Worksheet 8

Problem 8.1: Separable-variables ODEs

(a) Writing formally $y'(x) = \frac{\mathrm{d}y}{\mathrm{d}x}$ and $y(x) = y$, we obtain

$$\frac{1}{y^2}\,\mathrm{d}y \;=\; ax\,\mathrm{d}x, \qquad -\frac{1}{y} + C_1 \;=\; \frac{ax^2}{2} + C_2, \qquad y(x) \;=\; \frac{2}{C - ax^2},$$

where $C = 2(C_1 - C_2)$. **Check:**

$$y'(x) \;=\; \frac{\mathrm{d}}{\mathrm{d}x}\,\frac{2}{C - ax^2} \;=\; -\frac{2\cdot(-2ax)}{(C - ax^2)^2} \;=\; ax\,\frac{4}{(C - ax^2)^2} \;=\; ax\,y(x)^2. \quad \checkmark$$

(b) Writing formally $y'(x) = \frac{\mathrm{d}y}{\mathrm{d}x}$ and $y(x) = y$, we obtain

$$y\,\mathrm{d}y \;=\; x^2\,\mathrm{d}x, \qquad \frac{y^2}{2} + C_1 \;=\; \frac{x^3}{3} + C_2, \qquad y(x) \;=\; \pm\sqrt{\frac{2}{3}\,x^3 \,+\, C},$$

where $C = 2(C_2 - C_1)$. **Check:**

$$y'(x) \;=\; \pm\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2}{3}\,x^3 \,+\, C\right)^{1/2} \;=\; \pm\frac{1}{2}\left(\frac{2}{3}\,x^3 \,+\, C\right)^{-1/2} 2x^2 \;=\; \frac{x^2}{y(x)}. \quad \checkmark$$

(c) This ODE seems to be **not a separable-variables** one.
However, it is an (inhomogeneous) **linear ODE**,

$$y'(x) \,-\, y(x) \;=\; -x^2,$$

with inhomogeneity $-x^2$. One **particular** solution can be found by guessing,

$$y_{\text{part}}(x) \;=\; x^2 \,+\, 2x \,+\, 2.$$

Then, the **general** solution is obtained by adding to this **particular** solution the general solution of the corresponding **homogeneous** linear ODE $y'(x) - y(x) = 0$,

$$y_{\text{gen}}(x) \;=\; (x^2 \,+\, 2x \,+\, 2) \,+\, A\mathrm{e}^x \qquad (A \in \mathbb{R}).$$

(d) Writing formally $y'(x) = \frac{\mathrm{d}y}{\mathrm{d}x}$ and $y(x) = y$, we obtain

$$\frac{1}{1+y^2}\,\mathrm{d}y \;=\; 4x^3\,\mathrm{d}x, \qquad \arctan(y) + C_1 \;=\; x^4 + C_2, \qquad y(x) \;=\; \tan(x^4 + C),$$

where $C = C_2 - C_1$, and we have used the first hint. **Check:**

$$y'(x) \;\equiv\; \frac{\mathrm{d}}{\mathrm{d}x}\tan(x^4 + C) \;=\; \frac{1}{\cos^2(x^4 + C)}\,4x^3 \;=\; 4x^3\,\big(1 \,+\, \tan^2(x^4 + C)\big). \quad \checkmark$$

(e) Writing formally $y'(x) = \frac{\mathrm{d}y}{\mathrm{d}x}$ and $y(x) = y$, we obtain

$$\cos y\,\mathrm{d}y \;=\; 6x\,\mathrm{d}x, \qquad \sin y + C_1 \;=\; 3x^2 + C_2, \qquad y(x) \;=\; \arcsin(3x^2 + C),$$

where $C = C_2 - C_1$. **Check:** Using the second hint, we get

$$y'(x) \;\equiv\; \frac{\mathrm{d}}{\mathrm{d}x}\arcsin(3x^2 + C) \;=\; \frac{1}{\sqrt{1 - (3x^2 + C)^2}}\,6x.$$

