Fakultät Mikrosystemtechnik
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Vector calculus and numerical mathematics

Worksheet 10 (Update on 13 June: Problem 1f)

Problem 1: Separable-variables ODEs

For each one of the following ODEs, find a proper method to obtain **infinitely many** solutions analytically. **Hints:** $\frac{\mathrm{d}}{\mathrm{d}u}\arctan(u) = \frac{1}{1+u^2}$, $\frac{\mathrm{d}}{\mathrm{d}u}\arcsin(u) = \frac{1}{\sqrt{1-u^2}}$.

(a) The ODE $y'(x) = -2x\,y(x)^2$ from the lecture is a special case of

$$y'(x) \;=\; nx\,y(x)^2 \qquad (n \in \mathbb{Z}).$$

(b)

$$y'(x) \;=\; 5x^4 y(x)^3.$$

(c)

$$y'(x) \;=\; y(x) - x^2.$$

(d)

$$y'(x) \;=\; 4x^3\big[1 + y(x)^2\big].$$

(e)

$$y'(x) \;=\; \frac{6x}{\cos[y(x)]}.$$

(f) Here is a more sophisticated example:

$$y'(x) \;=\; \sin[y(x)].$$

Problem 2: Numerical solution of an ODE

(a) Apply a finite-differences method (FDM) with step size $h = 0.1$ to obtain an approximate solution $y_{\mathrm{app}}(x)$ to the ODE

$$y'(x) \;=\; -2x\,y(x)^2,$$

that satisfies the starting condition $y_{\mathrm{app}}(0) = 1$.

(b) Try to improve this approximation by considering the second derivative $y''(x)$. Compare your results with the exact solution $y_{\mathrm{ext}}(x)$, see problem 1!

Problem 3: Integral theorems

A cylinder Ω is in cartesian coordinates given by

$$\Omega \;=\; \left\{ \mathbf{r}(x,y,z) \,\middle|\, x^2+y^2 \le R^2,\; 0 \le z \le H \right\}$$

Its surface $\partial\Omega$ consists of floor (Φ), wall (W) and ceiling (Σ), $\partial\Omega = \Phi \cup W \cup \Sigma$. We consider the vector field

$$\mathbf{F}(\mathbf{r}) \;\equiv\; \mathbf{F}(x,y,z) \;=\; \begin{pmatrix} -y \\ x \\ z \end{pmatrix}.$$

(a) Find the value I_1 of the surface (or flux) integral

$$I_1 \;=\; \oint_{\partial\Omega} \mathrm{d}\mathbf{A} \cdot \mathbf{F}(\mathbf{r}),$$

choosing a method **as simple as possible**.

(b) Find the value I_2 of the line (or work) integral

$$I_2 \;=\; \oint_{\partial\Sigma} \mathrm{d}\mathbf{r} \cdot \mathbf{F}(\mathbf{r})$$

(circulation of $\mathbf{F}$ around the rim of Σ), choosing **two different** methods.

Problem 4

Consider two vectors $\mathbf{a}$ and $\mathbf{b}$ with $|\mathbf{a}| = |\mathbf{b}|$ and $\mathbf{a}\cdot\mathbf{b} = 0$ and, in addition, the vector $\vec{\omega}$ of an angular velocity, e.g.:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \qquad \vec{\omega} = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} \frac{2\pi}{\mathrm{sec}}.$$

(a) Give a physical interpretation for the vector field

$$\mathbf{v}(\mathbf{r}) \;=\; \vec{\omega} \times \mathbf{r} \;=\; \begin{pmatrix} \omega_2 z - \omega_3 y \\ \omega_3 x - \omega_1 z \\ \omega_1 y - \omega_2 x \end{pmatrix} \;=\; \begin{pmatrix} 0x + \;y\; + 2z \\ -x + 0y + 2z \\ -2x - 2y + 0z \end{pmatrix}.$$

(b) What is the geometrical shape of the curve $\partial\Sigma$ described by the parametrization

$$\mathbf{r}(\phi) \;=\; \mathbf{a}\,\cos\phi \;+\; \mathbf{b}\,\sin\phi \;=\; \begin{pmatrix} a_1\cos\phi \;+\; b_1\sin\phi \\ a_2\cos\phi \;+\; b_2\sin\phi \\ a_3\cos\phi \;+\; b_3\sin\phi \end{pmatrix}, \qquad (0 \le \phi < 2\pi)?$$

Describe a proper piece of surface Σ which has $\partial\Sigma$ as its rim.

(c) Evaluate the line integral $\oint_{\partial\Sigma} \mathrm{d}\mathbf{r}\cdot\mathbf{v}(\mathbf{r})$.

(d) For $\mathbf{F}(\mathbf{r}) = \nabla\times\mathbf{v}(\mathbf{r})$, evaluate the flux integral $\int_\Sigma \mathrm{d}\mathbf{A}\cdot\mathbf{F}(\mathbf{r}) = A_\Sigma \left\langle \vec{\mathcal{N}}(\mathbf{r})\cdot\mathbf{F}(\mathbf{r}) \right\rangle_{\mathbf{r}\in\Sigma}$.