Fakultät Mikrosystemtechnik
PD Dr. Michael Seidl

Sommersemester 2019
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Vector calculus and numerical mathematics

Worksheet 5

Problem 1

Cylindrical polar coordinates (s, ϕ, z) are given by

$$\mathbf{r}(\alpha) \;\equiv\; \mathbf{r}(s,\phi,z) \;=\; \begin{pmatrix} s\cos\phi \\ s\sin\phi \\ z \end{pmatrix}.$$

(a) Find the tangential vectors $\mathbf{h}_s(\alpha)$, $\mathbf{h}_\phi(\alpha)$, $\mathbf{h}_z(\alpha)$ to the coordinate lines.

(b) Draw the intersection curve of the surface

$$\Sigma \;=\; \left\{ \mathbf{r}(s_0,\phi,z) \,\middle|\, \frac{\pi}{4} \le \phi \le \frac{\pi}{2},\;\; -a \le z \le a \right\} \qquad (s_0, a > 0)$$

with the xy-plane. (Choose the value $s_0 = 3$.)

(c) Evaluate the flux integral $\int_\Sigma \mathrm{d}\mathbf{A}\cdot\mathbf{F}(\mathbf{r})$ of the vector field $\mathbf{F}(\mathbf{r})$ through Σ,

$$\mathbf{F}(\mathbf{r}) \;=\; \begin{pmatrix} 5x - 7z \\ 3x + 5y \\ 6y + 2z \end{pmatrix}.$$

Problem 2

(a) Show that the following vectors are mutually orthogonal.

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

Find the magnitudes $|\mathbf{a}|$, $|\mathbf{b}|$, and $|\mathbf{c}|$.

(b) A special set of “curvilinear” coordinates (u, v, w) is defined by

$$\mathbf{r}(\alpha) \;\equiv\; \mathbf{r}(u,v,w) \;\equiv\; \begin{pmatrix} x(u,v,w) \\ y(u,v,w) \\ z(u,v,w) \end{pmatrix} \;=\; u\,\mathbf{a} \;+\; v\,\mathbf{b} \;+\; w\,\mathbf{c}.$$

Find the tangential vectors $\mathbf{h}_u(\alpha)$, $\mathbf{h}_v(\alpha)$, $\mathbf{h}_w(\alpha)$ to the coordinate lines.

(c) What is the geometrical shape of the following piece of surface Σ?

$$\Sigma \;=\; \left\{ \mathbf{r}(u,v,w) \,\middle|\, 0 \le u \le 1,\;\; 0 \le v \le 1,\;\; w = 0 \right\}.$$

(d) For the vector field $\mathbf{F}(\mathbf{r})$ of problem 1, evaluate the flux integral

$$I \;=\; \int_\Sigma \mathrm{d}\mathbf{A}\cdot\mathbf{F}(\mathbf{r}).$$

Problem 3: Divergence

(a) Evaluate the divergence $\nabla \cdot \mathbf{J}(\mathbf{r})$ of the 2D vector field (with constants $a, c > 0$)

$$\mathbf{J}(\mathbf{r}) = \begin{pmatrix} J_1(x,y) \\ 0 \end{pmatrix}, \qquad J_1(x,y) = \frac{cx}{a^2 + x^2 + y^2}.$$

(b) Choose $a = c = 1$ and sketch the vectors $\mathbf{J}(\mathbf{r}) \equiv \mathbf{J}(x,y)$ for $x \in \{-3,-2,-1,0,1,2,3\}$ and $y \in \{0,1,2\}$ in the xy-plane (unit: 2 cm).

(c) What are the physical dimensions of the constants $a > 0$ and $c > 0$, when $\mathbf{J}(\mathbf{r})$ is the 2D (mass) current density of CO_2-gas?

(d) Where in the xy-plane are CO_2 sources, where are sinks?

Problem 4: Gauss' theorem

(a) Evaluate the flux integral $\int_\Sigma \mathrm{d}\mathbf{A} \cdot \mathbf{F}(\mathbf{r})$ of the vector field

$$\mathbf{F}(\mathbf{r}) \;=\; \begin{pmatrix} 2x + 3y \\ 5x - 4y \\ 7z \end{pmatrix}$$

out of the following piece Σ of a spherical surface,

$$\Sigma \;=\; \Big\{ \mathbf{r}(r,\theta,\phi) \,\Big|\, r = R,\;\; 0 \le \phi \le \phi_0,\;\; 0 \le \theta \le \pi \Big\} \qquad (R, \phi_0 > 0).$$

What do you find in the special case $\phi_0 = 2\pi$?

