

Worksheet 5

Problem 5.1: Flux integral

(a) In cylindrical coordinates $\alpha = (s, \phi, z)$, we have

$$\mathbf{h}_s(\alpha) \equiv \frac{\partial \mathbf{r}(\alpha)}{\partial s} = \begin{pmatrix} \cos \phi \\ \sin \phi \\ 0 \end{pmatrix}, \quad \mathbf{h}_\phi(\alpha) = \begin{pmatrix} -s \sin \phi \\ +s \cos \phi \\ 0 \end{pmatrix}, \quad \mathbf{h}_z(\alpha) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(b) Drawing: One eighth of a circle with radius $s_0 = 3$.

(c) Due to the formula given in the lecture, we have (possibly apart from a sign)

$$\begin{aligned} \int_{\Sigma} d\mathbf{A} \cdot \mathbf{F}(\mathbf{r}) &= \int_{\pi/4}^{\pi/2} d\phi \int_{-a}^a dz \left[\mathbf{h}_\phi(\alpha) \times \mathbf{h}_z(\alpha) \right] \cdot \mathbf{F}(\mathbf{r}(\alpha)) \Big|_{s=s_0} \\ &= \int_{\pi/4}^{\pi/2} d\phi \int_{-a}^a dz \begin{pmatrix} s_0 \cos \phi \\ s_0 \sin \phi \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 5s_0 \cos \phi - 7z \\ 3s_0 \cos \phi + 5s_0 \sin \phi \\ 6s_0 \sin \phi + 2z \end{pmatrix} \\ &= \int_{\pi/4}^{\pi/2} d\phi \int_{-a}^a dz \left(5s_0^2 \cos^2 \phi - 7s_0 z \cos \phi + 3s_0^2 \sin \phi \cos \phi + \right. \\ &\quad \left. + 5s_0^2 \sin^2 \phi + 0 + 0 \right) \\ &= \int_{\pi/4}^{\pi/2} d\phi \int_{-a}^a dz \left(5s_0^2 - 7s_0 z \cos \phi + 3s_0^2 \sin \phi \cos \phi \right) \\ &= \int_{\pi/4}^{\pi/2} d\phi \left[5s_0^2 z - \frac{7}{2} s_0 z^2 \cos \phi + 3s_0^2 z \sin \phi \cos \phi \right]_{z=-a}^{z=a} \\ &= \int_{\pi/4}^{\pi/2} d\phi \left(10s_0^2 a + 0 + 6s_0^2 a \sin \phi \cos \phi \right) \\ &= \left[10s_0^2 a \phi + 3s_0^2 a \sin^2 \phi \right]_{\phi=\pi/4}^{\phi=\pi/2} \\ &= \frac{5\pi + 3}{2} s_0^2 a. \end{aligned}$$

Problem 5.2: Flux integral (II)

(a) We have $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c} = 0$, and $|\mathbf{a}| = \sqrt{2^2 + 1^2 + (-2)^2} = 3 = |\mathbf{b}| = |\mathbf{c}|$.

(b) Writing

$$\mathbf{r}(u, v, w) \equiv \mathbf{r}(\alpha) = \begin{pmatrix} x(u, v, w) \\ y(u, v, w) \\ z(u, v, w) \end{pmatrix} = \begin{pmatrix} 2u + v + 2w \\ u + 2v - 2w \\ -2u + 2v + w \end{pmatrix},$$

we find

$$\mathbf{h}_u \equiv \frac{\partial \mathbf{r}(\alpha)}{\partial u} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \mathbf{a}, \quad \mathbf{h}_v = \mathbf{b}, \quad \mathbf{h}_w = \mathbf{c}.$$

(c) Σ is the **square** with the corners

$$\mathbf{r}_1 = \mathbf{0}, \quad \mathbf{r}_2 = \mathbf{a}, \quad \mathbf{r}_3 = \mathbf{b}, \quad \mathbf{r}_4 = \mathbf{a} + \mathbf{b}.$$

