HIOB: Gromov-Hausdorff convergence and metric geometry

Summer semester 2024

The main subject of this seminar is the notion of Gromov-Hausdorff convergence. This method, first defined by Edwards (compare [Edw75] and [Tuz16]) and popularised by Gromov in the French version of the book [Gro07], is an extremely versatile method in metric geometry, Riemannian geometry as well as in geometric group theory. This seminar is divided into three parts, the second and third being (almost) independent. In the first part, we deal with the basic notions of metric geometry, as well as the method of Gromov-Hausdorff convergence. In the second part, we turn to applications in Riemannian geometry and study to which extent curvature is preserved under Gromov-Hausdorff limits. A main result here is the Cheeger finiteness theorem [Che70]. In the third part, we turn to geometric group theory. After an introduction to this area, we will focus on applications of Gromov-Hausdorff convergence in that field. One example is a recently proved result about asymptotic growth rates in hyperbolic groups [FS23].

Introduction

Talk 1 (Hausdorff and Gromov-Hausdorff distance).
Date: 15 April
Speaker: Andrea Panontin
Main reference: [Pet20, Ch. I.4 and I.5.A-C], [BBI01, Ch. 7.0-7.3]

- Introduce the Hausdorff distance and present some easy examples of your own.
- Provide a complete proof of Blaschke's selection theorem.
- Define the Gromov-Hausdorff distance and show that it satisfies the axioms of a metric [Pet20, Thm. 5.2]. You may or may not discuss the

Last updated: April 12, 2024

set-theoretic issues associated with forming a space of isometry classes of compact spaces [BBI01, Rem. 7.2.5].

- Discuss some examples, e. g. [Pet20, Excs. 5.3, 5.5, 5.13].
- It is left up to you whether and to what extent you present the introductory examples for convergence in [BBI01, Ch. 7.1] and the proof of the isoperimetric inequality [Pet20, Ch. I.4.C].

Talk 2 (Gromov-Hausdorff convergence)**.**
Date: 22 April
Speaker: Lukas Krinner
Main reference: [Pet20, Ch. I.5.D-F], [BBI01, Ch. 7.4-7.5]

- Discuss, how Gromov-Hausdorff convergence can be understood in terms of ϵ-isometries [BBI01, Cor. 7.3.28], resp. [Pet20, Ch. I.5.D]. Compare it to at least one stronger form of convergence as [BBI01, Ch. 7.4.1]. (Such stronger notions will play a role in talk 7.)
- Prove Gromov's compactness theorem [Pet20, Thm. 5.22].
- Introduce geodesics and length spaces [Pet20, Ch I.1.I-K], cf. also [Gro07, Ch. 1], and prove Menger's lemma [Pet20, Thm. 1.27].
- Show that the property of being a length space is preserved by Gromov-Hausdorff limits [BBI01, Thm. 7.5.1].
- Discuss more examples of limits of your choice as in [BBI01, Ch. 7.5]. It would be desirable to see what can happen topologically, cf. also [BBI01, Figs. 7.4-7.5] – noting also that the stronger notions of convergence discussed there fix the underlying topology.

Talk 3 (Universal spaces)**.**
Date: 29 April
Speaker: Debam Biswas
Main reference: [Pet20, Ch. I.2 and I.5.G], alternative reference: [Gro07, Sec. 3.11$\frac{2}{3}{}_+$]

- Explain how the Gromov-Hausdorff distance can be understood using universal spaces [Pet20, Prop. 5.27].
- Discuss the example of ℓ^∞, also in the version of [Pet20, Exc. 2.3].
- Construct the Urysohn space $\mathcal{U}_\infty$ [Pet20, Thm. 2.10], discuss its universality [Pet20, Prop. 2.12] and uniqueness [Pet20, Thms. 2.17, 2.20].
- Conclude [Pet20, Exc. 5.28], i. e. the equality of Gromov-Hausdorff distance and Hausdorff distance up to isometry within the Urysohn space. It might be interesting to compare this with [Pet20, Exc. 5.4] (note that there is a typo concerning the direction of the strict inequality).

