Seminar on String Topology

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Winter semester 2024/25

This seminar will explore *string topology*, a field in mathematics that intersects topology, geometry, and algebra. String topology, as founded by Chas and Sullivan in their seminal work [CS99], aims to study the space of loops on manifolds through algebraic structures, like the Chas-Sullivan product. A good survey is given by the work of Cohen, Jones, and Voronov [CHV06] or [NRW23].

The initial motivation behind string topology was the quest to discover invariants for smooth manifolds, specifically to find invariants that could capture not only the topological features of a manifold but also the finer details of its smooth structure[1]. Sullivan even said

> "...it is the question that has fascinated me since grad school: what is the algebraic chain level meaning of a space being a combinatorial or smooth manifold?" ([NRW23, p. 4, Naturality and Invariance])

Over time, string topology has developed into a broad area with numerous applications and has given rise to many new ideas that have influenced both mathematical theory and its applications in physics. The seminar is divided into four parts.

The *first part* (Talk 1-3) of the seminar will provide a comprehensive introduction to the fundamental operations of string topology and the algebraic structures that arise from it. We will cover key concepts such as BV (Batalin-Vilkovisky) algebras, Gerstenhaber algebras, and Hochschild homology. These structures play a central role in understanding the operations in string topology and their connections to various algebraic frameworks. In Talk 3, we will delve into the relationship between the topological picture of loop spaces and their corresponding algebraic counterparts.

In the *second part* (Talk 4-7), we transition from classical algebraic structures to the modern framework of operads and their generalizations, known as PROPs (PROducts and Permutations). These algebraic tools are fundamental in understanding moduli spaces and their connections to field theories. We will

Last updated: October 14, 2024

[1] Later it was recognised that the simpler string operations are homotopy invariant.

explore how operads organize algebraic operations and how they are applied in various contexts, including field theories (Talk 12, in preparation). Talks 4 to 7 will cover the theory of operads in depth, their generalizations, and how these structures are used to describe intricate patterns in both algebra and topology.

The *third part* (Talk 8-9) of the seminar will present a Morse-theoretic point of view to string topology. In particular, we will focus on the free loop space of a manifold, which is the central object of the energy functional, which has geodesics as its critical points. Moreover, we will give a short introduction to Morse and Floer homology and the connection to string topology.

The *final part* (Talk 11-12) of the seminar will explore the dependency of the string coproduct (Goresky-Hingston product) for the lens space $L(k, 7)$ for several values of k. It will also highlight the relationship between this coproduct and Reidemeister/Whitehead torsion. Moreover we will focus on the three known (co)products on free loop space homology and how two of them can be unified in a single algebraic structure. Additionally, we will introduce differential Frobenius algebras and Tate-Hochschild complexes, explaining their relevance to string operations.

1. Introduction String topology

Talk 1 (Foundations)**.**
Date: 15.10.2024
Speaker: `Samuel Lockman`
Main reference: Own Preference

- Explain the following concepts (constructions): Thom class, Thom collapse map, Thom isomorphism, Pontryagin Tower, intersection theory of compact manifolds, infinite dimensional manifolds, free loop space, orientation of a homology theory over a space M, multiplicative homology theory ,...

Talk 2 (Structure on the loop homology)**.**
Date: 22.10.2024
Speaker: `Stefan Bastl`
Main reference: [CHV06],[Maz22],[CS99],[CKS08]

- Give the Definition of Gerstenhaber algebra and Batalin-Vilkovisky algebra [Maz22, Chapter 3.4]. Show that a BV algebra induces the structure of a Gerstenhaber algebra.
- Give the construction of the Chas-Sullivan product on the loop homology [CS99, Section 1-5].
- Give an equivalent construction of the product via Thom collapse map and the intersection product [CHV06, Section 1].

- Sketch a proof of the homotopy invariance of the loop product [CKS08].

Talk 3 (Hochschild homology, Jones isomorphism and the approximation of the free loop space)**.**
Date: 29.10.2024
Speaker: `Jonathan Glöckle`
Main reference: [CHV06],[Ger63],[LOA15]

- Give a Definition of Hochschild homology [LOA15, Chapter 4, section 1]
- Explain the Gerstenhaber algebra structure on Hochschild homology [Ger63].
- Give the Jones isomorphism $\mathbb{H}_*(LM) \cong HH_*(C^*(M))$ in the oriented, simply-connected case and sketch the proof [LOA15, Chapter 4],[CHV06, Section 1.5]. If time permits explain the de Rahm case $\mathbb{H}_*(LM) \cong HH_*(\Omega^*(M))$.

