Lecture notes on Quantum Field Theory III: Advanced topics and the Standard Model

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Lecture 1

1 Canonical Quantization

Aim of this section is to make a connection of QFT to Quantum Mechanics and explain QFT methods on a simple example:

1.1 Anharmonic Oscillator as a $0+1$-dimensional quantum field theory

Consider the following system:

- Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - V(x) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}\omega^2 x^2 - \frac{\lambda}{4!}x^4 \tag{1.1}$$

In the following $m \mapsto 1$; Here $4! = 4\cdot 3\cdot 2\cdot 1 = 24$

- Euler-Lagrange Eq. (= Newton's II Law)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \qquad \Leftrightarrow \qquad \ddot{x} = -\omega^2 x - \frac{\lambda}{3!}x^3 \tag{1.2}$$

- Canonically conjugated momentum

$$p = \frac{\partial L}{\partial \dot{x}} = \dot{x} \tag{1.3}$$

- Hamiltonian

$$H = p\dot{x} - L = \frac{1}{2}p^2 + \frac{\omega^2}{2}x^2 + \frac{\lambda}{4!}x^4 \tag{1.4}$$

Change of notation

$$\boxed{x \mapsto \phi\,, \qquad p \mapsto \pi} \qquad\qquad !!\ \pi \neq 3.14\ldots \tag{1.5}$$

so that

$$L = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\omega^2\phi^2 - \frac{\lambda}{4!}\phi^4 \qquad\qquad H = \frac{1}{2}\pi^2 + \frac{1}{2}\omega^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{1.6}$$

— Lagrangian (Hamiltonian) of a 0+1 dimensional scalar quantum field theory

- so far everything classical...

Canonical Quantization

The standard procedure, usually called "Canonical Quantization":

- Promote ϕ and π to operators in Hilbert space of quantum states

$$\phi \mapsto \hat{\phi}\,, \qquad \pi \mapsto \hat{\pi} \tag{1.7}$$

- Postulate canonical commutation relations

$$[\hat{\phi}, \hat{\pi}] = i \qquad\qquad \boxed{c = \hbar = 1} \tag{1.8}$$

• The state, say $|1\rangle$, is described by the wave function (WF), in Dirac notation $\Psi_1(\phi) = \langle\phi|1\rangle$. In this representation

$$\langle\phi|\hat{\phi}|1\rangle = \phi\Psi_1(\phi)$$
$$\langle\phi|\hat{\pi}|1\rangle = -i\frac{\partial}{\partial\phi}\Psi_1(\phi) \tag{1.9}$$

Time evolution of the quantum system can be described in two different ways:

Schrödinger picture

The operators are time-independent, the state vector depends on time so that the wave function satisfies the (Schrödinger) Eq.:

$$i\frac{\partial}{\partial t}\Psi(\phi,t) = \hat{H}\,\Psi(\phi,t) \tag{1.10}$$

with formal solution, schematically

$$\Psi(\phi,t) = e^{-i\hat{H}t}\Psi(\phi,t=0) \tag{1.11}$$

(One needs of course to define more accurately what is $e^{-i\hat{H}t}$)

Heisenberg picture

The states are time-independent, the operators depend on time:

$$\Psi(\phi) = \Psi(\phi,t=0) = e^{i\hat{H}t}\Psi(\phi,t)$$
$$\hat{\phi}(t) = e^{i\hat{H}t}\hat{\phi}e^{-i\hat{H}t}$$
$$\hat{\pi}(t) = e^{i\hat{H}t}\hat{\pi}e^{-i\hat{H}t} \tag{1.12}$$

Heisenberg Eqs.:

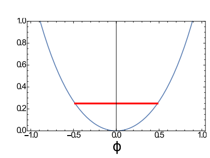
$$\frac{d}{dt}\hat{\phi}(t) = i[H,\hat{\phi}(t)] \qquad\qquad \frac{d}{dt}\hat{\pi}(t) = i[H,\hat{\pi}(t)] \tag{1.13}$$

Perturbation theory

In serious applications of QM one usually cannot find exact WF (analytically) and makes use of *perturbation theory* to find some approximation. In our toy model — expansion in powers of the anharmonicity parameter λ.

A typical problem: Find average value of ϕ^2 in the ground state $|\Omega\rangle$

$$\langle\Omega|\hat{\phi}^2|\Omega\rangle \;=\; ?$$



In the QFT jargon, the ground state of the Hamiltonian is called "vacuum state", and the corresponding energy is called "vacuum energy":

$$\hat{H}|\Omega\rangle = E_{\rm vac}|\Omega\rangle \tag{1.14}$$

The standard QM-solution of this problem is well known. We split the Hamiltonian in the "unperturbed" part (harmonic oscillator) and "perturbation" (unharmonicity)

$$\hat{H} = \hat{H}_0 + \hat{H}_1 \qquad \hat{H}_0 = \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\omega^2\hat{\phi}^2\,, \qquad \hat{H}_1 = \frac{\lambda}{4!}\hat{\phi}^4 \tag{1.15}$$

The energy levels and wave functions of the harmonic oscillator are known exactly:

$$\hat{H}_0|n\rangle = E_n|n\rangle\,, \qquad E_n = \left(n+\frac{1}{2}\right)\omega\,, \qquad \Psi_n(\phi) = \langle\phi|n\rangle = \text{ known} \tag{1.16}$$

The ground state of the unperturbed Hamiltonian is called "perturbative vacuum" in the QFT jargon:

$$\hat{H}_0|0\rangle = E_0|0\rangle\,, \qquad E_0 = \frac{\omega}{2}\,, \qquad \Psi_0(\phi) \equiv \langle\phi|0\rangle = \mathcal{N}e^{-\frac{\omega}{2}\phi^2} \tag{1.17}$$

where $\mathcal{N}$ is the proper normalization factor.

To the first order in λ one finds (see any QM textbook)

$$\begin{aligned} E_{\text{vac}} &= E_0 + \langle 0|\widehat{H}_1|0\rangle + \mathcal{O}(\lambda^2)\,, \\ |\Omega\rangle &= |0\rangle - \sum_{n\neq 0}|n\rangle\frac{1}{E_n - E_0}\langle n|\widehat{H}_1|0\rangle + \mathcal{O}(\lambda^2)\,, \end{aligned} \tag{1.18}$$

and then obviously

$$\langle\Omega|\hat{\phi}^2|\Omega\rangle = \langle 0|\hat{\phi}^2|0\rangle - 2\sum_{n\neq 0}\langle 0|\hat{\phi}^2|n\rangle\frac{1}{E_n - E_0}\langle n|\hat{H}_1|0\rangle + \mathcal{O}(\lambda^2) \tag{1.19}$$

Using the known expressions for E_n and Ψ_n one can calculate the remaining matrix elements easily and get the answer. This will gradually become more difficult if we want to calculate higher-order corrections in λ, terms $\mathcal{O}(\lambda^2)$ and beyond.

Note:

The QM expressions involve matrix elements between different energy states $\langle n|\ldots|0\rangle$. Why does this happen since naively one might think that the energy is conserved? The answer is of course in the Heisenberg's uncertainty principle: Since $\delta E \delta t \geq \hbar$ the energy is not conserved *locally* in time: a particle can "jump" to a higher energy level and then "fall" back, as soon as the whole process takes a short time of the order of $\hbar/(E_n - E_0)$. In QFT we will develop a different mathematical formalism to incorporate the uncertainty principle, stay tuned!

Our aim in this chapter is to illustrate how the same problem can be solved using QFT methods. So, let us take a deep breath and start...

1.1.1 Interaction representation

- Definition 1: Operators in the interaction representation

$$\begin{aligned} \hat{\phi}_I(t) &= e^{i\hat{H}_0 t}\hat{\phi}e^{-i\hat{H}_0 t}\,, \\ \hat{\pi}_I(t) &= e^{i\hat{H}_0 t}\hat{\pi}e^{-i\hat{H}_0 t}\,. \end{aligned} \tag{1.20}$$

- Def. 2: Time-ordered product of operators

$$T\{\widehat{O}_1(t_1)\widehat{O}_2(t_2)\} = \theta(t_1 - t_2)\widehat{O}_1(t_1)\widehat{O}_2(t_2) + \theta(t_2 - t_1)\widehat{O}_2(t_2)\widehat{O}_1\phi(t_1) \tag{1.21}$$

- Def. 3: Evolution operator

$$\widehat{U}(t,0) \stackrel{\text{def}}{=} e^{i\widehat{H}_0 t}e^{-i\widehat{H}t} \qquad \Rightarrow \qquad \widehat{U}^\dagger(t,0) = e^{i\widehat{H}t}e^{-i\widehat{H}_0 t}$$

$$\widehat{U}(t_1,t_2) \stackrel{\text{def}}{=} \widehat{U}(t_1,0)\widehat{U}^\dagger(t_2,0) = e^{i\widehat{H}_0 t_1}e^{-i\widehat{H}(t_1-t_2)}e^{-i\widehat{H}_0 t_2} \tag{1.22}$$

"Transitivity" property: (easy to check from definition)

$$\widehat{U}(t_1,t_2)\,\widehat{U}(t_2,t_3) = \widehat{U}(t_1,t_3) \tag{1.23}$$

- Important result (Dyson series):

$$\widehat{U}(t,0) = \text{Texp}\left[-i\int_0^t dt'\,\widehat{H}_I(t')\right], \qquad \widehat{H}_I(t) = \frac{\lambda}{4!}\hat{\phi}_I^4(t) \tag{1.24}$$

where the time-ordered exponential is defined as

$$\text{Texp}\left[-i\int_0^t dt'\,\widehat{H}_I(t')\right] \stackrel{\text{def}}{=} 1 - i\int_0^t dt'\,\widehat{H}_I(t') + (-i)^2\int_0^t dt'\int_0^{t'} dt''\,\widehat{H}_I(t')\widehat{H}_I(t'') + \ldots \tag{1.25}$$

This result should be known to you but let us prove it once again to illustrate a general method how such results can be obtained.
Note: here and in the following I will use the symbols ▶ … ◀ to indicate begin/end of a proof.

▶

I will show that the both sides of (1.24) satisfy the same first-order differential equation in time, and the same boundary condition at $t = 0$. Since the solution is unique, we will obtain the desired statement: l.h.s. = r.h.s. (left-hand-side = right-hand-side).

The boundary condition is trivial:

$$\widehat{U}(t,0)\big|_{t=0} = 1 \;=\; \text{Texp}[\ldots]\big|_{t=0} \tag{1.26}$$

Differential equation, l.h.s.:

$$\begin{aligned}\frac{d}{dt}\widehat{U}(t,0) &= e^{i\widehat{H}_0 t}i\widehat{H}_0 e^{-i\widehat{H}t} + e^{i\widehat{H}_0 t}(-i\widehat{H})e^{-i\widehat{H}t} = -ie^{i\widehat{H}_0 t}\Big(\widehat{H} - \widehat{H}_0\Big)e^{-i\widehat{H}t}\\ &= -ie^{i\widehat{H}_0 t}\Big(\frac{\lambda}{4!}\hat{\phi}^4\Big)e^{-i\widehat{H}t} = -i\,\underbrace{e^{i\widehat{H}_0 t}\Big(\frac{\lambda}{4!}\hat{\phi}^4\Big)e^{-i\widehat{H}_0 t}}\,\underbrace{e^{i\widehat{H}_0 t}e^{-i\widehat{H}t}}\\ &= \qquad -i\widehat{H}_I(t) \qquad \cdot \qquad \widehat{U}(t,0)\end{aligned} \tag{1.27}$$

Differential equation, r.h.s.:
In each term, time only appears as the upper limit of the first integral so a derivative removes it:

$$\begin{aligned}\frac{d}{dt}\text{Texp}\left[-i\int_0^t dt'\,\widehat{H}_I(t')\right] &= \frac{d}{dt}\left\{1 - i\int_0^t dt'\,\widehat{H}_I(t') + (-i)^2\int_0^t dt'\int_0^{t'} dt''\,\widehat{H}_I(t')\widehat{H}_I(t'') + \ldots\right\}\\ &= -i\widehat{H}_I(t) + (-i)\widehat{H}_I(t)(-i)\int_0^t dt''\,\widehat{H}_I(t'') + \ldots\\ &= -i\widehat{H}_I(t)\text{Texp}\left[-i\int_0^t dt'\,\widehat{H}_I(t')\right]\end{aligned} \tag{1.28}$$

which is the same equation. This completes the proof.

◀

- A similar representation can be proven for the general case

$$\boxed{\widehat{U}(t_1,t_2) = \mathrm{Texp}\left[-i\int_{t_1}^{t_2} dt'\,\widehat{H}_I(t')\right].} \tag{1.29}$$

Lecture 2

1.1.2 Master equation

Intuitive idea: if we take the unharmonic oscillator in the ground state of the harmonic oscillator $|0\rangle$, (which has higher energy than the true ground state Ω) and allow some energy loss (dissipation), then the system will gradually loose energy and eventually come to the true ground state!

Let us put this idea in formulas. In general, the time dependence of the WF is given by

$$\Psi(\phi,t) = e^{-i\widehat{H}t}\Psi(\phi,t=0) \tag{1.30}$$

We want to consider the limit

$$\lim_{T\to\infty} e^{-i\widehat{H}T(1-i\epsilon)}|0\rangle = ? \tag{1.31}$$

where ϵ is an infinitesimal number which we will put to zero at the end of the calculation, e.g. $\epsilon \to 0$ limit is taken *after* $T\to\infty$. The role of the $i\epsilon$ addition will be clear soon.

Let $|N\rangle$ be the complete set of eigenstates of the *complete* Hamiltonian

$$\widehat{H}|N\rangle = E_N|N\rangle\,, \qquad \widehat{H}|\Omega\rangle = E_{\mathrm{vac}}|\Omega\rangle\,, \qquad \sum_N |N\rangle\langle N| = \mathbb{1} \tag{1.32}$$

The last Eq. is the completeness condition, $\mathbb{1}$ is the unity operator. The sum includes $|\Omega\rangle$ and all excited states.

Then

$$\begin{aligned}
\lim_{T\to\infty} e^{-i\widehat{H}T(1-i\epsilon)}|0\rangle &= \lim_{T\to\infty} e^{-i\widehat{H}T(1-i\epsilon)}\sum_N |N\rangle\langle N|0\rangle \\
&= \lim_{T\to\infty}\left[e^{-iE_{\mathrm{vac}}T(1-i\epsilon)}|\Omega\rangle\langle\Omega|0\rangle + \sum_{N\neq\Omega} e^{-iE_N T(1-i\epsilon)}|N\rangle\langle N|0\rangle\right] \\
&= \lim_{T\to\infty} e^{-iE_{\mathrm{vac}}T(1-i\epsilon)}\left[|\Omega\rangle\langle\Omega|0\rangle + \cancel{\sum_{N\neq\Omega} e^{-i(E_N-E_{\mathrm{vac}})T(1-i\epsilon)}|N\rangle\langle N|0\rangle}\right]
\end{aligned} \tag{1.33}$$

The contribution of excited states can be neglected because each term is exponentially suppressed as $\exp[-(E_N - E_{\mathrm{vac}})T\epsilon]$ with $E_N - E_{\mathrm{vac}} > 0$ and $T\epsilon \to \infty$ with the assumed order of limits: first $T\to\infty$, then $\epsilon\to 0$.

Note that $\langle\Omega|0\rangle$ is the overlap integral between the WFs of the ground states of the unharmonic and harmonic oscillators, which we tacitly assume to be non-zero.

