# Worksheet 6

## Problem 6.1: Gauss' Theorem

In the present case of a sphere $\Omega$, it is appropriate to use spherical (polar) coordinates,

$$\mathbf{r}(\alpha) \;\equiv\; \mathbf{r}(r,\theta,\phi) \;=\; \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix}.$$

• On the **LHS of Gauss' Theorem**, we have the **flux integral**

$$\oint_{\partial\Omega} \mathrm{d}\mathbf{S}\cdot\mathbf{F}(\mathbf{r}) \;=\; \int_0^{\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\,\Big[\mathbf{h}_\theta(\alpha)\times\mathbf{h}_\phi(\alpha)\Big]\cdot\mathbf{F}\big(\mathbf{r}(\alpha)\big)\Big|_{r=R}$$

**out of** the spherical surface $\partial\Omega$. Consequently, we choose the cross product $\mathbf{h}_\theta(\alpha)\times\mathbf{h}_\phi(\alpha)$, as it (is normal to the surface $\partial\Omega$ and) points **out of** the sphere $\Omega$.[1] Using the vectors

$$\mathbf{h}_\theta(\alpha) \equiv \frac{\partial\mathbf{r}(\alpha)}{\partial\theta} = \begin{pmatrix} r\cos\theta\cos\phi \\ r\cos\theta\sin\phi \\ -r\sin\theta \end{pmatrix}, \qquad \mathbf{h}_\phi(\alpha) \equiv \frac{\partial\mathbf{r}(\alpha)}{\partial\phi} = \begin{pmatrix} -r\sin\theta\sin\phi \\ =R\sin\theta\cos\phi \\ 0 \end{pmatrix},$$

and using $x^2y = (r\sin\theta\cos\phi)^2(r\sin\theta\sin\phi) = r^3\sin^3\theta\cos^2\phi\sin\phi$, we then obtain

$$\begin{aligned}
\oint_{\partial\Omega}\mathrm{d}\mathbf{S}\cdot\mathbf{F}(\mathbf{r}) \;&=\; \int_0^{\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\begin{pmatrix} R^2\sin^2\theta\cos\phi \\ R^2\sin^2\theta\sin\phi \\ R^2\cos\theta\sin\theta \end{pmatrix}\cdot\begin{pmatrix} 0 \\ R^3\sin^3\theta\cos^2\phi\sin\phi \\ 0 \end{pmatrix} \\
&=\; \int_0^{\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\,\Big[R^5\sin^5\theta\cos^2\phi\sin^2\phi\Big] \\
&=\; R^5\underbrace{\int_0^{\pi}\mathrm{d}\theta\,\sin^5\theta}_{16/15}\underbrace{\int_0^{2\pi}\mathrm{d}\phi\,\cos^2\phi\sin^2\phi}_{\pi/4} \;=\; \frac{4\pi}{15}R^5.
\end{aligned}$$

• For the **RHS of Gauss' Theorem**, we first evaluate the divergence

$$\nabla\cdot\mathbf{F}(\mathbf{r}) \;\equiv\; \nabla\cdot\begin{pmatrix} 0 \\ x^2y \\ 0 \end{pmatrix} \;=\; \frac{\partial\,0}{\partial x} + \frac{\partial\,x^2y}{\partial y} + \frac{\partial\,0}{\partial z} \;=\; x^2 \;=\; (r\sin\theta\cos\phi)^2.$$

In the volume integral on the RHS, the Jacobian $J(r,\theta,\phi) = r^2\sin\theta$ is required,

$$\begin{aligned}
\int_\Omega \mathrm{d}^3r\,\big[\nabla\cdot\mathbf{F}(\mathbf{r})\big] \;&=\; \int_0^R\mathrm{d}r\int_0^{\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\,J(r,\theta,\phi)\,(r\sin\theta\cos\phi)^2 \\
&=\; \underbrace{\int_0^R\mathrm{d}r\,r^4}_{R^5/5}\underbrace{\int_0^{\pi}\mathrm{d}\theta\,\sin^3\theta}_{4/3}\underbrace{\int_0^{2\pi}\mathrm{d}\phi\,\cos^2\phi}_{\pi} \;=\; \frac{4\pi}{15}R^5.
\end{aligned}$$

• In summary, we find the same result on both LHS and RHS, confirmimg Gauss' theorem.

[1] The alternative choice $\mathbf{h}_\phi(\alpha)\times\mathbf{h}_\theta(\alpha) = -\mathbf{h}_\theta(\alpha)\times\mathbf{h}_\phi(\alpha)$ would yield the wrong sign!

