightarrow 0$$

is an exact sequence of vector bundles on X, then:

- *If V, W, U are real vector bundles, then $w_k(W) = \sum_{i=0}^{k} w_i(V)w_{k-i}(W)$, where $w_0 := 1$ (and the product is the cup-product).*
- *If V, W, U are complex vector bundles, then $c_k(W) = \sum_{i=0}^{k} c_i(V)c_{k-i}(W)$, where $c_0 := 1$.*

(No proof here.)

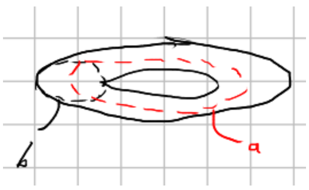
4.11 Characteristic classes of line bundles

Let M be a smooth oriented connected closed manifold. Denote $[M] \in H_{\dim(M)}(M;\mathbb{Q}) \cong \mathbb{Z}\cdot[M]$ the fundamental class of M. Let $N \subset M$ be an oriented submanifold; denote $i : N \hookrightarrow M$ the inclusion and let $[M] \in H_{\dim(N)}(N;\mathbb{Q})$ be its fundamental class. By the universal coefficient theorem 5.1.5, there exists an element corresponding to $i_*([N])$ in $H^{\dim(N)}(M;\mathbb{Q})$ (by abuse of notation still called $i_*([N])$), and by Poincaré duality (theorem 5.3.9), there exists $[N]^* \in H^{\dim(M)-\dim(N)}(M;\mathbb{Q})$ that is dual to $i_*([N])$.

Theorem 4.11.1*. *In the situation described above, let $N_1, N_2 \subset M$ be submanifolds that "intersect transversally", i.e. for all $x \in N_1 \cap N_2$, the map $T_xN_1 \oplus T_xN_2 \to T_xM$ is surjective. Then $[N_1]^* \cup [N_2]^* =$ "$\pm$"$[N_1 \cap N_2]^*$.*[51]

(No proof here.)

Example 4.11.2*. – Consider $T := S^1 \times S^1$.



We have $[S^1 \times \mathrm{pt}]^* \cup [\mathrm{pt} \times S^1]^* = [\mathrm{pt} \times \mathrm{pt}]^*$, which is the generator of $H^2(T;\mathbb{Q})$.[52] On the other hand, if $\mathrm{pt}, \mathrm{pt}'$ are different points in S^1, then $[\mathrm{pt} \times S^1]^* \cup [\mathrm{pt}' \times S^1]^* = [\emptyset]^* = 0$.

We know that $H^*(T;\mathbb{Q}) \cong \mathbb{Q}[a,b]/(a^2,b^2)$ (recall example 5.3.11). By the above observation, we can take $a = [S^1 \times \mathrm{pt}]^*$, $b = [\mathrm{pt} \times S^1]^*$.

---

[50] To be precise, the natural transformations are given by the Yoneda lemma, $f \mapsto f^*(w_i)$ (resp. $f \mapsto f^*(c_i)$, where $f \in [X, BO_k]$ resp. $f \in [X, BU_k]$ is the map corresponding to some given vector bundle via corollary 4.9.2.

[51] Note that $N_1 \cap N_2$ is a smooth manifold in its own right, and it carries an induced orientation. We use "$\pm$" because we do not want to keep track of the signs.

[52] We omit the i_*.

– Consider $\mathbb{C}P^n$: Recall that the inclusion $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ is induced by the diagram (which commutes by definition)

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, hook] & \mathbb{C}^{n+1} \\ \mathbb{C}^n\backslash\{0\} \arrow[r, hook] \arrow[u, hook] \arrow[d, two heads] & \mathbb{C}^{n+1}\backslash\{0\} \arrow[u, hook] \arrow[d, two heads] \\ \mathbb{C}P^{n-1} \arrow[r, hook] & \mathbb{C}P^n \end{tikzcd}.$$

This yields an element $[\mathbb{C}P^{n-1}]^* \in H^2(\mathbb{C}P^n;\mathbb{Q})$. One can show that this does not depend on the choice of the embedding $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ (by homotopy invariance; different embeddings yield homotopic inclusion maps $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$). Consider a different diagram

$$\begin{tikzcd} (\mathbb{C}^n)' \arrow[r, hook] & \mathbb{C}^{n+1} \\ (\mathbb{C}^n)'\backslash\{0\} \arrow[r, hook] \arrow[u, hook] \arrow[d, two heads] & \mathbb{C}^{n+1}\backslash\{0\} \arrow[u, hook] \arrow[d, two heads] \\ (\mathbb{C}P^{n-1})' \arrow[r, hook] & \mathbb{C}P^n \end{tikzcd},$$

where $(\mathbb{C}^n)' = \mathbb{C}^n$, $(\mathbb{C}P^{n-1})' = \mathbb{C}P^{n-1}$, but the inclusion maps are understood to be different ones. Then by the previous theorem we have $[\mathbb{C}P^{n-1}]^* \cup [(\mathbb{C}P^{n-1})']^* = [\mathbb{C}P^{n-2}]^*$, because $\mathbb{C}^n \cap (\mathbb{C}^n)' \cong \mathbb{C}^{n-1}$ (the intersection of two complex hyperplanes, if we specify for simplicity to linear maps) and thus $\mathbb{C}P^{n-1} \cap (\mathbb{C}P^{n-1})' \cong \mathbb{C}P^{n-2}$. On the other hand, we observed above $[(\mathbb{C}P^{n-1})']^* = [\mathbb{C}P^{n-1}]^*$. Thus $[\mathbb{C}P^{n-1}]^* \cup [\mathbb{C}P^{n-1}]^* = [\mathbb{C}P^{n-2}]^*$.

Inductively, we obtain $([\mathbb{C}P^{n-1}]^*)^{\cup k} = [\mathbb{C}P^{n-k}]$ for $k < n$, and $(\mathbb{C}P^{n-1}]^*)^{\cup n} = [\mathrm{pt}]^*$. This is the generator of $H^{2n}(\mathbb{C}P^n;\mathbb{Q})$. By example 5.3.11, we already know $H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[h]/(h^{n+1})$, where $\deg(h) = 2$, and the above observations show that we can take $h = [\mathbb{C}P^{n-1}]^*$.[53]

The inclusion $j : \mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n-1}$ yields a map

$$\begin{aligned} j^* : \quad & \mathbb{Z}[h]/(h^{n+2}) \cong H^*(\mathbb{C}P^{n+1};\mathbb{Z}) && \to H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[h]/(h^{n+1}), \\ & h && \mapsto h. \end{aligned}$$

Thus

$$H^*(\underbrace{\mathbb{C}P^\infty}_{\mathrm{Gr}^{\mathbb{C}}(1,\infty)};\mathbb{Z}) = H^*(\mathrm{colim}_n\, \mathbb{C}P^n;\mathbb{Z}) = \lim_n H^*(\mathbb{C}P^n;\mathbb{Z}) = \lim_n \mathbb{Z}[h]/(h^{n+1}) \cong \mathbb{Z}[h],$$

where $\deg(h) = 2$. (Note that this means that we can take $c_1 = h$, the first Chern class.)[54]

– Similarly, $H^*(\mathbb{R}P^\infty;\mathbb{Z}/2) \cong (\mathbb{Z}/2)[a]$, where $\deg(a) = 1$ and we may choose a such that $a|_{\mathbb{R}P^n} = [\mathbb{R}P^{n-1}]^*$.

Definition 4.11.3. For a topological space X, we define the *Picard group* of X:

$$\mathrm{Pic}^{\mathbb{C}}(X) := \{\text{complex line bundles on } X\},$$

where the group multiplication is given by $\otimes$, the unit is the trivial line bundle and the inverse of

[53] h: "hyperplane".

[54] There are subtle difficulties with $\mathbb{Z}[[h]]$ and $\mathbb{Z}[h]$ in this case. The definition of a graded ring could also be made using the direct product instead of the direct sum, and the two options yield different results (i.e. the limits in "the category of graded rings defined with the product" and "the category of graded rings defined with the direct sum" do not coincide). We do not go into detail here.

a bundle is its dual bundle.[55]

This is a functor $\mathrm{Top}^{\mathrm{op}} \to \mathrm{Grp}$: A map $f : X \to Y$ yields a group homomorphism $f^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ (defined using $\mathrm{Pic}^{\mathbb{C}}(X) \cong [X, \mathbb{C}P^\infty]$ as a set by corollary 4.9.2; then $f^*(g) := g \circ f$ for $g \in [X, \mathbb{C}P^\infty]$).

Remark 4.11.4. *Fact:* $\mathrm{Pic}^{\mathbb{C}}(\mathbb{C}P^\infty)$ is a free abelian group generated by $\mathcal{O}(-1)$, the tautological line bundle on $\mathbb{C}P^\infty$.

Lemma 4.11.5. *For $n, m \in \mathbb{N}$, let*

$$\begin{array}{rll} s_{n,m}: & \mathbb{C}P^n \times \mathbb{C}P^m & \to \mathbb{C}P^{(n+1)(m+1)-1}, \\ & (Z_1 \subset \mathbb{C}^{n+1}, Z_2 \subset \mathbb{C}^{m+1}) & \mapsto (Z_1 \otimes Z_2 \subset \underbrace{\mathbb{C}^{n+1} \otimes \mathbb{C}^{m+1}}_{\cong \mathbb{C}^{(n+1)(m+1)}}) \end{array}$$

be the Segre map. *Then the induced map $s : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ (still called Segre map) classifies tensor products of complex line bundles.*

Proof (sketch). Let Z_1, Z_2 be complex line bundles on X. Since $\mathbb{C}P^\infty$ classifies complex line bundles by corollary 4.9.2, for $i = 1, 2$ there exist maps $f_i : X \to \mathbb{C}P^\infty$ such that $Z_i \cong f_i^*(\mathcal{O}(-1))$. We need to show that for the composition $s \circ (f_1 \times f_2) : X \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$, we have $(f_1 \times f_2)^* s^*(\mathcal{O}(-1)) \cong Z_1 \otimes Z_2$.

