Fakultät Mikrosystemtechnik
PD Dr. Michael Seidl

Sommersemester 2019
1 April 2019

Vector calculus and numerical mathematics

Worksheet 3

Problem 1

(a) Find the area A_R of a circle Ω_R with radius R in the xy-plane by evaluating the integral $\int_{\Omega_R} \mathrm{d}^2 r\, 1$, using

(i) cartesian coordinates (x, y) [hint: $\int \mathrm{d}x \sqrt{a^2 - x^2} = \frac{1}{2}(x\sqrt{a^2 - x^2} + a^2 \arcsin \frac{x}{a})$],

(ii) planar polar coordinates (r, ϕ).

(b) Draw a sketch of the region

$$\Omega \;=\; \left\{ \mathbf{r}(r, \phi) \in \mathbb{R}^2 \,\middle|\, 0 \leq \phi < 2\pi, \quad 0 \leq r \leq 3 + \cos 4\phi \right\}$$

in the xy-plane. Find its area $\int_\Omega \mathrm{d}^2 r\, 1$.

Problem 2: Gaussian integrals

(a) For which values of $n \in \{0, 1, 2, 3, ...\}$ can you evaluate the definite integral

$$I_n \;=\; \int_{-\infty}^{\infty} \mathrm{d}x\, x^n \mathrm{e}^{-x^2}$$

in an elementary way?

(b) Using planar polar coordinates (r, ϕ), evaluate the double integral

$$K_R \;=\; \int_{\Omega_R} \mathrm{d}^2 r\, f(\mathbf{r})$$

of the scalar field $f(\mathbf{r}) \equiv f(x, y) = \mathrm{e}^{-(x^2+y^2)}$ over the disk

$$\Omega_R \;=\; \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 \leq R^2 \right\}.$$

(c) Using cartesian coordinates (x, y), show that

$$\sqrt{\lim_{R\to\infty} K_R} \;=\; \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-x^2} \;=\; I_0.$$

(d) Using the result of part (c), evaluate the integral

$$I_2 \;=\; \int_{-\infty}^{\infty} \mathrm{d}x\, x^2 \mathrm{e}^{-x^2}.$$

Problem 3: Spherical polar coordinates

Consider the cube $\Omega = \{(x|y|z) \in \mathbb{R}^3 \,|\, 0 \leq x, y, z \leq a\}$ with side a.
Its floor $ABCD$ has the corners $A(0|0|0)$, $B(a|0|0)$, C, and $D(0|a|0)$,
its ceiling $EFGH$ has the corners $E(0|0|a)$, $F(a|0|a)$, G, and $H(0|a|a)$.
Complete the following table of the cartesian (x, y, z) and the spherical polar coordinates (r, θ, ϕ) of each corner of the cube.

	x	y	z	r	θ	ϕ
A						
B						
$\vdots$						
H						

Problem 4: Moments of inertia

Consider a solid, consisting of N point masses m_n $(n = 1, ..., N)$ that are connected by massless rigid rods. Rotating about a given axis, the moment of inertia of this solid is

$$I = \sum_{n=1}^{N} m_n a_n^2,$$

where a_n is the distance of m_n from the axis. For a continuous solid with mass density $\rho(\mathbf{r})$, this expression becomes a triple integral,

$$I = \int_\Omega \mathrm{d}^3 r \, \rho(\mathbf{r}) \, a(\mathbf{r})^2,$$

where Ω is the volume region covered by the solid and $a(\mathbf{r})$ is the distance of point $\mathbf{r}$ from the axis.

(a) Express $a(\mathbf{r})$ in cartesian coordinates in the case of rotation about the z-axis.

(b) Find I for a sphere with uniform mass distribution, $\rho(\mathbf{r}) \equiv \rho_0 = \frac{3M}{4\pi R^3}$.

(c) Find I for a cylinder (radius R, height H, centered at the origin) for rotation about an axis **perpendicular** to its axis of symmetry.

Problem 5

(Cf. problem 7 of worksheet 2!) In 3D xyz-space, we consider the volume region

$$\Omega = \left\{ (x, y, z) \,\middle|\, x, y, z \geq 0 \quad \text{and} \quad x^2 + y^2 + z^2 \leq a^2 \right\}.$$

Evaluate the triple integrals $\int_\Omega \mathrm{d}^3 r \, f(\mathbf{r})$ for the following scalar fields, using spherical polar coordinates (r, θ, ϕ).

(a) $f(x, y, z) = xz$.

(b) $f(x, y, z) = xyz$.