Thus we get, in a simplified notation

$$e^{-i\hat{H}T}|0\rangle \overset{T\to\infty}{\approx} |\Omega\rangle\langle\Omega|0\rangle e^{-iE_{\text{vac}}T(1-i\epsilon)} \tag{1.34}$$

or

$$|\Omega\rangle = e^{-i\hat{H}T}|0\rangle \cdot \left[\frac{1}{e^{-iE_{\text{vac}}T}\langle\Omega|0\rangle}\right] \qquad \text{if} \qquad T\to\infty(1-i\epsilon) \tag{1.35}$$

This is what we expected: The state $|0\rangle$ evolves to $|\Omega\rangle$ with time, up to a numerical factor (the expression in brackets).

NB: Instead of writing $T(1-i\epsilon)$ with real time T, here and below I write simply T assuming that it has a small imaginary part, it is a convenient shorthand notation.

Let us rewrite this expression slightly using

$$e^{-i\hat{H}T}|0\rangle = \underbrace{e^{-i\hat{H}T(1-i\epsilon)}e^{i\hat{H}_0T}}_{\hat{U}(0,-T)}\underbrace{e^{-i\hat{H}_0T}|0\rangle}_{e^{-iE_0T}|0\rangle} \tag{1.36}$$

and add the corresponding result for the relation of the ket-vectors, $\langle\Omega|$ and $\langle 0|$. Obtain

$$|\Omega\rangle = \hat{U}(0,-T)|0\rangle \cdot \left[e^{-i(E_{\text{vac}}-E_0)T}\langle\Omega|0\rangle\right]^{-1} \qquad \text{if} \qquad T\to\infty(1-i\epsilon)\,,$$

$$\langle\Omega| = \langle 0|\,\hat{U}(T,0)\cdot\left[e^{-i(E_{\text{vac}}-E_0)T}\langle 0|\Omega\rangle\right]^{-1} \qquad \text{if} \qquad T\to\infty(1-i\epsilon)\,. \tag{1.37}$$

Note that the second equation is *not* a hermitian conjugate of the first one because under hermitian conjugation (HC) $T(1-i\epsilon)\mapsto T(1+i\epsilon)$ and this is not what we want.

Now remember that we actually want to calculate the average value of ϕ^2 in the ground state. Thus we use the above expressions to write

$$\langle\Omega|\hat{\phi}^2|\Omega\rangle = \frac{\langle 0|\hat{U}(T,0)\hat{\phi}^2\hat{U}(0,-T)|0\rangle}{\langle 0|\Omega\rangle e^{-2i(E_{\text{vac}}-E_0)T}\langle\Omega|0\rangle}\Bigg|_{T\to\infty(1-i\epsilon)} \tag{1.38}$$

Note that here $\hat{\phi}^2$ is an operator in Schrödinger picture (time independent) which is the same as an operator in Heisenberg picture at time zero.

The last step is to go over to the interaction representation. In the numerator we can rewrite

$$\begin{aligned}\hat{U}(T,0)\hat{\phi}^2\hat{U}(0,-T) = \hat{U}(T,0)\hat{\phi}_I^2(0)\hat{U}(0,-T) \;&=\; \mathrm{T}\Big\{\hat{\phi}_I^2(0)\hat{U}(T,-T)\Big\}\\ &= \mathrm{T}\Big\{\hat{\phi}_I^2(0)\exp\Big[-i\int_{-T}^{T}dt'\,\hat{H}_I(t')\Big]\Big\}\end{aligned} \tag{1.39}$$

whereas the denominator can be written (for $T\to\infty$) as the ground state average of $\hat{U}(T,-T)$:

$$\begin{aligned}\langle 0|\hat{U}(T,-T)|0\rangle = \langle 0|e^{i\hat{H}_0T}e^{-2i\hat{H}T}e^{i\hat{H}_0T}|0\rangle &= e^{2iE_0T}\langle 0|e^{-2i\hat{H}T}|0\rangle\\ &\overset{T\to\infty}{\to} e^{2iE_0T}\langle 0|\Omega\rangle e^{-2iE_{\text{vac}}T}\langle\Omega|0\rangle\end{aligned} \tag{1.40}$$

In this way we get

$$\langle\Omega|\hat\phi^2|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\mathrm{T}\big\{\hat\phi_I^2(0)\exp\big[-i\int_{-T}^{T}dt'\,\widehat H_I(t')\big]\big\}|0\rangle}{\langle 0|\mathrm{T}\big\{\exp\big[-i\int_{-T}^{T}dt'\,\widehat H_I(t')\big]\big\}|0\rangle}$$
$$\stackrel{!}{=} \frac{\langle 0|\mathrm{T}\big\{\hat\phi_I^2(0)\exp\big[-i\int_{-\infty}^{\infty}dt'\,\widehat H_I(t')\big]\big\}|0\rangle}{\langle 0|\mathrm{T}\exp\big[-i\int_{-\infty}^{\infty}dt'\,\widehat H_I(t')\big]|0\rangle} \tag{1.41}$$

• A similar representation can be proven for the general case of a product of arbitrary number of Heisenberg operators (derivation → exercises):

$$\boxed{\langle\Omega|\hat\phi(t_1)\ldots\hat\phi(t_n)|\Omega\rangle \;=\; \frac{\langle 0|\mathrm{T}\big\{\hat\phi(t_1)\ldots\hat\phi(t_n)\exp\big[-i\int_{-\infty}^{\infty}dt'\,\widehat H_I(t')\big]\big\}|0\rangle}{\langle 0|\mathrm{T}\exp\big[-i\int_{-\infty}^{\infty}dt'\,\widehat H_I(t')\big]|0\rangle}} \tag{1.42}$$

— the master equation for calculations in the interaction representation.

At this point you certainly may ask — what is the point to rewrite the short expression on the l.h.s. in such a cumbersome and contrive way? Well, this is the next part of our program — develop effective tools to evaluate the expression on the r.h.s. in perturbation theory. This will take some time and effort, however, after the tools are at hand, you will see that the actual calculation becomes simple, in fact simpler than a calculation using conventional QM methods, especially if one progresses to higher orders in λ.

1.1.3 Creation and annihilation operators; the propagator

• To the first order in λ we can use

$$\exp\Big[-i\int_{-\infty}^{\infty}dt\,\widehat H_I(t)\Big] = 1 - i\frac{\lambda}{4!}\int_{-\infty}^{\infty}dt\,\hat\phi_I^4(t) + \mathcal{O}(\lambda^2) \tag{1.43}$$

so that from (1.41)

$$\langle\Omega|\hat\phi^2|\Omega\rangle = \langle 0|\hat\phi^2|0\rangle - i\frac{\lambda}{4!}\int_{-\infty}^{\infty}dt\Big[\langle 0|\mathrm{T}\{\hat\phi_I^4(t)\hat\phi_I^2(0)|0\rangle - \langle 0|\hat\phi_I^2(0)|0\rangle\langle 0|\hat\phi_I^4(t)|0\rangle\Big] + \mathcal{O}(\lambda^2) \tag{1.44}$$

where the first term in [...]comes from the expansion of the numerator and the second term from the denominator.
Note that $\hat\phi^2 = \hat\phi_I^2(0)$ and I will later use that $\langle 0|\hat\phi_I^4(t)|0\rangle = \langle 0|\hat\phi_I^4(0)|0\rangle$ thanks to the time-translation invariance.

It is clear that if I continue the perturbative expansion to higher-order in λ, I will generally get multiple time-ordered integrals of the matrix elements of the type

$$G_n(t_1\ldots t_n) = \langle 0|\mathrm{T}\{\hat\phi_I(t_1)\hat\phi_I(t_2)\ldots\hat\phi_I(t_n)\}|0\rangle \tag{1.45}$$

(*The times t_k may coincide; e.g., in the leading order (above) we encounter a matrix element with four fields at time t and two fields at time zero*)

In the QFT jargon, such objects are called *Green functions*: they are matrix elements of the time-ordered products of field operators in the interaction representation over the ground state of the unperturbed Hamiltonian (perturbative vacuum state). The simplest nontrivial Green function is called the *propagator*

$$\boxed{G(t-t') = \langle 0|\mathrm{T}\{\hat{\phi}_I(t)\hat{\phi}_I(t')\}|0\rangle} \tag{1.46}$$

We need to develop a general technique how to calculate such objects.

• We are going to profit from the standard formalism of creation and annihilation operators that you should know from QM lectures. Our unperturbed Hamiltonian is

$$\hat{H}_0 = \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\omega^2\hat{\phi}^2 \,. \tag{1.47}$$

Define

$$\begin{aligned}
\hat{a} &= \frac{1}{\sqrt{2\omega}}[\omega\hat{\phi} + i\hat{\pi}] && \text{annihilation operator} \\
\hat{a}^\dagger &= \frac{1}{\sqrt{2\omega}}[\omega\hat{\phi} - i\hat{\pi}] && \text{creation operator}
\end{aligned} \tag{1.48}$$

Then (easy to derive using def.)

- $[\hat{\phi}, \hat{\pi}] = i \qquad \Rightarrow \qquad [\hat{a}, \hat{a}^\dagger] = 1$
- $\widehat{H}_0 = \omega\big[\hat{a}^\dagger\hat{a} + \frac{1}{2}\big]$
- $\hat{a}|0\rangle = 0$
- $(\hat{a}^\dagger)^N|0\rangle = \mathit{const}\cdot|N\rangle$
- $[\widehat{H}_0, \hat{a}] = -\omega\hat{a}\,, \qquad [\widehat{H}_0, \hat{a}^\dagger] = +\omega\hat{a}^\dagger$ (1.49)

Using these elementary properties one can show that

$$\begin{aligned}
e^{i\widehat{H}_0 t}\hat{a}e^{-i\widehat{H}_0 t} &= \hat{a}\,e^{-i\omega t}\,, \\
e^{i\widehat{H}_0 t}\hat{a}^\dagger e^{-i\widehat{H}_0 t} &= \hat{a}^\dagger\,e^{i\omega t}
\end{aligned} \tag{1.50}$$

(e.g., using Heisenberg eq. $(d/dt)\hat{a}(t) = i[\widehat{H}_0, a(t)]$)
and therefore

$$\begin{aligned}
\phi_I(t) = e^{i\widehat{H}_0 t}\hat{\phi}e^{-i\widehat{H}_0 t} &= \frac{1}{\sqrt{2\omega}}e^{i\widehat{H}_0 t}(\hat{a}+\hat{a}^\dagger)e^{-i\widehat{H}_0 t} \\
&= \frac{1}{\sqrt{2\omega}}\Big(\hat{a}\,e^{-i\omega t} + \hat{a}^\dagger\,e^{i\omega t}\Big)
\end{aligned} \tag{1.51}$$

This representation plays the central role; let us use it to calculate the propagator:

$$\langle 0|\mathrm{T}\{\hat{\phi}_I(t)\hat{\phi}_I(t')\}|0\rangle = \frac{1}{2\omega}\theta(t-t')\langle 0|\Big(\hat{a}\,e^{-i\omega t} + \hat{a}^\dagger\,e^{i\omega t}\Big)\Big(\hat{a}\,e^{-i\omega t'} + \hat{a}^\dagger\,e^{i\omega t'}\Big)|0\rangle + (t\leftrightarrow t') \tag{1.52}$$

Here use that $\hat{a}|0\rangle = 0$ and $\langle 0|\hat{a}^\dagger = 0$. Thus a nonzero contribution can only come from the term $\propto \hat{a}\hat{a}^\dagger$ and we can replace $\hat{a}\hat{a}^\dagger = [\hat{a}, \hat{a}^\dagger] + \cancel{\hat{a}^\dagger\hat{a}} = \mathbb{1}$. Thus

$$\begin{aligned} \ldots &= \frac{1}{2\omega}\theta(t-t')\langle 0|[\hat{a}, \hat{a}^\dagger]e^{i\omega(t'-t)}|0\rangle + (t \leftrightarrow t') \\ &= \frac{1}{2\omega}\theta(t-t')e^{i\omega(t'-t)} + \frac{1}{2\omega}\theta(t'-t)e^{i\omega(t-t')} = \frac{1}{2\omega}e^{-i\omega|t-t'|} \end{aligned} \tag{1.53}$$

so that

$$\boxed{G(t-t') = \frac{1}{2\omega}e^{-i\omega|t-t'|}} \tag{1.54}$$

Lecture 3

1.1.4 Feynman diagrams

- Consider now the more complicated matrix element that we encountered in Eq. (1.44):

$$\begin{aligned} \langle 0|\mathrm{T}\{\hat{\phi}_I^4(t)\phi_I^2(0)\}|0\rangle = &\frac{1}{8\omega^3}\theta(t)\langle 0|\left(\hat{a}\,e^{-i\omega t} + \hat{a}^\dagger\,e^{i\omega t}\right)^4(\hat{a}+\hat{a}^\dagger)^2|0\rangle \\ &+ \frac{1}{8\omega^3}\theta(-t)\langle 0|(\hat{a}+\hat{a}^\dagger)^2\left(\hat{a}\,e^{-i\omega t} + \hat{a}^\dagger\,e^{i\omega t}\right)^4|0\rangle \end{aligned} \tag{1.55}$$

This produces a sum of terms containing products of creation and annihilation operators, and we can evaluate each term moving all $\hat{a}$ to the right and/or all $\hat{a}^\dagger$ to the left, where they annihilate the ground state. In order to get a nonzero contribution, all $\hat{a}, \hat{a}^\dagger$ must end up in the commutators. Thus we only need terms with three $\hat{a}$ and three $\hat{a}^\dagger$ (in this example) and we need to transform their product to the product of three commutators $[\hat{a}, \hat{a}^\dagger][\hat{a}, \hat{a}^\dagger][\hat{a}, \hat{a}^\dagger]$ decorated by the proper $e^{i\omega t}$ factors that remember from which fields these pairs originated from. This procedure of indicating the pair of $\hat{a}$ and $\hat{a}^\dagger$ that originate from particular fields and end up in a commutator is called *a Wick contraction*. One can show that summing up the terms with different time ordering ($\theta(t)$ and $\theta(-t)$) will quite generally produce the expression for the propagator (see any textbook on QFT). We define a contraction as

$$\overbrace{\phi_I(t)\phi_I}(t') \stackrel{!}{=} G(t-t') \tag{1.56}$$

and a nonzero contribution is obtained when all fields are contracted, e.g.

$$\phi_I(0)\phi_I(0)\phi_I(t)\phi_I(t)\phi_I(t)\phi_I(t) \tag{1.57}$$

The answer is obtained by summing up contributions of all possible contractions. E.g. another one is

$$\phi_I(0)\phi_I(0)\phi_I(t)\phi_I(t)\phi_I(t)\phi_I(t) \tag{1.58}$$

Since some fields are taken at the same time position, interchanging them does not make a difference. E.g. a Wick contraction

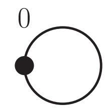
$$\phi_I(0)\phi_I(0)\phi_I(t)\phi_I(t)\phi_I(t)\phi_I(t) \tag{1.59}$$

gives the same contribution as above. Thus we can choose one and take in account other equivalent contractions by the appropriate combinatorial factor.

• Following Feynman, it has become standard to denote different contributions by pictures where different time positions are shown as points (called vertices) and each contraction (propagator) is depicted as a line — the Feynman diagrams.