It suffices to show that $s^*(\mathcal{O}(-1)) \cong \mathrm{pr}_1^*(\mathcal{O}(-1)) \otimes \mathrm{pr}_2^*(\mathcal{O}(-1))$, because

$$\begin{aligned} (f_1 \times f_2)^*(\mathrm{pr}_1^*(\mathcal{O}(-1)) \otimes \mathrm{pr}_2^*(\mathcal{O}(-1))) &\cong (f_1 \times f_2)^*(\mathrm{pr}_1^*(\mathcal{O}(-1))) \otimes (f_1 \times f_2)^*(\mathrm{pr}_2^*(\mathcal{O}(-1))) \\ &\cong f_1^*(\mathcal{O}(-1)) \otimes f_2^*(\mathcal{O}(-1)) \cong Z_1 \otimes Z_2. \end{aligned}$$

Furthermore, it suffices to prove this for the maps $s_{n,m}$, i.e. $s_{n,m}^*(\mathcal{O}(-1)) \cong \mathrm{pr}_1^*(\mathcal{O}(-1)) \otimes \mathrm{pr}_2^*(\mathcal{O}(-1))$. Over a point $(l_1, l_2) \in \mathbb{C}P^\infty \times \mathbb{C}P^\infty$, we have $(\mathrm{pr}_1^*(\mathcal{O}(-1)) \otimes \mathrm{pr}_2^*(\mathcal{O}(-1)))_{(l_1,l_2)} \cong l_1 \otimes l_2$, because $\mathcal{O}(-1)$ is the tautological bundle. On the other hand, we also have $(s_{n,m}^*(\mathcal{O}(-1)))_{(l_1,l_2)} = (\mathcal{O}(-1))_{s_{n,m}(l_1,l_2)} = (\mathcal{O}(-1))_{l_1 \otimes l_2} \cong l_1 \otimes l_2$.

One can find a map $s_{n,m}^*(\mathcal{O}(-1)) \to \mathrm{pr}_1^*(\mathcal{O}(-1)) \otimes \mathrm{pr}_2^*(\mathcal{O}(-1))$ that induces the identity on $l_1 \otimes l_2$ on the fibers. □

Proposition 4.11.6. *For all topological spaces X, the map $c_1 : \mathrm{Pic}^{\mathbb{C}}(X) \xrightarrow{\sim} H^2(X; \mathbb{Z})$ induced by the first Chern class $h = c_1 \in H^2(\mathbb{C}P^\infty; \mathbb{Z})$ is a homomorphism of abelian groups (i.e. $c_1(Z_1 \otimes Z_2) = c_1(Z_1) + c_1(Z_2)$ for all complex line bundles Z_1, Z_2 over X).*

Proof (sketch). It suffices to show that (where s is the Segre map from the previous lemma)

$$\begin{array}{rll} s^*: & H^2(\mathbb{C}P^\infty; \mathbb{Z}) & \to H^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty; Z) \qquad \text{maps} \\ & h & \mapsto \mathrm{pr}_1^*(h) + \mathrm{pr}_2^*(h), \end{array}$$

since then

$$\begin{aligned} c_1(Z_1 \otimes Z_2) &= (f_1 \times f_2)^* s^*(h) = (f_1 \times f_2)^*(\mathrm{pr}_2^*(h) + \mathrm{pr}_1^*(h)) \\ &= f_1^*(h) + f_2^*(h) = c_1(Z_1) + c_1(Z_2) \end{aligned}$$

for all complex line bundles $Z_i = f_i^*(\mathcal{O}(-1))$ on X. Furthermore, it suffices to show that $s_{1,1}^* : H^2(\mathbb{C}P^3; \mathbb{Z}) \to H^2(\mathbb{C}P^1 \times \mathbb{C}P^1; \mathbb{Z})$ satisfies the formula above (since s^* is defined as a map into a limit). By the Künneth formula and $\mathbb{C}P^1 \cong S^2$, we have

$$H^*(\mathbb{C}P^1 \times \mathbb{C}P^1; \mathbb{Z}) \cong H^*(\mathbb{C}P^1; \mathbb{Z}) \otimes H^*(\mathbb{C}P^1; \mathbb{Z}) \cong \mathbb{Z}[h_1]/(h_1^2) \otimes \mathbb{Z}[h_2]/(h_2^2) \cong \mathbb{Z}[h_1, h_2]/(h_1^2, h_2^2),$$

where $h_i = \mathrm{pr}_i^*(h)$ for $i = 1, 2$; $\deg(h_i) = 2$. Thus we have $s_{1,1}^*(h) = ah_1 + bh_2$ for $a, b \in \mathbb{Z}$ (since $s_{1,1}^*$ preserves degrees). We need to show that $a = b = 1$.

Since $h_1^2 = h_2^2 = 0$, we have $s_{1,1}^*(h)h_1 = bh_1h_2 = b[\mathrm{pt}]^*$, $s_{1,1}^*(h)h_2 = ah_1h_2 = a[\mathrm{pt}]^*$ (where the multiplication is the cup-product).

[55] If $V \to X$ is a vector bundle, the isomorphism $V^* \otimes V \to \underline{\mathbb{C}}$ is given by $(\phi \otimes v)_x \mapsto (\phi(v))_x$ for all $x \in X$. This is injective, hence an isomorphism for dimension reasons.

Fact: If $f : M \to N$ is a continuous map of smooth manifolds and $N_1 \subset N$ is a submanifold of codimension k such that $f^{-1}(N_1) \subset M$ is again a submanifold of codimension k, then $f^*([N_1]^*) = [f^{-1}(N_1)]^*$.

Thus we have $s_{1,1}^*(\underbrace{h}_{[\mathbb{C}P^2 \subset \mathbb{C}P^3]^*}) = [\underbrace{s_{1,1}^{-1}(\mathbb{C}P^2)}_{\subset \mathbb{C}P^1 \times \mathbb{C}P^1}]^*$ and then by theorem 4.11.1

$$s_{1,1}^*(h)h_1 = [s_{1,1}^{-1}(\mathbb{C}P^2)]^* \cup \underbrace{[\mathrm{pt} \times \mathbb{C}P^1]^*}_{=[\mathrm{pr}_1^{-1}(\mathrm{pt})]^* = \mathrm{pr}_1^*([\mathrm{pt}]^*) = \mathrm{pr}_1^*(h) = h_1} = [s_{1,1}^{-1}(\mathbb{C}P^2) \cap (\mathrm{pt} \times \mathbb{C}P^1)]^* = [\mathrm{pt}]^*.$$

$(s_{1,1}^{-1}(\mathbb{C}P^2) \cap (\mathrm{pt} \times \mathbb{C}P^1) = \mathrm{pt}$ can be verified geometrically, choosing explicit isomorphisms $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$. In the end, by homotopy invariance the resulting cohomology class will be independent of such choices.) This yields $b = 1$ by comparision. Similarly one obtains $a = 1$, yielding $s_{1,1}(h) = h_1 + h_2$ as required. □

Analogously, the first Stiefel-Whitney class $w_1 : \mathrm{Pic}^{\mathbb{R}}(X) \xrightarrow{\sim} H^1(X; \mathbb{Z}/2)$ is an isomorphism of abelian groups (where $\mathrm{Pic}^{\mathbb{R}}(X)$ is defined analogously to $\mathrm{Pic}^{\mathbb{C}}(X)$).

4.12 Properties of characteristic classes

Lemma 4.12.1. *For $k \leq n \in \mathbb{N}$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, there exist injective maps $i_{k,n} : \mathrm{Gr}^{\mathbb{K}}(k,n) \hookrightarrow \mathrm{Gr}^{\mathbb{K}}(k+1, n+1)$ that fit into a commutative diagram*

$$\begin{tikzcd} \ldots \arrow[r, hook] & \mathrm{Gr}^{\mathbb{K}}(k,n) \arrow[r, hook] \arrow[d, hook, "i_{k,n}"] & \mathrm{Gr}^{\mathbb{K}}(k,n+1) \arrow[r, hook] \arrow[d, hook, "i_{k,n+1}"] & \ldots, \\ \ldots \arrow[r, hook] & \mathrm{Gr}^{\mathbb{K}}(k+1,n+1) \arrow[r, hook] & \mathrm{Gr}^{\mathbb{K}}(k+1,n+2) \arrow[r, hook] & \ldots \end{tikzcd}$$

where the horizontal inclusions are induced by $\mathbb{K}^n \hookrightarrow \mathbb{K}^{n+1}$.

Thus there is an induced map of colimits $\mathrm{Gr}^{\mathbb{K}}(k,\infty) \xrightarrow{i_k} \mathrm{Gr}^{\mathbb{K}}(k+1,\infty)$ for all $k \in \mathbb{N}$.

Proof. We define $i_{k,n} : \mathrm{Gr}^{\mathbb{K}}(k,n) \to \mathrm{Gr}^{\mathbb{K}}(k+1,n+1)$, $\mathbb{K}^n \subset V \mapsto (V \oplus \mathbb{K}) \subset \mathbb{K}^n \oplus \mathbb{K}$. Then the diagram above commutes, because we get squares like

$$\begin{tikzcd} (V \subset \mathbb{K}^n) \arrow[r, mapsto] \arrow[d, mapsto, "i_{k,n}"] & (V \subset \mathbb{K}^n \oplus \mathbb{K}) \arrow[d, mapsto, "i_{k,n+1}"] \\ (V \oplus \mathbb{K} \subset \mathbb{K}^n \oplus \mathbb{K}) \arrow[r, mapsto] & (V \oplus \mathbb{K} \subset \mathbb{K}^n \oplus \mathbb{K} \oplus \mathbb{K}) \end{tikzcd}.$$

□

Remark 4.12.2. – *Fact:* With i_k as in the lemma before, the map

$$\begin{array}{llll} i_k^* : & \mathbb{Z}[c_1, \ldots, c_{k+1}] \cong H^*(\mathrm{Gr}^{\mathbb{C}}(k+1,\infty)) & \to H^*(\mathrm{Gr}^{\mathbb{C}}(k,\infty)) \cong \mathbb{Z}[c_1, \ldots, c_k] & \text{maps} \\ & c_i & \mapsto c_i, \quad \text{if } i < k+1, & \\ & c_{k+1} & \mapsto 0. & \end{array}$$

Furthermore, denoting the tautological vector bundle over $\mathrm{Gr}^{\mathbb{C}}(k+1,\infty)$ by τ_{k+1}, we have $i_k^* \tau_{k+1} \cong \tau_k \oplus \underline{\mathbb{C}}$.