Since $y = \arcsin(3x^2 + C)$, we have $3x^2 + C = \sin y$, and therefore

$$y'(x) \;=\; \frac{1}{\sqrt{1 - (\sin y)^2}}\,6x \;=\; \frac{6x}{\cos y}. \quad \checkmark$$

(f) Writing formally $y'(x) = \frac{\mathrm{d}y}{\mathrm{d}x}$ and $y(x) = y$, we obtain

$$\frac{\mathrm{d}y}{\sin y} \;=\; \mathrm{d}x.$$

We integrate the LHS, using the identity $\sin y = 2\sin\frac{y}{2}\cos\frac{y}{2}$,

$$\begin{aligned}\int \mathrm{d}y\,\frac{1}{\sin y} \;=\; \int \mathrm{d}y\,\frac{1}{2\sin\frac{y}{2}\cos\frac{y}{2}} \;&=\; \int \mathrm{d}y\,\frac{\frac{1}{2}\frac{1}{(\cos\frac{y}{2})^2}}{\tan\frac{y}{2}} \\ &=\; \int \mathrm{d}y\,\frac{t'(y)}{t(y)} \;=\; \ln\big|t(y)\big| + C_1,\end{aligned}$$

where $t(y) = \tan\frac{y}{2}$, with the derivative $t'(y) = \frac{1}{2}\frac{1}{(\cos\frac{y}{2})^2}$.
Integrating the RHS, $\int \mathrm{d}\,x \;=\; x + C_2$ and setting $C_2 - C_1 = C$, we obtain

$$\ln\left|\tan\frac{y}{2}\right| \;=\; x \,+\, C.$$

Resolving for y, we eventually find[1]

$$y(x) \;=\; 2\,\arctan\big(\pm\,\mathrm{e}^{x+C}\big) \;=\; 2\,\arctan\big(a\mathrm{e}^{x}\big) \qquad (a = \pm\mathrm{e}^{C} \in \mathbb{R}).$$

Check: Since $\frac{\mathrm{d}}{\mathrm{d}u}\arctan u = \frac{1}{1+u^2}$, the chain rule yields

$$\begin{aligned} y'(x) \;=\; 2\,\frac{a\mathrm{e}^{x}}{1\,+\,(a\mathrm{e}^{x})^2} \;&=\; 2\,\frac{\tan\frac{y(x)}{2}}{1\,+\,\big(\tan\frac{y(x)}{2}\big)^2} \\ &=\; 2\,\sin\frac{y(x)}{2}\,\cos\frac{y(x)}{2} \;\equiv\; \sin\big[y(x)\big].\end{aligned}$$

[1]Notice that $\arctan(x)$ is the **inverse function** of $\tan(y)$:

$$\arctan(A) \;=\; B \quad\Leftrightarrow\quad \tan(B) \;=\; A.$$

Problem 8.2

Problem 8.3

(a) This is a simple exercise in partial differentiation, see section 9.1.5.

(b) This is a simple volume integral, see section 9.1.5.

Problem 8.4

Using the ansatz $T(x,y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$ in the given PDE,

$$\nabla^2 T(x,y) \;\equiv\; \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \;=\; -\frac{s_0}{\lambda},$$

we find a condition on the values of A and C,

$$2(A+C) \;=\; -\frac{s_0}{\lambda}. \tag{1}$$

Holding the rim of the disk at $T = T_0$ corresponds to the (Dirichlet) boundary condition

$$T(x,y) \;=\; T_0 \qquad \text{(for all } x, y \text{ with } x^2 + y^2 = R^2\text{)}. \tag{2}$$

Setting $B = D = E = 0$, we see that conditions (1) and (2) can be satisfied by the choice

$$A = C = -\frac{s_0}{4\lambda}, \qquad F = T_0 + \frac{s_0}{4\lambda}\,R^2 \qquad (B = D = E = 0),$$

which turns the above ansatz into the (unique!) solution

$$T(x,y) \;=\; T_0 + \frac{s_0}{4\lambda}\Big[R^2 - \big(x^2 + y^2\big)\Big].$$

Remark: For completeness, we derive the given PDE (this is not part of Problem 8.4): In the medium, density ρ and current density $\mathbf{J}$ of heat satisfy the **continuity equation**

$$\frac{\partial}{\partial t}\rho(\mathbf{r},t) \;+\; \nabla\cdot\mathbf{J}(\mathbf{r},t) \;=\; s(\mathbf{r},t).$$

