

Vector calculus and numerical mathematics

Worksheet 8

Problem 0: Nabla-Operator (completing problem 6 on WS 7)

Let $g(\mathbf{r})$ be a differentiable scalar field and \mathbf{a} be a constant vector. Show that divergence and curl of the vector field $\mathbf{F}(\mathbf{r}) = \mathbf{a}g(\mathbf{r})$ are given by

$$\nabla \cdot [\mathbf{a}g(\mathbf{r})] = \mathbf{a} \cdot \nabla g(\mathbf{r}), \quad \nabla \times [\mathbf{a}g(\mathbf{r})] = -\mathbf{a} \times \nabla g(\mathbf{r})$$

Problem 1: Laplacian and radial symmetry

A radially symmetric vector field $\mathbf{E}(\mathbf{r})$,

$$\mathbf{E}(\mathbf{r}) = E(r) \frac{\mathbf{r}}{r}, \quad r = |\mathbf{r}| \equiv (x^2 + y^2 + z^2)^{1/2},$$

is always a gradient field,

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r}), \quad \phi(\mathbf{r}) = f(r),$$

with a simple function $f : \mathbb{R}_0^+ \rightarrow \mathbb{R}$, $r \mapsto f(r)$.

- (a) Express the function $f(r)$ in terms of the given function $E(r)$.
- (b) Assuming that $\lim_{r \rightarrow 0} rf(r) = 0$, show that $\phi(\mathbf{r})$ has the Laplacian

$$\nabla^2\phi(\mathbf{r}) = f''(r) + \frac{2}{r}f'(r).$$

- (c) For a ball Ω (radius R , center at $\mathbf{r} = \mathbf{0}$, surface $\partial\Omega$), show that

$$\begin{aligned} \text{(i)} \quad \oint_{\partial\Omega} d\mathbf{S} \cdot \mathbf{E}(\mathbf{r}) &= -4\pi R^2 f'(R), \\ \text{(ii)} \quad \oint_{\partial\Omega} d\mathbf{S} \cdot \mathbf{E}(\mathbf{r}) &= -\int_0^R dr (4\pi r^2) \left[f''(r) + \frac{2}{r} f'(r) \right]. \end{aligned}$$

Hints: $\oint_{\Sigma} d\mathbf{A} \cdot \mathbf{E}(\mathbf{r}) = A_{\Sigma} \langle \mathcal{N}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) \rangle_{\mathbf{r} \in \Sigma}$. Have Gauss' theorem in mind.

- (d) Integrating by parts, show directly that

$$\int_0^R dr (4\pi r^2) \left[f''(r) + \frac{2}{r} f'(r) \right] = 4\pi R^2 f'(R).$$

Problem 2: Linear ODEs

Find the general real-valued solutions $f(x)$ of the following ODEs.

- (a) $f'''(x) + 9f''(x) + 26f'(x) + 24f(x) = 0$,
- (b) $f'''(x) - 11f''(x) + 55f'(x) - 125f(x) = 0$,
- (c) $f'''(x) - 2f''(x) - 5f'(x) + 10f(x) = 0$,
- (d) $f''(x) - f(x) = 0$,
- (e) $f''(x) + f(x) = 0$.

Hint: In all cases (a–e), the exponential ansatz $f(x) = e^{\lambda x}$ leads to an algebraic equation for $\lambda \in \mathbb{C}$. In cases (a–c), this equation is third-order: one of its three zeros can be guessed to be $\lambda_1 = -2$ (a), $\lambda_1 = 5$ (b) and $\lambda_1 = 2$ (c); the remaining two zeros λ_2 and λ_3 are then obtained via polynomial division. Cases (d) and (e) are (almost) trivial.

Problem 3: Wave equation

In vacuum, the electromagnetic fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ satisfy the 3D wave equation,

$$\nabla^2 u(\mathbf{r}, t) \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) u(\mathbf{r}, t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} u(\mathbf{r}, t), \quad (1)$$

where $u(\mathbf{r}, t) = u(x, y, z, t)$ is one of the six components

$$E_1(\mathbf{r}, t), \quad E_2(\mathbf{r}, t), \quad E_3(\mathbf{r}, t), \quad B_1(\mathbf{r}, t), \quad B_2(\mathbf{r}, t), \quad B_3(\mathbf{r}, t).$$

- (a) Consider the 1D version of the wave equation,

$$\frac{\partial^2}{\partial x^2} u(x, t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} u(x, t).$$

Given two differentiable functions $f(x)$ and $g(x)$, show that a solution is

$$u(x, t) = f(x - ct) + g(x + ct).$$

Choose f and g properly to obtain a solution of the form

$$u(x, t) = A \sin(kx - \omega t).$$

What is the value of the constant ω ?

- (b) For a given constant (wave) vector

$$\mathbf{k} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

and an arbitrary function $f(s)$ with first and second derivatives $f'(s)$ and $f''(s)$, choose a proper constant ω in such a way that the function

$$u(\mathbf{r}, t) = f(\mathbf{k} \cdot \mathbf{r} - \omega t) = f(k_1 x + k_2 y + k_3 z - \omega t)$$

is a solution of Eq. (1).