The simplest one:

$$\langle 0|\mathrm{T}\{\phi_I^2(0)\}|0\rangle \quad =$$



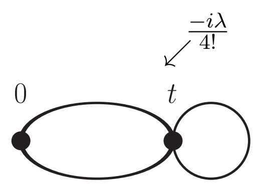
$$= \quad G(0) \tag{1.60}$$

What we need for Eq. (1.44), apart from time integration:

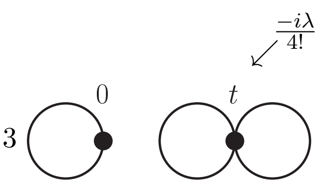
$$-i\frac{\lambda}{4!}\Big[\langle 0|\mathrm{T}\{\hat\phi_I^4(t)\hat\phi_I^2(0)|0\rangle - \langle 0|\hat\phi_I^2(0)|0\rangle\langle 0|\hat\phi_I^4(t)|0\rangle\Big] \tag{1.61}$$

• The first term comes from the numerator of the master-equation:

$$\left(-\frac{i\lambda}{4!}\right)\langle 0|\mathrm{T}\{\hat\phi_I^4(t)\hat\phi_I^2(0)\}|0\rangle \quad = \quad 12$$



$$+ \quad 3$$

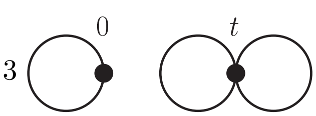


$$= 12\left(-\frac{i\lambda}{4!}\right)G^2(t)G(0) + 3\left(-\frac{i\lambda}{4!}\right)G(0)G^2(0) \tag{1.62}$$

Here "12" and "3" are the combinatorial factors. Note that it is customary and convenient to associate the factor $-\frac{i\lambda}{4!}$ with the interaction vertex (here at time t).

• The second term comes from the denominator of the master-equation:

$$\langle 0|\mathrm{T}\{\phi_I^2(0)\}|0\rangle\left(-\frac{i\lambda}{4!}\right)\langle 0|\mathrm{T}\{\phi_I^4(t)\}|0\rangle \quad = \quad 3$$



$$\tag{1.63}$$

We see that this contribution is exactly the same as the second term in (1.62) and it enters in (1.61) with a minus sign so that they cancel each other!

This is an example of the general statement: all contributions that can be separated in parts without cutting any propagator lines (called disconnected diagrams) cancel exactly (to all orders) between the contributions of the numerator and the denominator of the master-equation: The role of the denominator is to cancel contributions of all disconnected Feynman diagrams. This is a general result:

Green functions are given by a sum of *connected* Feynman diagrams

Disconnected diagrams can simply be ignored (always).

- Returning to our original problem (1.44) we get

$$\begin{aligned}\langle\Omega|\hat{\phi}^2|\Omega\rangle &= G(0) + 12\left(-\frac{i\lambda}{4!}\right)\int_{-\infty}^{\infty} dt\, G^2(t)G(0) + \mathcal{O}(\lambda^2) \\ &= \frac{1}{2\omega}\left[1 - \frac{i\lambda}{2}\frac{1}{4\omega^2}\int_{-\infty}^{\infty} dt\, e^{-2i\omega|t|}\right] + \mathcal{O}(\lambda^2) \\ &= \frac{1}{2\omega}\left[1 - \frac{\lambda}{8\omega^3} + \mathcal{O}(\lambda^2)\right] \qquad (1.64)\end{aligned}$$

which is the desired result ☺

Note that, after the machinery is there, the calculation becomes simple and can be extended to higher orders in λ in a straightforward manner…

1.2 Klein-Gordon Field

- The formalism developed above may seem to be too heavy for such a simple QM problem, but the main point is that it can be generalized from one quantum particle to a *field*, which can be viewed as a collection of oscillators sitting at different positions in space. The only thing we need to do is to add an additional argument

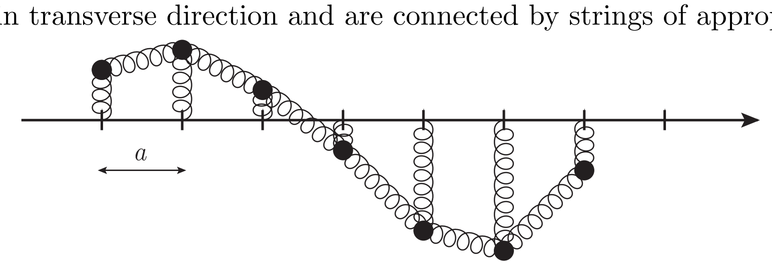
$$\text{particle}: \quad \phi(t) \quad \Rightarrow \quad \text{field}: \quad \phi(\vec{x},t) \equiv \phi(x)\,, \quad x^\mu = \{ct, \vec{x}\} \qquad (1.65)$$

If we require relativistic invariance of the action, time derivatives and space derivatives have to enter the Lagrangian (density) in a symmetric way so that a suitable generalization is

$$\begin{aligned}L &= \int d^3x\, \mathcal{L}(\dot{\phi}(\vec{x}), \phi(\vec{x}), t)\,, \\ \mathcal{L} &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \;=\; \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \qquad (1.66)\end{aligned}$$

If we throw away the unharmonicity — the ϕ^4 term – this becomes the standard Klein-Gordon Lagrangian.

The simplest physical realization — a one-dimensional chain of oscillators separated by the distance a, that can swing in transverse direction and are connected by strings of appropriate strength:



$$L(t) = a\sum_n \left[\frac{1}{2}\dot{\phi}_n^2(t) - \frac{1}{2a^2}(\phi_{n+1}(t) - \phi_n(t))^2 - \frac{1}{2}m^2\phi_n^2(t)\right] + \text{ anharmonicity} \qquad (1.67)$$

In the limit $a \to 0$ identifying $\phi_n(t) = \phi(x_n)$, $x_n = na$, $a\sum_n \to \int dx$, we obtain the Klein-Gordon Lagrangian (for one space dimension).

The corresponding Euler-Lagrange Eq. is the Klein-Gordon (KG) equation

$$(\Box + m^2)\phi = 0 \tag{1.68}$$

which has a general solution

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left(a(\vec{p})e^{-ipx} + a^*(\vec{p})e^{ipx}\right) \tag{1.69}$$

where $px \equiv p_\mu x^\mu$ with $p_0 = E_p = \sqrt{\vec{p}^2 + m^2}$. We can view this general solution as a superposition of waves with given momentum.

• So far we have classical oscillators, but we can quantize this system in the same way as above. The canonical momenta of course also acquire a coordinate dependence

$$\pi(\vec{x}, t) = \dot{\phi}(\vec{x}, t) \tag{1.70}$$

so that the KG Hamiltonian reads

$$H_0^{\rm KG} = \int d^3x \left[\frac{1}{2}\pi^2(\vec{x}, t) + \frac{1}{2}(\vec{\nabla}\phi(\vec{x}, t))^2 + \frac{1}{2}m^2\phi^2(\vec{x}, t)\right] \tag{1.71}$$

Using the representation in Eq. (1.69) it is easy to show that the Hamiltonian can be written as

$$H_0^{\rm KG} = \int \frac{d^3p}{(2\pi)^3}\, E_p\, a^*(\vec{p})a(\vec{p}) \tag{1.72}$$

so it describes a system of *noninteracting* (classical) waves with given momentum.

To quantize this system we promote $\phi(\vec{x})$, $\pi(\vec{x})$ to operators and postulate, for Schrödinger operators

$$\begin{aligned} [\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})] &= i\delta^{(3)}(\vec{x} - \vec{y})\,, \\ [\hat{\phi}(\vec{x}), \hat{\phi}(\vec{y})] &= [\hat{\pi}(\vec{x}), \hat{\pi}(\vec{y})] = 0 \end{aligned} \tag{1.73}$$

The coefficients $a(\vec{p})$ in (1.69) also become operators: $a(\vec{p}) \mapsto \hat{a}(\vec{p})$, $a^\star(\vec{p}) \mapsto \hat{a}^\dagger(\vec{p})$, with the commutation relations

$$\begin{aligned} [\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{p'})] &= (2\pi)^3\delta^{(3)}(\vec{p} - \vec{p'})\,, \\ [\hat{a}(\vec{p}), \hat{a}(\vec{p'})] &= [\hat{a}^\dagger(\vec{p}), \hat{a}^\dagger(\vec{p'})] = 0 \end{aligned} \tag{1.74}$$

and the quantum Hamiltonian corresponding to (1.72) is

$$\widehat{H}_0^{\rm KG} = \int \frac{d^3p}{(2\pi)^3}\, E_p\, \hat{a}^\dagger(\vec{p})\hat{a}(\vec{p}) + \text{ const} \tag{1.75}$$

Note that going over from (1.72) to (1.75) involves some ambiguity since the operators $\hat{a}^\dagger$ and $\hat{a}$ do not commute, so that the order in which they are written does matter. For this (still simple) system, this ambiguity reduces to the possibility to add a constant which is usually fixed by the requirement the the ground state energy of the system is taken to be zero. This is different from the usual QM choice to add 1/2, cf. (1.49) but apart from that the similarity is obvious.

Note that the field operator $\hat{\phi}(x)$ is naturally the Heisenberg operator, indeed

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left(\hat{a}(\vec{p})e^{-ipx} + \hat{a}^\dagger(\vec{p})e^{ipx}\right) \qquad \boxed{px \equiv E_p t - \vec{p}\vec{x}}$$

$$= e^{i\widehat{H}_0^{\text{KG}}t}\left[\int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left(\hat{a}(\vec{p})e^{i\vec{p}\vec{x}} + \hat{a}^\dagger(\vec{p})e^{-i\vec{p}\vec{x}}\right)\right]e^{-i\widehat{H}_0^{\text{KG}}t} \tag{1.76}$$

cf. Eq. (1.51).

• We see that the difference to the case of a single oscillator is minimal, and therefore if we add a small anharmonicity to the Lagrangian/Hamiltonian (term $\sim \hat{\phi}^4$), we can develop the perturbation theory in the same way as before. Thus we end up with very much the same master equation

$$\boxed{\langle\Omega|\hat{\phi}(x_1)\dots\hat{\phi}(x_n)|\Omega\rangle \;=\; \frac{\langle 0|\mathrm{T}\big\{\hat{\phi}(x_1)\dots\hat{\phi}(x_n)\exp\left[i\int d^4x\,\widehat{\mathcal{L}}_I(x)\right]\big\}|0\rangle}{\langle 0|\mathrm{T}\exp\left[i\int d^4x\,\widehat{\mathcal{L}}_I(x)\right]|0\rangle}} \tag{1.77}$$

In order to arrive at this expression I used

$$\int_{-\infty}^{\infty} dt\,\widehat{H}_I(t) = -\int_{-\infty}^{\infty} dt\int d^3\vec{x}\,\widehat{\mathcal{L}}_I(\vec{x},t) \tag{1.78}$$

• Let us calculate the propagator of the KG field:

$$\begin{aligned}
G_0(x-y) &= \langle 0|\mathrm{T}\,\{\hat{\phi}(x),\hat{\phi}(y)\}|0\rangle \\
&= \theta(x_0-y_0)\langle 0|\int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\left(\hat{a}(\vec{p})e^{-ipx} + \hat{a}^\dagger(\vec{p})e^{ipx}\right) \\
&\qquad\times\int \frac{d^3p'}{(2\pi)^3\sqrt{2E_{p'}}}\left(\hat{a}(\vec{p'})e^{-ip'y} + \hat{a}^\dagger(\vec{p'})e^{ip'y}\right)|0\rangle + (x\leftrightarrow y) \\
&= \theta(x_0-y_0)\int \frac{d^3p\,d^3p'}{(2\pi)^6\sqrt{2E_p}\sqrt{2E_{p'}}}e^{-ipx+ip'y}\langle 0|\hat{a}(\vec{p})\hat{a}^\dagger(\vec{p'})|0\rangle + (x\leftrightarrow y) \\
&= \theta(x_0-y_0)\int \frac{d^3p}{(2\pi)^3 2E_p}e^{-iE_p(x_0-y_0)+i\vec{p}(\vec{x}-\vec{y})} + (x\leftrightarrow y)
\end{aligned} \tag{1.79}$$

This is again almost the same formula as for the oscillator, with E_p playing the role of frequency and added integration over all possible momenta.

As well known, this expression can be rewritten in another form, introducing an auxiliary integration over p_0

$$G_0(x-y) = \int_{-\infty}^{\infty} \frac{dp_0}{2\pi i} \int \frac{d^3p}{(2\pi)^3} e^{-ip_0(x_0-y_0)+i\vec{p}(\vec{x}-\vec{y})} \frac{1}{m^2 - p_0^2 + \vec{p}^2 - i\epsilon}$$
$$\stackrel{!}{=} \int \frac{d^4p}{(2\pi)^4 i} e^{-ip(x-y)} \frac{1}{m^2 - p^2 - i\epsilon} \tag{1.80}$$

The above expression is recovered doing the p_0 integration with the help of Cauchy's residue theorem after which in the exponent $p_0 \mapsto E_p$. This representation is standard in QFT applications and is actually nothing but a different way to implement the Heisenberg's uncertainty principle. In conventional QM formalism energy is not conserved (for short period of time) so that the particle can jump to higher levels and fall back. In the QFT formalism, the four-momentum variable p_μ is strictly conserved at any interaction point, but the zero component p_0 only becomes "true" energy $p_0 \to \sqrt{m^2 + \vec{p}^2}$ in the integral sense. It can deviate from "true" energy locally in time during the interaction process. We call such particles, with $p_0 \neq \sqrt{m^2 + \vec{p}^2}$, *virtual* in contradistinction to *real* particles with $p_0 = \sqrt{m^2 + \vec{p}^2}$.

A final comment: I have used here the usual nonrelativistic QM convention for the normalization of states, $\langle N|N'\rangle = \delta_{NN'}$, or $\langle p|p'\rangle = \delta^{(3)}(\vec{p} - \vec{p'})$. In a relativistic field theory it is customary to use a different normalization $\langle p|p'\rangle = 2E_p\delta^{(3)}(\vec{p} - \vec{p'})$ to ensure good Lorentz trafo properties. Changing to the relativistic convention corresponds to the redefinition $\hat{a}(\vec{p}) \mapsto \sqrt{2E_p}\hat{a}(\vec{p})$, $\hat{a}^\dagger(\vec{p}) \mapsto \sqrt{2E_P}\hat{a}^\dagger(\vec{p})$. The basic commutation relation (1.74) becomes $[\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{p'})] = (2\pi)^3 2E_p \delta^{(3)}(\vec{p} - \vec{p'})$ and the integration measure in the expansion of the field over annihilation and creation operators changes to $\int d^3p/\sqrt{2E_p} \mapsto \int d^3p/(2E_p)$.