(d) Due to the formula given in the lecture, we have (possibly apart from a sign)

$$\begin{aligned} I &= \int_0^1 du \int_0^1 dv [\mathbf{h}_u \times \mathbf{h}_v] \cdot \mathbf{F}(\mathbf{r}(u, v, 0)) \\ &= \int_0^1 du \int_0^1 dv \left[\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right] \cdot \begin{pmatrix} 5(2u+v) - 7(-2u+2v) \\ 3(2u+v) + 5(u+2v) \\ 6(u+2v) + 2(-2u+2v) \end{pmatrix} \\ &= \int_0^1 du \int_0^1 dv \begin{pmatrix} 6 \\ -6 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 24u - 9v \\ 11u + 13v \\ 2u + 16v \end{pmatrix} \\ &= \int_0^1 du \int_0^1 dv (84u - 84v) \\ &= \int_0^1 du [84uv - 42v^2]_{v=0}^{v=1} \\ &= \int_0^1 du (84u - 42) = 42 - 42 = 0. \end{aligned}$$

Remark: This **zero result** is quite accidental! (Murphy's law is at work here...) To see this, we repeat the evaluation with a slightly more general vector field

$$\mathbf{F}_C(\mathbf{r}) = \begin{pmatrix} Cx - 7z \\ 3x + 5y \\ 6y + 2z \end{pmatrix}$$

(the original field is recovered by the choice $C = 5$),

$$\begin{aligned} I &= \int_0^1 du \int_0^1 dv [\mathbf{h}_u \times \mathbf{h}_v] \cdot \mathbf{F}_C(\mathbf{r}(u, v, 0)) \\ &= \int_0^1 du \int_0^1 dv \left[\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right] \cdot \begin{pmatrix} C(2u+v) - 7(-2u+2v) \\ 3(2u+v) + 5(u+2v) \\ 6(u+2v) + 2(-2u+2v) \end{pmatrix} \\ &= \int_0^1 du \int_0^1 dv \begin{pmatrix} 6 \\ -6 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} C(2u+v) + 14(u-v) \\ 11u + 13v \\ 2u + 16v \end{pmatrix} \\ &= \int_0^1 du \int_0^1 dv (6C(2u+v) + 24u - 114v) \\ &= \int_0^1 du \int_0^1 dv ((12C + 24)u + (6C - 114)v) \\ &= \int_0^1 du [(12C + 24)uv + (3C - 57)v^2]_{v=0}^{v=1} \\ &= \int_0^1 du ((12C + 24)u + (3C - 57)) \\ &= 9C - 45 \\ &= 9(C - 5). \end{aligned}$$

We see that the result $I = 0$ (for $C = 5$) has no deep reason, but occurs by chance.

Problem 5.3: Divergence

(a) As $J_2(x, y) = 0$, the divergence of the vector field $\mathbf{J}(x, y)$ in this case is

$$\begin{aligned}\nabla \cdot \mathbf{J}(\mathbf{r}) &\equiv \frac{\partial J_1(x, y)}{\partial x} + \frac{\partial J_2(x, y)}{\partial y} \\ &= \frac{\partial J_1(x, y)}{\partial x} \\ &= c \frac{a^2 - x^2 + y^2}{(a^2 + x^2 + y^2)^2}.\end{aligned}$$

(b) Table of values for $J_1(\mathbf{r}) = \frac{x}{1+x^2+y^2}$:

$y \backslash x$	-3	-2	-1	0	1	2	3
2	-0.21	-0.22	-0.17	0	0.17	0.22	0.21
1	-0.27	-0.33	-0.33	0	0.33	0.33	0.27
0	-0.30	-0.40	-0.50	0	0.50	0.40	0.30

In addition to the data of this table (which are asked for on the worksheet), Fig. 1 also shows data for half-integer values of x and y :