**Problem 6.2: Stokes' Theorem**

In case of a planar disk $\Sigma$, it is appropriate to use cylindrical coordinates,

$$\mathbf{r}(\alpha) \;\equiv\; \mathbf{r}(s,\phi,z) \;=\; \begin{pmatrix} s\cos\phi \\ s\sin\phi \\ z \end{pmatrix}.$$

• On the **LHS of Stokes' Theorem**, we have the **line (or curve) integral**

$$\oint_{\partial\Sigma} \mathrm{d}\mathbf{r}\cdot\mathbf{G}(\mathbf{r}) \;=\; \int_0^{2\pi} \mathrm{d}\phi\,\dot{\mathbf{r}}(\phi)\cdot\mathbf{G}\big(\mathbf{r}(\phi)\big)$$

along the rim $\partial\Sigma$ of the disk $\Sigma$. Choosing for $\partial\Sigma$ the parametrization

$$\mathbf{r}(\phi) \;=\; \begin{pmatrix} R\cos\phi \\ R\sin\phi \\ 0 \end{pmatrix} \qquad\Rightarrow\qquad \dot{\mathbf{r}}(\phi) \;\equiv\; \frac{\partial\mathbf{r}(\phi)}{\partial\phi} \;=\; \begin{pmatrix} -R\sin\phi \\ +R\cos\phi \\ 0 \end{pmatrix} \tag{1}$$

and using $xy^2 = s\cos\phi\,(s\sin\phi)^2$, we obtain

$$\begin{aligned}
\oint_{\partial\Sigma} \mathrm{d}\mathbf{r}\cdot\mathbf{G}(\mathbf{r}) \;&=\; \int_0^{2\pi} \mathrm{d}\phi \begin{pmatrix} -R\sin\phi \\ +R\cos\phi \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ (R\cos\phi)(R\sin\phi)^2 \\ 0 \end{pmatrix} \\
&=\; R^4 \int_0^{2\pi} \mathrm{d}\phi\,\cos^2\phi\sin^2\phi \;=\; \frac{\pi}{4}\,R^4.
\end{aligned}$$

• For the **RHS of Stokes' Theorem**, we first evaluate the curl (rotation)

$$\nabla\times\mathbf{G}(\mathbf{r}) \;\equiv\; \nabla\times\begin{pmatrix} 0 \\ xy^2 \\ 0 \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \\ y^2 \end{pmatrix} \;\equiv\; \begin{pmatrix} 0 \\ 0 \\ (s\sin\phi)^2 \end{pmatrix}.$$

In the flux integral on the RHS, we must choose the normal vector $\mathbf{h}_s(\alpha)\times\mathbf{h}_\phi(\alpha)$, as the alternative choice $\mathbf{h}_\phi(\alpha)\times\mathbf{h}_s(\alpha) = -\mathbf{h}_s(\alpha)\times\mathbf{h}_\phi(\alpha)$ would yield the wrong sign[2]

$$\begin{aligned}
\int_\Sigma \mathrm{d}\mathbf{S}\cdot\big[\nabla\times\mathbf{G}(\mathbf{r})\big] \;&=\; \int_0^R \mathrm{d}s \int_0^{2\pi} \mathrm{d}\phi \Big[\mathbf{h}_s(\alpha)\times\mathbf{h}_\phi(\alpha)\Big]\cdot\begin{pmatrix} 0 \\ 0 \\ y^2 \end{pmatrix} \\
&=\; \int_0^R \mathrm{d}s \int_0^{2\pi} \mathrm{d}\phi \left[\begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \times \begin{pmatrix} -s\sin\phi \\ +s\cos\phi \\ 0 \end{pmatrix}\right]\cdot\begin{pmatrix} 0 \\ 0 \\ y^2 \end{pmatrix} \\
&=\; \int_0^R \mathrm{d}s \int_0^{2\pi} \mathrm{d}\phi \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 0 \\ y^2 \end{pmatrix} \\
&=\; \int_0^R \mathrm{d}s \int_0^{2\pi} \mathrm{d}\phi\, s^3\sin^2\phi \;=\; \frac{\pi}{4}\,R^4.
\end{aligned}$$

[2]The correct choice is fixed (via the right-hand rule) by the parametrization $\mathbf{r}(\phi)$ chosen in Eq. (1).

**Problem 6.3: Another example for Stokes' Theorem**

The circulation of $\mathbf{H}(\mathbf{r})$ around $\partial\Sigma \equiv \{\mathbf{r}(r,\theta,\phi)\,|\,r=R,\ \theta=\frac{\pi}{2},\ 0\le\phi\le 2\pi\}$ is

$$\begin{aligned}
\oint_{\partial\Sigma} \mathrm{d}\mathbf{r}\cdot\mathbf{H}(\mathbf{r}) &= \int_0^{2\pi}\mathrm{d}\phi \begin{pmatrix} -R\sin\phi \\ +R\cos\phi \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 4R\cos\phi - 8R\sin\phi + 7\cdot 0 \\ 3R\cos\phi + 5R\sin\phi - 9\cdot 0 \\ \ldots \end{pmatrix} \\
&= \int_0^{2\pi}\mathrm{d}\phi\Big[-4R^2\sin\phi\cos\phi + 8R^2\sin^2\phi + 3R^2\cos^2\phi + 5R^2\cos\phi\sin\phi\Big] \\
&= \quad 0 \quad + \quad 8R^2\pi \quad + \quad 3R^2\pi \quad + \quad 0 \quad = \quad 11R^2\pi.
\end{aligned}$$

