Fakultät Mikrosystemtechnik
PD Dr. Michael Seidl

Sommersemester 2019
18 March 2019

Vector calculus and numerical mathematics

Worksheet 1

Problem 1: Vectors

The position vectors of the four corners of a regular tetrahedron (e.g., a methane molecule CH_4) are given as

$$\mathbf{r}_1 = \begin{pmatrix} x_1 \\ 0 \\ z_1 \end{pmatrix}, \quad \mathbf{r}_{2,3} = \begin{pmatrix} x_2 \\ \pm y_2 \\ z_1 \end{pmatrix}, \quad \mathbf{r}_4 = \begin{pmatrix} 0 \\ 0 \\ \ell \end{pmatrix} \qquad (\ell = 1.087\,\text{Å}, \quad x_1 > 0).$$

(a) Given that $\ell, x_1 > 0$, what are the signs of the constants x_2 and z_1 ?

(b) Utilizing the tetrahedral symmetry, find the values of x_1, x_2, y_2, and z_1.

(c) Find side length a and bond angle θ of a CH_4 molecule.

Problem 2: Vector product

Writing $\mathbf{a} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3$ and $\mathbf{b} = b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + b_3\mathbf{u}_3$, show that

$$\mathbf{a} \times \mathbf{b} \equiv \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

Fakultät Mikrosystemtechnik
PD Dr. Michael Seidl

Sommersemester 2019
25 March 2019

Vector calculus and numerical mathematics

Worksheet 2

Problem 1: Vectors of velocity and acceleration

The motion of a particle is described by the vector function

$$\mathbf{r}(t) \equiv \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} R\cos(\frac{1}{2}\alpha t^2) \\ R\sin(\frac{1}{2}\alpha t^2) \\ 0 \end{pmatrix},$$

representing the position vector $\mathbf{r}$ of the particle as a function of time t.

(a) What are the physical dimensions (m, s, kg, etc.) of the two constant parameters R and α ?

(b) What is the geometrical shape of this particle's orbit ?
By geometrical intuition, find the cartesian coordinates of unit vectors $\boldsymbol{\mathcal{T}}(t)$ tangential and $\boldsymbol{\mathcal{N}}(t)$ normal to the orbit at the time t. Are these vectors defined uniquely ?

(c) Find the velocity vector $\mathbf{v}(t)$ of the particle at the time t.

(d) Find the accelration vector $\mathbf{a}(t)$ of the particle at the time t.
What are the tangential and the normal components of $\mathbf{a}(t)$?
Discuss the force $\mathbf{F}(t)$ acting on the particle, according to Newton's law $\mathbf{F} = m\mathbf{a}$.

Problem 2: Scalar field

A rectangular metal plate covers the region $\{|x| \le a,\ 0 \le y \le b\}$ on the xy-plane, with two given lengths a and b. As a model for the temperature distribution on the plate, consider the function

$$T(x,y) = T_0 \frac{y}{y + c(a^2 - x^2)(b - y)} \qquad (T_0, c > 0). \tag{1}$$

(a) What are the physical units of T_0 and $c = \frac{1}{\ell^2}$?

(b) Show that $T(x,y)$ is constant on each one of the four edges of the plate.

(c) Draw a contour plot of $T(x,y)$.

(d) Evaluating partial derivatives, find the gradient of the 2D scalar field $T(x,y)$,

$$\mathbf{G}_T(x,y) \equiv \nabla T(x,y) = \begin{pmatrix} T_x(x,y) \\ T_y(x,y) \end{pmatrix}.$$

Evaluate the vector $\mathbf{G}_T(x,y)$ for $(x,y) = (0.3a, 0.5b)$, $(0.5a, 0.5b)$, and $(0.8a, 0.5b)$. Enter these vectors as arrows at the corresponding positions in the contour plot.

Problem 3: Surface integral

Consider the 2D scalar field (with constants a, b, c),

$$f(x,y) \;=\; c - ax^2 - by^2.$$

(a) Sketch the triangle Ω with the corners $A(-2|0)$, $B(1|3)$ and $C(1|-3)$ in the xy-plane.