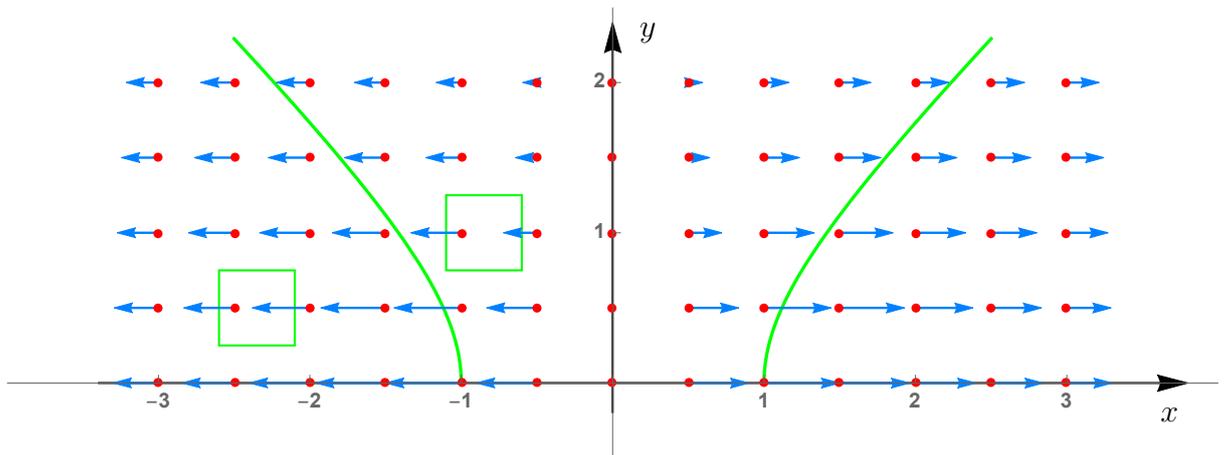


Figure 1: The vector field $\mathbf{J}(\mathbf{r})$ (blue arrows) in the xy plane (red dots!), using $a = c = 1$. Interpretation of divergence (green curves and green squares): See the text of part (d).

(c) A current density in 2D, rigorously, has the units: $[\mathbf{J}] = 1 \frac{\text{mg}}{\text{ms}}$.
 Consequently, $[a] = 1 \text{ m}$, $[c] = 1 \frac{\text{mg}}{\text{s}}$.
 Alternatively, taking \mathbf{J} for a current density in 3D, with $[\mathbf{J}] = 1 \frac{\text{mg}}{\text{m}^2 \text{ s}}$,
 we have: $[a] = 1 \text{ m}$, $[c] = 1 \frac{\text{mg}}{\text{sm}}$.

(d) Interpretation of the divergence $\nabla \cdot \mathbf{J}(x, y)$:

Points (in the xy -plane) with $\nabla \cdot \mathbf{J}(x, y) = 0$ satisfy the equation

$$a^2 - x^2 + y^2 = 0 \quad \Leftrightarrow \quad y = \pm \sqrt{x^2 - a^2},$$

defining two branches ($x \leq -a$, $x \geq a$) of a hyperbola (green curves in Fig. 1).

The area between these branches (including the y -axis) has $\nabla \cdot \mathbf{J}(x, y) > 0$ (sources).

The two remaining areas outside these branches have $\nabla \cdot \mathbf{J}(x, y) < 0$ (sinks).

Notice: In regions with $\nabla \cdot \mathbf{J}(x, y) > 0$, the net flow of \mathbf{J} out of a small square (green in Fig. 1) is positive, while in regions with $\nabla \cdot \mathbf{J}(x, y) < 0$, it is negative.

Problem 5.4: Gauss' theorem

(a) Since the vector $\mathbf{h}_\theta(\alpha) \times \mathbf{h}_\phi(\alpha)$ is pointing **out** of the spherical surface, we have

$$\begin{aligned} \int_{\Sigma} d\mathbf{A} \cdot \mathbf{F}(\mathbf{r}) &= \int_0^\pi d\theta \int_0^{\phi_0} d\phi [\mathbf{h}_\theta(\alpha) \times \mathbf{h}_\phi(\alpha)] \cdot \mathbf{F}(\mathbf{r}(\alpha)) \\ &= \int_0^\pi d\theta \int_0^{\phi_0} d\phi R^2 \begin{pmatrix} \sin^2 \theta \cos \phi \\ \sin^2 \theta \sin \phi \\ \cos \theta \sin \theta \end{pmatrix} \cdot \begin{pmatrix} R \sin \theta (2 \cos \phi + 3 \sin \phi) \\ R \sin \theta (5 \cos \phi - 4 \sin \phi) \\ 7R \cos \theta \end{pmatrix} \\ &= R^3 \int_0^\pi d\theta \int_0^{\phi_0} d\phi \left\{ \sin^3 \theta \left(2 \cos^2 \phi + (3+5) \cos \phi \sin \phi - 4 \sin^2 \phi \right) \right. \\ &\quad \left. + 7 \cos^2 \theta \sin \theta \right\}. \end{aligned}$$

