Worksheet 1

Problem 1.1

Symmetry implies that

$$\sum_{n=1}^{4} \mathbf{r}_n \equiv \begin{pmatrix} x_1 + 2x_2 \\ 0 \\ 3z_1 + \ell \end{pmatrix} = \mathbf{0},$$

resulting in $z_1 = -\frac{1}{3}\ell$ and $x_2 = -\frac{1}{2}x_1$,

$$\mathbf{r}_1 = \begin{pmatrix} x_1 \\ 0 \\ -\frac{1}{3}\ell \end{pmatrix}, \quad \mathbf{r}_{2,3} = \begin{pmatrix} -\frac{1}{2}x_1 \\ \pm y_2 \\ -\frac{1}{3}\ell \end{pmatrix}.$$

To find x_1 and y_2, notice that

$$|\mathbf{r}_n| = \ell \quad (n = 1, 2, 3, 4).$$

Consequently,

$$x_1 = \sqrt{\ell^2 - \frac{1}{9}\ell^2} = \frac{2\sqrt{2}}{3}\,\ell = 0.9428\,\ell, \qquad y_2 = \sqrt{\ell^2 - \frac{1}{4}x_1^2 - \frac{1}{9}\ell^2} = \sqrt{\frac{2}{3}}\,\ell = 0.8165\,\ell.$$

For the side length, we get

$$k = |\mathbf{r}_4 - \mathbf{r}_1| = \sqrt{x_1^2 + \left(\frac{4}{3}\,\ell\right)^2} = \frac{2\sqrt{6}}{3}\,\ell = 1.633\,\ell.$$

The bond angle θ is obtained by

$$\cos\theta = \frac{\mathbf{r}_4 \cdot \mathbf{r}_1}{|\mathbf{r}_4|\,|\mathbf{r}_1|} = -\frac{1}{3} \qquad \Rightarrow \qquad \theta = 109.471^\circ.$$

Problem 1.2

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= (a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3) \times (b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + b_3\mathbf{u}_3) \\
&= a_1b_1\underbrace{(\mathbf{u}_1 \times \mathbf{u}_1)}_{\mathbf{0}} + a_1b_2\underbrace{(\mathbf{u}_1 \times \mathbf{u}_2)}_{\mathbf{u}_3} + a_1b_3\underbrace{(\mathbf{u}_1 \times \mathbf{u}_3)}_{-\mathbf{u}_2} + \\
&+ a_2b_1\underbrace{(\mathbf{u}_2 \times \mathbf{u}_1)}_{-\mathbf{u}_3} + a_2b_2\underbrace{(\mathbf{u}_2 \times \mathbf{u}_2)}_{\mathbf{0}} + a_2b_3\underbrace{(\mathbf{u}_2 \times \mathbf{u}_3)}_{\mathbf{u}_1} + \\
&+ a_3b_1\underbrace{(\mathbf{u}_3 \times \mathbf{u}_1)}_{\mathbf{u}_2} + a_3b_2\underbrace{(\mathbf{u}_3 \times \mathbf{u}_2)}_{-\mathbf{u}_1} + a_3b_3\underbrace{(\mathbf{u}_3 \times \mathbf{u}_3)}_{\mathbf{0}} \\
&= (a_2b_3 - a_3b_2)\mathbf{u}_1 \;+\; (a_3b_1 - a_1b_3)\mathbf{u}_2 \;+\; (a_1b_2 - a_2b_1)\mathbf{u}_3.
\end{aligned}$$

Worksheet 2

Problem 2.1

(b)

$$\boldsymbol{\mathcal{T}}(t) \;=\; \begin{pmatrix} -\sin(\frac{1}{2}\alpha t^2) \\ +\cos(\frac{1}{2}\alpha t^2) \\ 0 \end{pmatrix}, \qquad \boldsymbol{\mathcal{N}}(t) \;=\; \begin{pmatrix} -\cos(\frac{1}{2}\alpha t^2) \\ -\sin(\frac{1}{2}\alpha t^2) \\ 0 \end{pmatrix}.$$

(c)

$$\mathbf{v}(t) \;\equiv\; \frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{r}(t) \;=\; \begin{pmatrix} -R\alpha t\sin(\frac{1}{2}\alpha t^2) \\ +R\alpha t\cos(\frac{1}{2}\alpha t^2) \\ 0 \end{pmatrix} \;=\; R\alpha t\,\boldsymbol{\mathcal{T}}(t).$$