1.3 Pauli's spin-statistics theorem

Let us look at the motivation for the commutation relations (1.73), (1.74) more closely. The major physical assumption behind them:

- **Microcausality:**

$$[\hat{\phi}(x), \hat{\phi}(y)] = 0 \qquad \text{if} \qquad (x-y)^2 < 0 \tag{1.81}$$

(The fields at space-like separations cannot influence each other and can be measured simultaneously; thus in QM formalism they are described by commuting operators)
In particular for equal times

$$\forall \vec{x}, \vec{y} \qquad [\hat{\phi}(\vec{x}, t), \hat{\phi}(\vec{y}, t)] = 0 \tag{1.82}$$

Write (here relativistic normalization convention)

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \left(\hat{a}(\vec{p}) e^{-iE_pt + i\vec{p}\vec{x}} + \hat{a}^\dagger(\vec{p}) e^{iE_pt - i\vec{p}\vec{x}} \right)$$
$$= \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-i\vec{p}\vec{x}} \left(\hat{a}(-\vec{p}) e^{-iE_pt} + \hat{a}^\dagger(\vec{p}) e^{iE_pt} \right) \tag{1.83}$$

Then

$$\hat{a}(-\vec{p})e^{-iE_pt} + \hat{a}^\dagger(\vec{p})e^{iE_pt} = 2E_p \int d^3x\, e^{i\vec{p}\vec{x}}\hat{\phi}(\vec{x},t) \tag{1.84}$$

and therefore

$$\begin{aligned}[\hat{a}(-\vec{p})e^{-iE_pt} + \hat{a}^\dagger(\vec{p})e^{iE_pt}, \hat{a}(-\vec{p'})e^{-iE_{p'}t} + \hat{a}^\dagger(\vec{p'})e^{iE_{p'}t}] &= 2E_p 2E_{p'} \int d^3x d^3y\, e^{i\vec{p}\vec{x}} e^{i\vec{p'}\vec{y}} [\phi(\vec{x},t), \phi(\vec{y},t)] \\ &= 0 \end{aligned} \tag{1.85}$$

Thus $\forall p, p', t$

$$\begin{aligned}\forall p, p', t \qquad 0 = {} & e^{i(E_p+E_{p'})t}[\hat{a}^\dagger(\vec{p}), \hat{a}^\dagger(\vec{p'})] + e^{i(E_p-E_{p'})t}[\hat{a}^\dagger(\vec{p}), \hat{a}(-\vec{p'})] \\ & + e^{i(-E_p+E_{p'})t}[\hat{a}(-\vec{p}), \hat{a}^\dagger(\vec{p'})] + e^{i(-E_p-E_{p'})t}[\hat{a}(-\vec{p}), \hat{a}(-\vec{p'})]\end{aligned} \tag{1.86}$$

Since $E_p > 0$ (we do not want states with negative energy), the first and the last term must vanish separately as they have a different time dependence from the rest. Thus

$$\forall \vec{p}, \vec{p'} \qquad [\hat{a}^\dagger(\vec{p}), \hat{a}^\dagger(\vec{p'})] = [\hat{a}(\vec{p}), \hat{a}(\vec{p'})] = 0 \tag{1.87}$$

whereas nonzero $[a(\vec{p}), \hat{a}^\dagger(\vec{p'})] \neq 0$ is not forbidden. Proceeding in the similar way as above one can derive

$$[a(\vec{p}), \hat{a}^\dagger(\vec{p'})] = e^{i(E_p-E_{p'})t} \int d^3x d^3y\, e^{-i\vec{p}\vec{x}} e^{i\vec{p'}\vec{y}} \Big\{ iE_{p'} \left[\tfrac{d}{dt}\hat{\phi}(\vec{x},t), \hat{\phi}(\vec{y},t)\right] - iE_p \left[\hat{\phi}(\vec{x},t), \tfrac{d}{dt}\hat{\phi}(\vec{y},t)\right] \Big\} \tag{1.88}$$

Here $\left[\frac{d}{dt}\hat{\phi}(\vec{x},t), \hat{\phi}(\vec{y},t)\right]$ must vanish for $\vec{x} \neq \vec{y}$ from microcausality, but does not need to vanish at $\vec{x} = \vec{y}$. The minimum assumption is

$$\left[\hat{\phi}(\vec{x},t), \tfrac{d}{dt}\hat{\phi}(\vec{y},t)\right] = i\delta^{(3)}(\vec{x}-\vec{y}) \qquad \Rightarrow \qquad [\hat{a}(p), \hat{a}^\dagger(p')] = (2\pi)^3 2E_p\, \delta^{(3)}(\vec{p}-\vec{p'}) \tag{1.89}$$

This is how the canonical commutation rules arise.

• From the commutation relations and the expression for the Hamiltonian it is easy to see that applying $a^\dagger(\vec{p})$ to an arbitrary state increases its energy by E_p and applying $a(\vec{p})$ decreases energy of the state by E_p. Thus if we start with some eigenstate of H with finite (positive) energy and apply $a(\vec{p})$ many times, we will eventually come to a state with negative energy that we do not want to exist. The only way to avoid this is to assume that, applying $a(\vec{p})$ many times, at some point this procedure has to break up: we annihilate the state (but not get another state with lower energy). We call this state vacuum, $|0\rangle$, and by definition $\forall \vec{p}\ a(\vec{p})|0\rangle = 0$. Energy of the vacuum state can be usually set to zero (convention).

All other states (with higher energy) can be obtained by applying creation operators to the vacuum state. The one particle state is defined as $|\vec{p}\rangle = \hat{a}^\dagger(\vec{p})|0\rangle$, two-particle states are $|\vec{p}_1\vec{p}_2\rangle = \hat{a}^\dagger(\vec{p}_1)\hat{a}^\dagger(\vec{p}_2)|0\rangle$, etc. Since the creation operators for different momenta commute, one obtains

$$|\vec{p}_1\vec{p}_2\rangle = \hat{a}^\dagger(\vec{p}_1)\hat{a}^\dagger(\vec{p}_2)|0\rangle = \hat{a}^\dagger(\vec{p}_2)\hat{a}^\dagger(\vec{p}_1)|0\rangle = |\vec{p}_2\vec{p}_1\rangle \tag{1.90}$$

so that interchanging particle momenta we obtain the same state — with the same wave function. This is the statement of Pauli's spin-statistics theorem: particles quantized using the canonical commutation relation (1.89) (scalars) have a symmetric wave function under the interchange of their momenta (bosons).

• What about fermions?

Similar to bosons we to consider solutions of the Dirac equation (Dirac spinors) and promote them to field operators. Again similar to bosons, we expand Dirac spinors in plane waves and have to promote the to operators:

$$\hat{\psi}(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_{s=\pm 1/2} \left\{ e^{ipx} v_s(p) \hat{b}_s^\dagger(p) + e^{-ipx} u_s(p) \hat{a}_s(p) \right\} \tag{1.91}$$

and postulate the relativistic generalization of Heisenberg equations (the same we do for scalars, I skipped this for brevity)

$$\frac{\partial \hat{\psi}(x)}{\partial x^\mu} = i[\widehat{\mathbf{P}}_\mu, \hat{\psi}(x)]\,, \qquad \widehat{\mathbf{P}}_\mu = \begin{pmatrix} \widehat{H} \\ \widehat{\vec{P}} \end{pmatrix} \tag{1.92}$$

where $\widehat{\mathbf{P}}_\mu$ is the four-energy-momentum operator. It follows that

$$\begin{aligned} &[\widehat{\mathbf{P}}_\mu, \hat{a}_s^\dagger(\vec{p})] = p_\mu \hat{a}_s^\dagger(\vec{p}) && [\widehat{\mathbf{P}}_\mu, \hat{b}_s^\dagger(\vec{p})] = p_\mu \hat{b}_s^\dagger(\vec{p}) \\ &[\widehat{\mathbf{P}}_\mu, \hat{a}_s(\vec{p})] = -p_\mu \hat{a}_s(\vec{p}) && [\widehat{\mathbf{P}}_\mu, \hat{b}_s(\vec{p})] = -p_\mu \hat{b}_s(\vec{p}) \end{aligned} \tag{1.93}$$

and similar as for scalars we have to assume existence of the vacuum state that is annihilated by both $\hat{a}_s(\vec{p})$ and $\hat{b}_s(\vec{p})$ (otherwise negative energy states):

$$\hat{a}_s(\vec{p})|0\rangle = 0 \qquad \hat{b}_s(\vec{p})|0\rangle = 0 \tag{1.94}$$

and all physical states are obtained by the application of $\hat{a}_s^\dagger(\vec{p})$ (electrons, quarks) or $\hat{b}_s^\dagger(\vec{p})$ (positrons, antiquarks).

So far so good, but now we have to decide what commutation relations we want to postulate. The first idea is of course to copy-paste the same equations as we had above for the KG field:

$$(?) \qquad [\hat{a}_s(\vec{p}), \hat{a}_{s'}^\dagger(\vec{p'})] = [\hat{b}_s(\vec{p}), \hat{b}_{s'}^\dagger(\vec{p'})] \;=\; \delta_{ss'}(2\pi)^3 2E_p \delta^{(3)}(\vec{p} - \vec{p'}) \tag{1.95}$$

This does not work, however, because using these relations one obtains a non-vanishing equal time commutator

$$[\hat{\psi}(\vec{x},t), \hat{\bar{\psi}}(\vec{y},t)] \neq 0 \qquad \text{for} \qquad \vec{x} \neq \vec{y} \qquad (?!) \tag{1.96}$$

(Here $\bar{\psi} = \psi^\dagger \gamma_0$ is a Dirac conjugate spinor). One may try to escape by observing that the Dirac field itself is not directly observable, and all physical quantities are expressed in terms of bilinear expressions of the type $\bar{\psi}\Gamma\psi$. Thus one needs to enforce microcausality for them and not for ψ itself. This is a good point, but using (1.95) one obtains non-vanishing equal-time commutators also for Dirac bilinears, so it does no help. Thus we have to conclude that the assumption in

(1.95) is wrong and therefore quantum spin-1/2 particles (quantized solutions of Dirac Eq.) are *not* bosons. What are they? Lecture 4

The solution was found in 1927 by Jordan and Wigner who observed that the microcausality condition for bilinear products of Dirac fields

$$[\hat{\bar{\psi}}(\vec{x},t)\Gamma_1\hat{\psi}(\vec{x},t),\hat{\bar{\psi}}(\vec{y},t)\Gamma_2\hat{\psi}(\vec{y},t)]=0 \qquad \text{for} \qquad \vec{x}\neq\vec{y} \tag{1.97}$$

can be fulfilled by assuming *anticommutativity* of the creation and annihilation operators:

$$\begin{aligned}
\{\hat{a}_s(\vec{p}),\hat{a}^\dagger_{s'}(\vec{p'})\} &= \{\hat{b}_s(\vec{p}),\hat{b}^\dagger_{s'}(\vec{p'})\} = \delta_{ss'}(2\pi)^3 2E_p\delta^{(3)}(\vec{p}-\vec{p'})\\
\{\hat{a}_s(\vec{p}),\hat{a}_{s'}(\vec{p'})\} &= \{\hat{b}_s(\vec{p}),\hat{b}_{s'}(\vec{p'})\} = \{\hat{a}^\dagger_s(\vec{p}),\hat{a}^\dagger_{s'}(\vec{p'})\} = \{\hat{b}^\dagger_s(\vec{p}),\hat{b}^\dagger_{s'}(\vec{p'})\} = 0\\
\{\hat{a}^\dagger_s(\vec{p}),\hat{b}^\dagger_{s'}(\vec{p'})\} &= \{\hat{a}^\dagger_s(\vec{p}),\hat{b}_{s'}(\vec{p'})\} = \{\hat{a}_s(\vec{p}),\hat{b}^\dagger_{s'}(\vec{p'})\} = \{\hat{a}_s(\vec{p}),\hat{b}_{s'}(\vec{p'})\} = 0
\end{aligned} \tag{1.98}$$

With this assumption one gets for the fields

$$\{\hat{\psi}(\vec{x},t),\hat{\psi}(\vec{y},t)\}=\{\hat{\bar{\psi}}(\vec{x},t),\hat{\bar{\psi}}(\vec{y},t)\}=0\,, \qquad \{\hat{\psi}(\vec{x},t),\hat{\bar{\psi}}(\vec{y},t)\}=\gamma_0\delta^{(3)}(\vec{x}-\vec{y}) \tag{1.99}$$

and one can check (1.97) using

$$[AB,CD]=A\{B,C\}D-AC\{B,D\}-C\{A,D\}B+\{C,A\}DB$$

For the two-particle state one obtains in this case (e.g., two electrons with spins s_1,s_2)

$$|(\vec{p}_1,s_1);(\vec{p}_2,s_2)\rangle=\hat{a}^\dagger_{s_1}(\vec{p}_1)\hat{a}^\dagger_{s_2}(\vec{p}_2)|0\rangle=-\hat{a}^\dagger_{s_2}(\vec{p}_2)\hat{a}^\dagger_{s_1}(\vec{p}_1)|0\rangle=-|(\vec{p}_2,s_2);(\vec{p}_1,s_1)\rangle \tag{1.100}$$

Thus Dirac particles are fermions!

1.4 Quantum Electrodynamics

- The QED Lagrangian is

$$\mathcal{L}(x)=-\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)+\bar{\psi}(x)(i\not{D}-m)\psi(x) \tag{1.101}$$

where the first term describes the electromagnetic field (Euler-Lagrange Eqs. = Maxwell equations).

We can try to quantize the electromagnetic field as above. Choose Coulomb gauge:

$$\vec{\nabla}\cdot\vec{A}=\partial_k A_k=0\,, \qquad k=1,2,3 \tag{1.102}$$

In this gauge there are no time derivatives in $\mathcal{L}$ of the scalar potential $A_\mu=\{\Phi,\vec{A}\}$ so that the corresponding Maxwell Eq. is the same as in electrostatics

$$\Delta\Phi=\rho(\vec{x},t) \tag{1.103}$$

and is solved by

$$\Phi(\vec{x},t)=-\int d^3x'\,\frac{\rho(\vec{x'},t)}{4\pi|\vec{x}-\vec{x'}|} \qquad \text{Coulomb potential} \tag{1.104}$$

Thus Φ in Coulomb gauge it is a given function (for a given charge distribution) and not a dynamic variable. Let us assume there are no external charges; then $\Phi = 0$ and we are left with three components of the vector potential $\vec{A}$ which we can take as generalized coordinates.

The corresponding generalized momenta are the electric fields

$$\pi_k = \frac{\partial \mathcal{L}}{\partial \dot{A}_k} = E_k \tag{1.105}$$

and we get the Hamiltonian for the Maxwell field in vacuum

$$\begin{aligned} &\mathcal{L}_F = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) = \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) \\ \Rightarrow \\ &H = \int d^3x\,(p_k\dot{A}_k - \mathcal{L}) = \int d^3x\,\frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right) \end{aligned} \tag{1.106}$$

So far everything is classical, let us now quantize the theory.

- To start with, we promote the vector potential and electric field to operators in Hilbert space

$$A(\vec{x},t) \mapsto \widehat{A}(\vec{x},t)\,, \qquad\qquad E(\vec{x},t) \mapsto \widehat{E}(\vec{x},t) \tag{1.107}$$

and have to assume some commutation relations. A natural option would be to try (like for KG field)

$$\begin{aligned} &[\widehat{A}_i(\vec{x},t), \widehat{A}_j(\vec{y},t)] = [\widehat{\pi}_i(\vec{x},t), \widehat{\pi}_j(\vec{y},t)] = 0 \\ &[\widehat{\pi}_i(\vec{x},t), \widehat{A}_j(\vec{y},t)] = [\widehat{E}_i(\vec{x},t), \widehat{A}_j(\vec{y},t)] = -i\delta_{ij}\delta^{(3)}(\vec{x}-\vec{y}) \end{aligned} \tag{1.108}$$

Problem: the last relation is in contradiction with the Gauss law:

$$\vec{\nabla}\cdot\vec{E} = 0 \tag{1.109}$$

Indeed

$$\begin{aligned} &[\widehat{E}_i(\vec{x},t), \widehat{A}_j(\vec{y},t)] = -i\delta_{ij}\delta^{(3)}(\vec{x}-\vec{y}) \\ \Rightarrow \\ &[\nabla_i\widehat{E}_i(\vec{x},t), \widehat{A}_j(\vec{y},t)] = -i\frac{\partial}{\partial x_j}\delta^{(3)}(\vec{x}-\vec{y}) \qquad (?!) \end{aligned} \tag{1.110}$$

This is a real difficulty.