Using for ρ the **specific-heat relation**

$$\rho(\mathbf{r},t) \;=\; c\,\mu\,\big[T(\mathbf{r},t) - T_1\big]$$

(with some reference temperature T_1), and for $\mathbf{J}$ the **heat conduction law**,

$$\mathbf{J}(\mathbf{r},t) \;=\; -\lambda\,\nabla\,T(\mathbf{r},t),$$

we obtain a PDE for the temperature distribution $T(\mathbf{r},t)$ in the medium,

$$c\,\mu\,\frac{\partial}{\partial t}T(\mathbf{r},t) \;-\; \lambda\,\nabla^2 T(\mathbf{r},t) \;=\; s(\mathbf{r},t).$$

In the case $s(\mathbf{r},t) = s(\mathbf{r})$, steady-state equilibrium will be reached after some time, when $\frac{\partial T}{\partial t} = 0$. The resulting static distribution $T(\mathbf{r},t) = T(\mathbf{r})$ will satisfy the given PDE,

$$\nabla^2 T(\mathbf{r}) \;=\; -\frac{1}{\lambda}\,s(\mathbf{r}).$$

Worksheet 9

Problem 9.1: Finite difference method (FDM) for a PDE

(a) In case of the **disk** Σ_1, the exact solution is obtained in Problem 4 of Worksheet 8, when we choose $T_0 = 0$ and $\frac{s_0}{\lambda} = S$ there,

$$T(x,y) \;=\; \frac{S}{4}\Big[R^2 - (x^2 + y^2)\Big].$$

(It is easy to check that this is really the solution !)

(b) The five regions $\Sigma_1, \ldots, \Sigma_5$ in the xy-plane are shown in Fig. 1 (Σ_3 is shown twice), along with a properly chosen mesh of discrete points in each case.