(d)

$$\begin{aligned} \mathbf{a}(t) \;\equiv\; \frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{v}(t) \;&=\; R\alpha\,\boldsymbol{\mathcal{T}}(t) \;+\; R(\alpha t)^2\boldsymbol{\mathcal{N}}(t) \\ &=\; (R\alpha)\,\boldsymbol{\mathcal{T}}(t) \;+\; \frac{\mathbf{v}(t)^2}{R}\,\boldsymbol{\mathcal{N}}(t). \end{aligned}$$

This is the vector sum of linear and centripetal accelerations.

Problem 2.2

(a) In the case $a = b = \ell = 1\mathrm{m}$, $T_0 = 50°\ \mathrm{C}$, we at $(x|y) = (-0.3\,\mathrm{m}\,|\,0.7\,\mathrm{m})$ find

$$T(x,y) \;=\; (50°\ \mathrm{C})\cdot\frac{0.7}{0.7\;+\;\big(1-(-0.3)^2\big)(1-0.7)} \;=\; (50°\ \mathrm{C})\cdot 0.7194 \;=\; 35.97°\ \mathrm{C}.$$

(b) It is easy to see that

$$T(x,0) \;=\; 0, \qquad T(a,y) \;=\; T(x,b) \;=\; T(-a,y) \;=\; T_0.$$

(c) The condition $T(x,y) = T_1$ (with $T_1 = \text{const.}$) yields a linear equation for y,

$$y \;=\; \frac{T_1}{T_0}\left[y \;+\; \frac{1}{\ell^2}(a^2-x^2)(b-y)\right] \qquad (0 \le T_1 \le T_0).$$

Writing $\frac{T_1}{T_0} = \theta$, resolving for y yields the contours $y = y_\theta(x)$ in the xy-plane,

$$y_\theta(x) \;=\; b\,\frac{1}{1\;+\;\frac{1-\theta}{\theta}\,\frac{\ell^2}{a^2-x^2}} \qquad (0 \le \theta \le 1).$$

(d) The required partial derivatives are (with $c = \frac{1}{\ell^2}$)

$$\begin{aligned} T_x(x,y) \;&\equiv\; \frac{\partial T(x,y)}{\partial x} \;=\; T_0\,\frac{2c\,xy(b-y)}{[y+c(a^2-x^2)(b-y)]^2}, \\ T_y(x,y) \;&\equiv\; \frac{\partial T(x,y)}{\partial y} \;=\; T_0\,\frac{bc\,(a^2-x^2)}{[y+c(a^2-x^2)(b-y)]^2}. \end{aligned}$$

With $a, b, \ell = 1$ m, we find for the gradient $\mathbf{G}_T(x,y) = \binom{T_x(x,y)}{T_y(x,y)}$ (in units of 1 m)

$$\frac{\mathbf{G}_T(0.2,0.5)}{T_0} = \begin{pmatrix} 0.1041 \\ 0.9996 \end{pmatrix}, \quad \frac{\mathbf{G}_T(0.5,0.5)}{T_0} = \begin{pmatrix} 0.3265 \\ 0.9796 \end{pmatrix}, \quad \frac{\mathbf{G}_T(0.8,0.5)}{T_0} = \begin{pmatrix} 0.8651 \\ 0.7785 \end{pmatrix}.$$