(b) Evaluate the surface integral

$$\int_\Omega \mathrm{d}^2r\, f(\mathbf{r}).$$

Problem 4

(a) Evaluate the 2D surface integral $\int_\Sigma \mathrm{d}^2r\, f(\mathbf{r})$ of the scalar field

$$f(x,y) \;=\; ax - by$$

(with constants $a, b > 0$) over the domain

$$\Sigma \;=\; \Big\{(x,y) \in \mathbb{R}^2 \,\Big|\, 0 \le x \le 1, \qquad 0 \le y \le 1 - x\Big\}.$$

(b) Find the volume $V = \int_\Omega \mathrm{d}^3r 1$ of the 3D region

$$\Omega \;=\; \Big\{(x,y,z) \in \mathbb{R}^3 \,\Big|\, (x,y) \in \Sigma, \qquad 0 \le z \le f(x,y)\Big\}.$$

Problem 5

Let $\Omega \subset \mathbb{R}^2$ be the triangle with corners $A(4|0)$, $B(4|3)$, and $C(0|1)$ in the xy-plane.

(a) Evaluate the surface integral

$$\int_\Omega \mathrm{d}^2r\, f(x,y)$$

for the (linear!) function $f(x,y) = ax + by$, where $a, b \in \mathbb{R}$ are constant numbers. To find the proper integration limits, first sketch the triangle Ω in the xy-plane.

(b) To check your result, mark in your sketch the points $P_i(x_i|y_i)$ with the coordinates

$$(x_1|y_1) = (1|1), \qquad (x_2|y_2) = (\tfrac{7}{2}|\tfrac{1}{2}), \qquad (x_3|y_3) = (3|2),$$

and evaluate the function values $f(x_i, y_i)$ for $i \in \{1,2,3\}$. Why can we expect that

$$\int_\Omega \mathrm{d}^2r\, f(x,y) \;\approx\; A_\Omega \cdot \frac{1}{3}\sum_{i=1}^{3} f(x_i, y_i),$$

where A_Ω is the area of the triangle Ω?

(c) In terms of the position vectors $\mathbf{r}_A$, $\mathbf{r}_B$, and $\mathbf{r}_C$ of its corners, the position vector $\mathbf{r}_M$ of the triangle's center of mass M is given by the vector sum $\mathbf{r}_M = \frac{1}{3}(\mathbf{r}_A + \mathbf{r}_B + \mathbf{r}_C)$, with coordinates x_M and y_M. Compare the value of the integral $\int_\Omega \mathrm{d}^2 r\, f(x, y)$ with

$$A_\Omega \cdot f(x_M, y_M).$$

(d) Repeat parts (a), (b), and (c) for the quadratic function $f(x, y) = ax^2 + bxy + cy^2$. Hint: You might need the binomic formula $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

Problem 6

In the xy-plane, we consider the region

$$\Omega \;=\; \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; 0 \leq y \leq 4 - x^2, \quad |x| \leq 2 \right\}.$$

(a) Draw a sketch of Ω. By elementary integration, find the area A_Ω of Ω.

(b) Evaluate the surface integral

$$\int_\Omega \mathrm{d}^2 r\, f(\mathbf{r})$$

for the function $f(x, y) = x^2 + y^2$.

(c) What is the average value $\langle f(\mathbf{r}) \rangle_{\mathbf{r} \in \Omega}$ of this function in Ω?

(d) **Bonus:** Find the minimum and maximum values of $f(\mathbf{r})$ for $\mathbf{r} \in \Omega$ without performing any calculation. Hint: Make a rough sketch of the 3D-plot of f!

Problem 7

In 3D xyz-space, we consider the volume region ($\frac{1}{8}$ of a sphere with radius a!)

$$\Omega \;=\; \left\{ (x, y, z) \;\middle|\; x, y, z \geq 0 \quad \text{and} \quad x^2 + y^2 + z^2 \leq a^2 \right\}.$$

Evaluate the volume integrals $\int_\Omega \mathrm{d}^3 r\, f(\mathbf{r})$ for the following scalar fields.

(a) $f(x, y, z) = xz$.

(b) $f(x, y, z) = xyz$.