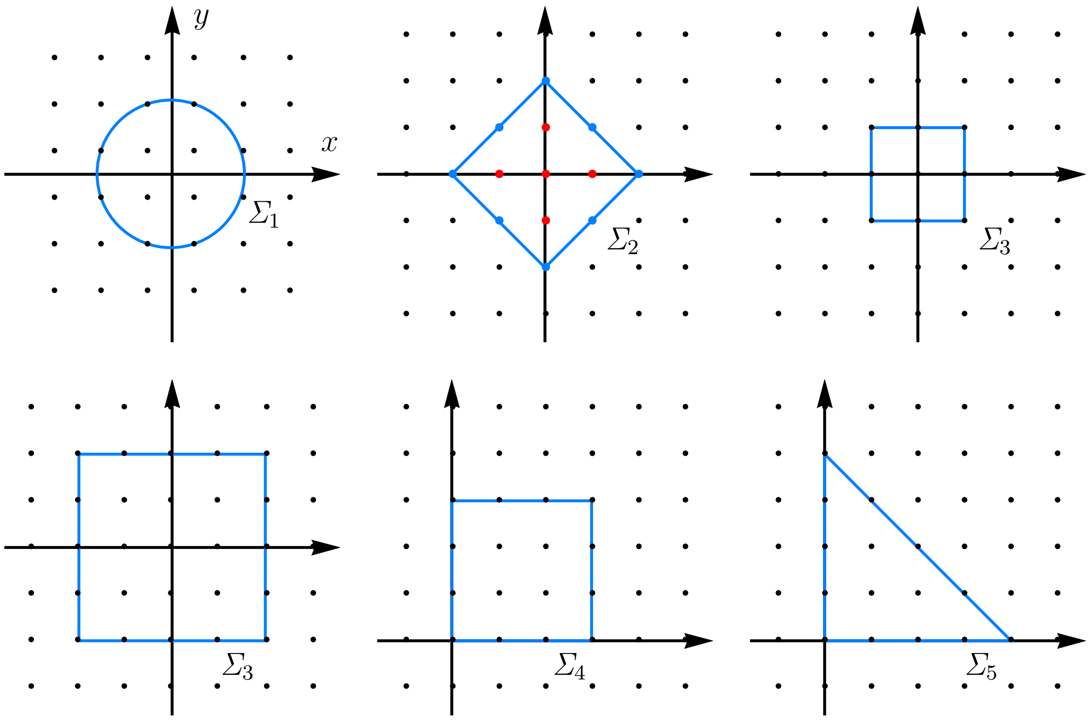

Figure 1: **Upper row:** Σ_1 (with $h = \frac{R}{5}\sqrt{10}$), Σ_2 (with $h = \frac{D}{4}$), Σ_3 (with $h = \frac{L}{2}$), **Lower row:** Σ_3 (once more, but with $h = \frac{L}{4}$), Σ_4 (with $h = \frac{L}{3}$), and Σ_5, (with $h = \frac{L}{4}$). The mesh points **inside** Σ_2 are highlighted in red, the ones **on the rim** $\partial\Sigma_2$ in blue.

Remark 1: Each panel in Fig. 1 has a certain number N of mesh points **inside** Σ_n (marked in red in the case of Σ_2) plus a considerable number of extra mesh points **on the rim** $\partial\Sigma_n$ (marked in blue in the case of $\partial\Sigma_2$).

Remark 2: The four different panels with $\Sigma_2, \Sigma_3, \Sigma_4$ all belong to the same physical situation (when we choose $D = L\sqrt{2}$), but to different levels of approximation: With $N = 5$, 1, 9, or 4 mesh points inside Σ_n (the choice $N = 1$ being a poor one).

(c) In the lecture (section 12.2 "Finite difference methods"), we have obtained Eq. (448) ("a set of **iteration equations**") which in the present notation reads

$$T_{n,m} \;=\; \frac{1}{4}\Big[T_{n+1,m} + T_{n-1,m} + T_{n,m+1} + T_{n,m-1}\Big] \;+\; \frac{h^2}{4}\,S_{n,m}\,.$$

In the present case $S(x,y) = S$ is a constant, $S_{n,m} = S$, independently of n and m. In further simplified notation, we may write

$$T_{n,m} \;=\; \frac{T_{\rightarrow} \,+\, T_{\leftarrow} \,+\, T_{\uparrow} \,+\, T_{\downarrow}}{4} \,+\, \frac{h^2}{4}\,S\;, \tag{3}$$

where "$T_{\rightarrow}$" means the temperature at the next-neighbor mesh point **to the right** ("$\rightarrow$") of $\mathbf{r}_{n,m}$, etc.

Iteration method: Using approximate values $T_{\rightarrow}$, $T_{\leftarrow}$, $T_{\uparrow}$, $T_{\downarrow}$ on the RHS of Eq. (3), the LHS should yield improved temperatures $T_{n,m}$.

As an example, we here pick the square Σ_2 (second panel in upper row of Fig. 1): Given the point mesh of Fig. 1, we have 5 points inside Σ_2 (marked in red) plus 8 ones on the rim $\partial\Sigma_2$ (marked in blue). For short, we re-label the "red points" $\mathbf{r}_{n,m}$ as follows

$$\begin{array}{ccc} & \mathbf{r}_{0,1} & \\ \mathbf{r}_{-1,0} & \mathbf{r}_{0,0} & \mathbf{r}_{1,0} \\ & \mathbf{r}_{0,-1} & \end{array} \quad = \quad \begin{array}{ccc} & \mathbf{r}_{2} & \\ \mathbf{r}_{3} & \mathbf{r}_{5} & \mathbf{r}_{1} \\ & \mathbf{r}_{4} & \end{array}$$

While the "blue points" are held at $T = 0$, we approximate the unknown temperatures $T_n = T(\mathbf{r}_n)$ at the "red points" $(n = 1,2,3,4,5)$ by guessed values $T_n^{(0)} \approx T_n$. For example, choosing "heating strength" S and step size h as

$$S \;=\; 100, \qquad h \;=\; 1,$$

a proper choice of these "initial guess" (or zeroth-order) temperatures $T_n^{(0)}$ could be

$$\begin{array}{ccccc} & & 0 & & \\ & 0 & T_2^{(0)} & 0 & \\ 0 & T_3^{(0)} & T_5^{(0)} & T_1^{(0)} & 0 \\ & 0 & T_4^{(0)} & 0 & \\ & & 0 & & \end{array} \quad = \quad \begin{array}{ccccc} & & 0 & & \\ & 0 & 50 & 0 & \\ 0 & 50 & 80 & 50 & 0 \\ & 0 & 50 & 0 & \\ & & 0 & & \end{array}$$