Analogous statements hold in the real case.

– Thus an abuse of notation is justified: Cohomology classes named c_k occur in $H^*(\mathrm{Gr}^{\mathbb{C}}(k+i,\infty))$ for all $i \geq 0$. But these yield the same characteristic classes: Let V be a complex vector bundle of rank m over a topological space X. Then, since $\mathrm{Gr}^{\mathbb{C}}(m,\infty)$ classifies such vector bundles (cf. corollary 4.9.2), there exists $f : X \to \mathrm{Gr}^{\mathbb{C}}(m,\infty)$ such that $f^* \tau_m \cong V$.

The i'th Chern class of V is defined to be $\underbrace{c_i(V)}_{\in H^{2i}(X;\mathbb{Z})} := f^*(\underbrace{c_i}_{\in H^{2i}(\mathrm{Gr}^{\mathbb{C}}(m,\infty);\mathbb{Z})})$. To justify our abuse of notation, we want to show $c_i(V \oplus \underline{\mathbb{C}}) = c_i(V)$. This follows by the Whitney sum formula, theorem 4.10.3: We compute (using $f^*\underline{\mathbb{C}} = \underline{\mathbb{C}}$ and naturality of the Chern classes)

$$c_i(V \oplus \underline{\mathbb{C}}) = c_i((i_m \circ f)^*\tau_{m+1}) = c_i(f^* i_m^* \tau_{m+1}) = f^* c_i(\tau_m \oplus \underline{\mathbb{C}})$$

$$= f^*(c_i(\tau_m)1 + c_{i-1}(\tau_m)c_i(\underline{\mathbb{C}}) + \dots) = f^*(c_i(\tau_m)) = c_i(V).$$

[56] Here we have used $c_0(\underline{\mathbb{C}}) = 1$ (because the induced maps on cohomology are ring homomorphisms, and $c_0 = 1$), and furthermore over any space Y, we have $c_i(\underline{\mathbb{C}}) = 0$ for $i > 0$: We have $\underline{\mathbb{C}} \cong \mathrm{pr}^*\underline{\mathbb{C}}_*$, where $\mathrm{pr} : Y \to *$ is the constant map to the one-point space and $\underline{\mathbb{C}}_*$ denotes the trivial bundle over $*$ (with a subscript to distinguish it from $\underline{\mathbb{C}}$ over Y). Then, using that the Chern classes are natural transformations,

$$c_i(\underline{\mathbb{C}}) = c_i(\mathrm{pr}^*\underline{\mathbb{C}}_*) = \mathrm{pr}^* c_i(\underline{\mathbb{C}}_*) \mathrm{pr}^*(0) = 0,$$

because $c_i(\underline{\mathbb{C}}_*) \in H^{2i}(*, \mathbb{Z}) = 0$. This justifies our abuse of notation.

- The inclusions above allow for the definition of $c_i(V)$ for $i > \mathrm{rk}(V)$; it follows in this case $c_i(V) = 0$.

Proposition 4.12.3*. *Let V be a vector bundle on a topological space X of rank r. Assume that there exists a nowhere vanishing section $s : X \to V$ of V. Then $c_r(V) = 0$ if V is a complex vector bundle and $w_r(V) = 0$ if V is a real vector bundle.*

Proof (sketch). The proof of the complex case is sketched. The section s induces an exact sequence of vector bundles

$$0 \longrightarrow \underline{\mathbb{C}} \xrightarrow{\tilde{s}} V \longrightarrow \mathrm{coker}(\tilde{s}) \longrightarrow 0\ ,$$

where $\tilde{s}(\lambda_x) := \lambda \cdot s(x)$ for $\lambda_x \in \underline{C}_x$, $x \in X$. We have $\mathrm{rk}(\mathrm{coker}(\tilde{s})) = r - 1$. By the Whitney sum formula 4.10.3, the computation from the previous remark ($c_i(\underline{\mathbb{C}}) = 0$ for $i > 0$) and $c_r(W) = 0$ for $r > \mathrm{rk}(W)$ (see also the previous remark)

$$c_r(V) = \sum_{i=0}^{r} c_i(\underline{\mathbb{C}}) c_{k-i}(\mathrm{coker}(\tilde{s})) = \underbrace{c_0(\underline{\mathbb{C}})}_{=1} \underbrace{c_r(\mathrm{coker}(\tilde{s}))}_{=0} = 0.$$

□

Corollary 4.12.4*. *Let V be a vector bundle of rank r. If V is a complex vector bundle and $c_r \neq 0$, or V is a real vector bundle and $w_r \neq 0$, then V does not admit a nowhere vanishing section.*

The statement is equivalent to the one from the proposition before.

Example 4.12.5*. Let V be a complex vector space of dimension $n+1$ and let its projective space be $\mathbb{P}(V)$.[57]

Fact: The following sequence of vector bundles is exact (Euler exact sequence):

$$0 \longrightarrow \mathcal{O} \xrightarrow{f} \mathcal{O}(1) \otimes V \xrightarrow{e} T\mathbb{P}(V) \longrightarrow 0\ .$$

Here $\mathcal{O} = \underline{\mathbb{C}}$ is the trivial line bundle, $\mathcal{O}(1) = \mathcal{O}(-1)^*$ is the dual of the tautological line bundle defined earlier, V is regarded here as a trivial vector bundle (one takes the tensor product with V on each fiber) and $T\mathbb{P}(V)$ is the tangent bundle of $\mathbb{P}(V)$.[58]

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[56] $c_i(\tau_m) = \mathrm{id}^*(c_i) = c_i$ by definition.

[57] This is homeomorphic to $\mathbb{C}P^n$.

[58] The maps f, e are defined as follows: Let $x \in \mathbb{P}(V)$, corresponding to a one-dimensional subspace of V. Let $\mathbf{v} \in V \backslash \{0\}$ such that $x = [\mathbf{v}]$. Over x, the map f is defined by $f_x : \mathcal{O} \to \mathcal{O}(1) \otimes V$, $a \mapsto a\mathbf{v}^* \otimes \mathbf{v}$, where $\mathbf{v}^*$ is the

We define the *total Chern class* of a complex vector bundle $W \to X$ (for a topological space X):

$$c_{\text{tot}}(W) := \underbrace{c_0(W)}_{=1\in H^0(X)} + \underbrace{c_2(W)}_{\in H^2(X)} + \cdots + \underbrace{c_{\text{rk}(W)}(W)}_{\in H^{2\text{rk}(W)}(X)}.$$

Then the Whitney sum formula 4.10.3 may be reformulated as follows: If

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence of complex vector bundles, then $c_{\text{tot}}(B) = c_{\text{tot}}(A)c_{\text{tot}}(C)$ (where the multiplication is the cup-product).

Since from the computation in remark 4.12.2 it follows that $c_{\text{tot}}(\mathcal{O}) = 1$, it follows from the Euler exact sequence that $c_{\text{tot}}(T\mathbb{P}(V)) = c_{\text{tot}}(\mathcal{O}(1) \otimes V)$. Now since $V \cong \mathbb{C}^{n+1}$, we have $\mathcal{O}(1) \otimes V \cong \mathcal{O}(1)^{\oplus(n+1)}$. Furthermore, again from the Whitney sum formula it follows that if $B \cong A \oplus B$ for vector bundles A, B, C, then $c_{\text{tot}}(B) = c_{\text{tot}}(A)c_{\text{tot}}(C)$. Thus $c_{\text{tot}}(\mathcal{O}(1)^{\oplus n}) = c_{\text{tot}}(\mathcal{O}(1))^n = (1+h)^{n+1}$, where $h := c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}(V);\mathbb{Z})$.[59] We conclude $c_{\text{tot}}(T\mathbb{C}P^n) = (1+h)^{n+1}$ with $h := c_1(\mathcal{O}(1)) \in H^2(\mathbb{C}P^n;\mathbb{Z})$.

Similarly in the real case we have for the total Stiefel-Whitney class w_{tot} (defined analogously to the total Chern class): $w_{\text{tot}}(T\mathbb{R}P^n) = (1+a)^{n+1}$, where $a := c_1(\mathcal{O}(1)) \in H^1(\mathbb{R}P^n;\mathbb{Z}/2)$.

In particular, for $n = 1$ we have $c_{\text{tot}}(T\mathbb{C}P^1) = (1+h)^2 = 1 + 2h + h^2$, where $h^2 = 0$ in $H^*(\mathbb{C}P^1;\mathbb{Z})$ (recall $\mathbb{C}P^1 \cong S^2$ as real manifolds). Thus the complex line bundle $T\mathbb{C}P^1$ over $\mathbb{C}P^1$ satisfies $c_1(T\mathbb{C}P^1) = 2h \neq 0$. By corollary 4.12.4, $T\mathbb{C}P^1$ has no nowhere vanishing section. Since $\mathbb{C}P^1 \cong S^2$, $T\mathbb{C}P^1 \cong TS^2$ (as real manifolds resp. real vector bundles), we conclude that TS^2 has no nowhere vanishing section. This is the "hedgehog theorem".

Another conclusion we can draw from the above observations: Unless $n + 1 = 2^k$ for $k \in \mathbb{N}$, $w_{\text{tot}}(T\mathbb{R}P^n) \neq 1$, and so $T\mathbb{R}P^n$ is not a trivial vector bundle.[60]

Proposition 4.12.6*. *For $k \in \mathbb{N}$, $\mathbb{R}P^{2^k}$ can not be immersed in $\mathbb{R}^m$ unless $m \geq 2^{k+1} - 1$.*

Proof. Let $m \in \mathbb{N}$ and $f : \mathbb{R}P^{2^k} \to \mathbb{R}^m$ be an immersion. Then

$$0 \longrightarrow T\mathbb{R}P^{2^k} \longrightarrow f^*T\mathbb{R}^m \longrightarrow N_f \longrightarrow 0$$

is exact. Here N_f denotes the normal bundle for f (with rank $m - 2^k$). The map $T\mathbb{R}P^{2^k} \to f^*T\mathbb{R}^m$ is injective because f is an immersion.

