Vector calculus and numerical mathematics

Worksheet 11

Problem 1: Linear functions

We consider two linear functions $f, g : \mathbb{R}^2 \to \mathbb{R}$, defined on the $xy$-plane $\mathbb{R}^2$,

$$f(x, y) = x + y, \quad g(x, y) = 2x - y,$$

and a region $\Sigma$ in the $xy$-plane,

$$\Sigma = \{(x, y) \mid x, y \geq 0 \text{ and } y \leq h(x) \equiv 3 - \frac{1}{2}x \}.$$  

(a) Draw the contour lines $f(r) = n$, $g(r) = n$ for $n \in \{-2, 0, 2, 4, 6\}$.

(b) Evaluate the integrals

$$I_1 = \int_\Sigma d^2r \ f(r), \quad I_2 = \int_\Sigma d^2r \ g(r).$$

What is the average value $\langle f(r) \rangle_{r \in \Sigma}$ of $f$ within $\Sigma$?

(c) For the constant vector $\mathbf{k} = \begin{bmatrix} a \\ b \end{bmatrix}$, evaluate the integral

$$I_3 = \int_\Sigma d^2r \ \mathbf{k} \cdot \nabla Q(r), \quad Q(r) = g(r)^2.$$  

(d) Evaluate the integrals

$$I_4 = \int_\Sigma d^2r \ [\nabla f(r)] \cdot [\nabla g(r)], \quad I_5 = \int_\Sigma d^2r \ |\nabla f(r)|^2,$$

$$I_6 = \int_\Sigma d^2r \ [\nabla f(r)] \cdot [\nabla Q(r)], \quad Q(r) = g(r)^2.$$  

Problem 2: Tent functions

We consider 7 points O, M, A, B, C, D, E in the $xy$-plane:

<table>
<thead>
<tr>
<th></th>
<th>O</th>
<th>M</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>1</td>
<td>2</td>
<td>-1</td>
<td>-2</td>
<td>-3</td>
<td>-4</td>
<td>-5</td>
</tr>
<tr>
<td>$r_n$</td>
<td>(0</td>
<td>0)</td>
<td>(1</td>
<td>0)</td>
<td>(2</td>
<td>0)</td>
<td>(1</td>
</tr>
</tbody>
</table>

The region $\Omega \subset \mathbb{R}^2$, with $\partial \Omega = ABCDEA$, is divided up into 7 triangles (green lines),

MAB, MBO, MOE, MEA, OBC, OCD, ODE.
Defining $r_1 = r_O$ and $r_2 = r_M$, we obtain two tent functions $w_1(r)$ and $w_2(r)$. Compute the following integrals:

$$
\eta_1 = \int d^2r w_1(r),
$$

$$
A_{11} = \int_\Omega d^2r |\nabla w_1(r)|^2,
$$

$$
A_{12} = \int_\Omega d^2r [\nabla w_1(r)] \cdot [\nabla w_2(r)].
$$