Using these trial values on the RHS of Eq. (3), we find the improved temperatures

$$\begin{aligned}
T_1^{(1)} &= \frac{0 + 0 + T_5^{(0)} + 0}{4} + \frac{100}{4} = 45, \\
T_2^{(1)} &= \frac{0 + 0 + 0 + T_5^{(0)}}{4} + \frac{100}{4} = 45, \\
T_3^{(1)} &= \frac{T_5^{(0)} + 0 + 0 + 0}{4} + \frac{100}{4} = 45, \\
T_4^{(1)} &= \frac{0 + T_5^{(0)} + 0 + 0}{4} + \frac{100}{4} = 45, \\
T_5^{(1)} &= \frac{T_1^{(0)} + T_2^{(0)} + T_3^{(0)} + T_4^{(0)}}{4} + \frac{100}{4} = 75.
\end{aligned}$$

These values, obtained upon the **first iteration**, comprise our **first-order result**,

$$\begin{array}{ccccc} & & 0 & & \\ & 0 & T_2^{(1)} & 0 & \\ 0 & T_3^{(1)} & T_5^{(1)} & T_1^{(1)} & 0 \\ & 0 & T_4^{(1)} & 0 & \\ & & 0 & & \end{array} \quad = \quad \begin{array}{ccccc} & & 0 & & \\ & 0 & 45 & 0 & \\ 0 & 45 & 75 & 45 & 0 \\ & 0 & 45 & 0 & \\ & & 0 & & \end{array}$$

A **second iteration**, using these first-order temperatures $T_n^{(1)}$ as new input on the RHS of Eq. (3), yields an even better set of **second-order temperatures**

$$T_1^{(2)} \;=\; T_2^{(2)} \;=\; T_3^{(2)} \;=\; T_4^{(2)} \;=\; 43.75, \qquad\qquad T_5^{(2)} \;=\; 70.$$

One can (quite easily!) show that, after an infinite number of iterations, this procedure will converge towards the solution

$$T_1 \;=\; T_2 \;=\; T_3 \;=\; T_4 \;=\; \frac{5}{12}\,S \;\approx\; 41.67, \qquad\qquad T_5 \;=\; \frac{2}{3}\,S \;\approx\; 66.67.$$

Problem 9.2:

(a) This is an inhomogeneous linear ODE, with the general solution

$$\begin{aligned} f_{\rm gen}^{\rm inh}(x) \;&=\; f_{\rm gen}^{\rm hom}(x) \;+\; f_{\rm part}^{\rm inh}(x) \\ &=\; \Big[A\cos 7x + B\sin 7x\Big]\mathrm{e}^{-4x} \;-\; \frac{1}{5}, \end{aligned}$$

where $f_{\rm gen}^{\rm hom}(x)$ can be found from an exponential ansatz $f(x) = \mathrm{e}^{\lambda x}$,

$$\lambda^2 + 8\lambda + 65 = 0, \qquad \lambda_{1,2} = \frac{-8 \pm \sqrt{64 - 240}}{2} = \frac{-8 \pm 14\mathrm{i}}{2} = -4 \pm 7\mathrm{i},$$

and $f_{\rm part}^{\rm inh}(x) = -\frac{1}{5}$ (= const.) can be guessed.

(b) This is another inhomogeneous linear ODE, with the general solution

$$\begin{aligned} f_{\rm gen}^{\rm inh}(x) \;&=\; f_{\rm gen}^{\rm hom}(x) \;+\; f_{\rm part}^{\rm inh}(x) \\ &=\; \Big[A + B\mathrm{e}^{-8x}\Big] \;-\; \frac{65}{8}\,x, \end{aligned}$$

where $f_{\rm gen}^{\rm hom}(x)$ can be found from an exponential ansatz $f(x) = \mathrm{e}^{\lambda x}$,

$$\lambda^2 + 8\lambda = 0, \qquad \lambda_{1,2} \in \{0, -8\},$$

and $f_{\rm part}^{\rm inh}(x) = -\frac{65}{8}\,x$ can be guessed.