Using $\int_0^\pi d\theta \sin^3 \theta = \int_0^\pi d\theta \sin \theta (1 - \cos^2 \theta) = [-\cos \theta + \frac{\cos^3 \theta}{3}]_0^\pi = \frac{4}{3}$, we obtain

$$\begin{aligned} \int_{\Sigma} d\mathbf{A} \cdot \mathbf{F}(\mathbf{r}) &= R^3 \int_0^{\phi_0} d\phi \left\{ \frac{4}{3} \left(2 \cos^2 \phi + 8 \cos \phi \sin \phi - 4 \sin^2 \phi \right) + 7 \left[-\frac{\cos^3 \theta}{3} \right]_0^\pi \right\} \\ &= R^3 \int_0^{\phi_0} d\phi \left\{ \frac{4}{3} \left(6 \cos^2 \phi + 8 \cos \phi \sin \phi - 4 \right) + \frac{14}{3} \right\}, \end{aligned}$$

where we have used $\sin^2 \phi = 1 - \cos^2 \phi$ in the last step;

$$\begin{aligned} \int_{\Sigma} d\mathbf{A} \cdot \mathbf{F}(\mathbf{r}) &= R^3 \left\{ \frac{4}{3} \left(6 \left[\frac{\phi + \cos \phi \sin \phi}{2} \right]_0^{\phi_0} + 8 \left[\frac{\sin^2 \phi}{2} \right]_0^{\phi_0} - 4\phi_0 \right) + \frac{14}{3} \phi_0 \right\} \\ &= \frac{10}{3} R^3 \left\{ \phi_0 + \frac{6}{5} \cos \phi_0 \sin \phi_0 + \frac{8}{5} \sin^2 \phi_0 \right\}. \end{aligned}$$

• In the **particular case** $\phi_0 = 2\pi$, this becomes

$$\int_{\Sigma} d\mathbf{A} \cdot \mathbf{F}(\mathbf{r}) = \frac{20\pi}{3} R^3.$$

(b) Since $\nabla \cdot \mathbf{F}(\mathbf{r}) = 2 - 4 + 7 = 5$ is a **constant**, we obtain

$$\int_{\Omega} d^3r [\nabla \cdot \mathbf{F}(\mathbf{r})] \equiv V_{\Omega} \cdot 5 = \frac{20\pi}{3} R^3$$

(where $V_{\Omega} = \frac{4\pi}{3} R^3$ is the volume of the sphere Ω), the same result!

Problem 5.5: Gauss' theorem (II)

Given are the vector field $\mathbf{F}(\mathbf{r})$ and, in spherical coordinates (r, θ, ϕ) , the volume region Ω ("upper" hemisphere with radius R),

$$\mathbf{F}(\mathbf{r}) = \begin{pmatrix} 5x - 3y \\ 2x + 8y \\ \frac{4}{L} x^2 \end{pmatrix}, \quad \Omega = \left\{ \mathbf{r}(r, \theta, \phi) \mid 0 \leq r \leq R, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq 2\pi \right\}.$$

(a) The flux integral of $\mathbf{F}(\mathbf{r})$ out of the surface $\partial\Omega$ of Ω is

$$\oint_{\partial\Omega} d\mathbf{A} \cdot \mathbf{F}(\mathbf{r}) = \int_{\Sigma_1} d\mathbf{A} \cdot \mathbf{F}(\mathbf{r}) + \int_{\Sigma_2} d\mathbf{A} \cdot \mathbf{F}(\mathbf{r}).$$

• Σ_1 is the upper half of the curved spherical surface,

$$\begin{aligned} \int_{\Sigma_1} d\mathbf{A} \cdot \mathbf{F}(\mathbf{r}) &= \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi [\mathbf{h}_\theta \times \mathbf{h}_\phi] \cdot \mathbf{F}(\mathbf{r}(r, \theta, \phi)) \Big|_{r=R} \\ &= \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi \left[\begin{pmatrix} R \cos \theta \cos \phi \\ R \cos \theta \sin \phi \\ -R \sin \theta \end{pmatrix} \times \begin{pmatrix} -R \sin \theta \sin \phi \\ R \sin \theta \cos \phi \\ 0 \end{pmatrix} \right] \cdot \mathbf{F} \\ &= \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi \begin{pmatrix} R^2 \sin^2 \theta \cos \phi \\ R^2 \sin^2 \theta \sin \phi \\ R^2 \sin \theta \cos \theta \end{pmatrix} \cdot \begin{pmatrix} R \sin \theta (5 \cos \phi - 3 \sin \phi) \\ R \sin \theta (2 \cos \phi + 8 \sin \phi) \\ \frac{4}{L} R^2 \sin^2 \theta \cos^2 \phi \end{pmatrix} \\ &= R^3 \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi \left\{ \sin^3 \theta (5 \cos^2 \phi - \sin \phi \cos \phi + 8 \sin^2 \phi) + \right. \\ &\quad \left. + \frac{4R}{L} (\sin^3 \theta \cos \theta) \cos^2 \phi \right\}. \end{aligned}$$

