Fakultät Mikrosystemtechnik
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Vector calculus and numerical mathematics

Worksheet 4

Problem 1: Double integral, area, volume, and average value

For the region $\Omega = \{(x,y)\,|\,0 \le x \le a,\ 0 \le y \le b\}$ in the xy-plane, with two constants $a > 0$ and $b > 0$, evaluate the double integral

$$I \;=\; \int_\Omega \mathrm{d}^2 r\, f(\mathbf{r})$$

for the following functions, and give a geometrical interpretation.

(a) $f(\mathbf{r}) \equiv f(x,y) = 1$,

(b) $f(\mathbf{r}) \equiv f(x,y) = c$, with a constant $c > 0$,

(c) $f(\mathbf{r}) \equiv f(x,y) = \frac{x}{1+y}$.

Problem 2: Gradient

Which ones of these vector fields are gradients of a scalar field?

$$\text{(a)}\quad \mathbf{v}(\mathbf{r}) = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}, \qquad \text{(b)}\quad \mathbf{p}(\mathbf{r}) = \begin{pmatrix} y \\ x \\ z \end{pmatrix}, \qquad \text{(c)}\quad \mathbf{q}(\mathbf{r}) = \begin{pmatrix} xy \\ yz \\ zx \end{pmatrix}.$$

Occasionally, can you guess the explicit expression for such a scalar field?

Problem 3: Curl

Evaluate the curl

$$\nabla \times \mathbf{B}(\mathbf{r}) = \begin{pmatrix} \partial_2 B_3 - \partial_3 B_2 \\ \partial_3 B_1 - \partial_1 B_3 \\ \partial_1 B_2 - \partial_2 B_1 \end{pmatrix}$$

(short-hand notation: $\partial_n f := \frac{\partial f}{\partial x_n}$, $x_1 := x, x_2 := y, x_3 := z$) of the following vector fields,

$$\text{(a)}\quad \mathbf{B}(\mathbf{r}) \;=\; \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}, \qquad \text{(b)}\quad \mathbf{B}(\mathbf{r}) \;=\; \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \;\equiv\; \begin{pmatrix} -\frac{y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \\ 0 \end{pmatrix}.$$

Problme 4: Line integrals

Consider the 2D vector field $\mathbf{F}(\mathbf{r})$ and the 2D vector function $\mathbf{r}(\phi)$,

$$\mathbf{F}(\mathbf{r}) = \begin{pmatrix} F_1(x,y) \\ F_2(x,y) \end{pmatrix} = \begin{pmatrix} \frac{y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{pmatrix}, \qquad \mathbf{r}(\phi) = \begin{pmatrix} x(\phi) \\ y(\phi) \end{pmatrix} = \begin{pmatrix} R\cos\phi \\ R\sin\phi \end{pmatrix}.$$

(a) Compute the vectors $\mathbf{F}(\mathbf{r})$ for the eight points

$$\mathbf{r} \;=\; (x|y) \;=\; (1|0),\ (2|0),\ (0|1),\ (1|1),\ (2|1),\ (0|2),\ (1|2),\ (2|2),$$

and draw them as arrows at the coresponding positions of the xy-plane.

(b) In the diagram of part (a), sketch the curve Γ_1

$$\Gamma_1 \;=\; \left\{\mathbf{r}(\phi)\,\middle|\, 0 \le \phi \le \frac{\pi}{4}\right\}, \qquad R = 2.$$

Decide qualitatively whether the line integral $\int_{\Gamma_1} \mathrm{d}\mathbf{r}\cdot\mathbf{F}(\mathbf{r})$ is > 0, $= 0$, or < 0.

(c) Evaluate this line integral exactly, using the formula from the lecture,

$$\int_{\Gamma_1} \mathrm{d}\mathbf{r}\cdot\mathbf{F}(\mathbf{r}) \;=\; \int_0^{\pi/4} \mathrm{d}\phi\,\Big[\dot{\mathbf{r}}(\phi)\cdot\mathbf{F}(\mathbf{r}(\phi))\Big].$$

(d) Repeat parts (b) and (c) for the curve

$$\Gamma_2 \;=\; \left\{\mathbf{r}(u)\,\middle|\, 1 \le u \le 2\right\}, \qquad \mathbf{r}(u) \;=\; \begin{pmatrix} u \\ u \end{pmatrix}.$$

(e) Repeat parts (b) and (c) for the curve

$$\Gamma_3 \;=\; \left\{\mathbf{r}(u)\,\middle|\, \frac{1}{2} \le u \le 2\right\}, \qquad \mathbf{r}(u) \;=\; \begin{pmatrix} u \\ 1/u \end{pmatrix}.$$

Problem 5: Line integrals (II)

(a) Show that the vector field

$$\mathbf{G}(\mathbf{r}) = \begin{pmatrix} x^2+y^2 \\ 2xy \end{pmatrix}$$

is the gradient of the 2D scalar field $f(\mathbf{r}) = \frac{1}{3}x^3 + xy^2$.

(b) Evaluate the line integrals $\int_\Gamma \mathrm{d}\mathbf{r}\cdot\mathbf{G}(\mathbf{r})$ of the vector field $\mathbf{G}(\mathbf{r})$ over the three curves $\Gamma = \{\mathbf{r}(u)\,|\,u\in[0,1]\}$ parametrized by $\mathbf{r}(u) = \binom{x(u)}{y(u)}$,

$$\text{(i)}\ \ \mathbf{r}(u) = \begin{pmatrix} 3u^2 \\ 2u^3 \end{pmatrix}, \qquad \text{(ii)}\ \ \mathbf{r}(u) = \begin{pmatrix} -6u^3 \\ 5u \end{pmatrix}, \qquad \text{(iii)}\ \ \mathbf{r}(u) = \begin{pmatrix} 3u \\ 8u^2 \end{pmatrix}.$$

(c) What are the values of the scalar field $f(\mathbf{r})$ at the starting and end points of these three curves? Compare with the line integrals from part (b)!