Vector calculus and numerical mathematics

Worksheet 6

Problem 1: Gauss’ Theorem

Verify Gauss’ theorem,
\[ \oint_{\partial \Omega} \mathbf{S} \cdot \mathbf{F}(\mathbf{r}) = \int_{\Omega} \mathbf{F}(\mathbf{r}) \cdot \nabla \mathbf{F}(\mathbf{r}) \]
with the following vector field \( \mathbf{F}(\mathbf{r}) \) and a spherical volume region \( \Omega \),
\[ \mathbf{F}(\mathbf{r}) \equiv \begin{pmatrix} F_1(\mathbf{r}) \\ F_2(\mathbf{r}) \\ F_3(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} 0 \\ x^2 y \\ 0 \end{pmatrix}, \quad \Omega = \left\{ \mathbf{r}(x, y, z) \mid x^2 + y^2 + z^2 \leq R^2 \right\}. \]

Problem 2: Stokes’ Theorem

Verify Stokes’ theorem,
\[ \int_{\partial \Sigma} \mathbf{r} \cdot \mathbf{G}(\mathbf{r}) = \int_{\Sigma} \mathbf{S} \cdot \left[ \nabla \times \mathbf{G}(\mathbf{r}) \right], \]
with the following vector field \( \mathbf{G}(\mathbf{r}) \) and a planar disk \( \Sigma \),
\[ \mathbf{G}(\mathbf{r}) \equiv \begin{pmatrix} G_1(\mathbf{r}) \\ G_2(\mathbf{r}) \\ G_3(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} 0 \\ xy^2 \\ 0 \end{pmatrix}, \quad \Sigma = \left\{ \mathbf{r}(x, y, 0) \mid x^2 + y^2 \leq R^2 \right\}. \]

Problem 3: Another example for Stokes’ Theorem

Verify Stokes’ theorem,
\[ \int_{\partial \Sigma} \mathbf{r} \cdot \mathbf{H}(\mathbf{r}) = \int_{\Sigma} \mathbf{S} \cdot \left[ \nabla \times \mathbf{H}(\mathbf{r}) \right], \]
with the vector field
\[ \mathbf{H}(\mathbf{r}) \equiv \begin{pmatrix} H_1(\mathbf{r}) \\ H_2(\mathbf{r}) \\ H_3(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} 4x - 8y + 7z \\ 3x + 5y - 9z \\ 8x - 2y + 6z \end{pmatrix} \]
and with the curved piece of surface \( \Sigma \) (a semi-sphere), given by
\[ \Sigma = \left\{ \mathbf{r}(r, \theta, \phi) \mid r = R, \ 0 \leq \theta \leq \frac{\pi}{2}, \ 0 \leq \phi \leq 2\pi \right\}, \]
where \((r, \theta, \phi)\) are the usual spherical polar coordinates.

Hint: In this case, \( \partial \Sigma = \left\{ \mathbf{r}(R, \frac{\pi}{2}, \phi) \mid 0 \leq \phi \leq 2\pi \right\} \) is the equator of the semi-sphere.
Problem 4: Chain rule for partial derivatives

Let \( f(\mathbf{r}) \equiv f(x, y, z) \) be a scalar field and \( \mathbf{r}(\phi) \) be a vector function of one variable \( \phi \),

\[
\mathbf{r}(\phi) = \begin{pmatrix} x(\phi) \\ y(\phi) \\ z(\phi) \end{pmatrix}.
\]

We define the simple function \( \bar{f}(\phi) \) of one variable \( \phi \) by

\[
\bar{f}(\phi) = f(\mathbf{r}(\phi)) = f(x(\phi), y(\phi), z(\phi)).
\]

**Chain rule for partial derivatives:** This function \( \bar{f}(\phi) \) has the derivative

\[
\bar{f}'(\phi) = \frac{d}{d\phi} f(\mathbf{r}(\phi)) = \mathbf{r}'(\phi) \cdot \nabla f(\mathbf{r}) \bigg|_{\mathbf{r}=\mathbf{r}(\phi)} = \left[ \dot{x}(\phi) \frac{\partial f(\mathbf{r})}{\partial x} + \dot{y}(\phi) \frac{\partial f(\mathbf{r})}{\partial y} + \dot{z}(\phi) \frac{\partial f(\mathbf{r})}{\partial z} \right]_{\mathbf{r}=\mathbf{r}(\phi)}.
\]

(a) Verify this rule for the two scalar fields \( f(x, y, z) = xy^2 \) and \( g(x, y, z) = x^2 + y^2 \) and the vector function

\[
\mathbf{r}(\phi) = \begin{pmatrix} R \cos \phi \\ R \sin \phi \\ 0 \end{pmatrix}.
\]

(b) Consider the special case of a vector field \( \mathbf{F}(\mathbf{r}) \) that is the gradient of a scalar field \( f(\mathbf{r}) \),

\[
\mathbf{F}(\mathbf{r}) = \nabla f(\mathbf{r}).
\]

Use the above chain rule for partial derivatives to prove for the line integral \( \int_{\Gamma} \mathbf{dr} \cdot \mathbf{F}(\mathbf{r}) \) of this vector field the formula

\[
\int_{\Gamma} \mathbf{dr} \cdot \nabla f(\mathbf{r}) = f(\mathbf{r}_B) - f(\mathbf{r}_A),
\]

where \( \mathbf{r}_A \) is the initial point and \( \mathbf{r}_B \) is the final point of the curve \( \Gamma \).