- If there is time remaining, you may think of presenting some of the properties of $\mathcal{U}_\infty$ discussed in the exercises, e. g. [Pet20, Excs. 2.13, 2.16, 2.19, 2.21], and answering Gromov's *question to the reader* [Gro07, p. 82]. It probably leads to far to discuss that the Urysohn space is the generic separable complete space in a certain sense [Ver04].

Gromov-Hausdorff convergence and curvature

Talk 4 (Alexandrov geometry I).
Date: 6 May
Speaker: Niklas Kipp
Main reference: [Pet20, Ch. II.7],[BBI01, Ch. 4, Ch. 10]

- Introduce notions like triangles, model triangles, hinges and angles [Pet20, Ch. II.7], which are needed for the rest of the seminar. If necessary, repeat notions from the first talks.
- Define the notions CAT and CBB [Pet20, Ch. II.7] and indicate the fundamental difference between upper and lower curvature bounds.
- Give many examples and non-examples. Give an example of a two-dimensional locally-finite polyhedron that is not an Alexandrov space [BBI01, Example 4.1.3, Example 4.2.3].
- Show that the conditions CAT/CBB are closed under Gromov-Hausdorff limits [Pet20, Ch. II.7.C, 7.8 Proposition].
- Show Alexandrov's Lemma and conclude that angles are well-defined in CAT/CBB spaces [BBI01, Proposition 4.3.2].

Talk 5 (Alexandrov geometry II).
Date: 27 May
Speaker: Johannes Gloßner
Main reference: [Pet20, Ch. II.9],[AKP19, Sections 3.1-3.3]

- Introduce the notions of locally CAT and prove a characterization of being locally CAT for a proper length space.
- Illustrate the concept of patchworks along geodesics and give a sketch of a proof. Conclude necessary corollaries for CAT(0) and CAT(1) spaces.
- Use the previous statements to show Exercise 9.5 in [Pet20] and point out why the length 2π is critical.
- Sketch a proof of the globalization theorem [Pet20, 9.6 Globalization Theorem] and illustrate the necessity of the assumptions by presenting counterexamples.

- If time permits: Present the analogous globalization theorem for CBB spaces.

Talk 6 (Pointed Gromov-Hausdorff convergence and C^k-convergence).
Date: 3 June
Speaker: Benjamin Dünzinger
Main reference: [Pet16, Ch. 11.1-11.3.4],[Jan17],[Pet20, Ch. I.5H]

- Define pointed Gromov-Hausdorff distance and give an example why this concept is necessary if we want to speak about non-compact metric spaces. One could motivate as in [Pet20, Ch. I.5H].
- Indicate that the pointed notion in the compact setting coincides with the notion of talk 1 in the compact case [Jan17, Corollary 2.5].
- Introduce the $C^{m,\alpha}$-norms for pointed Riemannian manifolds [Pet16, Ch. 11.3.1] and state/prove various properties of these norms. Indicate how these norms relate to Gromov-Hausdorff convergence.
- State the fundamental theorem of convergence theory [Pet16, Theorem 11.3.6].

Talk 7 (Cheeger finiteness theorem).
Date: 10 June
Speaker:
Main reference: [Pet16, Ch. 11.3.5-11.5],[Che70]

- Introduce and explain the various Riemannian notions, which are used in the Cheeger finiteness theorem and the fundamental theorem of convergence theory [Pet16, Corollary 11.4.10, Theorem 11.3.6]
- Give a proof sketch of the fundamental theorem. Focus on the Gromov-Hausdorff aspects.
- State and prove Cheeger's lemma [Pet16, Lemma 11.4.9]. One should give the necessary properties of injectivity radius and Klingenberg's estimate.
- Conclude from the previous statement Cheeger's theorem.

