Fakultät Mikrosystemtechnik
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Vector calculus and numerical mathematics

Worksheet 9

Problem 1: Linear ODEs

We consider an inhomogeneous linear ODE for a wanted function $f(x)$,

$$5f''(x) \;+\; 8f'(x) \;+\; 5f(x) \;=\; g(x),$$

where $g(x)$ is some given function.

(a) By educated guess, find some special solution $f^{\text{inh}}_{\text{spec}}(x)$ for each of the following cases.

  (i) $g(x) = 10$.

  (ii) $g(x) = 8\cos x$.

(b) Using the ansatz $f(x) = \mathrm{e}^{\lambda x}$, find the general solution to the corresponding homogeneous ODE.

(c) What is, as a consequence, the general solution of the inhomogeneous ODE?
Find the particular solution that satisfies the initial conditions $f(0) = f'(0) = 0$.

Problem 2: Harmonic oscillator

A harmonic oscillator with mass m, damping constant γ and spring constant k has the equation of motion

$$m\ddot{x}(t) \;=\; -k\,x(t) \;-\; \gamma\,\dot{x}(t) \;+\; F_0\cos(\Omega t),$$

where $F(t) = F_0\cos(\Omega t)$ is an external driving force. Choosing numerical values for the constants m, k, γ, F_0, how can the position $x(t)$ of the mass be found at a given time t, when the initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$ are given?
Compare with the solution of Problem 9.1(a)ii !

Problem 3: Divergence in spherical coordinates

In spherical coordinates (r, θ, ϕ), a vector field $\mathbf{F}(\mathbf{r})$ is given by

$$\mathbf{F}(\mathbf{r}) \;=\; F_r(r,\theta,\phi)\mathbf{u}_r \;+\; F_\theta(r,\theta,\phi)\mathbf{u}_\theta \;+\; F_\phi(r,\theta,\phi)\mathbf{u}_\phi,$$

where $(\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_\phi)$ are the unit vectors tangential to the coordinate lines running through the point $\mathbf{r}(r,\theta,\phi)$. Note that these "local basis vectors" are not the same at different points $\mathbf{r}(r,\theta,\phi)$ in space.

(a) A vector field $\mathbf{F}(\mathbf{r})$ is in cartesian coordinates given by

$$\mathbf{F}(\mathbf{r}) \;=\; x\mathbf{u}_x \,+\, y\mathbf{u}_y \,+\, z\mathbf{u}_z \;\equiv\; \mathbf{r}.$$

What is the divergence $\nabla \cdot \mathbf{F}(\mathbf{r})$ of this field?

(b) Express $\mathbf{F}(\mathbf{r})$ in terms of spherical coordinates (r, θ, ϕ), and verify the formula for the divergence in spherical coordinates,

$$\nabla \cdot \mathbf{F}(\mathbf{r}) \;=\; \frac{1}{r^2}\frac{\partial}{\partial r}\big[r^2 F_r\big] \,+\, \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\big[(\sin\theta)F_\theta\big] \,+\, \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}.$$

Problem 4: Diffusion equation in 3D

A simple solution of the diffusion equation $(\frac{\partial}{\partial t} - D\nabla^2)\rho(\mathbf{r}, t) = 0$ is (see lecture notes)

$$\rho(\mathbf{r}, t) \;\equiv\; \rho(x, y, z, t) \;=\; \frac{M}{(4\pi Dt)^{3/2}}\,\mathrm{e}^{-(x^2+y^2+z^2)/4Dt}.$$

In spherical coordinates,

$$\rho(\mathbf{r}, t) \;\equiv\; \tilde{\rho}(r, \theta, \phi, t) \;=\; \frac{M}{(4\pi Dt)^{3/2}}\,\mathrm{e}^{-r^2/4Dt}.$$

Using the continuity equation, find a possible current density $\mathbf{J}(\mathbf{r}, t)$ associated with this time-dependent density distribution.

Problem 5: Fourier Analysis

Find the coefficients α_n and β_n in the Fourier series

$$f_{\mathrm{S}}(z) \;=\; \frac{\alpha_0}{2} \,+\, \sum_{n=1}^{\infty}\Big[\alpha_n\cos(nz) \,+\, \beta_n\sin(nz)\Big]$$

for the function $f(z) = z$ (with $-\pi \le z \le \pi$).
Hint: $\int_a^b \mathrm{d}x\, x\,\sin(mx) \;=\; \Big[\frac{\sin(mx)}{m^2} \,-\, \frac{x\,\cos(mx)}{m}\Big]_a^b$.

Problem 6: Temperature Distribution (Lecture, section 10.2)

Consider a heat conducting plate covering the rectangle Σ on the xy-plane,

$$\Sigma \;=\; \Big\{(x, y) \in \mathbb{R}^2 \,\Big|\, 0 \le x \le a, \quad 0 \le y \le b\Big\}.$$

The steady-state temperature distribution $T(x, y)$ on this plate obeys Laplace's equation in 2D,

$$\nabla^2 T(x, y) \;=\; 0.$$

Using a separation ansatz $T(x, y) = X(x)Y(y)$, find the particular solution $T(x, y)$ of this partial differential equation (PDE) that satisfies the following boundary conditions,

$$T(x, b) = T_0 = 80, \qquad T(0, y) = T(x, 0) = T(a, y) = 0.$$

Problem 7: Vibrating String (Lecture, section 10.3)

The y-coordinate $u(x,t)$ of a vibrating string, fixed at $x = 0$ and at $x = L$ on the x-axis, satisfies the PDE (wave equation in 1D)

$$\frac{\partial^2}{\partial x^2} u(x,t) \;-\; \frac{1}{c^2}\frac{\partial^2}{\partial t^2} u(x,t) \;=\; 0.$$

(a) Using the separation ansatz $u_{\text{sep}}(x,t) = X(x)T(t)$, derive the two ODEs

$$X''(x) \;=\; \gamma\, X(x), \qquad \ddot{T}(t) \;=\; \gamma\, c^2\, T(t),$$

where $\gamma \in \mathbb{R}$ is some constant.

(b) Find the general solutions of these ODEs. Decide whether $\gamma > 0$ or $\gamma < 0$.

(c) Consider a superposition of solutions $u_{\text{sep}}(x,t)$ for different values of γ to find a Fourier expansion for the exact solution $u(x,t)$, fixed by the conditions

$$\begin{aligned} & & u(0,t) = u(L,t) = 0 \qquad & \text{(for all } t\text{)}, \\ u(x,0) = Y(x), & & \frac{\partial}{\partial t} u(x,t)\Big|_{t=0} = 0 \qquad & \text{(for all } x\text{)}, \end{aligned}$$

where $Y(x)$ is a given function with $Y(0) = Y(L) = 0$.