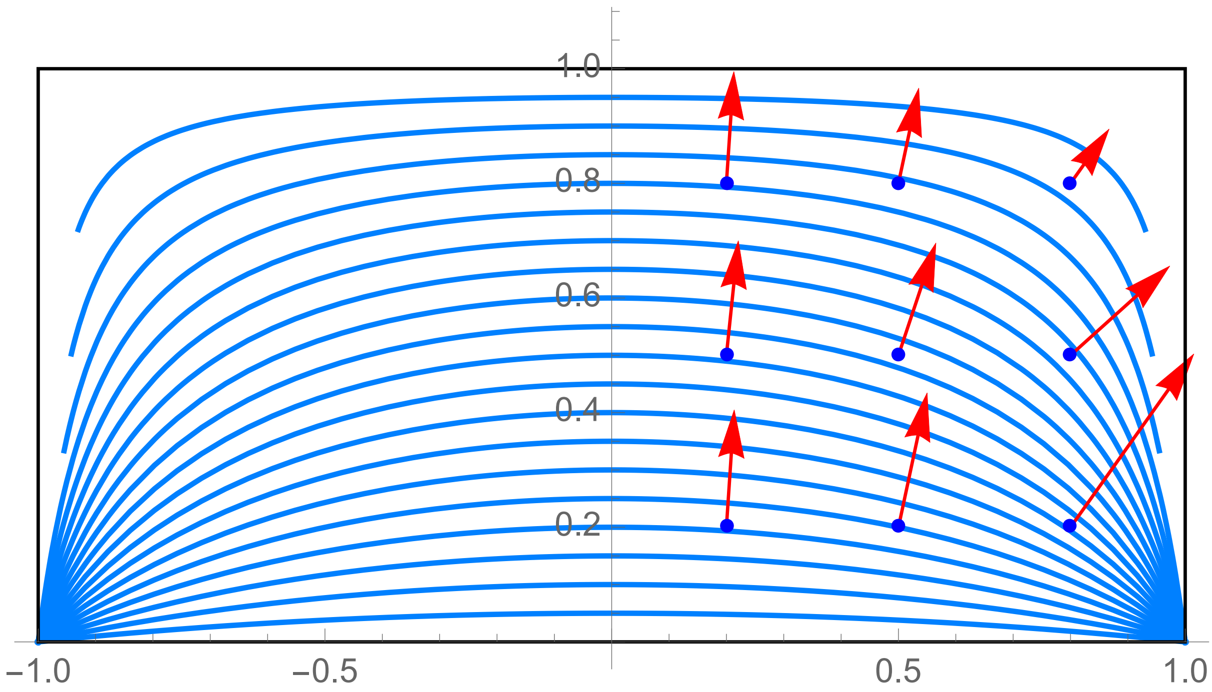

Figure 1: The contour plot for problem 2.2c.