Gromov-Hausdorff convergence in geometric group theory

Talk 8 (Introduction to geometric group theory).
Date: 17 June
Speaker: Chiara Sabadin
Main reference: [Löh17, Chapters 3.2, 5, 7]

- Define Cayley graphs and quasi-isometries.
- Define geometric properties and give examples. [Löh17, Example 5.2.11, Theorem 4.2.14, Example 5.6.7, Proposition 5.2.5, Definition 5.6.6]
- State that finite index subgroups have the same quasi-isometry type as the ambient group [Löh17, Corollary 5.4.5].
- Introduce growth of groups and give examples. [Löh17, Chapter 6.1, 6.2]
- Growth is a geometric property [Löh17, Corollary 6.2.6]

Talk 9 (Gromov's theorem on polynomial growth)**.**
Date: 1 July
Speaker:
Main reference: [DW84], [Gro81]

- State Gromov's theorem: Finitely generated groups have polynomial growth if and only if they contain a nilpotent subgroup of finite index [Löh17, Theorem 6.3.1].
- State Milnor-Wolf's theorem. [Löh17, Theorem 6.3.8].
- In this talk, we focus on the 'hard' implication (polynomial growth implies virtual nilpotence).
- Sketch how to reduce to reduce the statement to the theorem that there exists a finite-index subgroup admitting an epimorphism to $\mathbb{Z}$ [DW84, Lemma 2.1, Theorem 2.5].
- Let G be a finitely generated group of polynomial growth, $S \subset G$ be a finite generating set. Use the compactness theorem to show that a subsequence of $(G, \frac{1}{n} \cdot d_S)$ converges in the Gromov-Hausdorff metric.
- Time permitting: Explain which properties can be proved about the Gromov-Hausdorff limit.

Talk 10 (The Bestvina-Paulin method)**.**
Date: 8 July
Speaker:
Main reference: [BS94]

- Introduce hyperbolic groups and give examples [Löh17, Chapter 7.3]. Stress differences with the definition of the CAT-conditions.
- Prove Paulin's theorem [BS94].
- Survey some consequences and results of your choice that can be proved using similar methods. Explain the occuring terms that might be unknown to the audience [BS94, Theorem 4.2] [Bes02, Section 7.2] [FS23, Theorem 2.2, Corollary 2.9] (see also [Löh23]).

Bonus material

Talk B (Applied topology in viral evolution).
Date: 13 May or 24 June[1]
Speaker:
Main reference: [CCR13], [RB20, Chapter 5]

This talk is meant to be more expository, highlighting applications of Gromov-Hausdorff distance and persistent homology in applied topology.

- Introduce persistent homology of finite metric spaces. [Ghr08]
- Introduce the bottleneck distance, state the stability theorem [RB20, Theorem 2.4.10] [CCR13, Appendix, Theorem 2.2] and explain the practical relevance.
- Explain the connection with evolutionary relationships between organisms and state some of the biological results [CCR13], [RB20, Chapter 5].
- Time permitting: Sketch another application of persistent homology of your choice [RB20, Chapters 6–9] [TPJ23] [Vit+23].

Talk D (Discussion for the next HIOB).
Date: 15 July
Speaker: everyone
Main reference:

References

[AKP19] Stephanie Alexander, Vitali Kapovitch, and Anton Petrunin. "An invitation to Alexandrov geometry: CAT(0) spaces". English. In: (2019). DOI: `https://doi.org/10.48550/arXiv.1701.03483`.

[BBI01] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry*. 1st ed. Vol. 33. Graduate Studies in Mathematics. 2001. DOI: `10.1090/gsm/033`.

[Bes02] Mladen Bestvina. "$\mathbb{R}$-trees in topology, geometry, and group theory". English. In: *Handbook of geometric topology*. Amsterdam: Elsevier, 2002, pp. 55–91. ISBN: 0-444-82432-4.

[BS94] M. R. Bridson and G. A. Swarup. "On Hausdorff-Gromov convergence and a theorem of Paulin". English. In: *Enseign. Math. (2)* 40.3-4 (1994), pp. 267–289. ISSN: 0013-8584.

[1] We will vote on the date during the first or second session of the seminar

[CCR13] Joseph Minhow Chan, Gunnar Carlsson, and Raul Rabadan. "Topology of viral evolution". English. In: *Proc. Natl. Acad. Sci. USA* 110.46 (2013), pp. 18566–18571. ISSN: 0027-8424. DOI: 10.1073/pnas.1313480110.