Now $T\mathbb{R}^m \cong \underline{\mathbb{R}^m}$ is a trivial vector bundle, and $f^*\underline{\mathbb{R}^m} \cong \underline{\mathbb{R}^m}$. Thus by the Whitney sum formula 4.10.3 and observations from the previous example, we have

$$w_{\text{tot}}(\underline{\mathbb{R}^m}) = 1 = w_{\text{tot}}(T\mathbb{R}P^{2^k})w_{\text{tot}}(N_f) = (1+a)^{2^k+1}w_{\text{tot}}(N_f).$$

Since $(1+a)^{2^k+1} = (1+a^{2^k})(1+a) = 1+a+a^{2^k}$ (since $H^{2k+1}(\mathbb{C}P^{2^k};\mathbb{Z}/2) \ni a^{2^k+1} = 0$), it follows that $w_{\text{tot}}(N_f) = (1+a+a^{2^k})^{-1} = 1 + a + a^2 + \cdots + a^{2^k-1}$. In particular $w_{2^k-1}(N_f) \neq 0$.

Now by remark 4.12.2, this requires $m - 2^k = \text{rk}(N_f) \geq 2^k - 1$ resp. $m \geq 2^{k+1} - 1$. □

dual of $\mathbf{v}$. (This lies in $(\mathcal{O}(1))_x = x^*$. Note that since x is one-dimensional, there is a canonical way to define a dual vector, and furthermore one verifies that $a\mathbf{v}^* \otimes \mathbf{v}$ is independent of the choice of $\mathbf{v}$.)

The map e is defined as follows: Over x, let $e_x : Z_x^* \otimes V \to V/Z_x$, $\mathbf{v}^* \otimes \mathbf{w} \mapsto d\pi|_{\mathbf{v}}(\mathbf{w})$. (Here $\mathbf{v}^*$ is as before, $\mathbf{w} \in V$ is arbitrary, $\pi : V\backslash\{0\} \to \mathbb{P}(V)$ is the canonical projection and $d\pi$ its differential. Note that vectors of this form are a basis of $Z_x^* \otimes V$ and again the result is independent of the choice of $\mathbf{v}$.)

[59] This equals h from example 4.11.2 up to the isomorphism $\mathbb{P}(V) \cong \mathbb{C}P^n$ and a sign: With h as in example 4.11.2, we have $c_{\text{tot}}(\mathcal{O}(1)) = c_0(\mathcal{O}(1)) + c_1(\mathcal{O}(1)) = 1 + (-c_1\mathcal{O}(1)) = 1 - h$ by proposition 4.11.6 and $c_1(\mathcal{O}(1) \otimes \mathcal{O}(-1)) = c_1(\underline{\mathbb{C}}) = 0$ by a computation from remark 4.12.2. The sign is not very relevant.

[60] Since we are dealing with $\mathbb{Z}/2$-coefficients, inductively one proves $(1+a)^{2^k} = 1 + a^{2^k}$. However if $n+1 = 2^k$, then $a^{2^k} \in H^{n+1}(\mathbb{R}P^n;\mathbb{Z}/2) = 0$, thus in this case $w_{\text{tot}}(T\mathbb{R}P^n) = 1$. On the other hand, if $n+1 = 2^k m$ with $m > 1$ odd, then $(1+a)^{n+1} = (1+a^{2^k})^m = 1 + a^{2^k} + \ldots \neq 1$ (calculating mod 2 and using $H^{2^k}(\mathbb{R}P^n;\mathbb{Z}/2) \ni a^{2^k} \neq 0$).

4.13 Orientability

Definition 4.13.1. Let B be a topological space and $V \to B$ be a real vector bundle of rank n over B. We call V *oriented*, if for all $b \in B$, V_b is given an orientation such that for all $b \in B$, there exists an open neighbourhood $U \subset B$ of b trivializing V by a fiberwise orientation-preserving homeomorphism. I.e. there is a commutative diagram

$$\begin{tikzcd} V|_U \arrow[rr, "\rho_U", "\cong"'] \arrow[dr] & & \mathbb{R}^n \times U \arrow[dl] \\ & U & \end{tikzcd} ,$$

where ρ_U is fiberwise linear and a homeomorphism and the map between the fibers $(\rho_U)_b : V_b \to \mathbb{R}^n$ preserves orientation for all $b \in U$. (Here $\mathbb{R}^n$ shall be oriented with its standard orientation; i.e. $(e_1, \ldots, e_n)$ is positively oriented.)

Lemma 4.13.2*. *Let W be a complex vector space. Then the underlying real vector space $\tilde{W}$ has a canonical orientation, i.e. if W' is another complex vector space with underlying real vector space $\tilde{W}'$ and $W \to W'$ is a map, then the corresponding map $\tilde{W} \to \tilde{W}'$ preserves orientation.*

Proof (sketch). Let $e_1, \ldots, e_n$ be a $\mathbb{C}$-basis of W. The orientation on $\tilde{W}$ is defined by declaring $(e_1, ie_1, \ldots, e_n, ie_n)$ to be positively oriented. We need to check: This orientation is independent of the choice of $e_1, \ldots, e_n$. To this end, we verify that this orientation is preserved by transformations $e_i \mapsto e_i + \lambda e_k, e_k \mapsto e_k$ (for $i \neq k \in \{1, \ldots, n\}, \lambda \in \mathbb{C}$) or $e_i \mapsto \lambda e_i$, (for $i \in \{1, \ldots, n\}, \lambda \in \mathbb{C}$).[61] $\square$

Corollary 4.13.3*. *Let B be a topological space and $V \to B$ be a complex vector bundle over B. Then the underlying real vector bundle $\tilde{V} \to B$ is canonically oriented.*

This follows directly from the previous lemma.

Proposition 4.13.4*. *Let $SO_n := \ker(\det : O_n \to O_1 \cong \mathbb{Z}/2)$ be the special orthogonal group (in dimension n). Then*

1. *The diagram*

$$\begin{tikzcd} BSO_n \arrow[r] \arrow[d] & BO_n \arrow[d, "B\det"] \\ * \arrow[r] & BO_1 \end{tikzcd}$$

   is a homotopy pullback.[62]

2. *For X a topological space, we have a commutative diagram*

$$\begin{tikzcd} {[X, BSO_n]_*} \arrow[r, "\cong"] \arrow[d, "\mathrm{incl}^*"'] & \{\text{oriented real v.b. of rank } n\}/\text{isomorphism of oriented real v.b.} \arrow[d, "\text{forget orientation}"] \\ {[X, BO_n]_*} \arrow[r, "\cong"'] & \{\text{real v.b. of rank } n\}/\text{isomorphism of real v.b.} \end{tikzcd} ,$$

   where the lower bijection is the one from corollary 4.9.2 and also the upper map is a bijection. Thus BSO_n classifies oriented real vector bundles.

[61] Every change of basis vectors in $\mathbb{C}$ is a composition of transformations of this form; cf. Linear Algebra 1.

[62] Given a group homomorphism $f : G_1 \to G_2$, one obtains a map $Bf : BG_1 \to BG_2$ from the Yoneda lemma and the natural transformation $[\cdot, BG_1] \cong \mathrm{Bun}_{G_1}(\cdot) \Rightarrow \mathrm{Bun}_{G_2}(\cdot) \cong [\cdot, BG_2]$, where the middle natural transformation is induced by f as in example 4.4.2. One verifies that this construction turns $B : \mathrm{Grp} \to \mathrm{hCW}$ into a functor. Applying this to det yields the map $B\det : BO_n \to BO_1$ and applying it to the inclusion $SO_n \hookrightarrow O_n$ yields the map $BSO_n \to BO_n$.

Proof (Sketch). 1. It suffices to show that BSO_n is a weak homotopy pullback in the diagram in 1. Since $\det : O_n \to O_1$ is a fiber bundle (hence a Serre fibration), by corollary 2.4.10, the diagram

$$\begin{tikzcd} SO_n \arrow[r] \arrow[d] & O_n \arrow[d, "\det"] \\ * \arrow[r] & O_1 \end{tikzcd}$$

is a homotopy pullback. We thus have a Puppe exact sequence 2.4.12 forming the lower row in

$$\begin{tikzcd} \ldots \arrow[r] & \pi_{i+1}(BO_n) \arrow[r] & \pi_{i+1}(BO_1) \arrow[r] & \pi_i(BSO_n) \arrow[r] & \pi_i(BO_n) \arrow[r] & \pi_i(BO_1) \arrow[r] & \ldots. \\ \ldots \arrow[r] & \pi_i(O_n) \arrow[r] \arrow[u, "\cong"] & \pi_i(O_1) \arrow[r] \arrow[u, "\cong"] & \pi_{i-1}(SO_n) \arrow[r] \arrow[u, "\cong"] & \pi_{i-1}(O_n) \arrow[r] \arrow[u, "\cong"] & \pi_{i-1}(O_1) \arrow[r] \arrow[u, "\cong"] & \ldots \end{tikzcd}$$

The vertical maps are isomorphisms due to corollary 4.3.5 (and the $\Sigma' - \Omega$-adjunction 2.4.11). The diagram is commutative by naturality of the constructions. Thus also the top row is exact.

Then let Y be the homotopy pullback of $(BO_n) * (BO_1)$; there is a map $BSO_n \to Y$. We obtain a commutative diagram of exact sequences (where the bottom row is again the Puppe long exact sequence, now for the homotopy pullback Y)

$$\begin{tikzcd} \ldots \arrow[r] & \pi_{i+1}(BO_n) \arrow[r] \arrow[d, equal] & \pi_{i+1}(BO_1) \arrow[r] \arrow[d, equal] & \pi_i(BSO_n) \arrow[r] \arrow[d] & \pi_i(BO_n) \arrow[r] \arrow[d, equal] & \pi_i(BO_1) \arrow[r] \arrow[d, equal] & \ldots. \\ \ldots \arrow[r] & \pi_{i+1}(BO_n) \arrow[r] & \pi_{i+1}(BO_1) \arrow[r] & \pi_i(Y) \arrow[r] & \pi_i(BO_n) \arrow[r] & \pi_i(BO_1) \arrow[r] & \ldots \end{tikzcd}$$

The five-lemma yields that the middle map is an isomorphism, hence 1. follows.