2. Operads, field theories and string topology

Talk 4 (PROPs and operads)**.**
Date: 5.11.2024
Speaker: `Raphael Schmidpeter`
Main reference: [CHV06, Chapter 2]

- Give a brief presentation of the definition of symmetric monoidal categories and describe the terms that are not generally known. ([Bae01], an easy example is the category of vector spaces over a fixed field).
- Present [CHV06, Section 2.1.1 to 2.1.3] for a thorough exposition of PROPs, operads and algebras over them. Give an idea of the remark, that an operad is the "$\mathrm{Mor}(n, 1)$, $n \geq 0$"-part of a PROP (and note that in Def. 2.1.3 (1.) an error has occurred with the subscript indices). Proceed with great care on account of the abstractness of the material.
- Give definition 2.1.4 of [CHV06] and define the notion of an algebra X over an operad $\mathcal{O}$ in the category of topological spaces, the notion of topological operad and solve as an example Ex. 2 on page 29. As far as time permits use the following examples and exercises to develop the concept, that algebraic structures can equivalently be defined as algebra structures over the right operads.

Talk 5 (Little Disks and Cacti operads)**.**
Date: 12.10.2024
Speaker: `Roman Schießl`
Main reference: [CHV06, Chapter 2]

- Treat the part of [CHV06, Section 2.1.4] that is headlined: "The little disks operads and GBV-algebras" and especially state the theorems 2.1.1 an 2.1.2 therein. You can accept the definition of $H_*(fD;k)(n) := H_*(fD(n);k)$ as a black box.
- Present of [CHV06] section 2.2 until the statement of theorem 2.2.1 and section 2.3.

Talk 6 (Introduction to field theory and chord diagrams)**.**
Date: 19.10.2024
Speaker:
Main reference: [CHV06, Chapter 3],[CG04]

- Give a short recap of the definition of topological quantum field theories and the equivalence of a 2-dim. TQFT to a Frobenius algebra structure. Pass then on to discuss the description of a Frobenius algebra without counit [CR18, Sections 2.2 and 3.3] and [CG04, Page 2].
- State and explain Thm. 1 of [CG04]. In case of unfamiliarity with generalized (co)homology theories, it is always possible to simply think of singular (co)homology.
- Follow [CG04, section 1] through the various definitions related to chord diagrams to get to the definition of the space of marked metric chord diagrams.
- State Thm. 2 of [CG04] and possibly give a sketchy proof supported by drawings. The book [CHV06, Section 3.2] can give a survey and may inspire some shortcuts (The results 3.2.1 and 3.2.3 of the book are not to be part of the talk).

Talk 7 (Construction of string field theory)**.**
Date: 26.10.2024
Speaker:
Main reference: [CHV06, Chapter 3.2-3.3],[CG04]

- Give a thorough account of [CG04, Section 2], and wrap the two talks on field theories up by commenting especially on the last paragraph of the said section.

3. Morse and Floer theoretic aspects of string topology

Talk 8 (Morse theoretic viewpoint on string topology)**.**
Date: 3.12.2024
Speaker:
Main reference: [CHV06, Chapter 4],[Oan14]

- A talk about a Morse theoretic perspective on the field theory structure of $h_*(LM)$. *In preparation.*

Talk 9 (Further Morse and Floer theoretic topics)**.**
Date: 10.12.2024
Speaker:
Main reference: [CHV06, Chapter 4]

- A talk on further Morse and Floer theoretic aspects and some trains of thoughts leading to conjecture a deeper relation between the string topology of a closed manifold and the symplectic topology of its cotangent bundle. *In preparation.*

4. Applications

Talk 10 (The String coproduct "Knows" Reidemeister/Whitehead Torsion)**.**
Date: 7.1.2025
Speaker: `Bernd Ammann`
Main reference: [Nae21]

- Explain the decomposition of LM into its 7 components for $M = L(k, 7)$ [Nae21, section 2.1]. In particular point out the relevance of the class $[\rho_{lm}]$.
- Give the definition of the string coproduct (Goresky-Hingston product) [Nae21, section 2.2], [GH09] or [Sul04] and show the explicit form in the case of the lens spaces $L(k, 7)$.
- Relate the coproduct with Reidemeister and Whitehead torsion in Proposition 5 [Nae21, Section 2.4].

Talk 11 (Invariant properties of string operations)**.**
Date: 14.1.2025
Speaker: `Julian Seipel`
Main reference: [NRW23]

- Define all three known (co)products on free loop space homology. Sketch how two of them can be unified in a single structure, i.e. Manin triples on the Tate-Hochschild complex [NRW23, Thm. 3.17].
- Define Differential graded Frobenius Algebras [NRW23, Definition 3.2] and give Examples [NRW23, Example 3.3 and 3.5].
- Define the Tate-Hochschild complex [NRW23, Definition 3.11] and give the algebraic versions of the string operations. Explain the notion of Manin triples [NRW23, Remark 3.18].
- Sketch a proof of the Theorem [NRW23, Theorem 4.2], which connects the algebraic and topological versions of the string operations (either [NRW23] or the original source [NW19])

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