- QED is in fact a simple theory (we will later see why) and can be saved by a trick. It is known that only two transverse polarizations of the electromagnetic field are physically relevant, If we expand the field in plane waves, we can keep these two polarizations only

$$\widehat{\vec{A}}(x) = \int \frac{d^3k}{(2\pi)^3 2k_0}\sum_{\lambda=1,2}\left[\vec{e}_{(\lambda)}(\vec{k})\hat{a}_\lambda(\vec{k})e^{ikx} + \vec{e}^{\,*}_{(\lambda)}(\vec{k})\hat{a}^\dagger_\lambda(\vec{k})e^{-ikx}\right] \tag{1.111}$$

where $(\vec{k}\cdot\vec{e}_{(1,2)}(\vec{k})) = 0$, $k_0 = E_k = |\vec{k}|$, and impose nontrivial commutation relations for the creation/annihilation operators of physical polarizations only:

$$[\hat{a}_\lambda(\vec{k})\hat{a}^\dagger_{\lambda'}(\vec{k'})] = (2\pi)^3 2k_0\delta_{\lambda\lambda'}\delta^{(3)}(\vec{x}-\vec{y}) \tag{1.112}$$

This will lead to

$$[\widehat{E}_i(\vec{x},t), \widehat{A}_j(\vec{y},t)] = -i\delta^{\perp}_{ij}\delta^{(3)}(\vec{x}-\vec{y}) \tag{1.113}$$

where the object on the r.h.s. is defined as

$$\delta^{\perp}_{ij}\delta^{(3)}(\vec{x}-\vec{y}) \stackrel{!}{=} \int \frac{d^3k}{(2\pi)^3}\left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}\right) e^{-i\vec{k}(\vec{x}-\vec{y})} \tag{1.114}$$

On can show that in this way the Gauss law will be satisfied. This does not work for more complicated (nonabelian) gauge theories, however, (like QCD) so we need another method.

• To summarize this Section: canonical quantization of field theories has its roots in usual quantum mechanics and is rather intuitive. It has its limits, however, so we will now proceed to develop a different approach, called path-integral quantization.

2 Path Integral in Quantum Mechanics

Let us consider a simple system with one degree of freedom (particle) as example

$$q: \quad \text{coordinate} \qquad\qquad p: \quad \text{momentum} \tag{2.1}$$

(one-dimensional for simplicity)

I will use notations (used already)

$$\hat{q}|q\rangle = q|q\rangle \qquad \text{eigenstates of the position operator} \tag{2.2}$$

$$\hat{p}|p\rangle = p|p\rangle \qquad \text{eigenstates of the momentum operator} \tag{2.3}$$

Completeness condition:

$$\int dq\,|q\rangle\langle q| = \mathbb{1}\,, \qquad \int \frac{dp}{2\pi}\,|p\rangle\langle p| = \mathbb{1}\,, \tag{2.4}$$

Then, e.g.

$$\Psi_p(q) = \langle q|p\rangle = e^{iqp} \tag{2.5}$$

— coordinate space wave function of the state $|p\rangle$ (particle with momentum p)

2.1 Path-integral representation for the transition amplitude

• Transition amplitude is defined as a probability amplitude of the transition

$$|q_1\rangle_{\text{time}=t_1} \mapsto |q_2\rangle_{\text{time}=t_2} \tag{2.6}$$

and is given by the appropriate matrix element of the evolution operator

$$\langle q_2|e^{-i\widehat{H}(t_2-t_1)}|q_1\rangle \tag{2.7}$$

Indeed

$$|2\rangle = e^{-i\widehat{H}(t_2-t_1)}|1\rangle \qquad \Leftrightarrow \qquad \langle q_2|2\rangle = \langle q_2|e^{-i\widehat{H}(t_2-t_1)}|q_1\rangle \tag{2.8}$$

and inserting the unity operator this can be rewritten as

$$\langle q_2|2\rangle = \int dq_1\, \langle q_2|e^{-i\widehat{H}(t_2-t_1)}|q_1\rangle\langle q_1|1\rangle \tag{2.9}$$

We will derive a representation for the transition amplitude in the form of a *path integral*. This result was obtained by Feynman and had profound importance for the development of QFT and statistical physics.

Let

$$t_i = \text{ initial time} \qquad\qquad t_f = \text{ final time} \tag{2.10}$$

and cut the whole time interval $t_f - t_i$ in N equal slices

$$t_f \equiv t_N > t_{N-1} > t_{N-2} > \ldots > t_0 \equiv t_i \tag{2.11}$$

Let

$$\Delta t = t_k - t_{k-1} = \frac{t_f - t_i}{N} \qquad \forall k \tag{2.12}$$

Using transitivity property of the evolution operator we can write

$$e^{-i\widehat{H}(t_f-t_i)} = e^{-i\widehat{H}(t_N-t_{N-1})}e^{-i\widehat{H}(t_{N-1}-t_{N-2})}\ldots e^{-i\widehat{H}(t_1-t_0)} \tag{2.13}$$

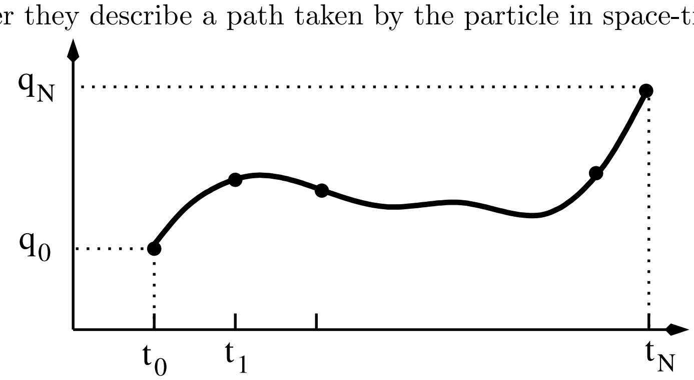
and therefore

$$\langle q_f|e^{-i\widehat{H}(t_f-t_i)}|q_i\rangle = \int dq_{N-1}\ldots dq_1\langle q_N|e^{-i\widehat{H}\Delta t}|q_{N-1}\rangle\langle q_{N-1}|e^{-i\widehat{H}\Delta t}|q_{N-2}\rangle\ldots\langle q_1|e^{-i\widehat{H}\Delta t}|q_0\rangle \tag{2.14}$$

We can identify the integration variables as particle positions at intermediate times

$$q_k \stackrel{!}{=} q(t_k) \tag{2.15}$$

so that taken together they describe a path taken by the particle in space-time



In the limit $N \to \infty$ the path becomes continuous.

The multidimensional integral $\int dq_{N-1} \ldots dq_1$ can then be viewed as an integral over all possible paths (trajectories) that the particle can take on its way from q_i to q_f:

$$\mathcal{D}q(t) \stackrel{!}{=} \prod_{k=1}^{N-1} dq_k \tag{2.16}$$

— **the path integral**

• We will now show that the integrand in the path integral can be written as e^{iS} where S is the classical action calculated along the path.

• To this end, write

$$\langle q_k|e^{-i\widehat{H}\Delta t}|q_{k-1}\rangle = \int \frac{dp_k}{2\pi} \langle q_k|p_k\rangle\langle p_k|e^{-i\widehat{H}\Delta t}|q_{k-1}\rangle\,, \qquad \langle q_k|p_k\rangle = e^{iq_kp_k} \tag{2.17}$$

Let $H(p,q)$ be the classical Hamiltonian. The quantum Hamiltonian operator is obtained by the substitution

$$q \mapsto \hat{q}\,, \qquad p \mapsto \hat{p}\,, \qquad [\hat{q},\hat{p}] = i\,. \tag{2.18}$$

Since $\hat{q}$ and $\hat{p}$ do not commute, this procedure is not unique. Let us assume that $\widehat{H}$ is written in such a form that all momentum operators always stand before the coordinate operators. [This can always be done, rewriting "wrongly ordered" terms as "correctly ordered" plus commutators]. In this case

$$\langle p_k|e^{-i\widehat{H}(\hat{p},\hat{q})\Delta t}|q_{k-1}\rangle = e^{-iH(p_k,q_{k-1})\Delta t}\langle p_k|q_{k-1}\rangle = e^{-iH(p_k,q_{k-1})\Delta t}e^{-ip_kq_{k-1}} \tag{2.19}$$

and therefore

$$\langle q_k|e^{-i\widehat{H}\Delta t}|q_{k-1}\rangle = \int \frac{dp_k}{2\pi} e^{-iH(p_k,q_{k-1})\Delta t}e^{ip_k(q_k-q_{k-1})} \tag{2.20}$$

To evaluate (2.14) we need a product of such factors for $k = 1, \ldots, N-1$ so we will get a multiple integral over the momenta p_k. Following the same logic as above, we can identify $p_k = p(t_k)$ and introduce a path integral in momentum space

$$\mathcal{D}p(t) \stackrel{!}{=} \prod_{k=1}^{N-1} \frac{dp_k}{2\pi} \tag{2.21}$$

In this way we get

$$\begin{aligned}\langle q_f|e^{-i\widehat{H}(t_f-t_i)}|q_i\rangle = \int\mathcal{D}q(t)\int\mathcal{D}p(t)\,\exp\Big[ip_N(q_N-q_{N-1})+\ldots+ip_1(q_1-q_0)\Big]\\ \times\exp\Big[-i\Delta t\Big(H(p_N,q_{N-1})+\ldots+H(p_1,q_0)\Big)\Big]\end{aligned} \tag{2.22}$$

and in the limit $N \to \infty$ (whether this limit exists is a very delicate question) one obtains

$$\boxed{\langle q_f|e^{-i\widehat{H}(t_f-t_i)}|q_i\rangle = \int\mathcal{D}q(t)\int\mathcal{D}p(t)\,\exp\Big\{i\int_{t_i}^{t_f}dt\Big[p(t)\dot{q}(t)-H\big(p(t),q(t)\big)\Big]\Big\}} \tag{2.23}$$

—— **path integral in phase space**

This is the most general form of the path integral which should be used as a starting point in complicated situations.

- The most important special case is

$$H(p,q) = \frac{1}{2}p^2 + V(q) \tag{2.24}$$

(quadratic dependence on momentum). In this case the path integral over momenta can be taken explicitly.

By definition

$$\int \mathcal{D}p(t)\,\exp\Big\{i\int_{t_i}^{t_f} dt\Big[p(t)\dot q(t) - \frac{1}{2}p^2(t)\Big]\Big\} = \lim_{N\to\infty}\prod_{k=1}^{N}\int_{-\infty}^{\infty}\frac{dp_k}{2\pi}\exp\Big\{i\Delta t\sum_k [p_k\dot q_k - \frac{1}{2}p_k^2]\Big\}\,,$$

$$p_k \equiv p(t_k)\,,\quad \dot q_k \equiv \dot q(t_k) \tag{2.25}$$

which is nothing but a product of a large number of Gaussian integrals:

$$\int_{-\infty}^{\infty} dx\, e^{\frac{1}{2}Ax^2 - bx} = \sqrt{\frac{2\pi}{A}}e^{\frac{b^2}{2A}} \qquad \begin{array}{l} x \leftrightarrow p_k \\ A \leftrightarrow i\Delta t \\ b \leftrightarrow -i\Delta t\dot q_k \end{array} \tag{2.26}$$

so that get

$$\ldots = \lim_{N\to\infty}\prod_{k=1}^{N}\frac{1}{2\pi}\sqrt{\frac{2\pi}{i\Delta t}}e^{-\frac{1}{2}(\Delta t)^2\dot q_k^2/(i\Delta t)}$$

$$= \underbrace{\lim_{N\to\infty}\left(\frac{N}{2\pi i(t_f - t_i)}\right)^{N/2}}_{???}\exp\Big\{\frac{i}{2}\int_{t_i}^{t_f} dt\,\dot q^2(t)\Big\} \tag{2.27, 2.28}$$

The normalization factor looks horrible, let us denote it as $\mathcal{N}$ for the time being (we will see how to handle it in what follows). Thus obtain

$$\begin{aligned}\langle q_f|e^{-i\widehat H(t_f-t_i)}|q_i\rangle &= \mathcal{N}\int\mathcal{D}q(t)\exp\Big\{i\int_{t_i}^{t_f}dt\Big[\frac{1}{2}\dot q^2(t) - V\big(q(t)\big)\Big]\Big\}\\ &= \mathcal{N}\int\mathcal{D}q(t)\exp\Big\{i\int_{t_i}^{t_f}dt\,L(q,\dot q)\Big\}\\ &= \mathcal{N}\int\mathcal{D}q(t)\exp\Big\{iS[q]\Big\}\end{aligned} \tag{2.29}$$

which is the final result:

$$\boxed{\langle q_f|e^{-i\widehat H(t_f-t_i)}|q_i\rangle = \mathcal{N}\int_{q(t_i)=q_i}^{q(t_f)=q_f}\mathcal{D}q(t)\exp\{iS[q]\}} \tag{2.30}$$

where I added the boundary conditions: the path integral goes over all paths starting at q_i at time t_i and ending at q_f at time t_f.

Lecture 5

2.2 Euclidean version of the path integral

- Let $|N\rangle$ be the energy eigenstates

$$\widehat{H}|N\rangle = E_N|N\rangle \tag{2.31}$$

so that

$$\begin{aligned}\langle q_f|e^{-i\widehat{H}(t_f-t_i)}|q_i\rangle &= \sum_N \langle q_f|N\rangle e^{-iE_N(t_f-t_i)}\langle N|q_i\rangle \\ &= \sum_N \Psi_N(q_f)e^{-iE_N(t_f-t_i)}\Psi_N^*(q_i)\end{aligned} \tag{2.32}$$

and make a formal substitution

$$t \mapsto -i\tau\,, \qquad\qquad \tau = it \tag{2.33}$$

If $\tau \in \mathbb{R}$, t becomes imaginary — **imaginary time**

Then

$$\begin{aligned}\langle q_f|e^{-\widehat{H}(\tau_f-\tau_i)}|q_i\rangle &= \sum_N \Psi_N(q_f)e^{-E_N(\tau_f-\tau_i)}\Psi_N^*(q_i) \\ &\overset{\tau_f-\tau_i\to\infty}{\longrightarrow} \Psi_0(q_f)e^{-E_0(\tau_f-\tau_i)}\Psi_0^*(q_i)\end{aligned} \tag{2.34}$$

— only the ground state contribution survives at $\tau_f - \tau_i \to \infty$. This can be useful or useless depending on a particular problem.

- With this substitution, the relativistic interval becomes

$$q^2 = t^2 - \vec{q}^2 \mapsto -(\tau^2 + q_1^2 + q_2^2 + q_3^2) = -q_E^2 \tag{2.35}$$

— Euclidean metric (overall sign minus)

For this reason one usually speaks of going over to imaginary time as going over from Minkowski to Euclidean metric in space-time.

Euclidean version of the theory is useful if one is interested in properties of the ground state (or low-lying states, in general). We will see this soon.

- Euclidean version of the path integral is obviously

$$\begin{aligned}\langle q_f|e^{-\widehat{H}(\tau_f-\tau_i)}|q_i\rangle &= \mathcal{N}\int \mathcal{D}q(\tau)\exp\left\{\int_{\tau_i}^{\tau_f} d\tau \left[-\frac{1}{2}\dot{q}^2(\tau) - V(q(\tau))\right]\right\} \qquad ! \text{ here } \dot{q} = \frac{d}{d\tau}q(\tau) \\ &= \mathcal{N}\int \mathcal{D}q(\tau)\exp\left\{-\int_{\tau_i}^{\tau_f} d\tau\, \mathcal{L}_E(q,\dot{q})\right\} \\ &= \mathcal{N}\int \mathcal{D}q(\tau)\exp\left\{-\int_{\tau_i}^{\tau_f} d\tau\, S_E[q]\right\}\end{aligned} \tag{2.36}$$

— **Euclidean action**.