2. The proof is similar to the steps leading to corollary 4.9.2 (in particular the proof of theorem 4.9.1). Since BSO_n classifies principal SO_n-bundles, we need to find a bijection between principal SO_n-bundles on X (modulo isomorphism) and oriented real vector bundles with rank n on X (modulo isomorphism).
   - Given a principal SO_n-bundle $P \to X$, the associated vector bundle $P \times^{SO_n} \mathbb{R}^n$ is an oriented real vector bundle of rank n on X. [63]
   - On the other hand, given an oriented real vector bundle $V \to X$, one can show that $\coprod_{b \in B} \mathrm{Iso}_+(\mathbb{R}^n, V_b)$, equipped with a suitable topology, is a principal $\mathrm{Gl}_n^+(\mathbb{R})$-bundle on X. Here $\mathrm{Iso}_+(\mathbb{R}^n, V_b)$ denotes the group of orientation-preserving isomorphisms $\mathbb{R}^n \to V_b$ and $\mathrm{Gl}_n^+(\mathbb{R})$ is the group of orientation-preserving isomorphisms $\mathbb{R}^n \to \mathbb{R}^n$. This is "the same" as a principal SO_n-bundle, because the inclusion $SO_n \to \mathrm{Gl}_n^+(\mathbb{R})$ is a homotopy equivalence (compare remark 4.6.8).

□

[63] The orientation is defined as follows: For all $x \in X$, let $U \subset X$ be a trivializing neighbourhood for P of x; i.e. there exists a SO_n-equivariant homeomorphism $\rho_U : P|_U \xrightarrow{\sim} U \times SO_n$. Then $(P \times^{SO_n} \mathbb{R}^n)|_U \cong P|_U \times^{SO_n} \mathbb{R}^n \cong (U \times SO_n) \times^{SO_n} \mathbb{R}^n$; let the homeomorphism be denoted $\overline{\rho_U}$. We declare $(\overline{\rho_U}^{-1}([b, \mathrm{id}, e_1]), \ldots, \overline{\rho_U}^{-1}([b, \mathrm{id}, e_n]))$ to be positively oriented.

The resulting orientation does not depend on the choice of ρ_U, since any different homeomorphism $\rho'_U : P|_U \xrightarrow{\sim} U \times SO_n$ differs (by SO_n-equivariance) from ρ_U only by multiplication with an element $A \in SO_n$. Defining $\overline{\rho'_U}$ analogously to $\overline{\rho_U}$ before, one computes $(\overline{\rho'_U}^{-1}([b, \mathrm{id}, e_1]), \ldots, \overline{\rho'_U}^{-1}([b, \mathrm{id}, e_n])) = (\overline{\rho_U}^{-1}([b, A^{-1}, e_1]), \ldots, \overline{\rho'_U}^{-1}([b, A^{-1}, e_n])) = (\overline{\rho_U}^{-1}([b, \mathrm{id}, A^{-1}e_1]), \ldots, \overline{\rho'_U}^{-1}([b, \mathrm{id}, A^{-1}e_n]))$. Declaring this to be positively oriented is equivalent to declaring $(\overline{\rho_U}^{-1}([b, \mathrm{id}, e_1]), \ldots, \overline{\rho'_U}^{-1}([b, \mathrm{id}, e_n]))$, since $A^{-1} \in SO_n$.

Thus these locally defined orientations "glue together" to an orientation of $P \times^{SO_n} \mathbb{R}^n$.

Corollary 4.13.5*. *We have an exact sequence in* Sets$_*$*:*

$$\begin{tikzcd} {[X,BSO_n]} \arrow[r] & {[X,BO_n]} \arrow[r, "(B\det)\circ"] \arrow[dr, "w_1"'] & {[X,BO_1]} \arrow[d, "\cong"] \\ & & H^1(X;\mathbb{Z}/2) \end{tikzcd} \quad ,$$

where the triangle commutes. (The vertical isomorphism is, abusing notation, again w_1.)[64]

I.e. $w_1 \circ B\det = w_1 \in H^1(BO_n;\mathbb{Z}/2)$. Thus a vector bundle $V \to X$ is orientable if and only if $w_1(V) = 0$.

Proof. The sequence is exact by corollary 2.4.12 and the previous proposition. We need to show $(B\det\circ f)^*(w_1) = f^*((B\det)^*(w_1)) = f^*(w_1)$ for all $f \in [X,BO_n]$, i.e. $(B\det)^*(w_1) = w_1$, where $(B\det)^* : H^*(BO_1;\mathbb{Z}/2) \to H^*(BO_n;\mathbb{Z}/2)$ (abusing notation; these two w_1 are actually different).

By lemma 4.12.1 (applied inductively), there is an injection $Bi : \mathrm{Gr}^{\mathbb{R}}(1,\infty) \cong BO_1 \hookrightarrow BO_n \cong \mathrm{Gr}^{\mathbb{R}}(n,\infty)$. One verifies that this map can also be regarded as induced by the inclusion $i : O_1 \hookrightarrow O_n$, $a \mapsto \begin{pmatrix} a & 0 \\ 0 & \mathbb{1}_{(n-1)\times(n-1)} \end{pmatrix}$.[65] Note that i is a section of det; $\det\circ i = \mathrm{id}_{O_1}$.

The map $(Bi)^* : \underbrace{H^*(\mathrm{Gr}^{\mathbb{R}}(n,\infty);\mathbb{Z}/2)}_{\cong(\mathbb{Z}/2)[w_1,\dots,w_n]} \to \underbrace{H^*(\mathrm{Gr}^{\mathbb{R}}(1,\infty);\mathbb{Z}/2)}_{\cong(\mathbb{Z}/2)[w_1]}$ satisfies $w_1 \mapsto w_1$, $w_i \mapsto 0$ for $i > 1$ (by lemma 4.12.1). Due to functoriality, $\det\circ i = \mathrm{id}_{O_1}$ implies $B\det\circ Bi = \mathrm{id}_{BO_1}$. Then $(Bi)^*((B\det)^*(w_1)) = w_1$ and one concludes that one must already have had $(B\det)^*(w_1) = w_1$. □

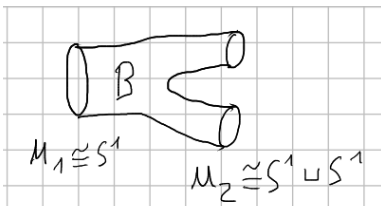
Remark 4.13.6. A smooth manifold M is called orientable, if its tangent bundle TM is orientable. Thus by the previous corollary and example 4.12.5, $\mathbb{R}P^n$ is orientable if and only if n is odd. (We have computed $w_{\mathrm{tot}}(T\mathbb{R}P^n) = (1+a)^{n+1} = 1 + (n+1)a + \dots$, where $(n+1)a = w_1(T\mathbb{R}P^n)$. This is zero in $H^1(\mathbb{R}P^n;\mathbb{Z}/2)$ if and only if n is odd.)

4.14 Characteristic numbers and bordism

Definition 4.14.1. Let M be a closed smooth manifold of dimension n, $[M] \in H_n(M;\mathbb{Z}/2)$ its fundamental class and TM its tangent bundle (a real vector bundle of rank n). The *Stiefel-Whitney numbers* of M are $w_1^{r_1}\dots w_n^{r_n}[M] := (w_1(TM)^{r_1}\dots w_n(TM)^{r_n}) \cap [M] \in \mathbb{Z}/2$, indexed by $(r_1,\dots,r_n)$ such that $\sum_{i=1}^n ir_i = n$ and where $\cap$ is the cap-product.

Theorem 4.14.2*. (Pontryagin) *If B is a compact smooth manifold of dimension $n+1$ with boundary $\partial B = M$ for a closed smooth manifold M of dimension n, then the Stiefel-Whitney numbers of M are all zero.*

Definition 4.14.3. Two closed smooth manifolds M_1, M_2 are *unorientedly bordant*, if $M_1 \sqcup M_2 = \partial B$ for B a smooth compact manifold of dimension $n+1$.



Theorem 4.14.4*. (Thom) *Let M_1, M_2 be two closed smooth manifolds. Then they are unorientedly bordant if and only if their corresponding Stiefel-Whitney numbers are the same.*

[64]Viewing w_1 as a cohomology class $w_1 \in H^1(BO_1;\mathbb{Z}/2)$, it is given by $f \mapsto f^*(w_1)$, analogously the diagonal map denoted w_1. The vertical map is an isomorphism because $BO_1 = \mathbb{R}P^\infty = K(\mathbb{Z}/2,1)$ is also the classifying space for H^1, cf. example 5.2.6.

[65]Hence the abuse of notation is consistent, calling it Bi.

In particular, a smooth manifold M is the boundary of a closed smooth manifold B if and only if all Stiefel-Whitney numbers of M are zero (taking $M_1 = M, M_2 = \emptyset$ above).

5 Cohomology of topological spaces

Literature:

- T. tom Dieck, *Algebraic Topology*
- A. Hatcher, *Algebraic Topology*

5.1 Singular and cellular cohomology

Recall the construction of singular homology: Let X be a topological space and define the singular chain complex $C_*^{\mathrm{sing}}(X)$ as in Algebraic Topology 1. By tensoring with a coefficient $\mathbb{Z}$-module (i.e. Abelian group) A, we get $C^{\mathrm{sing}}(X) \otimes A$ and then the singular homology groups in degree $i \in \mathbb{Z}$ is defined as $H_i^{\mathrm{sing}}(X; A) := H_i(C_*^{\mathrm{sing}}(X) \otimes A)$ (where H_i on the right-hand side denotes the algebraic homology of a chain complex, and the singular chain complex is by definition zero in degrees lower than zero).

Similarly, if X is a CW-complex, one can construct cellular homology: Define the cellular chain complex $C_*^{\mathrm{cell}}(X)$, tensor with A and then let $H_i^{\mathrm{cell}}(X; A) := H_i(C_*^{\mathrm{cell}}(X) \otimes A)$.

We found that $H_i^{\mathrm{sing}} : \mathrm{CW} \to \mathrm{Ab}$ and $H_i^{\mathrm{cell}} : \mathrm{CW} \to \mathrm{Ab}$ are isomorphic functors for all $i \in \mathbb{Z}$.

