Problem 1: Separable-variables ODEs

Which ones of the following ODEs are of the separable-variables type?  
In any case, find a proper method to obtain infinitely many solutions analytically.  

Hints: \( \frac{d}{du} \arctan(u) = \frac{1}{1+u^2}, \frac{d}{du} \arcsin(u) = \frac{1}{\sqrt{1-u^2}}. \)

(a) (See also Problem 2 below.)  
\[ y'(x) = ax y(x)^2 \quad (a \in \mathbb{R}). \]

(b)  
\[ y'(x) y(x) = x^2. \]

(c)  
\[ y'(x) = y(x) - x^2. \]

(d)  
\[ y'(x) = 4x^3[1 + y(x)^2]. \]

(e)  
\[ y'(x) = \frac{6x}{\cos[y(x)]}. \]

(f) Here is a more sophisticated example:  
\[ y'(x) = \sin[y(x)]. \]

Problem 2: Numerical solution of an ODE

Let \( y(x) = f(x) \) be that particular (exact) solution of the ODE  
\[ y'(x) = -2x y(x)^2 \]
(Problem 1a with \( a = -2 \)) that satisfies the starting condition \( y(0) = 1 \).

(a) Applying a finite-differences method (FDM) with step size \( h = 0.1 \), find approximate values \( y_1(x_n) \approx f(x_n) \) (with \( x_n = nh \) and \( n = 1, 2, 3, 4, 5 \)).

(b) Find better approximations \( y_2(x_n) \) by considering the second derivative \( y''(x) \).  

Compare your results with the exact solution \( f(x) = \frac{1}{1+x^2} \).
Problem 3: Diffusion

Consider the time dependent 3D density distribution (e.g., of ink in a water basin)

\[ \rho(r, t) = \frac{M}{(4\pi Dt)^{3/2}} e^{-r^2/4Dt} \]

\[ \equiv \frac{M}{(4\pi Dt)^{3/2}} e^{-(x^2+y^2+z^2)/4Dt} = \rho(x, y, z, t) \quad (t > 0), \]

where \( M \) is a constant with the dimension of a mass (kg) and \( D \) is a diffusion constant.

Figure 1: The function \( \rho(x, 0, 0, t) \) (in units of \( 1 \text{ g mm}^{-3} \)), plotted versus \( x \) (in mm) at different times \( t = 200 \text{ s} \) (red), \( 300 \text{ s} \) (yellow), \( 400 \text{ s} \) (green), \( 600 \text{ s} \) (blue), and \( 1000 \text{ s} \) (violet), using the values \( M = 0.001 \text{ kg} \equiv 1 \text{ g} \) and \( D = 0.01 \text{ mm}^2 \text{s}^{-1} \).

(a) Show that \( \rho(r, t) \) of Eq. (1) is a solution of the PDE (diffusion equation)

\[ \frac{\partial}{\partial t} \rho(r, t) = D \nabla^2 \rho(r, t) + s(r, t). \]

What do you find for the source density \( s(r, t) \)? Is ink being added to the water at any time \( t > 0 \)?

(b) Compute the total amount \( m(t) \) of mass at a given time \( t \),

\[ m(t) = \int d^3r \rho(r, t). \]

Hint: \( \int_{-\infty}^{\infty} du e^{-au^2} = \sqrt{\frac{\pi}{a}} \).

Problem 4: Heated Disk

The rim of a circular metal disk in the \( xy \)-plane (centered at \( x = y = 0 \) and with radius \( R \)) is held at a constant temperature \( T_0 \). Find the steady-state temperature distribution \( T(x, y) \) on this disk (with heat conductivity \( \lambda \)), when it is heated with a uniform heat source density \( s(x, y) = s_0 \),

\[ \nabla^2 T(x, y) = -\frac{1}{\lambda}s(x, y). \]

Hint: Use the ansatz \( T(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F \), with constants \( A, B, C, D, E, F \).