(b) Evaluate the volume integral

$$\int_\Omega \mathrm{d}^3 r \, \big[\nabla \cdot \mathbf{F}(\mathbf{r})\big]$$

of the divergence $\nabla \cdot \mathbf{F}(\mathbf{r})$ over the full sphere Ω with center $\mathbf{r} = \mathbf{0}$ and radius R.

Problem 5: Gauss' theorem (II)

Consider a vector field $\mathbf{F}(\mathbf{r})$ and, in spherical coordinates (r,θ,ϕ), a volume region Ω ("upper“ half of a ball with radius R),

$$\mathbf{F}(\mathbf{r}) \;=\; \begin{pmatrix} 5x - 3y \\ 2x + 8y \\ \frac{4}{L} x^2 \end{pmatrix}, \qquad \Omega = \Big\{ \mathbf{r}(r,\theta,\phi) \,\Big|\, 0 \le r \le R,\; 0 \le \theta \le \frac{\pi}{2},\; 0 \le \phi \le 2\pi \Big\}.$$

(a) Evaluate the flux integral of $\mathbf{F}(\mathbf{r})$ out of the surface $\partial\Omega$ of Ω: $\quad \oint_{\partial\Omega} \mathrm{d}\mathbf{A} \cdot \mathbf{F}(\mathbf{r})$.

(b) Verify Gauss' theorem by evaluating the volume integral: $\quad \int_\Omega \mathrm{d}^3 r \, \big[\nabla \cdot \mathbf{F}(\mathbf{r})\big]$.

*Problem 6

For a differentiable vector field $\mathbf{F}(\mathbf{r})$, show that

$$\lim_{a \to 0} \frac{1}{(2a)^3} \oint_{\partial\Omega_a} \mathrm{d}\mathbf{A} \cdot \mathbf{F}(\mathbf{r}) \;=\; \nabla \cdot \mathbf{F}(\mathbf{r})\Big|_{\mathbf{r}=\mathbf{r}_0},$$

where Ω_a is a cube with center at $\mathbf{r}_0$ and edges of length $2a$ in x-, y-, and z-directions.

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Worksheet 6

Problem 1: Chain rule for partial derivatives

Let $f(\mathbf{r}) \equiv f(x, y, z)$ be a **scalar field** and $\mathbf{r}(\phi)$ be a **vector function** of one variable ϕ,

$$\mathbf{r}(\phi) \;=\; \begin{pmatrix} x(\phi) \\ y(\phi) \\ z(\phi) \end{pmatrix}.$$

Chain rule for partial derivatives: The function $\tilde{f}(\phi) = f(\mathbf{r}(\phi))$ has the derivative

$$\begin{aligned} \tilde{f}'(\phi) &\equiv \frac{\mathrm{d}}{\mathrm{d}\phi} f\big(\mathbf{r}(\phi)\big) \\ &= \dot{\mathbf{r}}(\phi) \cdot \nabla f(\mathbf{r})\Big|_{\mathbf{r}=\mathbf{r}(\phi)} \;\equiv\; \left[\dot{x}(\phi)\frac{\partial f(\mathbf{r})}{\partial x} \;+\; \dot{y}(\phi)\frac{\partial f(\mathbf{r})}{\partial y} \;+\; \dot{z}(\phi)\frac{\partial f(\mathbf{r})}{\partial z}\right]_{\mathbf{r}=\mathbf{r}(\phi)}. \end{aligned}$$

(a) Verify this rule for the two scalar fields $f(x, y, z) = xy^2$ and $g(x, y, z) = x^2 + y^2$ and the vector function

$$\mathbf{r}(\phi) \;=\; \begin{pmatrix} R\cos\phi \\ R\sin\phi \\ 0 \end{pmatrix}.$$

(b) Use this chain rule to prove the formula: $\quad \int_{\Gamma} \mathrm{d}\mathbf{r} \cdot \nabla f(\mathbf{r}) \;=\; f(\mathbf{r}_B) - f(\mathbf{r}_A)$.

(c) Find a general proof for the chain rule.