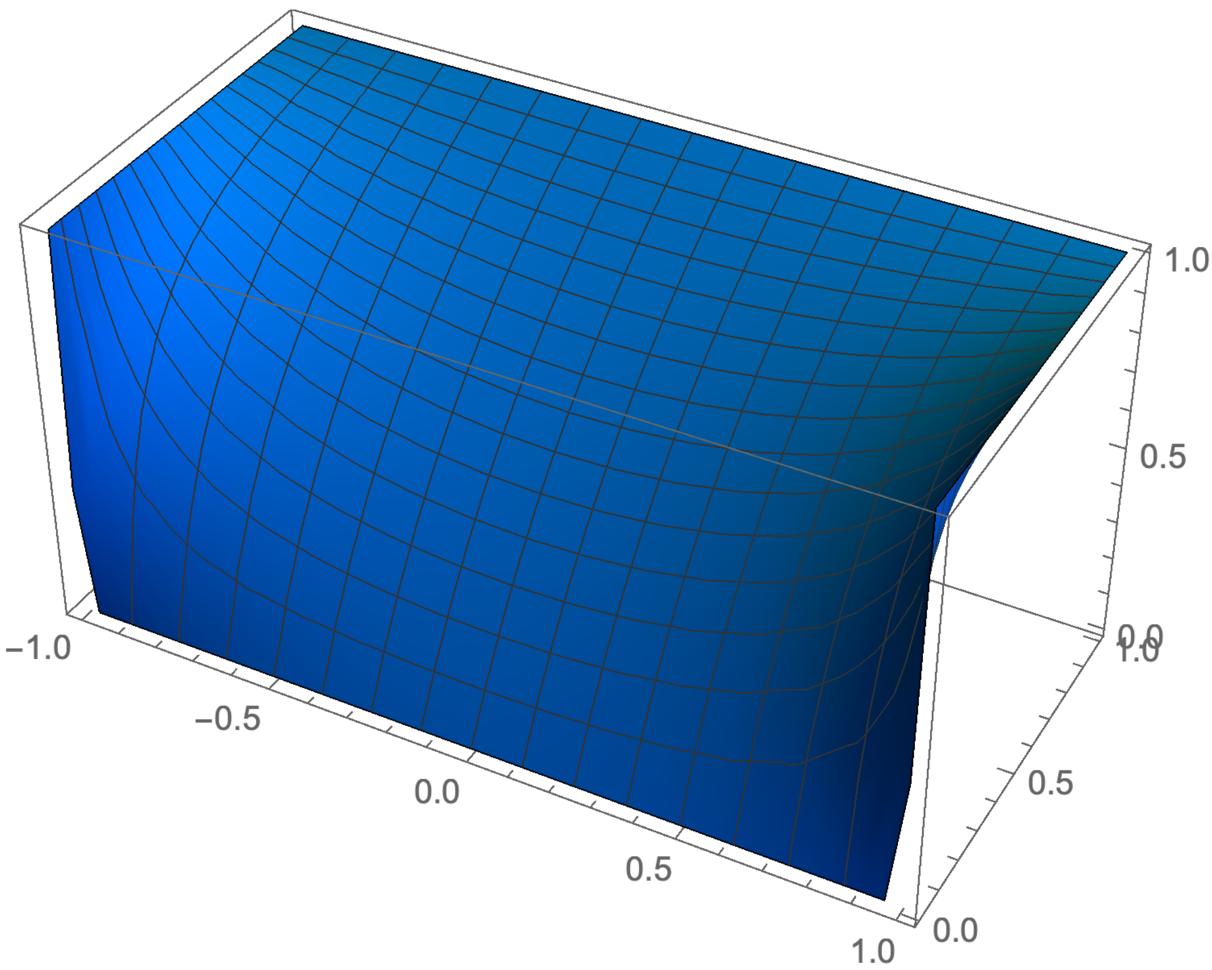

Figure 2: The 3D plot for problem 2.2c.

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Problem 2.3

(b)

$$\int_{\Omega} \mathrm{d}^2 r\, f(\mathbf{r}) \;=\; \int_{-2}^{1} \mathrm{d}x \int_{-(x+2)}^{x+2} \mathrm{d}y \big(c - ax^2 - by^2\big) \;=\; \ldots \;=\; -\frac{9}{2}\big(a + 3b - 2c\big).$$

Problem 2.4

(a)

$$\begin{aligned}\int_{\Sigma} \mathrm{d}^2 r\, f(x,y) \;&=\; \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\,(ax - by) \;=\; \int_0^1 \mathrm{d}x \left[axy - \frac{by^2}{2}\right]_{y=0}^{y=1-x} \\ &=\; \int_0^1 \mathrm{d}x \left\{ax(1-x) \;-\; \frac{b(1-x)^2}{2}\right\} \;=\; \ldots \;=\; \frac{a-b}{6}.\end{aligned}$$

(b) The result must be the same as in part (a)! Formally, we find

$$\begin{aligned}\int_{\Omega} \mathrm{d}^3 r\, 1 \;&=\; \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int_0^{f(x,y)} \mathrm{d}z\, 1 \\ &=\; \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \Big[\; z \;\Big]_{z=0}^{z=f(x,y)} \\ &=\; \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, f(x,y) \;\equiv\; \int_{\Sigma} \mathrm{d}^2 r\, f(x,y).\end{aligned}$$

Problem 2.5

(a)

$$\begin{aligned}\int_0^4 \mathrm{d}x \int_{1-\frac{x}{4}}^{1+\frac{x}{2}} \mathrm{d}y\,(ax + by) \;&=\; \int_0^4 \mathrm{d}x \left[axy + \frac{b}{2}y^2\right]_{y=1-\frac{x}{4}}^{y=1+\frac{x}{2}} \\ &=\; \int_0^4 \mathrm{d}x \;\left[\frac{3a}{4}x^2 + \frac{b}{2}\left\{\left(1+\frac{x}{2}\right)^2 - \left(1-\frac{x}{4}\right)^2\right\}\right] \\ &=\; \int_0^4 \mathrm{d}x \;\left[\frac{3a}{4}x^2 + \frac{b}{2}\left(\frac{3}{2}x + \frac{3}{16}x^2\right)\right] \\ &=\; \int_0^4 \mathrm{d}x \;\left[\left(\frac{3a}{4} + \frac{3b}{32}\right)x^2 + \frac{3b}{4}x\right] \;=\; 16a + 8b.\end{aligned}$$

