Vector calculus and numerical mathematics

Worksheet 7

Remark: "ODE" here means "Ordinary differential equation".

Problem 1: Linear ODEs

(a) Here is a **homogeneous** linear ODE (with constant coefficients),
\[ f''(x) - 8f'(x) + 15f(x) = 0. \]
Find the general solution, starting from the exponential ansatz \( f(x) = e^{\lambda x} \).

(b) Proceed in a similar way with the following ODEs,
\[ f''(x) - 6f'(x) + 58f(x) = 0, \]
\[ f'''(x) + 4f''(x) + f'(x) - 6f(x) = 0. \]

(c) We now consider an **inhomogeneous** linear ODE,
\[ f''(x) - 8f'(x) + 15f(x) = 15x^2 - 16x + 2. \]
Show that one particular solution is \( f_0(x) = x^2 \).
Using the result of part (a), give the general solution of this inhomogeneous ODE.

Problem 2: Homogeneous linear ODEs

Find the general real-valued solutions \( f(x) \) of the following ODEs.

(a) \( f'''(x) + 9f''(x) + 26f'(x) + 24f(x) = 0, \)

(b) \( f'''(x) - 11f''(x) + 55f'(x) - 125f(x) = 0, \)

(c) \( f'''(x) - 2f''(x) - 5f'(x) + 10f(x) = 0, \)

(d) \( f''(x) - f(x) = 0, \)

(e) \( f''(x) + f(x) = 0. \)

**Hint:** In all cases (a–e), the exponential ansatz \( f(x) = e^{\lambda x} \) leads to an algebraic equation for \( \lambda \in \mathbb{C} \). In cases (a–c), this equation is third-order: one of its three zeros can be guessed to be \( \lambda_1 = -2 \) (a), \( \lambda_1 = 5 \) (b) and \( \lambda_1 = 2 \) (c); the remaining two zeros \( \lambda_2 \) and \( \lambda_3 \) are then obtained via polynomial division. Cases (d) and (e) are (almost) trivial.
Problem 3: Example: Harmonic oscillator

The position \( x \) of a point mass, moving on the \( x \)-axis, is a function \( x(t) \) of time \( t \). The derivatives \( \dot{x}(t) \) and \( \ddot{x}(t) \), respectively, yield the instantaneous velocity and acceleration. As this mass is attached to a spring (with spring constant \( k > 0 \)), it experiences at time \( t \) the restoring force \( F_r(t) = -kx(t) \). In addition, there is a frictional force \( F_f(t) = -\gamma \dot{x}(t) \) (with a damping constant \( \gamma > 0 \)) plus an external driving force \( F_{ext}(t) = F_0 \cos(\Omega t) \) (with given amplitude \( F_0 \) and given frequency \( \Omega \)). The resulting equation of motion reads

\[
m \ddot{x}(t) = F_r(t) + F_f(t) + F_{ext}(t).
\]

Obviously, this is an \textbf{inhomogeneous} linear ODE for the unknown function \( x(t) \),

\[
m \ddot{x}(t) + \gamma \dot{x}(t) + k x(t) = F_0 \cos(\Omega t).
\]

(a) Find the general solution \( x(t) \) of the corresponding \textbf{homogeneous} linear ODE,

\[
m \ddot{x}(t) + \gamma \dot{x}(t) + k x(t) = 0.
\]

(b) By physical intuition, find a particular solution of the \textbf{inhomogeneous} ODE. What is the resulting general solution of the \textbf{inhomogeneous} ODE?

Problem 4: Laplacian (operator)

(a) For a differentiable scalar field \( g(\mathbf{r}) = g(x, y, z) \), show that

\[
\nabla \cdot [\nabla g(\mathbf{r})] = \nabla^2 g(\mathbf{r}),
\]

where the \textbf{Laplacian} \( \nabla^2 \) in cartesian coordinates reads \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \).

(b) Evaluate the Laplacian \( \nabla^2 g(\mathbf{r}) \) for the following scalar fields (where \( A, B, C, D, E, F, R, \alpha, \beta \) are constants).

\[
\begin{align*}
g_1(\mathbf{r}) &= Ax^2 + By^2 - (A + B)z^2 + Dx + Ey + Fz + C, \\
g_2(\mathbf{r}) &= A \sin(\alpha x) \sin(\beta y) e^{-\gamma z}, \quad \gamma = \sqrt{\alpha^2 + \beta^2}, \\
g_3(\mathbf{r}) &= \frac{A}{\sqrt{x^2 + y^2 + z^2}} \equiv A(x^2 + y^2 + z^2)^{-1/2} \quad (\mathbf{r} \neq 0), \\
g_4(\mathbf{r}) &= (Ax + By + Cz)(x^2 + y^2 + z^2)^{-3/2} \quad (\mathbf{r} \neq 0), \\
g_5(\mathbf{r}) &= \frac{Q}{4\pi \varepsilon_0} \begin{cases} 
\frac{3r^2-x^2}{2r^3} & (r \leq R), \\
\frac{2r^2-r^2}{r^3} & (r > R).
\end{cases}
\end{align*}
\]

(c) Use the result of part (b) for \( g_5(\mathbf{r}) \), to verify the identity

\[
\nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}_0|} = -4\pi \delta(\mathbf{r} - \mathbf{r}_0).
\]