Fakultät Mikrosystemtechnik
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Vector calculus and numerical mathematics

Worksheet 6

Problem 1: Gauss' Theorem

Verify Gauss' theorem,

$$\oint_{\partial\Omega} \mathrm{d}\mathbf{S}\cdot\mathbf{F}(\mathbf{r}) \;=\; \int_{\Omega} \mathrm{d}^3r\,\big[\nabla\cdot\mathbf{F}(\mathbf{r})\big]$$

with the following vector field $\mathbf{F}(\mathbf{r})$ and a spherical volume region Ω,

$$\mathbf{F}(\mathbf{r}) \;\equiv\; \begin{pmatrix} F_1(\mathbf{r}) \\ F_2(\mathbf{r}) \\ F_3(\mathbf{r}) \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ x^2y \\ 0 \end{pmatrix}, \qquad \Omega \;=\; \Big\{\mathbf{r}(x,y,z)\,\Big|\,x^2+y^2+z^2\le R^2\Big\}.$$

Problem 2: Stokes' Theorem

Verify Stokes' theorem,

$$\oint_{\partial\Sigma} \mathrm{d}\mathbf{r}\cdot\mathbf{G}(\mathbf{r}) \;=\; \int_{\Sigma} \mathrm{d}\mathbf{S}\cdot\big[\nabla\times\mathbf{G}(\mathbf{r})\big],$$

with the following vector field $\mathbf{G}(\mathbf{r})$ and a planar disk Σ,

$$\mathbf{G}(\mathbf{r}) \;\equiv\; \begin{pmatrix} G_1(\mathbf{r}) \\ G_2(\mathbf{r}) \\ G_3(\mathbf{r}) \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ xy^2 \\ 0 \end{pmatrix}, \qquad \Sigma \;=\; \Big\{\mathbf{r}(x,y,0)\,\Big|\,x^2+y^2\le R^2\Big\}.$$

Problem 3: Another example for Stokes' Theorem

Verify Stokes' theorem,

$$\oint_{\partial\Sigma} \mathrm{d}\mathbf{r}\cdot\mathbf{H}(\mathbf{r}) \;=\; \int_{\Sigma} \mathrm{d}\mathbf{S}\cdot\big[\nabla\times\mathbf{H}(\mathbf{r})\big],$$

with the vector field

$$\mathbf{H}(\mathbf{r}) \;\equiv\; \begin{pmatrix} H_1(\mathbf{r}) \\ H_2(\mathbf{r}) \\ H_3(\mathbf{r}) \end{pmatrix} \;=\; \begin{pmatrix} 4x-8y+7z \\ 3x+5y-9z \\ 8x-2y+6z \end{pmatrix}$$

and with the curved piece of surface Σ (a **semi**-sphere), given by

$$\Sigma \;=\; \Big\{\mathbf{r}(r,\theta,\phi)\,\Big|\,r=R,\;\; 0\le\theta\le\tfrac{\pi}{2},\;\; 0\le\phi\le 2\pi\Big\},$$

where (r,θ,ϕ) are the usual spherical polar coordinates.
Hint: In this case, $\partial\Sigma = \big\{\mathbf{r}(R,\frac{\pi}{2},\phi)\,\big|\,0\le\phi\le 2\pi\big\}$ is the **equator** of the semi-sphere.

Problem 4: Chain rule for partial derivatives

Let $f(\mathbf{r}) \equiv f(x, y, z)$ be a **scalar field** and $\mathbf{r}(\phi)$ be a **vector function** of one variable ϕ,

$$\mathbf{r}(\phi) \;=\; \begin{pmatrix} x(\phi) \\ y(\phi) \\ z(\phi) \end{pmatrix}.$$

We define the simple function $\bar{f}(\phi)$ of one variable ϕ by

$$\begin{aligned} \bar{f}(\phi) &= f(\mathbf{r}(\phi)) \\ &\equiv f\Big(x(\phi), x(\phi), x(\phi)\Big). \end{aligned}$$

Chain rule for partial derivatives: This function $\bar{f}(\phi)$ has the derivative

$$\begin{aligned} \bar{f}'(\phi) &\equiv \frac{\mathrm{d}}{\mathrm{d}\phi} f\big(\mathbf{r}(\phi)\big) \\ &= \dot{\mathbf{r}}(\phi) \cdot \nabla f(\mathbf{r})\Big|_{\mathbf{r}=\mathbf{r}(\phi)} \equiv \left[\dot{x}(\phi)\frac{\partial f(\mathbf{r})}{\partial x} \;+\; \dot{y}(\phi)\frac{\partial f(\mathbf{r})}{\partial y} \;+\; \dot{z}(\phi)\frac{\partial f(\mathbf{r})}{\partial z}\right]_{\mathbf{r}=\mathbf{r}(\phi)}. \end{aligned}$$

(a) Verify this rule for the two scalar fields $f(x, y, z) = xy^2$ and $g(x, y, z) = x^2 + y^2$ and the vector function

$$\mathbf{r}(\phi) \;=\; \begin{pmatrix} R\cos\phi \\ R\sin\phi \\ 0 \end{pmatrix}.$$

(b) Consider the special case of a vector field $\mathbf{F}(\mathbf{r})$ that is the gradient of a scalar field $f(\mathbf{r})$,

$$\mathbf{F}(\mathbf{r}) \;=\; \nabla f(\mathbf{r}).$$

Use the above **chain rule for partial derivatives** to prove for the line integral $\int_\Gamma \mathrm{d}\mathbf{r} \cdot \mathbf{F}(\mathbf{r})$ of this vector field the formula

$$\int_\Gamma \mathrm{d}\mathbf{r} \cdot \nabla f(\mathbf{r}) \;=\; f(\mathbf{r}_B) - f(\mathbf{r}_A),$$

where $\mathbf{r}_A$ is the initial point and $\mathbf{r}_B$ is the final point of the curve Γ.