(b) Since the points P_1, P_2 and P_3 are distributed within the triangle Ω rather uniformly, we have $\frac{1}{3}\sum_{i=1}^{3} f(x_i, y_i) \approx \langle f(\mathbf{r})\rangle_{\mathbf{r}\in\Omega}$, and thus

$$\begin{aligned}\int_{\Omega} \mathrm{d}^2 r\, f(x,y) \;\equiv\; A_{\Omega} \cdot \big\langle f(\mathbf{r})\big\rangle_{\mathbf{r}\in\Omega} \;&\approx\; A_{\Omega} \cdot \frac{1}{3}\sum_{i=1}^{3} f(x_i, y_i) \\ &=\; 6 \cdot \frac{1}{3}(7.5a + 3.5b) \;=\; 15a + 7b.\end{aligned}$$

(c) Since f is **linear**, we have exactly $\langle f(\mathbf{r})\rangle_{\mathbf{r}\in\Omega} = f(x_M, y_M)$. Therefore,

$$\int_\Omega \mathrm{d}^2 r\, f(x,y) \;=\; A_\Omega \cdot f(x_M, y_M) \;=\; 6\cdot f(\tfrac{8}{3},\tfrac{4}{3}) \;=\; 6\cdot(\tfrac{8}{3}a + \tfrac{4}{3}b) \;=\; 16a + 8b.$$

(d) Part (a):

$$\int_0^4 \mathrm{d}x \int_{1-\frac{x}{4}}^{1+\frac{x}{2}} \mathrm{d}y\,(ax^2 + bxy + cy^2) \;=\; \int_0^4 \mathrm{d}x \left[ax^2 y + \frac{b}{2}xy^2 + \frac{c}{3}y^3\right]_{y=1-\frac{x}{4}}^{y=1+\frac{x}{2}}$$

$$= \int_0^4 \mathrm{d}x \left[\frac{3a}{4}x^3 \;+\; \frac{b}{2}x\left\{\left(1+\frac{x}{2}\right)^2 - \left(1-\frac{x}{4}\right)^2\right\} \;+\; \frac{c}{3}\left\{\left(1+\frac{x}{2}\right)^3 - \left(1-\frac{x}{4}\right)^3\right\}\right].$$

Using the binomic formulae, we obtain

$$\begin{aligned}
\ldots \;&=\; \int_0^4 \mathrm{d}x \left[\frac{3a}{4}x^3 + \frac{b}{2}x\left\{\left(1+x+\frac{x^2}{4}\right) - \left(1-\frac{x}{2}+\frac{x^2}{16}\right)\right\}\right.\\
&\qquad\qquad \left. + \frac{c}{3}\left\{\left(1+\frac{3}{2}x+\frac{3}{4}x^2+\frac{x^3}{8}\right) - \left(1-\frac{3}{4}x+\frac{3}{16}x^2-\frac{x^3}{64}\right)^3\right\}\right]\\
&=\; \int_0^4 \mathrm{d}x \left[\left(\frac{3a}{4}+\frac{3b}{32}+\frac{3c}{64}\right)x^3 + \left(\frac{3b}{4}+\frac{3c}{16}\right)x^2 + \frac{3c}{4}x\right]\\
&=\; (48a + 6b + 3c) \;+\; (16b + 4c) \;+\; 6c\\
&=\; 48a + 22b + 13c.
\end{aligned}$$

Part (b) and (c), respectively:

$$\begin{aligned}
6\cdot\frac{1}{3}\sum_{i=1}^{3} f(x_i, y_i) \;&=\; 44.5a + 17.5b + 10.5c,\\
6\cdot f(x_M, y_M) \;&=\; \frac{128}{3}a + \frac{64}{3}b + \frac{32}{3}c.
\end{aligned}$$

Problem 2.6

(a) The area of Ω is

$$A_\Omega \;=\; \int_{-2}^{2} \mathrm{d}x\,(4-x^2) \;=\; 16 - \frac{16}{3} \;=\; \frac{32}{3}.$$

(b)

$$\begin{aligned}
\int_\Omega \mathrm{d}^2 r\, f(\mathbf{r}) \;&=\; \int_{-2}^{2}\mathrm{d}x \int_0^{4-x^2} \mathrm{d}y\,(x^2+y^2) \;=\; \int_{-2}^{2}\mathrm{d}x \Big[x^2 y + \frac{y^3}{3}\Big]_{y=0}^{y=4-x^2}\\
&=\; \int_{-2}^{2}\mathrm{d}x \Big[y\Big(x^2 + \frac{y^2}{3}\Big)\Big]_{y=0}^{y=4-x^2}\\
&=\; \int_{-2}^{2}\mathrm{d}x\,(4-x^2)\Big(x^2 + \frac{(4-x^2)^2}{3}\Big)\\
&=\; \frac{1}{3}\int_{-2}^{2}\mathrm{d}x\,(4-x^2)(x^4 - 5x^2 + 16)\\
&=\; \frac{1}{3}\int_{-2}^{2}\mathrm{d}x\,(-x^6 + 9x^4 - 36x^2 + 64) \;=\; \frac{1664}{35} \;=\; 47 + \frac{19}{35}.
\end{aligned}$$