!!! Notice sign change for the potential

$$S_M = \int_{t_i}^{t_f} dt \Big[\frac{1}{2}\left(\frac{dq}{dt}\right)^2 - V(q)\Big]$$

$$S_E = \int_{\tau_i}^{\tau_f} d\tau \Big[\frac{1}{2}\left(\frac{dq}{d\tau}\right)^2 + V(q)\Big] \tag{2.37}$$

The equations of motion (classical)

$$\begin{aligned} \delta S_E &= S_E[q(\tau) + \delta q(\tau)] - S_E[q(\tau)] \\ &= \int_{\tau_i}^{\tau_f} d\tau\, \delta q(\tau) \left[-\frac{d^2 q}{d\tau^2} + V'(q)\right] \stackrel{!}{=} 0 \end{aligned} \tag{2.38}$$

so that

$$\frac{d^2 q}{d\tau^2} = \ddot{q} = +V'(q) \qquad\qquad \text{! sign} \tag{2.39}$$

— Newton equation for the potential $\boxed{-V(q)}$

- **Feynman-Kac formula**

Consider a special case $q_i = q_f$ and integrate over all q_i. Obtain

$$\int dq_i \, \langle q_i | e^{-\widehat{H}(\tau_f - \tau_i)} | q_i \rangle = \int_{\circlearrowleft} \mathcal{D}q(\tau)\, e^{-S_E[q]} \tag{2.40}$$

The expression on the l.h.s. can be viewed as a definition of the trace of an operator, thus

$$\mathrm{Tr}\, \left\{ e^{-\widehat{H}(\tau_f - \tau_i)} \right\} = \int_{\circlearrowleft} \mathcal{D}q(\tau)\, e^{-S_E[q]} \tag{2.41}$$

Finally, make a formal substitution

$$\tau_f - \tau_i \mapsto \beta \equiv \frac{1}{k_B T} \tag{2.42}$$

and interpret k_B as Bolzmann's constant, T as temperature. Thus

$$\boxed{\mathrm{Tr}\, \left\{ e^{-\beta \widehat{H}} \right\} = \int_{\circlearrowleft} \mathcal{D}q(\tau)\, e^{-S_E[q]}} \tag{2.43}$$

The path integral goes over all closed paths that come back to the initial position after (imaginary) time $1/\beta$

— This is a **path-integral representation of a statistical sum**.

2.3 Semiclassical expansion

- We are working in the system of units where $\hbar = 1$ and restoring this factor

$$\langle q_f | e^{-\widehat{H}(\tau_f - \tau_i)} | q_i \rangle = \int \mathcal{D}q(t)\, e^{-\frac{1}{\hbar} S_E[q]} \tag{2.44}$$

Semiclassical approximation means $\hbar \to 0$, in other words the action is very large in units of $\hbar$. The path integral will then be dominated by the trajectories where the action is as small as possible.

For the time being everything is Euclidean and I do not write the subscript "E" for brevity

Take some fixed path, $q_0(\tau)$, and expand to the second order in the deviation:

$$\begin{aligned} S[q_0 + \delta q] = S[q_0] &+ \int_{\tau_i}^{\tau_f} d\tau\, \delta q(\tau) \left[-\frac{d^2 q_0}{d\tau^2} + V'(q_0) \right] \\ &+ \int_{\tau_i}^{\tau_f} d\tau \left[-\frac{1}{2} \delta q \frac{d^2}{d\tau^2} \delta q + \frac{1}{2} (\delta q)^2 V''(q_0) \right] + \mathcal{O}(\delta_q^3) \end{aligned} \tag{2.45}$$

Now assume that $q_0(\tau)$ satisfies classical equations of motion (EOM) $\Rightarrow$ the linear term in δq vanishes.

We want to take into account contributions of all trajectories in the path integral that are not far away from the classical path.

$$\begin{aligned} &q(\tau) = q_0(\tau) + \delta q(\tau)\,, \\ &q_0(\tau_i) = q_i\,, \qquad q_0(\tau_f) = q_f\,, \qquad \Rightarrow \quad \delta q(\tau_i) = \delta q(\tau_f) = 0 \end{aligned} \tag{2.46}$$

- How can one integrate over *all* such trajectories? One has to parametrize them in some way and integrate over the parameters.

Let $x_n(\tau)$ be a complete set of eigenfunctions of the equation

$$-\frac{d^2}{d\tau^2} x_n(\tau) + V''\big(q_0(\tau)\big) x_n(\tau) = \epsilon_n x_n(\tau) \tag{2.47}$$

with the above boundary conditions, $x_n(\tau_i) = x_n(\tau_f) = 0$. Assume that the solutions are normalized as

$$\int_{\tau_i}^{\tau_f} d\tau\, x_n(\tau) x_m(\tau) = \delta_{mn} \tag{2.48}$$

These solutions form a complete set of functions so that arbitrary function (path) $\delta q(\tau)$ can be written as a superposition of $x_n(\tau)$

$$\delta q(\tau) = \sum_n c_n x_n(\tau) \tag{2.49}$$

Thus in order to integrate over all $\delta q(\tau)$ we can integrate over all coefficients:

$$\mathcal{D}q(\tau) = \mathcal{D}\delta q(\tau) = \prod_n dc_n \tag{2.50}$$

On the other hand

$$\int_{\tau_i}^{\tau_f} d\tau \left[-\frac{1}{2}\delta q \frac{d^2}{d\tau^2}\delta q + \frac{1}{2}(\delta q)^2 V''(q_0)\right] = \sum_n \sum_m c_n c_m \int_{\tau_i}^{\tau_f} d\tau\, x_n(\tau)\left[-\frac{1}{2}\frac{d^2}{d\tau^2} + \frac{1}{2}V''(q_0)\right] x_m(\tau)$$
$$= \frac{1}{2}\sum_n \epsilon_n c_n^2 \tag{2.51}$$

Therefore if terms of higher order in δq can be neglected

$$\int \mathcal{D}q\, e^{-S[q]} = \int (\prod_n dc_n)\, e^{-S[q_0]-\frac{1}{2}\sum_n \epsilon_n c_n^2} \tag{2.52}$$

which splits in an (infinite) product of Gaussian integrals

$$\int_{-\infty}^{\infty} dc\, e^{-\frac{1}{2}\epsilon c^2} = \frac{\sqrt{2\pi}}{\epsilon^{1/2}} \tag{2.53}$$

Thus we get, to semiclassical accuracy

$$\langle q_f|e^{-\widehat{H}(\tau_f-\tau_i)}|q_i\rangle = e^{-S[q_0]}\mathcal{N}\left(\prod_n \epsilon_n\right)^{-1/2} \tag{2.54}$$

The product of eigenvalues defines a determinant of the differential operator

$$\det\left(-\frac{d^2}{d\tau^2} + V''(q_0(\tau))\right) \stackrel{!}{=} \prod_n \epsilon_n \tag{2.55}$$

so that

$$\boxed{\langle q_f|e^{-\widehat{H}(\tau_f-\tau_i)}|q_i\rangle = e^{-S_0}\mathcal{N}\left[\det\left(-\frac{d^2}{d\tau^2} + V''(q_0(\tau))\right)\right]^{-1/2}} \tag{2.56}$$

where $S_0 = S[q_0]$ is the action on the classical trajectory.

Example: Harmonic oscillator

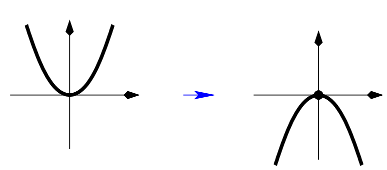
Take

$$V(q) = \frac{1}{2}\omega^2 q^2 \qquad \begin{array}{ll} \tau_i = 0\,, & \tau_f = T \\ q_i = 0\,, & q_f = 0 \end{array} \tag{2.57}$$

We want to calculate

$$\langle q_f = 0|e^{-\widehat{H}T}|q_i = 0\rangle \ = \ ? \tag{2.58}$$

Note that in transition to Euclidean space the potential changes sign



and the only possibility (classically) to start at the top of the mountain and return to the top after time T is just to stay at the top all the time!

Thus the solution of classical EOM is obviously

$$q_0(\tau) = 0 \tag{2.59}$$

It follows

$$S_E[q_0] = 0 \qquad\qquad V''(q_0) = \omega^2 \tag{2.60}$$

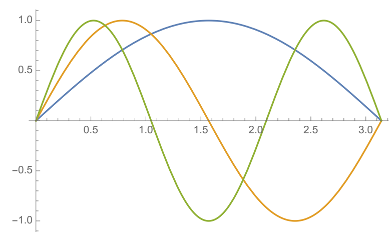
and therefore

$$\langle q=0|e^{-\widehat{H}T}|q=0\rangle = \mathcal{N}\left[\det\left(-\frac{d^2}{d\tau^2}+\omega^2\right)\right]^{-1/2} \tag{2.61}$$

To calculate the determinant we need solutions of

$$\begin{gathered}\left(-\frac{d^2}{d\tau^2}+\omega^2\right)x_n(\tau) = \epsilon_n x_n(\tau)\\ x_n(0) = x_n(T) = 0\end{gathered} \tag{2.62}$$

Fortunately this is simple:



$$x_n(\tau) \sim \sin\left(\frac{\pi n}{T}\tau\right), \qquad \epsilon_n = \frac{\pi^2 n^2}{T^2}+\omega^2, \qquad n = 1,2,3,\ldots \tag{2.63}$$

To proceed, split the determinant in a product of two factors:

$$\mathcal{N}\left[\det\left(-\frac{d^2}{d\tau^2}+\omega^2\right)\right]^{-1/2} = \underbrace{\mathcal{N}\left[\det\left(-\frac{d^2}{d\tau^2}\right)\right]^{-1/2}}_{(I)}\underbrace{\left[\frac{\det\left(-\frac{d^2}{d\tau^2}+\omega^2\right)}{\det\left(-\frac{d^2}{d\tau^2}\right)}\right]^{-1/2}}_{(II)} \tag{2.64}$$

so that

$$\begin{aligned}(I) &= \mathcal{N}\left[\prod_{n=1}^{\infty}\epsilon_n(\omega\to 0)\right]^{-1/2} = \mathcal{N}\left[\prod_{n=1}^{\infty}\frac{\pi^2 n^2}{T^2}\right]^{-1/2} \qquad ☹\\ (II) &= \left[\prod_{n=1}^{\infty}\frac{\epsilon_n}{\epsilon_n(\omega\to 0)}\right]^{-1/2} = \left[\prod_{n=1}^{\infty}\left(1+\frac{\omega^2 T^2}{\pi^2 n^2}\right)\right]^{-1/2} \qquad ☺\end{aligned} \tag{2.65}$$

Our representation for (I) is not useful, but we can calculate this factor by observing that it corresponds to the free motion of the particle without any potential:

$$
\begin{aligned}
(I) = \langle q_f = 0|e^{-\frac{1}{2}\hat{p}^2 T}|q_i = 0\rangle &= \int \frac{dp}{2\pi} \langle q_f = 0|e^{-\frac{1}{2}\hat{p}^2 T}|p\rangle\langle p|q_i = 0\rangle \\
&= \int\limits_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-\frac{1}{2}p^2 T} \underbrace{|\langle q = 0|p\rangle|^2}_{=1} = \frac{1}{\sqrt{2\pi T}}
\end{aligned} \tag{2.66}
$$

To calculate (II) use Eq. 1.431.2 from Gradshtein&Ryzhik book:

$$
\pi y \prod_{n=1}^{\infty} \left(1 + \frac{y^2}{n^2}\right) = \sinh(\pi y)\,, \qquad y \leftrightarrow \frac{\omega T}{\pi} \tag{2.67}
$$

Collecting everything

$$
\langle q_f = 0|e^{-\widehat{H}T}|q_i = 0\rangle = \frac{1}{\sqrt{2\pi T}} \left(\frac{\sinh \omega T}{\omega T}\right)^{-1/2} = \left(\frac{\omega}{\pi}\right)^{1/2} \Big(2\sinh \omega T\Big)^{-1/2} \tag{2.68}
$$

Expanding at $T \to \infty$

$$
\langle q_f = 0|e^{-\widehat{H}T}|q_i = 0\rangle = \left(\frac{\omega}{\pi}\right)^{1/2} e^{-\frac{1}{2}\omega T}\left[1 + \frac{1}{2}e^{-2\omega T} + \ldots\right] = \sum_N |\Psi_N(0)|^2 e^{-E_N T} \tag{2.69}
$$

Thus we read from our expansion

$$
\begin{aligned}
E_0 &= \frac{1}{2}\omega\,, & |\Psi_0(0)|^2 &= \left(\frac{\omega}{\pi}\right)^{1/2} \\
E_2 &= \left(\frac{1}{2} + 2\right)\omega\,, & |\Psi_2(0)|^2 &= \frac{1}{2}\left(\frac{\omega}{\pi}\right)^{1/2}
\end{aligned} \tag{2.70}
$$

etc. Note that odd N do not contribute for our choice $q_i = q_f = 0$ because the wave function at the origin vanishes.

Two comments are in order:
— the semiclassical result for harmonic oscillator is exact
— notable applications: double well potential, instantons, sphaleron transitions ('t Hooft) *Lecture 6*

2.4 Path integral representation for Green functions

In what follows I will use a compact notation for time differences

$$
t_{12} \stackrel{!}{=} t_1 - t_2\,, \tag{2.71}
$$

etc.
As an example consider a two-point function

$$
G(t_1, t_2) = \langle \Omega|\mathrm{T}\{\hat{\phi}(t_1)\hat{\phi}(t_2)\}|\Omega\rangle \tag{2.72}
$$

- As the first step, let us show that

$$G(t_1,t_2) = \theta(t_{12}) \lim_{\substack{t_f\to+\infty \\ t_i\to-\infty}} \frac{\langle 0|e^{-i\widehat{H}t_{f1}}\hat{\phi}\, e^{-i\widehat{H}t_{12}}\hat{\phi}\, e^{-i\widehat{H}t_{2i}}|0\rangle}{\langle 0|e^{-i\widehat{H}t_{fi}}|0\rangle} + (t_1 \leftrightarrow t_2) \tag{2.73}$$

►

Let $t_1 > t_2$.
Using the definition of Heisenberg operators the Green function can be written as

$$\langle\Omega|e^{i\widehat{H}t_1}\hat{\phi}e^{-i\widehat{H}t_1}e^{i\widehat{H}t_2}\hat{\phi}e^{-i\widehat{H}t_2}\}|\Omega\rangle = e^{iE_{vac}t_1}\langle\Omega|\hat{\phi}e^{-i\widehat{H}t_{12}}\hat{\phi}|\Omega\rangle e^{-iE_{vac}t_2} \tag{2.74}$$

For the expression on the r.h.s. use

$$\begin{aligned} e^{-i\widehat{H}t_{2i}}|0\rangle &\overset{t_i\to-\infty}{=} e^{-iE_{vac}t_{2i}}|\Omega\rangle\langle\Omega|0\rangle \\ \langle 0|e^{-i\widehat{H}t_{f1}} &\overset{t_f\to+\infty}{=} \langle 0|\Omega\rangle\langle\Omega|e^{-iE_{vac}t_{f1}} \end{aligned} \tag{2.75}$$

Then

$$\begin{aligned} \text{numerator} &= |\langle 0|\Omega\rangle|^2 e^{-iE_{vac}(t_{f1}+t_{2i})}\langle\Omega|\hat{\phi}e^{-i\widehat{H}t_{12}}\hat{\phi}|\Omega\rangle \\ \text{denominator} &= |\langle 0|\Omega\rangle|^2 e^{-iE_{vac}t_{fi}} \end{aligned} \tag{2.76}$$

and

$$\text{r.h.s.} = \frac{\text{numerator}}{\text{denominator}} = e^{iE_{vac}t_{12}}\langle\Omega|\hat{\phi}e^{-i\widehat{H}t_{12}}\hat{\phi}|\Omega\rangle \tag{2.77}$$

This is the same expression as in (2.74) so that our relation is correct for $t_1 > t_2$. For inverse time ordering the proof is the same.