Definition 5.1.1. Let X be a topological space. The chain complex $C^*_{\mathrm{sing}}(X) := \mathrm{Hom}(C_*^{\mathrm{sing}}(X), \mathbb{Z})$, i.e.

$$\ldots \longrightarrow 0 \longrightarrow \mathrm{Hom}_{\mathrm{Ab}}(C_0^{\mathrm{sing}}, \mathbb{Z}) \xrightarrow{\circ d} \mathrm{Hom}_{\mathrm{Ab}}(C_1^{\mathrm{sing}}, \mathbb{Z}) \xrightarrow{\circ d} \mathrm{Hom}_{\mathrm{Ab}}(C_2^{\mathrm{sing}}, \mathbb{Z}) \xrightarrow{\circ d} \ldots,$$

is called the *singular cochain complex* of X.

If X is a CW-complex, we define the *cellular cochain complex* of X by $C^*_{\mathrm{cell}}(X) := \mathrm{Hom}(C_*^{\mathrm{cell}}(X), \mathbb{Z})$.

For $A \in \mathrm{Ob}\,\mathrm{Ab}$ an Abelian group and $i \in \mathbb{N}$, define the *singular cohomology group* $H^i_{\mathrm{sing}}(X; A) := H^i(C^*_{\mathrm{sing}}(X) \otimes A)$ and, if X is a CW-complex, the *cellular cohomology group* $H^i_{\mathrm{cell}}(X; A) := H^i(C^*_{\mathrm{cell}}(X) \otimes A)$. Here H^i on the respective right-hand sides denotes the algebraic homology of a chain complex.

Remark 5.1.2. *Fact:* If $X \in \mathrm{Ob}\,\mathrm{CW}$, then for all $A \in \mathrm{Ob}\,\mathrm{Ab}$, for all $i \in \mathbb{N}$, $H^i_{\mathrm{sing}}(X; A) \cong H^i_{\mathrm{cell}}(X; A)$.[1]

Example 5.1.3. – Consider S^n with its CW-structure $\{e^0, e^n\}$. The cellular chain complex reads

$$C_*^{\mathrm{cell}}(S^n) = \left(\ldots \longrightarrow \underbrace{\mathbb{Z}}_{\text{(degree } n)} \xrightarrow{0} 0 \longrightarrow \ldots 0 \xrightarrow{0} \underbrace{\mathbb{Z}}_{\text{(degree } 0)} \right).$$

[2] Thus the cellular cochain complex is

$$C^*_{\mathrm{cell}}(S^n) = \left(\underbrace{\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}}_{\text{(degree } 0)} \xrightarrow{0} 0 \longrightarrow \ldots 0 \xrightarrow{0} \underbrace{\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}}_{\text{(degree } n)} \longrightarrow \ldots \right).$$

We obtain

$$H^i_{\mathrm{cell}}(S^n; A) \cong \begin{cases} A, & i = 0, n, \\ 0, & \text{else.} \end{cases}$$

[1] This is the same statement for cohomology as the one made for homology above.
[2] We omit the degrees lower than zero.

– Consider $\mathbb{R}P^\infty$, which has a CW-structure with one cell in each dimension: The cellular chain complex is given by

$$C_*^{\text{cell}}(\mathbb{R}P^\infty) = \left(\ldots \longrightarrow \underbrace{\mathbb{Z}}_{4} \xrightarrow{2} \underbrace{\mathbb{Z}}_{3} \xrightarrow{0} \underbrace{\mathbb{Z}}_{2} \xrightarrow{2} \underbrace{\mathbb{Z}}_{1} \xrightarrow{0} \underbrace{\mathbb{Z}}_{0} \right).$$

(The numbers over the arrows mean that the respective homomorphism is given by multiplication with 2 resp. 0. The numbers below the entries of the chain complex again denote the degree.) Then, using again $\mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}$,

$$C^*_{\text{cell}}(\mathbb{R}P^\infty) = \left(\underbrace{\mathbb{Z}}_{0} \xrightarrow{0} \underbrace{\mathbb{Z}}_{1} \xrightarrow{2} \underbrace{\mathbb{Z}}_{2} \xrightarrow{0} \underbrace{\mathbb{Z}}_{3} \xrightarrow{2} \underbrace{\mathbb{Z}}_{4} \longrightarrow \ldots \right).$$

Thus

$$H_i^{\text{cell}}(\mathbb{R}P^\infty;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & i = 0, \\ \mathbb{Z}/2, & i = 2k+1, k \in \mathbb{N}, \\ 0, & i = 2k+2, k \in \mathbb{N} \end{cases}$$

and

$$H^i_{\text{cell}}(\mathbb{R}P^\infty;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & i = 0, \\ \mathbb{Z}/2, & i = 2k+2, k \in \mathbb{N}, \\ 0, & i = 2k+1, k \in \mathbb{N}. \end{cases}$$

Remark 5.1.4. – In general, we have $H^0(X;A) = \mathrm{Hom}(H_0(X),A) \cong A^{\pi_0(X)}$: By CW-approximation (resp. corollary 1.8.3), we can assume that each connected component of X contains only one 0-cell. Then $C_0^{\text{cell}}(X) \cong \mathbb{Z}^{\pi_0(X)}$, the differential d_0 must necessarily be zero and the last part of the cellular chain complex thus reads

$$\ldots \longrightarrow C_1^{\text{cell}}(X) \xrightarrow{0} \mathbb{Z}^{\pi_0(X)} \longrightarrow 0\,.$$

Going over to the cochain complex, tensoring with A and taking the homology, we obtain the claim.

– We have $H^1(X;A) \cong \mathrm{Hom}_{\mathrm{Ab}}(H_1(X),A)$: By the same reasoning as above, we can assume that $d_0 : C_1^{\text{cell}}(X) \to C_0^{\text{cell}}(X) = 0$. Then we have $H_1(X) \cong C_1^{\text{cell}}(X)/d_1(C_2^{\text{cell}}(X))$ resp.

$$C_2^{\text{cell}}(X) \xrightarrow{d_1} C_1^{\text{cell}}(X) \xrightarrow{\pi} H_1(X) \longrightarrow 0$$

is exact. Since the Hom-functor is left exact, then

$$0 \longrightarrow \mathrm{Hom}(H_1(X),A) \xrightarrow{\circ\pi} \underbrace{\mathrm{Hom}(C_1^{\text{cell}}(X),A)}_{\cong \mathrm{Hom}(C_1^{\text{cell}}(X),\mathbb{Z})\otimes A} \xrightarrow{\circ d_1} \underbrace{\mathrm{Hom}(C_2^{\text{cell}}(X),A)}_{\cong \mathrm{Hom}(C_2^{\text{cell}}(X),\mathbb{Z})\otimes A}$$

is exact. I.e. $H^1(X;A)\ker(\circ d_1)/\mathrm{im}(\circ d_0) = \ker(\circ d_1)/\{0\} = \ker(\circ d_1) = (\circ\pi)(\mathrm{Hom}(H_1(X),A)) \cong \mathrm{Hom}(H_1(X),A)$ since $\circ\pi$ is injective.

– If X is connected, we can also rewrite this as $H^1(X;A) \cong \mathrm{Hom}_{\mathrm{Ab}}(H_1(X),A) \cong \mathrm{Hom}_{\mathrm{Grp}}(\pi_1(X),A)$ by Hurewicz' theorem 1.9.8 (the variant for non-simply connected spaces).

In general, we have:

Theorem 5.1.5 (Universal coefficient theorem)**.** *Let X be a topological space, R be a principal ideal domain and A be an R-module. Then there exists a natural exact sequence*

$$0 \longrightarrow \mathrm{Ext}^1(H_{i-1}(X;R),A) \longrightarrow H^i(X;A) \longrightarrow \mathrm{Hom}_R(H_i(X;R),A) \longrightarrow 0$$

that splits non-canonically.

(No proof here.)

Remark 5.1.6. Why is cohomology often more interesting?

1. We will see that cohomology is representable by Eilenberg-Mac Lane spaces $K(A,n)$.
2. $H^*(X;\mathbb{Z})$ has a natural graded ring structure.

5.2 Generalized cohomology theories

Definition 5.2.1. A *generalized reduced cohomology theory* consists of:

- A family of functors $\{\widetilde{E}^n : \mathrm{CW}_*^{\mathrm{op}} \to \mathrm{Ab}\}_{n\in\mathbb{Z}}$.
- Natural suspension isomorphisms $\sigma : \widetilde{E}^{n+1}(\Sigma X) \xrightarrow{\sim} \widetilde{E}^n(X)$ (for $X \in \mathrm{CW}_*$)

These are required to satisfy:

1. Homotopy invariance: $\widetilde{E}^n$ factor through $\mathrm{hCW}_*^{\mathrm{op}}$ for all $n \in \mathbb{Z}$.
2. Exactness/Mayer-Vietoris property/Excision: If

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & B \arrow[d, "v"] \\ C \arrow[r, "u"'] & X \end{tikzcd}$$

   is a homotopy pushout, then the sequence

$$\widetilde{E}^n(X) \xrightarrow{e\mapsto(v^*(e),u^*(e))} \widetilde{E}^n(B) \oplus \widetilde{E}^n(C) \xrightarrow{(b,c)\mapsto i^*b-j^*c} \widetilde{E}^n(A)$$

   is exact for all $n \in \mathbb{Z}$.

3. Additivity: If $\{X_\alpha\}_\alpha$ is a family in CW_*, then the inclusions induce an isomorphism $\widetilde{E}^n(\bigvee_\alpha X_\alpha) \cong \prod_\alpha \widetilde{E}^n(X_\alpha)$ for all $\alpha \in \mathbb{Z}$.

$\{\widetilde{E}^n, \sigma\}$ is called *ordinary* if it satisfies additionally

4. $\widetilde{E}^n(S^0) = 0$ for $n \neq 0$.

Theorem 5.2.2. *Reduced cellular cohomology $\{\widetilde{H}^n_{\mathrm{cell}}, \sigma\}$ is an ordinary generalized cohomology theory.*[3]

The theorem follows from the properties of H_i^{cell}, but we omit the details.