(c) The wanted average value is

$$\Big\langle f(\mathbf{r})\Big\rangle_{\mathbf{r}\in\Omega} = \frac{1}{A_\Omega}\int_\Omega \mathrm{d}^2r\, f(\mathbf{r}) = \frac{3}{32}\cdot\frac{1664}{35} = \frac{156}{35} \approx 4.457.$$

Estimate:

$$\frac{1}{12}\cdot 2\Big[f(0.5|0.5)+f(0.5|1.5)+f(0.5|2.5)+f(0.5|3.5)+f(1.5|0.5)+f(1.5|1.5)\Big] = \frac{1}{6}\Big[0.5+2.5+6.5+12.5+2.5+4.5\Big] = \frac{29}{6} \approx 4.833.$$

(d) **Bonus:** Since the value $f(\mathbf{r}) = x^2+y^2$ is a monotonical function of $|\mathbf{r}| = \sqrt{x^2+y^2}$, the maximum of $f(x,y)$ on Ω is $f(0,4) = 16$ and the minimum is $f(0,0) = 0$.

Problem 2.7

(a)

$$\begin{aligned}
\int_\Omega \mathrm{d}^3r\, xz &= \int_0^a \mathrm{d}x\, x \int_0^{\sqrt{a^2-x^2}} \mathrm{d}y \int_0^{\sqrt{a^2-x^2-y^2}} \mathrm{d}z\, z \\
&= \int_0^a \mathrm{d}x\, x \int_0^{\sqrt{a^2-x^2}} \mathrm{d}y \left[\frac{z^2}{2}\right]_{z=0}^{z=\sqrt{a^2-x^2-y^2}} \\
&= \frac{1}{2}\int_0^a \mathrm{d}x\, x \int_0^{\sqrt{a^2-x^2}} \mathrm{d}y\, (a^2-x^2-y^2) \\
&= \frac{1}{2}\int_0^a \mathrm{d}x\, x \left[(a^2-x^2)y - \frac{y^3}{3}\right]_{y=0}^{y=\sqrt{a^2-x^2}} \\
&= \frac{1}{2}\int_0^a \mathrm{d}x\, x \left[(a^2-x^2)^{3/2} - \frac{(a^2-x^2)^{3/2}}{3}\right] \\
&= \frac{1}{6}\int_0^a \mathrm{d}x\, 2x\, (a^2-x^2)^{3/2} = \frac{1}{6}\left[-\frac{2}{5}(a^2-x^2)^{5/2}\right]_{x=0}^{x=a} = \frac{a^5}{15}.
\end{aligned}$$

(b)

$$\begin{aligned}
\int_\Omega \mathrm{d}^3r\, xyz &= \int_0^a \mathrm{d}x\, x \int_0^{\sqrt{a^2-x^2}} \mathrm{d}y\, y \int_0^{\sqrt{a^2-x^2-y^2}} \mathrm{d}z\, z \\
&= \frac{1}{2}\int_0^a \mathrm{d}x\, x \int_0^{\sqrt{a^2-x^2}} \mathrm{d}y\, y(a^2-x^2-y^2) \qquad \text{[as in part (a)]} \\
&= \frac{1}{2}\int_0^a \mathrm{d}x\, x \left[(a^2-x^2)\frac{y^2}{2} - \frac{y^4}{4}\right]_{y=0}^{y=\sqrt{a^2-x^2}} \\
&= \frac{1}{2}\int_0^a \mathrm{d}x\, x \left[\frac{(a^2-x^2)^2}{2} - \frac{(a^2-x^2)^2}{4}\right] \\
&= \frac{1}{16}\int_0^a \mathrm{d}x\, 2x\, (a^2-x^2)^2 = \frac{1}{16}\left[-\frac{(a^2-x^2)^3}{3}\right]_{x=0}^{x=a} = \frac{a^6}{48}.
\end{aligned}$$