◄

- The second step is to derive a representation for (2.73) as a ratio of path integrals.

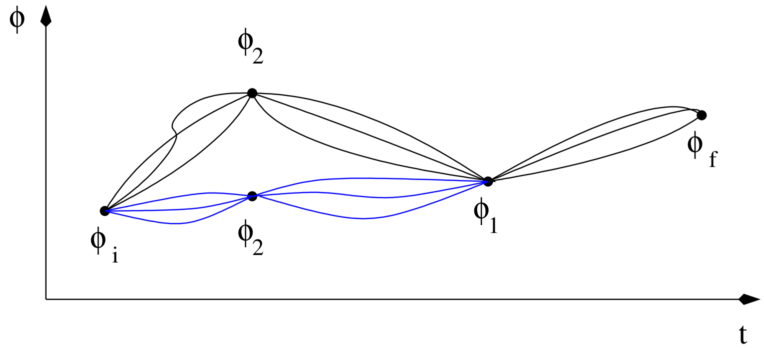
The denominator is simple:

$$\begin{aligned} \text{denominator}(t_1,t_2) &= \langle 0|\,\mathbb{1}\, e^{-i\widehat{H}t_{fi}}\,\mathbb{1}\,|0\rangle \\ &= \int d\phi_f \int d\phi_i \langle 0|\phi_f\rangle\langle\phi_f|e^{-i\widehat{H}t_{fi}}|\phi_i\rangle\langle\phi_i|0\rangle \\ &= \int d\phi_f\, e^{-\frac{\omega}{2}\phi_f^2} \int d\phi_i\, e^{-\frac{\omega}{2}\phi_i^2} \int\limits_{\phi(t_i)=\phi_i}^{\phi(t_f)=\phi_f} \mathcal{D}\phi(t)\, e^{iS[\phi]} \end{aligned} \tag{2.78}$$

The numerator is more lengthy:

$$\begin{aligned} \text{denominator}(t_1,t_2) &= \langle 0|e^{-i\widehat{H}t_{f1}}\hat{\phi}\,\mathbb{1}\, e^{-i\widehat{H}t_{12}}\hat{\phi}\,\mathbb{1}\, e^{-i\widehat{H}t_{2i}}|0\rangle \\ &= \int d\phi_1 \int d\phi_2\, \langle 0|e^{-i\widehat{H}t_{f1}}\hat{\phi}\,|\phi_1\rangle\langle\phi_1|e^{-i\widehat{H}t_{12}}\hat{\phi}\,|\phi_2\rangle\langle\phi_2|e^{-i\widehat{H}t_{2i}}|0\rangle \\ &= \int d\phi_1 \int d\phi_2\, \langle 0|e^{-i\widehat{H}t_{f1}}|\phi_1\rangle\phi_1\langle\phi_1|e^{-i\widehat{H}t_{12}}|\phi_2\rangle\phi_2\langle\phi_2|e^{-i\widehat{H}t_{2i}}|0\rangle \end{aligned} \tag{2.79}$$

Here we still need to rewrite the matrix elements over the ground state as integrals with the ground state wave function, and get a product of transition amplitudes for which the path integral rep. is known.



It is clear that after the integration over $\phi_1 = \phi(t_1)$ and $\phi_2 = \phi(t_2)$ everything combines into the single path integral from $\phi_i = \phi(t_i)$ to $\phi_f = \phi(t_f)$ (weighted with wave functions of the ground state) so we get

$$\text{numerator}(t_1, t_2) = \int d\phi_f\, \Psi_0^*(\phi_f) \int d\phi_i\, \Psi_0(\phi_i) \int\limits_{\phi(t_i)=\phi_i}^{\phi(t_f)=\phi_f} \mathcal{D}\phi(t)\, \phi(t_1)\phi(t_2) e^{iS[\phi]} \tag{2.80}$$

Thus we obtain

$$G(t_1, t_2) = \frac{\int\limits_{|0\rangle, t_i}^{|0\rangle, t_f} \mathcal{D}\phi(t)\, \phi(t_1)\phi(t_2) e^{iS[\phi]}}{\int\limits_{|0\rangle, t_i}^{|0\rangle, t_f} \mathcal{D}\phi(t)\, e^{iS[\phi]}} \tag{2.81}$$

where I used a shorthand notation

$$\int_{|0\rangle, t_i}^{|0\rangle, t_f} \mathcal{D}\phi \equiv \int d\phi_f\, \Psi_0^*(\phi_f) \int d\phi_i\, \Psi_0(\phi_i) \int_{\phi(t_i)=\phi_i}^{\phi(t_f)=\phi_f} \mathcal{D}\phi \tag{2.82}$$

This result can be generalized to an arbitrary number of fields

$$\boxed{\langle\Omega|\mathrm{T}\{\hat{\phi}(t_1)\ldots\hat{\phi}(t_n)\}|\Omega\rangle = \frac{\int\limits_{|0\rangle, t_i}^{|0\rangle, t_f} \mathcal{D}\phi(t)\, \phi(t_1)\ldots\phi(t_n) e^{iS[\phi]}}{\int\limits_{|0\rangle, t_i}^{|0\rangle, t_f} \mathcal{D}\phi(t)\, e^{iS[\phi]}}} \tag{2.83}$$

where it is assumed that $t_i \to -\infty$ and $t_f \to +\infty$.
Note:

- The fields $\phi(t_k)$ in the path integral are functions and can be written in any order.
- Divergent normalization factors cancel in the ratio of path integrals

2.5 Perturbation theory and the generating functional

Let

$$
\begin{aligned}
S &= S_0 + S_I \,, \\
S_0 &= \int dt \left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\omega^2\phi^2 \right] \qquad \text{(Harmonic oscillator)} \\
S_I &= \int dt \left[-\frac{\lambda}{4!}\phi^4(t) \right]
\end{aligned} \tag{2.84}
$$

Now we do not have any operators so forget time ordering and one only needs to Taylor-expand exponential of the action:

$$
e^{iS[\phi]} = e^{iS_0[\phi]}\left\{ 1 - \frac{i\lambda}{4!}\int dt'\, \phi^4(t') + \frac{1}{2}\left(-\frac{i\lambda}{4!}\right)^2 \int dt_1' \int dt_2' \phi^4(t_1')\phi^4(t_2') + \dots \right\} \tag{2.85}
$$

— a perturbation theory.
Thus everything we want to know can be calculated in terms of

$$
\left(-\frac{i\lambda}{4!}\right)^k \int\limits_{|0\rangle}^{|0\rangle} \mathcal{D}\phi(t)\, \phi(t_1)\dots\phi(t_n)\, \phi^4(t_1')\dots\phi^4(t_k')\, e^{iS_0[\phi]} \tag{2.86}
$$

— multiparticle Green functions with unperturbed action!

They can be calculated (all!) using a tool called

- **Generating functional**

$$
\boxed{ Z[J] \stackrel{!}{=} \frac{\int\limits_{|0\rangle,t_i}^{|0\rangle,t_f} \mathcal{D}\phi(t)\, \exp\left[iS_0[\phi] + i\int dt\, J(t)\phi(t) \right]}{\int\limits_{|0\rangle,t_i}^{|0\rangle,t_f} \mathcal{D}\phi(t)\, \exp\left[iS_0[\phi] \right]} } \tag{2.87}
$$

– a harmonic oscillator driven by a (weak) time-dependent external force.

Expanding at $J(t) \to 0$:

$$
\begin{aligned}
Z[J] = 1 &+ i\int dt\, J(t)\, G_0(t) + \frac{i^2}{2}\int dt_1 dt_2\, J(t_1)J(t_2)\, G_0(t_1,t_2) \\
&+ \frac{i^3}{6}\int dt_1 dt_2 dt_3\, J(t_1)J(t_2)J(t_3)\, G_0(t_1,t_2,t_3) + \dots
\end{aligned} \tag{2.88}
$$

where

$$
\begin{aligned}
G_0(t) &= \langle 0|\hat{\phi}(t)|0\rangle \,, \\
G_0(t_1,t_2) &= \langle 0|\mathrm{T}\{\hat{\phi}(t_1)\hat{\phi}(t_2)\}|0\rangle \,, \\
G_0(t_1,t_2,t_3) &= \langle 0|\mathrm{T}\{\hat{\phi}(t_1)\hat{\phi}(t_2)\hat{\phi}(t_3)\}|0\rangle \,,
\end{aligned} \tag{2.89}
$$

etc. One can write

$$G_0(t_1,\dots,t_n) = (-i)^n \frac{\delta}{\delta J(t_1)} \cdots \frac{\delta}{\delta J(t_n)} Z[J]\big|_{J(t)=0} \tag{2.90}$$

where the functional derivatives are defined as

$$\frac{\delta}{\delta J(t_1)} J(t_2) = \delta(t_1 - t_2) \tag{2.91}$$

- How can one calculate $Z[J]$?

Idea:

$$\frac{\int dx\, e^{-Ax^2+Jx}}{\int dx\, e^{-Ax^2}} = \frac{\int dx\, e^{-A(x-\frac{J}{2A})^2+\frac{1}{4}\frac{J^2}{A}}}{\int dx\, e^{-Ax^2}} = \exp\left(\frac{1}{4}\frac{J^2}{A}\right) \tag{2.92}$$

Complications:

- We have functions $x \to x(t)$, $J \to J(t)$ not numbers
- We have operators $A \to \hat{A}$ and need $\hat{A}^{-1}$. Does it exist?

So, let us proceed slowly and carefully.

- We start with the expression in the numerator

$$\int d\phi_f \int d\phi_i\, e^{-\frac{\omega}{2}\phi_f^2 - \frac{\omega}{2}\phi_i^2} \int\limits_{\phi(t_i)=\phi_i}^{\phi(t_f)=\phi_f} \mathcal{D}\phi(t)\, \exp\left\{ i \int\limits_{t_i}^{t_f} dt\, \left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\omega^2\phi^2 + J\phi\right]\right\} \tag{2.93}$$

and make a shift of the integration "variable" in the path integral

$$\phi(t) \mapsto \phi(t) + \eta(t) \tag{2.94}$$

where $\eta(t)$ is some function. Also

$$\phi_{f,i} \mapsto \phi_{f,i} + \eta_{f,i}\,, \qquad \eta_f = \eta(t_f)\,, \quad \eta_i = \eta(t_i) \tag{2.95}$$

We will eventually choose it in such a way that the linear term in $\phi(t)$ vanishes, but wait.

We have in the exponent:

$$\begin{aligned}
&-\frac{\omega}{2}\phi_f^2 - \frac{\omega}{2}\phi_i^2 + i\int\limits_{t_i}^{t_f} dt\, \left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\omega^2\phi^2 + J\phi\right] \quad \mapsto \\
&-\frac{\omega}{2}(\phi_f+\eta_f)^2 - \frac{\omega}{2}(\phi_i+\eta_i)^2 + i\int\limits_{t_i}^{t_f} dt\, \left[\frac{1}{2}(\dot{\phi}+\dot{\eta})^2 - \frac{1}{2}\omega^2(\phi+\eta)^2 + J(\phi+\eta)\right] \\
= &-\frac{\omega}{2}\eta_f^2 - \frac{\omega}{2}\eta_i^2 + iS_0[\eta] + i\int_{t_i}^{t_f} dt\, J\eta && (I) \\
&-\omega\phi_f\eta_f - \omega\phi_i\eta_i + i\int_{t_i}^{t_f} dt\left[\dot{\phi}\dot{\eta} - \omega^2\phi\eta + J\phi\right] && (II) \\
&-\frac{\omega}{2}\phi_f^2 - \frac{\omega}{2}\phi_i^2 + iS_0[\phi] && (III)
\end{aligned} \tag{2.96}$$

- In (II) rewrite integrating by parts

$$i\int_{t_i}^{t_f} dt\, \frac{d\phi}{dt}\frac{d\eta}{dt} = i\phi\dot{\eta}\Big|_{t_i}^{t_f} - i\int_{t_i}^{t_f} dt\, \phi(t)\frac{d^2}{dt^2}\eta(t)$$

$$= i\phi_f\dot{\eta}(t_f) - i\phi_i\dot{\eta}(t_i) - i\int_{t_i}^{t_f} dt\, \phi(t)\frac{d^2}{dt^2}\eta(t) \tag{2.97}$$

Then

$$(II) = \phi_f\Big[i\dot{\eta}(t_f) - \omega\eta_f\Big] + \phi_i\Big[-i\dot{\eta}(t_i) - \omega\eta_f\Big] + i\int_{t_i}^{t_f} dt\, \phi(t)\Big[\Big(-\frac{d^2}{dt^2} - \omega^2\Big)\eta(t) + J(t)\Big] \tag{2.98}$$

We want to choose η that $(II) = 0$ for arbitrary $\phi(t)$. This requires
A) Differential equation to be satisfied

$$\Big(\frac{d^2}{dt^2} + \omega^2\Big)\eta(t) = J(t) \tag{2.99}$$

B) Boundary conditions to be satisfied

$$i\dot{\eta}(t_f) = \omega\eta(t_f)\,,$$

$$i\dot{\eta}(t_i) = -\omega\eta(t_i) \tag{2.100}$$

This means that we need solutions such that

$$\eta(t) \sim e^{-i\omega t} \qquad \text{for} \quad t \to +\infty \qquad (t_f)$$

$$\eta(t) \sim e^{+i\omega t} \qquad \text{for} \quad t \to -\infty \qquad (t_i) \tag{2.101}$$

The solution is then

$$\eta(t) = i\int_{t_i}^{t_f} dt'\, D(t-t')J(t') \tag{2.102}$$

where $D(t-t')$ is the particular Green function of the differential operator (2.99)

$$\Big(\frac{d^2}{dt^2} + \omega^2\Big)D(t-t') = -i\delta(t-t') \tag{2.103}$$

that satisfies the same boundary conditions:

$$D(t-t') = \frac{1}{2\omega}\Big[\theta(t-t')e^{-i\omega(t-t')} + \theta(t'-t)e^{+i\omega(t-t')}\Big] = \frac{1}{2\omega}e^{-i\omega|t-t'|} \tag{2.104}$$

- Now we have to substitute our solution for $\eta(t)$ in (I). We have

$$(I) = -\frac{\omega}{2}\eta_f^2 - \frac{\omega}{2}\eta_i^2 + i\int_{t_i}^{t_f} dt\,\Big[\frac{1}{2}\dot{\eta}^2 - \frac{1}{2}\omega^2\eta^2\Big] + i\int_{t_i}^{t_f} dt\, J(t)\eta(t)$$

$$\stackrel{p.I}{=} \cancel{-\frac{\omega}{2}(\eta_f^2 + \eta_i^2) + \frac{i}{2}\eta\dot{\eta}\Big|_{t_i}^{t_f}} + i\int_{t_i}^{t_f} dt\, \eta(t)\underbrace{\Big[-\frac{1}{2}\frac{d^2}{dt^2} - \frac{1}{2}\omega^2\Big]\eta(t)}_{=-\frac{1}{2}J(t)} + i\int_{t_i}^{t_f} dt\, J(t)\eta(t)$$

$$= \frac{1}{2}i\int_{t_i}^{t_f} dt\, J(t)\eta(t) = -\frac{1}{2}\int dt dt'\, J(t)D(t-t')J(t') \tag{2.105}$$

