Fakultät Mikrosystemtechnik
PD Dr. Michael Seidl

Sommersemester 2020
24 June 2020

Vector calculus and numerical mathematics

Worksheet 7

Remark: "ODE" here means "Ordinary differential equation".

Problem 1: Linear ODEs

(a) Here is a **homogeneous** linear ODE (with constant coefficients),

$$f''(x) - 8f'(x) + 15f(x) = 0.$$

Find the general solution, starting from the exponential ansatz $f(x) = \mathrm{e}^{\lambda x}$.

(b) Proceed in a similar way with the following ODEs,

$$\begin{aligned} f''(x) - 6f'(x) + 58f(x) &= 0, \\ f'''(x) + 4f''(x) + f'(x) - 6f(x) &= 0. \end{aligned}$$

(c) We now consider an **inhomogeneous** linear ODE,

$$f''(x) - 8f'(x) + 15f(x) = 15x^2 - 16x + 2.$$

Show that **one particular** solution is $f_0(x) = x^2$.
Using the result of part (a), give the general solution of this inhomogeneous ODE.

Problem 2: Homogeneous linear ODEs

Find the general real-valued solutions $f(x)$ of the following ODEs.

(a) $f'''(x) + 9f''(x) + 26f'(x) + 24f(x) = 0,$

(b) $f'''(x) - 11f''(x) + 55f'(x) - 125f(x) = 0,$

(c) $f'''(x) - 2f''(x) - 5f'(x) + 10f(x) = 0,$

(d) $f''(x) - f(x) = 0,$

(e) $f''(x) + f(x) = 0.$

Hint: In all cases (a–e), the exponential ansatz $f(x) = \mathrm{e}^{\lambda x}$ leads to an algebraic equation for $\lambda \in \mathbb{C}$. In cases (a–c), this equation is third-order: one of its three zeros can be guessed to be $\lambda_1 = -2$ (a), $\lambda_1 = 5$ (b) and $\lambda_1 = 2$ (c); the remaining two zeros λ_2 and λ_3 are then obtained via polynomial division. Cases (d) and (e) are (almost) trivial.

Problem 3: Example: Harmonic oscillator

The position x of a point mass, moving on the x-axis, is a function $x(t)$ of time t. The derivatives $\dot{x}(t)$ and $\ddot{x}(t)$, respectively, yield the instantaneous velocity and acceleration. As this mass is attached to a spring (with spring constant $k > 0$), it experiences at time t the restoring force $F_{\mathrm{r}}(t) = -kx(t)$. In addition, there is a frictional force $F_{\mathrm{f}}(t) = -\gamma\dot{x}(t)$ (with a damping constant $\gamma > 0$) plus an external driving force $F_{\mathrm{ext}}(t) = F_0 \cos(\Omega t)$ (with given amplitude F_0 and given frequency Ω). The resulting equation of motion reads

$$m\ddot{x}(t) \;=\; F_{\mathrm{r}}(t) + F_{\mathrm{f}}(t) + F_{\mathrm{ext}}(t).$$

Obviously, this is an **inhomogeneous** linear ODE for the unknown function $x(t)$,

$$m\ddot{x}(t) + \gamma\,\dot{x}(t) + k\,x(t) \;=\; F_0 \cos(\Omega t).$$

(a) Find the general solution $x(t)$ of the corresponding **homogeneous** linear ODE,

$$m\ddot{x}(t) + \gamma\,\dot{x}(t) + k\,x(t) \;=\; 0.$$

(b) By **physical intuition**, find a particular solution of the **inhomogeneous** ODE. What is the resulting general solution of the **inhomogeneous** ODE?

Problem 4: Laplacian (operator)

(a) For a differentiable scalar field $g(\mathbf{r}) = g(x, y, z)$, show that

$$\nabla \cdot \big[\nabla g(\mathbf{r})\big] \;=\; \nabla^2 g(\mathbf{r}),$$

where the **Laplacian** ∇^2 in cartesian coordinates reads $\nabla^2 \;=\; \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

(b) Evaluate the Laplacian $\nabla^2 g(\mathbf{r})$ for the following scalar fields (where $A, B, C, D, E, F, R, \alpha, \beta$ are constants).

$$\begin{aligned}
g_1(\mathbf{r}) &= Ax^2 + By^2 - (A+B)z^2 \;+\; Dx + Ey + Fz \;+\; C, \\
g_2(\mathbf{r}) &= A\,\sin(\alpha x)\,\sin(\beta y)\,\mathrm{e}^{-\gamma z}, \qquad\qquad \gamma = \sqrt{\alpha^2+\beta^2}, \\
g_3(\mathbf{r}) &= \frac{A}{\sqrt{x^2+y^2+z^2}} \;\equiv\; A(x^2+y^2+z^2)^{-1/2} \qquad (\mathbf{r} \neq \mathbf{0}), \\
g_4(\mathbf{r}) &= (Ax + By + Cz)\,(x^2+y^2+z^2)^{-3/2} \qquad\qquad (\mathbf{r} \neq \mathbf{0}), \\
g_5(\mathbf{r}) &= \frac{Q}{4\pi\epsilon_0}\begin{cases} \frac{3R^2-r^2}{2R^3} & (r \le R), \\ \frac{1}{r} & (r > R). \end{cases}
\end{aligned}$$

(c) Use the result of part (b) for $g_5(\mathbf{r})$, to verify the identity

$$\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}_0|} \;=\; -4\pi\,\delta(\mathbf{r}-\mathbf{r}_0).$$