The flux of $\nabla\times\mathbf{H}(\mathbf{r}) = (7,-1,11)$ through $\Sigma$ is

$$\begin{aligned}
\int_\Sigma \mathrm{d}\mathbf{S}\cdot\big[\nabla\times\mathbf{H}\big] &= \int_0^{\pi/2}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\left[\begin{pmatrix} R\cos\theta\cos\phi \\ R\cos\theta\sin\phi \\ -R\sin\theta \end{pmatrix}\times\begin{pmatrix} -R\sin\theta\sin\phi \\ +R\sin\theta\cos\phi \\ 0 \end{pmatrix}\right]\cdot\begin{pmatrix} 7 \\ -1 \\ 11 \end{pmatrix} \\
&= \int_0^{\pi/2}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\begin{pmatrix} R^2\sin^2\theta\cos\phi \\ R^2\sin^2\theta\sin\phi \\ R^2\cos\theta\sin\theta \end{pmatrix}\cdot\begin{pmatrix} 7 \\ -1 \\ 11 \end{pmatrix} \\
&= \int_0^{\pi/2}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\Big[7R^2\sin^2\theta\cos\phi - R^2\sin^2\theta\sin\phi + 11R^2\cos\theta\sin\theta\Big] \\
&= \quad 0 \quad + \quad 0 \quad + \quad 2\pi\cdot 11R^2\Big[\frac{1}{2}\sin^2\theta\Big]_0^{\pi/2} \quad = \quad 11R^2\pi.
\end{aligned}$$

**Problem 6.4: Chain rule for partial derivatives**

(a) For the given vector function $\mathbf{r}(\phi)$, we have

$$x(\phi) = R\cos\phi, \qquad y(\phi) = R\sin\phi.$$

- For the first scalar field $f(\mathbf{r})$, we find

$$\bar{f}(\phi) = R^3\cos\phi\sin^2\phi \qquad\Rightarrow\qquad \bar{f}'(\phi) = R^3\big[-\sin^3\phi + 2\sin\phi\cos^2\phi\big].$$

On the other hand,

$$\begin{aligned}
\dot{\mathbf{r}}(\phi)\cdot\nabla f(\mathbf{r})\Big|_{\mathbf{r}=\mathbf{r}(\phi)} &= \dot{x}(\phi)\,y(\phi)^2 + \dot{y}(\phi)\,2x(\phi)y(\phi) \\
&= -R^3\sin^3\phi + 2R^3\cos^2\phi\sin\phi.
\end{aligned}$$

- For the second scalar field $g(\mathbf{r})$, we find: $\bar{g}(\phi) = R^2 = \text{const.} \quad\Rightarrow\quad \bar{g}'(\phi) = 0.$
On the other hand,

$$\begin{aligned}
\dot{\mathbf{r}}(\phi)\cdot\nabla g(\mathbf{r})\Big|_{\mathbf{r}=\mathbf{r}(\phi)} &= \dot{x}(\phi)\,2x(\phi) + \dot{y}(\phi)\,2y(\phi) \\
&= -2R^2\sin\phi\cos\phi + 2R^2\cos\phi\sin\phi \;=\; 0.
\end{aligned}$$

(b) Standard evaluation $\int_\Gamma \mathrm{d}\mathbf{r}\cdot\mathbf{F}(\mathbf{r}) = \int \mathrm{d}\phi\,\dot{\mathbf{r}}(\phi)\cdot\mathbf{F}(\mathbf{r}(\phi))$ of a line integral yields

$$\begin{aligned}
\int_\Gamma \mathrm{d}\mathbf{r}\cdot\nabla f(\mathbf{r}) &= \int_{\phi_A}^{\phi_B}\mathrm{d}\phi\;\dot{\mathbf{r}}(\phi)\cdot\Big[\nabla f(\mathbf{r})\Big]\Big|_{\mathbf{r}=\mathbf{r}(\phi)} \\
&= \int_{\phi_A}^{\phi_B}\mathrm{d}\phi\;\frac{\mathrm{d}}{\mathrm{d}\phi}f\big(\mathbf{r}(\phi)\big) \quad\equiv\quad f(\mathbf{r}_B) - f(\mathbf{r}_A),
\end{aligned}$$

where our **chain rule for partial derivatives** has been applied in the second step.