

Vector calculus and numerical mathematics

Worksheet 7

Problem 1: Proof of Gauss' theorem

(a) Considering the small cube

$$\Omega_a = \left\{ (x, y, z) \mid |x - x_0|, |y - y_0|, |z - z_0| \leq a \right\},$$

with side length $2a$ and centered at $\mathbf{r}_0 = (x_0, y_0, z_0)$, show that

$$\lim_{a \rightarrow 0} \frac{1}{V_{\Omega_a}} \oint_{\partial\Omega_a} d\mathbf{A} \cdot \mathbf{F}(\mathbf{r}) = \nabla \cdot \mathbf{F}(\mathbf{r}) \Big|_{\mathbf{r}=\mathbf{r}_0},$$

where $\mathbf{F}(\mathbf{r})$ is a differentiable vector field.

(b) In a second step, derive Gauss' theorem for any finite volume region Ω ,

$$\oint_{\partial\Omega} d\mathbf{A} \cdot \mathbf{F}(\mathbf{r}) = \int_{\Omega} d^3r \nabla \cdot \mathbf{F}(\mathbf{r}).$$

Problem 2: Proof of Stokes' Theorem

Find a strategy similar to the one of problem 1 to prove Stokes' Theorem.

Problem 3: Gauss and Stokes theorems

In cylindrical coordinates (s, ϕ, z) , we consider the cylinder (volume region)

$$\Omega = \left\{ \mathbf{r}(s, \phi, z) \mid s \leq R, \quad 0 \leq \phi \leq 2\pi, \quad 0 \leq z \leq H \right\}.$$

Let S be the side, T the top, and B the bottom surfaces of Ω .

(a) Which one of the following statements is true?

$$\partial\Omega = S \cup T \cup B, \quad B = \Omega \cap B, \quad S \cap T = \partial T, \quad \partial S = (T \cap S) \cup (B \cap S).$$

In addition, we consider the vector field

$$\mathbf{F}(\mathbf{r}) = \begin{pmatrix} Ax + By + Cz \\ Lx + My + Nz \\ Ux + Vy + Wz \end{pmatrix}.$$

(b) For this vector field $\mathbf{F}(\mathbf{r})$, verify Gauss' theorem for the volume region Ω .

(c) For the same field, verify Stokes' theorem for the pieces of surface B , T , and (!) S .

Problem 4: The scalar field $\phi(\mathbf{r}) = \frac{1}{|\mathbf{r}-\mathbf{r}_0|}$.

Use the explicit representation of $|\mathbf{r} - \mathbf{r}_0|$ in cartesian coordinates,

$$|\mathbf{r} - \mathbf{r}_0| = \left[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \right]^{1/2} \equiv f(x, y, z),$$

to prove the identities

$$\nabla \frac{1}{|\mathbf{r} - \mathbf{r}_0|} = -\frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3}, \quad \nabla \times \frac{\mathbf{a}}{|\mathbf{r} - \mathbf{r}_0|} = \frac{\mathbf{a} \times (\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^3},$$

where the differential operator ∇ is understood to act on the variables $\mathbf{r} = (x, y, z)$, while $\mathbf{r}_0 = (x_0, y_0, z_0)$ and $\mathbf{a} = (a_1, a_2, a_3)$ are constant vectors.

Problem 5: Laplacian (operator)

(a) For a differentiable scalar field $g(\mathbf{r}) = g(x, y, z)$, show that

$$\nabla \cdot [\nabla g(\mathbf{r})] = \nabla^2 g(\mathbf{r}),$$

where the **Laplacian** ∇^2 in cartesian coordinates reads $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

(b) Evaluate the Laplacian $\nabla^2 g(\mathbf{r})$ for the following scalar fields (where $A, B, C, D, E, F, R, \alpha, \beta$ are constants).

$$g_1(\mathbf{r}) = Ax^2 + By^2 - (A + B)z^2 + Dx + Ey + Fz + C,$$

$$g_2(\mathbf{r}) = A \sin(\alpha x) \sin(\beta y) e^{-\gamma z}, \quad \gamma = \sqrt{\alpha^2 + \beta^2},$$

$$g_3(\mathbf{r}) = \frac{A}{\sqrt{x^2 + y^2 + z^2}} \equiv A(x^2 + y^2 + z^2)^{-1/2} \quad (\mathbf{r} \neq \mathbf{0}),$$

$$g_4(\mathbf{r}) = (Ax + By + Cz)(x^2 + y^2 + z^2)^{-3/2} \quad (\mathbf{r} \neq \mathbf{0}),$$

$$g_5(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0} \begin{cases} \frac{3R^2 - r^2}{2R^3} & (r \leq R), \\ \frac{1}{r} & (r > R). \end{cases}$$

(c) Use the result of part (b) for $g_5(\mathbf{r})$, to verify the identity

$$\nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}_0|} = -4\pi \delta(\mathbf{r} - \mathbf{r}_0).$$

Problem 6: Second derivatives and integration by parts

(a) For a differentiable scalar field $g(\mathbf{r})$ and a differentiable vector field $\mathbf{F}(\mathbf{r})$, show that

$$\begin{aligned} \nabla \cdot [\nabla g(\mathbf{r})] &= \nabla^2 g(\mathbf{r}) \quad (\text{already shown!}), \\ \nabla \times [\nabla g(\mathbf{r})] &= \mathbf{0}, \\ \nabla \cdot [\nabla \times \mathbf{F}(\mathbf{r})] &= 0, \\ \nabla \times [\nabla \times \mathbf{F}(\mathbf{r})] &= \nabla[\nabla \cdot \mathbf{F}(\mathbf{r})] - \nabla^2 \mathbf{F}(\mathbf{r}). \end{aligned}$$

(b) Integrating by parts, show that

$$\int d^3r \mathbf{F}(\mathbf{r}) \cdot [\nabla g(\mathbf{r})] = - \int d^3r [\nabla \cdot \mathbf{F}(\mathbf{r})] g(\mathbf{r}),$$

provided that the vector field $\mathbf{P}(\mathbf{r}) = g(\mathbf{r}) \mathbf{F}(\mathbf{r})$ goes to zero as $|\mathbf{r}| \rightarrow \infty$.