Using $\int_0^{2\pi} d\phi \sin^2 \phi = \int_0^{2\pi} d\phi \cos^2 \phi = \pi$ and $\int_0^{2\pi} d\phi \sin \phi \cos \phi = 0$, this becomes

$$\int_{\Sigma_1} d\mathbf{A} \cdot \mathbf{F}(\mathbf{r}) = R^3 \left\{ \left(\int_0^{\pi/2} d\theta \sin^3 \theta \right) (5\pi - 0 + 8\pi) + \frac{4R}{L} \left[\frac{\sin^4 \theta}{4} \right]_0^{\pi/2} \pi \right\}.$$

Using $\int_0^{\pi/2} d\theta \sin^3 \theta = \int_0^{\pi/2} d\theta (1 - \cos^2 \theta) \sin \theta = [-\cos \theta + \frac{\cos^3 \theta}{3}]_0^{\pi/2} = \frac{2}{3}$, we obtain

$$\int_{\Sigma_1} d\mathbf{A} \cdot \mathbf{F}(\mathbf{r}) = R^3 \left\{ \frac{2}{3} \cdot 13\pi + \frac{4R}{L} \cdot \frac{1}{4} \cdot \pi \right\} \equiv R^3 \pi \left(\frac{26}{3} + \frac{R}{L} \right).$$

• Σ_2 is the planar disk with radius R that forms the "floor" of the hemisphere Ω ,

$$\begin{aligned} \int_{\Sigma_2} d\mathbf{A} \cdot \mathbf{F}(\mathbf{r}) &= \int_0^R dr \int_0^{2\pi} d\phi [\mathbf{h}_\phi \times \mathbf{h}_r] \cdot \mathbf{F}(\mathbf{r}(r, \theta, \phi)) \Big|_{\theta=\frac{\pi}{2}} \\ &= \int_0^R dr \int_0^{2\pi} d\phi \begin{pmatrix} 0 \\ 0 \\ -r \end{pmatrix} \cdot \begin{pmatrix} r(5 \cos \phi - 3 \sin \phi) \\ r(2 \cos \phi + 8 \sin \phi) \\ \frac{4}{L} r^2 \cos^2 \phi \end{pmatrix} \\ &= -\frac{4}{L} \int_0^R dr r^3 \pi = -\pi \frac{R^4}{L}. \end{aligned}$$

In summary,

$$\oint_{\partial\Omega} d\mathbf{A} \cdot \mathbf{F}(\mathbf{r}) \equiv \int_{\Sigma_1} d\mathbf{A} \cdot \mathbf{F}(\mathbf{r}) + \int_{\Sigma_2} d\mathbf{A} \cdot \mathbf{F}(\mathbf{r}) = \frac{26\pi}{3} R^3.$$

(b) The volume integral of the divergence of the vector field is

$$\int_{\Omega} d^3r [\nabla \cdot \mathbf{F}(\mathbf{r})] = V_{\Omega} \langle \nabla \cdot \mathbf{F}(\mathbf{r}) \rangle_{\mathbf{r} \in \Omega}.$$

Since the integrand is a constant, $\nabla \cdot \mathbf{F}(\mathbf{r}) = 5 + 8 + 0 = 13$, we simply have

$$\int_{\Omega} d^3r [\nabla \cdot \mathbf{F}(\mathbf{r})] = V_{\Omega} \cdot 13 = 13 \cdot \frac{2\pi}{3} R^3,$$

where $V_{\Omega} = \frac{2\pi}{3} R^3$ is the volume of the hemisphere.

Since the result is the same as in part (a), we have verified Gauss' theorem.