Problem 2: Estimating flux integrals

Consider the following vector fields,

$$\mathbf{B}(\mathbf{r}) \;=\; \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}, \qquad \mathbf{E}(\mathbf{r}) \;=\; \begin{pmatrix} 3 \\ -8 \\ 4 \end{pmatrix}, \qquad \mathbf{V}(\mathbf{r}) \;=\; \begin{pmatrix} x\,f(z) \\ y\,f(z) \\ g(z) \end{pmatrix}.$$

In cylindrical coordinates (s, ϕ, z), we define a piece Σ of curved surface,

$$\Sigma \;=\; \Big\{\mathbf{r}(s, \phi, z) \;\Big|\; s = R, \quad \alpha - \delta \le \phi \le \alpha + \delta, \quad 0 \le z \le H\Big\}.$$

(a) Estimate the value of the flux integral $\int_{\Sigma} \mathrm{d}\mathbf{A} \cdot \mathbf{F}(\mathbf{r})$ in these cases as a function of α, δ, and H, assuming that $\delta \ll \frac{\pi}{2}$.

(b) Evaluate these integrals exactly. Note that

$$\begin{aligned} \sin(\alpha + \delta) - \sin(\alpha - \delta) &= 2\cos\alpha\sin\delta, \\ \cos(\alpha + \delta) - \cos(\alpha - \delta) &= -2\sin\alpha\sin\delta. \end{aligned}$$

Problem 3: Velocity field of a rotating solid

Considering the velocity field $\mathbf{v}(\mathbf{r}) = \boldsymbol{\omega} \times \mathbf{r}$, with a given constant vector $\boldsymbol{\omega}$ of the angular velocity, show that the circulation around a planar disk Σ with radius a and unit normal vector $\boldsymbol{\mathcal{N}}_\Sigma$ is given by

$$\frac{1}{\pi a^2} \oint_{\partial\Sigma} \mathrm{d}\mathbf{r} \cdot \mathbf{v}(\mathbf{r}) \;=\; 2\boldsymbol{\omega} \cdot \boldsymbol{\mathcal{N}}_\Sigma.$$

Problem 4: Example for Stokes' Theorem

A piece of surface in 3D space is given by

$$\Sigma = \Big\{ \mathbf{r}_0 + \lambda\mathbf{p} + \mu\mathbf{q} \;\Big|\; |\lambda| \le a,\ |\mu| \le b \Big\},$$

with two numbers $a, b > 0$ and the vectors

$$\mathbf{r}_0 \;=\; \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}, \qquad \mathbf{p} \;=\; \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{q} \;=\; \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(a) Find area A_Σ and unit normal vector $\mathbf{n}_\Sigma$ of Σ.

(b) As a function of a and b, evaluate the circulation $\oint_{\partial\Sigma} \mathrm{d}\mathbf{r} \cdot \mathbf{F}(\mathbf{r})$ of the vector field

$$\mathbf{F}(\mathbf{r}) \;=\; \begin{pmatrix} F_1(\mathbf{r}) \\ F_2(\mathbf{r}) \\ F_3(\mathbf{r}) \end{pmatrix} \;=\; \begin{pmatrix} 4x - 8y + 7z \\ 3x + 5y - 9z \\ 8x - 2y + 6z \end{pmatrix}.$$

(c) Evaluate the following limit and give an interpretation

$$\lim_{A_\Sigma \to 0} \frac{1}{A_\Sigma} \oint_{\partial\Sigma} \mathrm{d}\mathbf{r} \cdot \mathbf{F}(\mathbf{r}).$$

(d) Evaluate $\nabla \times \mathbf{F}(\mathbf{r})$ and use the result of part (b) to verify Stokes' Theorem.

Problem 5: Another example for Stokes' Theorem

Verify Stokes' theorem,

$$\oint_{\partial\Sigma} \mathrm{d}\mathbf{r} \cdot \mathbf{F}(\mathbf{r}) \;=\; \int_\Sigma \mathrm{d}\mathbf{A} \cdot \big[\nabla \times \mathbf{F}(\mathbf{r})\big],$$

for the vector field $\mathbf{F}(\mathbf{r})$ of problem 4(b) and the **semi**-sphere given by

$$\Sigma \;=\; \Big\{ \mathbf{r}(r, \theta, \phi) \;\Big|\; r = R,\ \ 0 \le \theta \le \tfrac{\pi}{2},\ \ 0 \le \phi \le 2\pi \Big\},$$

where (r, θ, ϕ) are the usual spherical polar coordinates.