Fakultät Mikrosystemtechnik
PD Dr. Michael Seidl

Sommersemester 2019
3 July 2019

# Vector calculus and numerical mathematics

# Worksheet 12

---

## Problem 1: ODEs

Find infinitely many solutions to each one of the following ODEs.

(a) $f''(x) \, - \, 5f'(x) \, + \, 4f(x) \, = \, 0.$

(b) $f''(x) \, + \, 6f'(x) \, + \, 25f(x) \, = \, 0.$

(c) $f'(x) \, = \, x^2 \, f(x)^5.$

(d) $f'(x) \, = \, 3x^2 \big[ 1 \, + \, f(x)^2 \big].$

(e) $f'(x) \, = \, 3x^2 \, + \, f(x).$

(f) $f'''(x) \, = \, f(x).$

## Problem 2: FEM integrals

Given is a triangle $\Sigma$ in the $xy$-plane,

$$\Sigma \; = \; \Big\{ \mathbf{r} = (x,y) \; \Big| \; 0 \, \le \, x \, \le \, 4, \quad 0 \, \le \, y \, \le \, 4-x \Big\},$$

and three $\mathbb{R}^2 \to \mathbb{R}$ functions $w_1$, $w_2$, and $w_3$,

$$\begin{aligned} w_1(\mathbf{r}) &= 2x + 3y, \\ w_2(\mathbf{r}) &= 4x - 5y, \\ w_3(\mathbf{r}) &= 7xy. \end{aligned}$$

Compute the following integrals.

(a) $\displaystyle\int_\Sigma \mathrm{d}^2 r \, w_1(\mathbf{r}).$

(b) $\displaystyle\int_\Sigma \mathrm{d}^2 r \, \big[ \nabla w_1(\mathbf{r}) \big] \cdot \big[ \nabla w_2(\mathbf{r}) \big].$

(c) $\displaystyle\int_\Sigma \mathrm{d}^2 r \, \big[ \nabla w_1(\mathbf{r}) \big] \cdot \big[ \nabla w_3(\mathbf{r}) \big].$

## Problem 3: Numerical solution of Laplace's equation

Consider the 2D heat conduction problem

$$\nabla^2 T(x,y) \;\equiv\; \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) T(x,y) \;=\; 0,$$

on a rectangular plate

$$\Omega = \Big\{(x,y) \;\Big|\; 0 \le x \le a,\; 0 \le y \le b\Big\},$$

with the **boundary conditions** (BCs)

$$\left.\begin{array}{rcl} T(a,y) &=& T_1 \\ T(x,b) &=& T_2 \\ T(0,y) &=& T_3 \\ T(x,0) &=& T_4 \end{array}\right\} \qquad (0 \le x \le a, \quad 0 \le y \le b).$$

We choose $b = a$ or, alternatively, $b = \frac{3}{2}a$ or $b = 2a$.
Introducing a proper mesh of discrete points and applying the iteration method of the lecture, find approximate values of the true solution $T(x,y)$ for various choices of the temperatures $T_1, ..., T_4$.

## Problem 4: Theorems by Gauss and Stokes

Consider the vector field $\mathbf{F}(\mathbf{r})$ and the volume region $\Omega$,

$$\mathbf{F}(\mathbf{r}) = \begin{pmatrix} a_1 x \;+\; b_1 y \;+\; c_1 z \\ a_2 x \;+\; b_2 y \;+\; c_2 z \\ a_3 x \;+\; b_3 y \;+\; c_3 z \end{pmatrix}, \qquad \Omega = \Big\{\mathbf{r}(\alpha) \;\Big|\; r = R, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le \frac{\pi}{2}\Big\},$$

where $a_1, b_1, c_1;\; a_2, b_2, c_2;\; a_3, b_3, c_3$ are nine constants (some given numbers) and $\alpha = (r, \theta, \phi)$ are the usual spherical coordinates.

(a) Compute divergence $\nabla \cdot \mathbf{F}(\mathbf{r})$ and curl $\nabla \times \mathbf{F}(\mathbf{r})$ of the vector field $\mathbf{F}(\mathbf{r})$.

(b) Verify Gauss' theorem,

$$\oint_{\partial\Omega} \mathrm{d}\mathbf{S} \cdot \mathbf{F}(\mathbf{r}) \;=\; \int_{\Omega} \mathrm{d}^3 r \,\big[\nabla \cdot \mathbf{F}(\mathbf{r})\big].$$

(c) Verify Stokes' theorem,

$$\oint_{\partial\Sigma} \mathrm{d}\mathbf{r} \cdot \mathbf{F}(\mathbf{r}) \;=\; \int_{\Sigma} \mathrm{d}\mathbf{S} \cdot \big[\nabla \times \mathbf{F}(\mathbf{r})\big],$$

where $\Sigma \subset \partial\Omega$ is the **curved piece** of the full surface $\partial\Omega$ of $\Omega$.

**Hint:** $\int \mathrm{d}x \, \sin^2 x \;=\; \frac{1}{2}(x \;-\; \sin x \, \cos x), \quad \int \mathrm{d}x \, \sin x \, \cos x \;=\; \frac{1}{2} \sin^2 x.$