Proposition 5.2.3. *Let $\{\widetilde{E}^n, \sigma\}$ be a generalized reduced cohomology theory. Then for all $n \in \mathbb{Z}$, there exist pointed CW-complexes K_n, elements $a_n \in \widetilde{E}^n(K_n)$ and homotopy equivalences $\tau : K_n \xrightarrow{\simeq} \Omega K_{n+1}$ such that for all $X \in \mathrm{CW}_*$, the map $[X, K_n]_* \xrightarrow{\sim} \widetilde{E}^n(X)$, $f \mapsto f^*(a_n)$ is an isomorphism. Furthermore, the diagram*

$$\begin{tikzcd} {[X, K_n]_*} \arrow[r, "\sim"] \arrow[d, "\tau\circ"', "\sim"] & \widetilde{E}^n(X) \arrow[d, "\sim"', "\sigma"] \\ {[X, \Omega K_{n+1}]_* \cong [\Sigma X, K_{n+1}]_*} \arrow[r, "\sim"'] & \widetilde{E}^{n+1}(\Sigma X) \end{tikzcd}$$

commutes.[4]

[3] σ has been defined on homology in Algebraic Topology 1.

[4] Recall $\Sigma X \simeq \Sigma' X$, as was argued for somewhere in the proof of the Blakers-Massey theorem 2.7.1, in chapter II.11. We have $[X, \Omega K_{n+1}]_* \cong [\Sigma' X, K_{n+1}]_*$ by the $\Sigma' - \Omega$-adjunction, proposition 2.4.11.

Proof. We get K_n and a_n from the Brown representability theorem 3.1.1, and τ from the Yoneda lemma. □

The data $\{K_n, a_n, \tau\}$ are said to represent the generalized reduced cohomology theory $\{\widetilde{E}^n, \sigma\}$.

Remark 5.2.4. If $(\widetilde{E}^n, \sigma)$ is a generalized reduced cohomology theory, then we can define $E^n(X,Y) := \widetilde{E}^n((X/Y)_+)$ for a CW-subcomplex $Y \hookrightarrow X$, where the subscript "+" denotes the "adding basepoints"-functor already considered earlier. Then $E^n(\cdot,\cdot)$ satisfies the axioms of a generalized cohomology theory (which we do not state all here). In particular, there is a long exact sequence of a pair:

$$\dots \longrightarrow E^n(X,Y) \longrightarrow E^n(X) \longrightarrow E^n(Y) \longrightarrow E^{n+1}(X,Y) \longrightarrow \dots.$$

Remark 5.2.5. We can compute generalized cohomology theories of CW-complexes via "cellular cohomology". This follows from the representability, via Puppe sequences.

Example 5.2.6.

- The generalized cohomology theory $\{\widetilde{H}^n(\cdot, A), \sigma\}$ for $A \in \operatorname{Ob}\mathrm{Ab}$ (i.e. a $\mathbb{Z}$-module) is represented by $\{K(A,n), a_n, \sigma : K(A,n) \xrightarrow{\sim} \Omega K(A, n+1)\}$, where $K(A,n)$ are Eilenberg-Mac Lane spaces. We obtain these by the previous proposition, noting that the representing spaces K_n for $\{\widetilde{H}^n(\cdot, A), \sigma\}$ must satisfy

  $$\pi_m(K_n) = [S^m, K_n]_* = \widetilde{H}^n(S^m, A) = \begin{cases} A, & m = n, \\ 0, & \text{else}, \end{cases}$$

  for all $n \in \mathbb{N}$.[5] This implies $K_n = K(A,n)$.

  Furthermore, the Ext-term in the universal coefficient theorem 5.1.5 vanishes (since $\widetilde{H}_{i-1}(K(A,n);\mathbb{Z})$ vanishes because of the Hurewicz theorem 1.9.8). Thus we have $\widetilde{H}^n(K(A,n);A) \cong \operatorname{Hom}_{\mathrm{Ab}}(H_n(K(A,n);\mathbb{Z}), A) \cong \operatorname{Hom}_{\mathrm{Ab}}(A, A)$ (where in the last step again the Hurewicz theorem was used). The element $a_n \in \widetilde{H}^n(K(A,n);A)$ representing $\widetilde{H}^n$ corresponds under this isomorphism to $\mathrm{id}_A \in \operatorname{Hom}_{\mathrm{Ab}}(A, A)$.

- Complex K-theory KU^*, which satisfies $KU^*(*) \cong \mathbb{Z}[\beta, \beta^{-1}]$ with $\deg \beta = -1$, where $*$ is the one-point space.[6] This generalized cohomology theory is represented by

  $$K_n = \begin{cases} BU \times \mathbb{Z}, & n \equiv 0 \mod 4, \\ BU, & n \equiv 2 \mod 4, \\ U, & n \text{ is odd}. \end{cases}$$

  Here $U := \operatorname{colim}_n U_n$ and $BU := \operatorname{colim}_n BU_n \cong$ "$\mathrm{Gr}^{\mathbb{C}}(\infty,\infty)$". The Bott periodicity theorem states $\Omega^2 U \simeq U$.

  For example, we have $KU^0(X) = \mathbb{Z}\cdot[\{\text{complex vector bundles on } X\}]/(\text{if } V \cong U \oplus W, \text{ then } [V] = [U] \oplus [W])$.

- Real K-theory KO^*.

- Real/complex cobordism MO^*/MU^*.

5.3 Ring structure in singular cohomology

Definition 5.3.1. A *graded-commutative ring* R is an associative graded ring that decomposes into Abelian subgroups $R = \bigoplus_{n\in\mathbb{Z}} R^n$, such that for $x \in R^n$, $y \in R^m$, we have $x \cdot y = (-1)^{n+m} y \cdot x \in R^{n+m}$. If $x \in R^n$, we write also $\deg(x) = n$ and call $\deg(x)$ the *degree* of x.

(Sometimes, one could also write R^* instead of R.)

Let A be a commutative ring (e.g. $\mathbb{Z}$), $X \in \operatorname{Ob}\mathrm{Top}$. We define $C^k_{\mathrm{sing}}(X;A) := \operatorname{Hom}_{\mathrm{Ab}}(C_k^{\mathrm{sing}}(X), A) \cong \operatorname{Hom}_{\mathrm{Ab}}(C_k^{\mathrm{sing}}(X), \mathbb{Z}) \otimes A$ for $k \in \mathbb{Z}$.

[5] Here $[S^m, K_n]_* = \widetilde{H}^n(S^m, A)$ follows from representability.

[6] Here we already use notation from the next section: $\widetilde{E}^*(X) := \bigoplus_{n\in\mathbb{Z}} \widetilde{E}^n(X)$ admits a graded ring structure for a generalized cohomology theory $\{\widetilde{E}^n, \sigma\}$ and a topological space X.

Definition 5.3.2. The *cup-product* $\cup$ is defined by

$$\begin{aligned}&\cup : C^k_{\mathrm{sing}}(X;A)\otimes C^l_{\mathrm{sing}}(X;A)\to C^{k+l}_{\mathrm{sing}}(X;A),\\ &(\phi\cup\psi)(\sigma) := (-1)^{kl}\phi(\sigma|_{[e_1,\dots,e_k]})\psi(\sigma|_{[e_{k+1},\dots,e_{k+l}]})\qquad \text{for all singular } (k+l)\text{-simplices } \sigma:\Delta^{k+l}\to X.\end{aligned}$$

Here $[e_1,\dots,e_k]$ denotes the convex hull of the vertices $e_1,\dots,e_k$ of Δ^{k+l}, analogously $[e_{k+1},\dots,e_{k+l}]$.

Theorem 5.3.3*. *The cup-product on singular cochains defines a graded-commutative ring structure on $H^*_{\mathrm{sing}}(X;A)$, such that if $f: X\to Y$ is a map of topological spaces, then $f^*: H^*_{\mathrm{sing}}(Y;A)\to H^*_{\mathrm{sing}}(X;A)$ is a homomorphism of graded rings.*[7]

(No proof here.)

Remark 5.3.4. With this definition, it is hard to compute the ring structure. Alternative descriptions:

- We can use the representability of singular cohomology from example 5.2.6 and the definition of singular cohomology via its reduced variant from remark 5.2.4. We can construct a natural transformation of functors $[\cdot,K(A,n)]_*\times[\cdot,K(A,m)]_* = [\cdot,K(A,n)\times K(A,m)]_*\Rightarrow [\cdot,K(A,m+n)]_*$ in terms of the map

$$\begin{tikzcd} K(A,n)\times K(A,m) \arrow[rr, "\phi"] \arrow[dr] & & K(A,m+n)\ , \\ & K(A,n)\wedge K(A,m) \arrow[ur, dashed, "\overline{\alpha}"'] & \end{tikzcd}$$

  where "$\wedge$" denotes the smash product and ϕ is defined such that the diagram commutes.[8] The map $\overline{\alpha}$ is obtained as follows: $K(A,n)\wedge K(A,m)$ is $(n+m-1)$-connected. (Exercise; e.g. look at cells.) Thus by the Hurewicz theorem 1.9.8, the Ext-term in the universal coefficient theorem 5.1.5 vanishes, such that we get $H^{n+m}(K(A,n)\wedge K(A,m);A)\cong \mathrm{Hom}_{\mathbb{Z}}(H_{n+m}(K(A,n)\wedge K(A,m);\mathbb{Z}),A)$.[9] By the Künneth formula (not proven here) one has $H_{n+m}(K(A,n)\wedge K(A,m);\mathbb{Z})\cong A\otimes_{\mathbb{Z}}A$, such that $\mathrm{Hom}_{\mathbb{Z}}(H_{n+m}(K(A,n)\wedge K(A,m);A),A)\cong \mathrm{Hom}_{\mathbb{Z}}(A\otimes_{\mathbb{Z}}A,A)$. The map $\overline{\alpha}$ is then the map corresponding under these isomorphisms to $\alpha: A\otimes_{\mathbb{Z}}A\to A$, $a\otimes b\mapsto (-1)^{nm}ab$.[10]

- If X is a smooth (i.e. C^∞) manifold, then one can define the de Rham cohomology

$$H^*_{\mathrm{dR}}(X) := H^*\left(\Omega^0(X)\xrightarrow{d}\Omega^1(X)\xrightarrow{d}\Omega^2(X)\longrightarrow\dots\right),$$

  where $\Omega^k(X)$ denotes differential (k-)forms on X. This has a graded commutative ring structure induced from $\Omega^i(X)\otimes\Omega^j(X)\to\Omega^{i+j}(X)$, $\omega\otimes\eta\mapsto\omega\wedge\eta$ (where "$\wedge$" is the exterior product of differential forms). Moreover, by de Rham's theorem, integration over singular cycles gives an isomorphism $H^*_{\mathrm{sing}}(X)\xrightarrow{\sim}H^*_{\mathrm{dR}}(X)$ of graded rings.

