Fakultät Mikrosystemtechnik
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# Vector calculus and numerical mathematics

## Worksheet 8

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### Problem 1: Separable-variables ODEs

Which ones of the following ODEs are of the **separable-variables** type?
In any case, find a proper method to obtain **infinitely many** solutions analytically.
**Hints:** $\frac{\mathrm{d}}{\mathrm{d}u}\arctan(u) = \frac{1}{1+u^2}$, $\frac{\mathrm{d}}{\mathrm{d}u}\arcsin(u) = \frac{1}{\sqrt{1-u^2}}$.

(a) (See also Problem 2 below.)

$$y'(x) \;=\; ax\,y(x)^2 \qquad (a \in \mathbb{R}).$$

(b)

$$y'(x)\,y(x) \;=\; x^2.$$

(c)

$$y'(x) \;=\; y(x) - x^2.$$

(d)

$$y'(x) \;=\; 4x^3\big[1 + y(x)^2\big].$$

(e)

$$y'(x) \;=\; \frac{6x}{\cos[y(x)]}.$$

(f) Here is a more sophisticated example:

$$y'(x) \;=\; \sin[y(x)].$$

### Problem 2: Numerical solution of an ODE

Let $y(x) = f(x)$ be that particular (exact) solution of the ODE

$$y'(x) \;=\; -2x\,y(x)^2$$

(Problem 1a with $a = -2$) that satisfies the starting condition $y(0) = 1$.

(a) Applying a finite-differences method (FDM) with step size $h = 0.1$, find approximate values $y_1(x_n) \approx f(x_n)$ (with $x_n = nh$ and $n = 1, 2, 3, 4, 5$).

(b) Find better approximations $y_2(x_n)$ by considering the second derivative $y''(x)$.

Compare your results with the exact solution $f(x) = \frac{1}{1+x^2}$!

## Problem 3: Diffusion

Consider the time dependent 3D density distribution (e.g., of ink in a water basin)

$$\begin{aligned}\rho(\mathbf{r},t) &= \frac{M}{(4\pi Dt)^{3/2}}\,\mathrm{e}^{-\mathbf{r}^2/4Dt} \\ &\equiv \frac{M}{(4\pi Dt)^{3/2}}\,\mathrm{e}^{-(x^2+y^2+z^2)/4Dt} \;=\; \rho(x,y,z,t) \qquad (t>0), \end{aligned} \tag{1}$$

where $M$ is a constant with the dimension of a mass (kg) and $D$ is a diffusion constant.

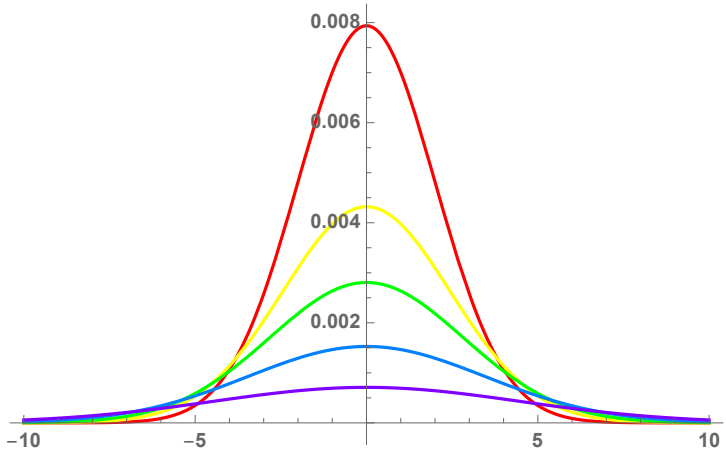


Figure 1: The function $\rho(x,0,0,t)$ (in units of $1\,\frac{\mathrm{g}}{\mathrm{mm}^3}$), plotted versus $x$ (in mm) at different times $t = 200\,\mathrm{s}$ (red), $300\,\mathrm{s}$ (yellow), $400\,\mathrm{s}$ (green), $600\,\mathrm{s}$ (blue), and $1000\,\mathrm{s}$ (violet), using the values $M = 0.001\,\mathrm{kg} \equiv 1\,\mathrm{g}$ and $D = 0.01\,\frac{\mathrm{mm}^2}{\mathrm{s}}$.

(a) Show that $\rho(\mathbf{r},t)$ of Eq. (1) is a solution of the PDE (diffusion equation)

$$\frac{\partial}{\partial t}\rho(\mathbf{r},t) \;=\; D\,\nabla^2\rho(\mathbf{r},t) \;+\; s(\mathbf{r},t).$$

What do you find for the source density $s(\mathbf{r},t)$ ?
Is ink being added to the water at any time $t>0$ ?

(b) Compute the total amount $m(t)$ of mass at a given time $t$,

$$m(t) \;=\; \int \mathrm{d}^3r\,\rho(\mathbf{r},t).$$

Hint: $\int_{-\infty}^{\infty}\mathrm{d}u\,\mathrm{e}^{-au^2} = \sqrt{\frac{\pi}{a}}$.

## Problem 4: Heated Disk

The rim of a circular metal disk in the $xy$-plane (centered at $x=y=0$ and with radius $R$) is held at a constant temperature $T_0$. Find the steady-state temperature distribution $T(x,y)$ on this disk (with heat conductivity $\lambda$), when it is heated with a uniform heat source density $s(x,y) = s_0$,

$$\nabla^2 T(x,y) \;=\; -\frac{1}{\lambda}\,s(x,y).$$

Hint: Use the ansatz $T(x,y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$, with constants $A,B,C,D,E,F$.