(c) $f''(x) + 8f'(x)^2 \;=\; 0$ is equivalent to a non-linear 1st-order ODE for $g(x) = f'(x)$,

$$g'(x) = -8g(x)^2, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = -8y^2, \qquad -\frac{1}{y^2}\,\mathrm{d}y = 8\,\mathrm{d}x, \qquad \frac{1}{y} = 8x + C,$$

with the solution

$$\begin{aligned} g(x) \;&=\; \frac{1}{8x + C} \;\equiv\; f'(x) \\ \Rightarrow \qquad f(x) \;&=\; \frac{1}{8}\ln(8x + C) \;+\; D \qquad (C, D \in \mathbb{R}). \end{aligned}$$

Problem 9.3: Average Temperature

(a) With $V_\Omega = \frac{\pi}{2} R^2 H$ and $T(\mathbf{r}) \equiv T(x,y,z) = T_0\,\frac{x}{R}$, we have

$$
\begin{aligned}
\Big\langle T(\mathbf{r}) \Big\rangle_{\mathbf{r}\in\Omega} &= \frac{1}{V_\Omega}\int_\Omega \mathrm{d}^3r\, T(\mathbf{r}) \\
&= \frac{1}{\frac{\pi}{2}R^2 H}\,\frac{T_0}{R}\int_0^H \mathrm{d}z \int_{-R}^{R}\mathrm{d}y \int_0^{\sqrt{R^2-y^2}} \mathrm{d}x\, x \\
&= \frac{2T_0}{\pi R^3 H}\underbrace{\int_0^H \mathrm{d}z}_{=H}\int_{-R}^{R}\mathrm{d}y\left[\frac{x^2}{2}\right]_{x=0}^{x=\sqrt{R^2-y^2}} \\
&= \frac{2T_0}{\pi R^3}\int_{-R}^{R}\mathrm{d}y\left[\frac{R^2-y^2}{2}\right] \\
&= \frac{2T_0}{\pi R^3}\cdot\frac{1}{2}\left[R^2 y - \frac{y^3}{3}\right]_{y=-R}^{y=R} \\
&= \frac{2T_0}{\pi R^3}\cdot\frac{1}{2}\left[2R^3 - \frac{2R^3}{3}\right] = \frac{4}{3\pi}T_0 \approx 0.424\,T_0.
\end{aligned}
$$

(b) In cylindrical coordinates, with $\widetilde{T}(s,\phi,z) = T_0\frac{s\cos\phi}{R}$ and the Jacobian $J(s,\phi,z) = s$,

$$
\begin{aligned}
\Big\langle T(\mathbf{r}) \Big\rangle_{\mathbf{r}\in\Omega} &= \frac{1}{V_\Omega}\int_0^H \mathrm{d}z\int_0^R \mathrm{d}s \int_{-\pi/2}^{\pi/2}\mathrm{d}\phi\, J(s,\phi,z)\,\widetilde{T}(s,\phi,z) \\
&= \frac{1}{\frac{\pi}{2}R^2 H}\,\frac{T_0}{R}\underbrace{\int_0^H \mathrm{d}z}_{=H}\int_0^R \mathrm{d}s\int_{-\pi/2}^{\pi/2}\mathrm{d}\phi\, s^2\cos\phi \\
&= \frac{2T_0}{\pi R^3}\left(\int_0^R \mathrm{d}s\, s^2\right)\left(\int_{-\pi/2}^{\pi/2}\mathrm{d}\phi\,\cos\phi\right) = \frac{2T_0}{\pi R^3}\cdot\frac{R^3}{3}\cdot 2 = \frac{4}{3\pi}T_0.
\end{aligned}
$$

Notice: As the limits of each inner integral are fixed numbers, not depending on any of the respective outer integration variables, this triple integral **factorises**.