[Che70] Jeff Cheeger. "Finiteness Theorems for Riemannian Manifolds". In: (1970). DOI: https://doi.org/10.2307/2373498.

[DW84] Lou van den Dries and A. J. Wilkie. "Gromov's theorem on groups of polynomial growth and elementary logic". English. In: *J. Algebra* 89 (1984), pp. 349–374. ISSN: 0021-8693. DOI: 10.1016/0021-8693(84)90223-0.

[Edw75] David A. Edwards. "The Structure of Superspace". In: *Studies in Topology, Academic Press* (1975).

[FS23] Koji Fujiwara and Zlil Sela. "The rates of growth in a hyperbolic group". English. In: *Invent. Math.* 233.3 (2023), pp. 1427–1470. ISSN: 0020-9910. DOI: 10.1007/s00222-023-01200-w.

[Ghr08] Robert Ghrist. "Barcodes: the persistent topology of data". English. In: *Bull. Am. Math. Soc., New Ser.* 45.1 (2008), pp. 61–75. ISSN: 0273-0979. DOI: 10.1090/S0273-0979-07-01191-3.

[Gro07] Misha Gromov. *Metric structures for Riemannian and Non-Riemannian spaces*. 1st ed. Birkhäuser Modern Classics. Originally published in French as: Structures Métriques des Variétés Riemanniennes. 2007. ISBN: 978-0-8176-4583-0. DOI: 10.1007/978-0-8176-4583-0.

[Gro81] Mikhael Gromov. "Groups of polynomial growth and expanding maps. Appendix by Jacques Tits". English. In: *Publ. Math., Inst. Hautes Étud. Sci.* 53 (1981), pp. 53–78. ISSN: 0073-8301. DOI: 10.1007/BF02698687.

[Jan17] Dorothea Jansen. "Notes on Pointed Gromov-Hausdorff Convergence". 2017. URL: https://doi.org/10.48550/arXiv.1703.09595.

[Löh17] Clara Löh. *Geometric group theory. An introduction*. English. Universitext. Cham: Springer, 2017. DOI: 10.1007/978-3-319-72254-2.

[Löh23] Clara Löh. *Exponential growth rates in hyperbolic groups (after Koji Fujiwara and Zlil Sela)*. 2023. arXiv: 2304.04424 [math.GR].

[Pet16] Peter Petersen. *Riemannian geometry*. 3rd ed. Vol. 171. Graduate Texts in Mathematics. New York: Springer, 2016.

[Pet20] Anton Petrunin. "Lectures on metric geometry". Unpublished lecture notes. 2020. URL: https://anton-petrunin.github.io/metric-geometry/tex/lectures.pdf.

[RB20] Raúl Rabadán and Andrew J. Blumberg. *Topological data analysis for genomics and evolution. Topology in biology*. English. Cambridge: Cambridge University Press, 2020. DOI: 10.1017/9781316671665.

[TPJ23] Mai Lan Tran, Changbom Park, and Jae-Hun Jung. "Topological data analysis of Korean music in Jeongganbo: a cycle structure". English. In: *J. Math. Music* 17.3 (2023), pp. 403–432. ISSN: 1745-9737. DOI: 10.1080/17459737.2022.2164626.

[Tuz16] Alexey A. Tuzhilin. "Who Invented the Gromov-Hausdorff Distance?" In: (2016). DOI: https://doi.org/10.48550/arXiv.1612.00728.

[Ver04] Anatoly M. Vershik. "Random and Universal Metric Spaces". In: *Dynamics and Randomness II*. Ed. by A. Maass, S. Martínez, and J. S. Martín. Dordrecht: Springer, 2004. DOI: 10.1007/978-1-4020-2469-6_6. URL: https://arxiv.org/abs/math/0205086.

[Vit+23] Martina Vittorietti, Ornella Giambalvo, Vincenzo Giuseppe Genova, and Fabio Aiello. "A new measure for the attitude to mobility of Italian students and graduates: a topological data analysis approach". English. In: *Stat. Methods Appl.* 32.2 (2023), pp. 509–543. ISSN: 1618-2510. DOI: 10.1007/s10260-022-00666-x.