Example 5.3.5.

- We have $H^*(S^n)\cong\mathbb{Z}[a]/(a^2)$, where $\deg(a)=n$. This means that $H^0(S^n)=\mathbb{Z}\cdot 1$ and $H^n(S^n)=\mathbb{Z}\cdot a$, and $a^2 := a\cup a = 0$, since it lies in $H^{2n}(S^n)$.[11]

- We have $H^*(S^1\times S^1)\cong\mathbb{Z}[a]/(a^2,b^2)$, where $\deg(a)=\deg(b)=1$. I.e. $H^0(S^1\times S^1)=\mathbb{Z}\cdot 1$, $H^1(S^1\times S^1)=\mathbb{Z}\cdot a\oplus\mathbb{Z}\cdot b$ and $H^2(S^1\times S^1)=\mathbb{Z}\cdot ab$ (where $ab := a\cup b$). This is due to the Künneth isomorphism $H^*(S^1)\otimes H^*(S^1)\xrightarrow{\sim}H^*(S^1\times S^1)$, $\alpha\otimes\beta\mapsto \mathrm{pr}_1^*(\alpha)\cup\mathrm{pr}_2^*(\beta)$.

[7] For clarity: $H^*_{\mathrm{sing}}(X;A) := \bigoplus_{n\in\mathbb{Z}} H^n_{\mathrm{sing}}(X;A)$.

[8] The desired natural transformation is then given by $\phi\circ$.

[9] A is a $\mathbb{Z}$-module.

[10] Recall that elements of $H^{n+m}(K(A,n)\wedge K(A,m))$ are in bijection with homotopy classes of maps $K(A,n)\wedge K(A,m)\to K(A,n+m)$.

[11] With "$\mathbb{Z}\cdot a$", we mean that as Abelian group, $H^n(S^n)$ is isomorphic to $\mathbb{Z}$ with some generator a. With $\mathbb{Z}[a]$ we mean the polynomial ring with the variable a. However sometimes also objects of the form $\mathbb{Z}\cdot[a]$ will occur, which means a group isomorphic to $\mathbb{Z}$ generated by the equivalence class $[a]$.

There is also a different type of product:

Definition 5.3.6. Let $X \in \operatorname{Ob}\mathrm{Top}$, A be a ring and M, N be A-modules. Then the *cap product* is defined on singular chains and cochains by

$$\begin{array}{llll} \cap : & C^k_{\mathrm{sing}}(X;M) & \otimes C^{\mathrm{sing}}_l(X;N) & \to C^{\mathrm{sing}}_{l-k}(X; M \otimes_A N), \\ & (\phi \otimes m) & \cap (\sigma \otimes n) & := (-1)^{kl}\left(\phi(\sigma|_{[e_1,\dots,e_k]})\right)\sigma|_{[e_{k+1},\dots,e_l]} \otimes (m \otimes n) \end{array}.$$

(Here $m \in M$, $n \in \mathbb{N}$; note that σ is a map $\Delta^{k+(l-k)} \to X$. The cap-product is not defined for $k > l$.)

Theorem 5.3.7*. *The cap-product on singular cochains defines a map*

$$\cap : H^k_{\mathrm{sing}}(X;M) \otimes H^{\mathrm{sing}}_l(X;N) \to H^{\mathrm{sing}}_{l-k}(X; M \otimes_A N).$$

(If $M = N = A$, then $M \otimes_A N = A$.) Moreover, this cap-product satisfies

$$(x \cup y) \cap a = x \cap (y \cap a)$$

(for cohomology classes x, y and a homology class a).

(No proof here.)

Remark 5.3.8. Recall that if M is an oriented smooth connected closed manifold of dimension n, then there exists $[M] \in H_n(M;\mathbb{Z})$ such that $H_n(M;\mathbb{Z}) \cong \mathbb{Z}\cdot[M]$. $[M]$ is also called the *fundamental class* of M.

If M is not oriented, there exists at least $[M] \in H_n(M;\mathbb{Z}/2)$ such that $H_n(M;\mathbb{Z}/2) \cong (\mathbb{Z}/2)\cdot[M]$.

Theorem 5.3.9*. (Poincaré duality) *Let M be an oriented smooth connected closed manifold of dimension n. Then the cap-product induces isomorphisms (for all k where this is well-defined) $(\cdot \cap [M]) : H^k(M;\mathbb{Z}) \xrightarrow{\sim} H_{n-k}(M;\mathbb{Z})$.*

If M is not oriented, the cap-product still induces an isomorphism $(\cdot \cap [M]) : H^k(M;\mathbb{Z}/2) \xrightarrow{\sim} H_{n-k}(M;\mathbb{Z}/2)$.

(No proof here.)

Corollary 5.3.10*. *In the situation of the previous theorem the bilinear form*

$$H^k(M;\mathbb{Q}) \times H^{n-k}(M;\mathbb{Q}) \xrightarrow{\cup} H^n(M;\mathbb{Q}) \xrightarrow[\cong]{\cdot\cap[M]} H_0(M;\mathbb{Q}) \cong \mathbb{Q}$$

is non-degenerate.

Proof. Consider the diagram

$$\begin{tikzcd} H^k(M;\mathbb{Q}) \times H^{n-k}(M;\mathbb{Q}) \arrow[r, "\cup"] \arrow[d, "\mathrm{id}\times(\cdot\cap[M])"', "\cong"] & H^n(M;\mathbb{Q}) \arrow[d, "\cong"', "\cdot\cap[M]"] \\ H^k(M;\mathbb{Q}) \times H_k(M;\mathbb{Q}) \arrow[r, "\cap"] \arrow[d, "\cong"'] & H_0(M;\mathbb{Q}) \arrow[d, "\cong"] \\ \mathrm{Hom}_{\mathbb{Z}}(H_k(M;\mathbb{Z}),\mathbb{Q}) \times H_k(M) \arrow[r, "\mathrm{eval}"'] & \mathbb{Q} \end{tikzcd} ,$$

where $\mathrm{eval}(\phi, a) := \phi(a)$. The upper square commutes by theorem 5.3.7. The lower left isomorphism comes from the universal coefficient theorem 5.1.5; the Ext-term in this theorem vanishes since $\mathbb{Q}$ is a field. The lower square commutes since the lower left isomorphism is natural.

Since the vertical maps are all isomorphisms and eval is always non-degenerate, also $\cup$ is non-degenerate. $\square$

Example 5.3.11*. – $H^*(S^1 \times S^1; \mathbb{Z})$ satisfies the conclusion of the corollary as well, even with $\mathbb{Z}$-coefficients (by example 5.3.5).

– We have $H^*(\mathbb{C}P^n, \mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1})$, where $\deg(x) = 2$.

– Similarly, $H^*(\mathbb{R}P^n, \mathbb{Z}/2) \cong (\mathbb{Z}/2)[a]/(a^{n+1})$ with $\deg(a) = 1$.

Proof (for $\mathbb{C}P^n$). By induction on n. For $n = 1$, we have $\mathbb{C}P^1 \cong S^2$; the statement follows.

Induction step $n \mapsto n+1$: Let $i : \mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+1}$ be the inclusion (induced by $\mathbb{C}^{n+1} \hookrightarrow \mathbb{C}^{n+2}$). The map $i^* : H^k(\mathbb{C}P^{n+1}) \xrightarrow{\sim} H^k(\mathbb{C}P^n)$ is an isomorphism for $k \leq n$.[12] Thus by induction assumption $H^{2k}(\mathbb{C}P^{n+1}) = \mathbb{Z} \cdot x^k$ for $k \leq n$.

Finally, consider $x \cup x^n \in H^{2n+2}(\mathbb{C}P^{n+1}; \mathbb{Z}) \cong \mathbb{Z}$, where the last isomorphism is due to Poincaré duality, theorem 5.3.9. Due to non-degeneracy of the bilinear form from the previous corollary, $x \cup x^n$ must generate $H^{2n+2}(\mathbb{C}P^{n+1}; \mathbb{Z})$.[13] □

[12] This follows from considering cellular cohomology; recall that $\mathbb{C}P^n$ has one cell in each even dimension, and $\mathbb{C}P^{n+1}$ is obtained from $\mathbb{C}P^n$ by attaching a $2n$-cell.

[13] Strictly speaking, one has to re-prove the previous corollary, now with $\mathbb{Z}$-coefficients. But this can be done (in this particular case), since the Ext-terms in the universal coefficient theorem still all vanish with $\mathbb{Z}$-coefficients. Afterwards, one needs to examine the proof and note that $\cup$ indeed maps x and x^n onto a generator of $H^{2n+2}(\mathbb{C}P^{n+1}; \mathbb{Z})$ and not just onto a nonzero element. Note here that $x^n \cap [\mathbb{C}P^{n+1}]$ is a generator of $H_2(\mathbb{C}P^{n+1}; \mathbb{Z})$ since the Poincaré duality map is an isomorphism.

What was not in this course but could have been?

- Group cohomology: Given a group G, $\mathbb{Z}[G]$ its group ring over $\mathbb{Z}$ and A a $\mathbb{Z}[G]$-module, consider $H^*(K(G,1);A) \cong \mathrm{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z},A)$. (Literature: K. Brown, *Cohomology of Groups*)
- Cohomology with coefficients in local systems.
- Serre spectral sequence. (Literature: A. Hatcher, *Algebraic Topology*, chapter 5)
- Obstruction theory: Given the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & X \\ W \arrow[ur, dashed] & \end{tikzcd},$$

  obstructions for the existence of the dashed map lie in $H^{n+1}(W,A;\pi_n(X))$. (Literature: A. Hatcher, *Algebraic Topology*)
- Steenrod operations: $H^*(\cdot,\mathbb{Z}/p) \to H^*(\cdot,\mathbb{Z}/p)$, for p a prime.