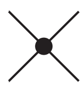
Finally, (III) gets cancelled by the denominator so we get

$$\boxed{Z[J] = \exp\left[-\frac{1}{2}\int dtdt'\, J(t)D(t-t')J(t')\right]} \tag{2.106}$$

From this answer for the generating functional we immediately get the Green functions:

$$\begin{aligned}
G_0(t_1,t_2) &= (-i)^2 \frac{\delta}{\delta J(t_1)}\frac{\delta}{\delta J(t_2)} Z[J]\big|_{J(t)=0} = D(t_1-t_2)\,, \\
G_0(t_1,t_2,t_3) &= 0 \\
G_0(t_1,t_2,t_3,t_4) &= G_0(t_1-t_2)G_0(t_3-t_4) + G_0(t_1-t_3)G_0(t_2-t_4) + G_0(t_1-t_4)G_0(t_2-t_3) \\
&\text{etc.}
\end{aligned} \tag{2.107}$$

$\Leftrightarrow$ Wick theorem $\Rightarrow$ Feynman Diagrams:

- ∘  $\quad \dfrac{-i\lambda}{4!}\displaystyle\int dt$ Vertex
- ∘ 

$\quad D(t_1-t_2)$ Propagator
- ∘ Combinatorial Factors (2.108)

$\Leftrightarrow$ same rules as in canonical quantization.
To summarize: path integral quantization in quantum mechanics is equivalent to canonical quantization, at least in perturbation theory. Offers new and powerful methods. *Lecture 7*

3 Scalar field theory

- We can overtake this formalism as a *definition* of a quantum field theory

$$\phi(t) \mapsto \phi(\vec{x},t) = \phi(x) \tag{3.1}$$

$$\boxed{\langle\Omega|\mathrm{T}\{\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)\}|\Omega\rangle = \frac{\int_{|0\rangle,t_i}^{|0\rangle,t_f} \mathcal{D}\phi\,\phi(x_1)\ldots\phi(x_n)e^{iS[\phi]}}{\int_{|0\rangle,t_i}^{|0\rangle,t_f} \mathcal{D}\phi\, e^{iS[\phi]}}} \tag{3.2}$$

For example the usual scalar theory

$$\begin{aligned}
S[\phi] &= \int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right] \\
&= S_0 + S_I
\end{aligned} \tag{3.3}$$

- Let us calculate the generating functional in this theory using a shortcut method:

$$Z[J] = \frac{\int \mathcal{D}\phi \, \exp\left\{i \int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \overbrace{\frac{1}{2}m^2\phi^2}^{\to\frac{1}{2}(m^2-i\epsilon)\phi^2} + J\phi\right]\right\}}{\int \mathcal{D}\phi \, \exp\left\{iS_0\right\}} \tag{3.4}$$

Write the fields in terms of the Fourier components:

$$\begin{aligned}
\phi(x) &= \int \frac{d^4p}{(2\pi)^4} e^{-ipx}\phi(p)\,, \\
J(x) &= \int \frac{d^4p}{(2\pi)^4} e^{-ipx}J(p)\,, \qquad\qquad \int d^4x\, e^{i(p-p')x} = (2\pi)^4\delta^{(4)}(p-p') \\
\partial_\mu\phi(x) &= \int \frac{d^4p}{(2\pi)^4} (-ip_\mu)e^{-ipx}\phi(p)
\end{aligned} \tag{3.5}$$

Putting everything together and combining some terms to get a full square get

$$\begin{aligned}
i\int d^4x[\ldots] = \frac{i}{2}\int \frac{d^4p}{(2\pi)^4}\Bigg[&\left(\phi(p) + \frac{J(p)}{p^2-m^2+i\epsilon}\right)\left[p^2-m^2+i\epsilon\right]\left(\phi(-p) + \frac{J(-p)}{p^2-m^2+i\epsilon}\right) \\
&- \frac{J(p)J(p')}{\left[p^2-m^2+i\epsilon\right]}\Bigg]
\end{aligned} \tag{3.6}$$

and introduce a new variable

$$\begin{aligned}
\phi'(x) &= \int \frac{d^4p}{(2\pi)^4} e^{-ipx}\left(\phi(p) + \frac{J(p)}{p^2-m^2+i\epsilon}\right) \\
\mathcal{D}\phi &= \mathcal{D}\phi'
\end{aligned} \tag{3.7}$$

Obtain

$$Z[J] = \exp\left\{-\frac{i}{2}\int \frac{d^4p}{(2\pi)^4}\frac{J(p)J(p')}{p^2-m^2+i\epsilon}\right\} \tag{3.8}$$

or, equivalently

$$Z[J] = \exp\left\{-\frac{1}{2}\int d^4xd^4y\, J(x)\Delta(x-y)J(y)\right\} \tag{3.9}$$

with

$$\Delta(x-y) = \int \frac{d^4p}{(2\pi)^4 i} e^{-ip(x-y)}\frac{1}{m^2-p^2-i\epsilon} \tag{3.10}$$

Here from get the Green functions in the free theory

$$G_0(x_1,\ldots,x_n) = (-i)^n \frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}\, Z[J]\Big|_{J(t)=0} \tag{3.11}$$

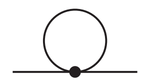
and for example

$$G_0(x_1, x_2) = \Delta(x_1 - x_2) \qquad \text{propagator} \tag{3.12}$$

Green functions in the interacting theory are then calculated as a perturbative expansion

$$\langle\Omega|\mathrm{T}\{\hat{\phi}(x_1)\hat{\phi}(x_2)\}|\Omega\rangle \;=\; \tag{3.13}$$

$+$



$+ \quad \cdots$

3.1 Equations of motion

- Klein-Gordon equation [= equation of motion (EOM)]

$$(\Box_x + m^2)\phi(x) = 0 \qquad \Leftarrow \text{ free field}$$

$$(\Box_x + m^2)\phi(x) + \frac{\lambda}{3!}\phi^3 = 0 \qquad \Leftarrow \text{ with interactions} \tag{3.14}$$

Does this mean that (promoting $\phi \to \hat{\phi}$) all matrix elements of such an object vanish?
Does this mean that all insertions of such an object in Green functions vanish?

$$\langle\Omega|\mathrm{T}\Big\{\Big[(\Box_x + m^2)\hat{\phi} + \frac{\lambda}{3!}\hat{\phi}^3\Big](x)\hat{\phi}(x_1)\ldots\phi(x_n)\Big\}|\Omega\rangle \;=\; ? \tag{3.15}$$

If

$$\langle\Omega|\mathrm{T}\Big\{\hat{\phi}(x)\hat{\phi}(x_1)\ldots\phi(x_n)\Big\}|\Omega\rangle = \mathcal{N}\int\mathcal{D}\phi\,\phi(x)\phi(x_1)\ldots\phi(x_n)e^{iS[\phi]} \tag{3.16}$$

(here $\mathcal{N}$ stands for the path integral in denominator), then

$$(\Box_x + m^2)\langle\Omega|\mathrm{T}\Big\{\hat{\phi}(x)\hat{\phi}(x_1)\ldots\phi(x_n)\Big\}|\Omega\rangle = \mathcal{N}\int\mathcal{D}\phi\,(\Box_x + m^2)\phi(x)\phi(x_1)\ldots\phi(x_n)e^{iS[\phi]} \tag{3.17}$$

We want to calculate this.

- A very important trick (comes now): Integration by parts in the path integral.

Consider

$$S[\phi + \delta\phi] - S[\phi] \equiv \int d^4x\,\delta\phi(x)\frac{\delta}{\delta\phi(x)}S[\phi] + \ldots$$

$$S[\phi + \delta\phi] - S[\phi] \overset{p.I}{=} \int d^4x\,\Big[-\Box_x\phi - m^2\phi - \frac{\lambda}{3!}\phi^3\Big]\delta\phi(x) \tag{3.18}$$

where the first line comes from the definition of functional derivative and the second line from explicit calculation. Thus

$$\boxed{\Big[(\Box_x + m^2)\phi(x) + \frac{\lambda}{3!}\phi^3(x)\Big]e^{iS[\phi]} = i\frac{\delta}{\delta\phi(x)}e^{iS[\phi]}} \tag{3.19}$$

Therefore (continue with three external fields as example)

$$\begin{aligned}
& \mathcal{N}\int \mathcal{D}\phi\,(\Box_x+m^2)\phi(x)\phi(x_1)\phi(x_2)\phi(x_3)e^{iS[\phi]} \\
= \;& \mathcal{N}\int \mathcal{D}\phi\,\phi(x_1)\phi(x_2)\phi(x_3)\Big[-\frac{\lambda}{3!}\phi^3(x)+i\frac{\delta}{\delta\phi(x)}\Big]e^{iS[\phi]} \\
= \;& -\frac{\lambda}{3!}\langle\Omega|\mathrm{T}\Big\{\hat\phi^3(x)\hat\phi(x_1)\hat\phi(x_2)\hat\phi(x_3)\Big\}|\Omega\rangle - i\mathcal{N}\int\mathcal{D}\phi\,e^{iS[\phi]}\frac{\delta}{\delta\phi(x)}\Big[\phi(x_1)\phi(x_2)\phi(x_3)\Big] \\
= \;& -\frac{\lambda}{3!}\langle\Omega|\mathrm{T}\Big\{\hat\phi^3(x)\hat\phi(x_1)\hat\phi(x_2)\hat\phi(x_3)\Big\}|\Omega\rangle - i\delta^{(4)}(x-x_1)\langle\Omega|\mathrm{T}\Big\{\phi(x_2)\hat\phi(x_3)\Big\}|\Omega\rangle \\
& - i\delta^{(4)}(x-x_2)\langle\Omega|\mathrm{T}\Big\{\phi(x_1)\hat\phi(x_3)\Big\}|\Omega\rangle - i\delta^{(4)}(x-x_3)\langle\Omega|\mathrm{T}\Big\{\phi(x_1)\hat\phi(x_2)\Big\}|\Omega\rangle
\end{aligned}\tag{3.20}$$

Thus

$$\begin{aligned}
\langle\Omega|\mathrm{T}\Big\{\underbrace{\Big[(\Box_x+m^2)\hat\phi+\frac{\lambda}{3!}\hat\phi^3\Big](x)}_{EOM}\,\hat\phi(x_1)\hat\phi(x_2)\hat\phi(x_3)\Big\}|\Omega\rangle = & -i\delta^{(4)}(x-x_1)\langle\Omega|\mathrm{T}\Big\{\phi(x_2)\hat\phi(x_3)\Big\}|\Omega\rangle \\
& - i\delta^{(4)}(x-x_2)\langle\Omega|\mathrm{T}\Big\{\phi(x_1)\hat\phi(x_3)\Big\}|\Omega\rangle \\
& - i\delta^{(4)}(x-x_3)\langle\Omega|\mathrm{T}\Big\{\phi(x_1)\hat\phi(x_2)\Big\}|\Omega\rangle
\end{aligned}\tag{3.21}$$

A generalization to an arbitrary number of extra fields is obvious.

Particular cases:

- No extra fields

$$\langle\Omega|\Big[(\Box_x+m^2)\hat\phi+\frac{\lambda}{3!}\hat\phi^3\Big](x)|\Omega\rangle = 0 \tag{3.22}$$

— Matrix elements of EOM operators vanish.

- Free theory $\lambda = 0$, one extra field

$$(\Box_x+m^2)\langle 0|\mathrm{T}\Big\{\hat\phi(x)\hat\phi(x_1)\Big\}|0\rangle = -i\delta^{(4)}(x-x_1) \tag{3.23}$$

— the propagator is a Green function of $(\Box_x + m^2)$ (as it should).

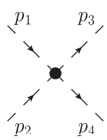
3.2 Reminder: UV divergences

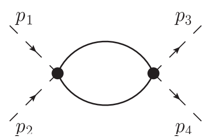
Going over from QM (finite number of degrees of freedom) to QFT (infinite number of d.o.f's) leads to a major problem — divergences at large momenta/small distances. A short reminder how they appear:

Consider a momentum-space Green function

$$\begin{aligned}
G(p_1,p_2,p_3,p_4) &= i^4\int d^4x_1\ldots d^4x_4\,e^{-ix_1p_1-ix_2p_2+ix_3p_3+ix_4p_4}\langle\Omega|\mathrm{T}\{\hat\phi(x_1)\ldots\hat\phi(x_4)\}|\Omega\rangle \\
&= \frac{(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)i\mathcal{M}}{(p_1^2-m^2)(p_2^2-m^2)(p_3^2-m^2)(p_4^2-m^2)}
\end{aligned}\tag{3.24}$$

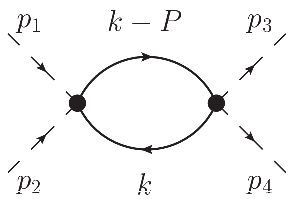
$i\mathcal{M}$ is usually called scattering amplitude as is given by a sum of Feynman diagrams with "amputated legs"

$i\mathcal{M} =$ 

$+$ 

$+ \quad \cdots \quad = i\lambda \; + \;$ quantum corrections (3.25)

The first nontrivial diagram $\boxed{p = p_1 + p_2 = p_3 + p_4}$

$T \; =$ 

$$= (-i\lambda)^2 \overbrace{\frac{1}{2}}^{\text{symmetry factor}} \int \frac{d^4k}{(2\pi)^4} \frac{i}{[k^2 - m^2 + i\epsilon]} \frac{i}{[(p-k)^2 - m^2 + i\epsilon]} \tag{3.26}$$

Use Feynman's formula to combine the denominators:

$$\frac{1}{AB} = \int_0^1 d\alpha \frac{1}{[\alpha A + \bar{\alpha} B]^2} \qquad \boxed{\bar{\alpha} = 1 - \alpha} \tag{3.27}$$

to get

$$T = \frac{1}{2}\lambda^2 \int_0^1 d\alpha \int \frac{d^4k}{(2\pi)^4} \frac{1}{[\alpha k^2 + \bar{\alpha}(k-p)^2 - m^2 + i\epsilon]^2} \tag{3.28}$$

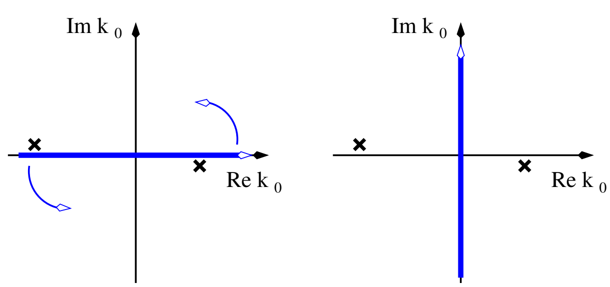
Rewrite

$$\alpha k^2 + \bar{\alpha}(k-p)^2 = (k - \bar{\alpha}p)^2 + \alpha\bar{\alpha}p^2$$

and make a shift in the integration variable $k \to k + \bar{\alpha}p$. Obtain

$$T = \frac{1}{2}\lambda^2 \int_0^1 d\alpha \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 + \alpha\bar{\alpha}p^2 - m^2 + i\epsilon]^2} \tag{3.29}$$

Calculate this integral using analytic continuation to the complex k_0 plane (Minkowski $\to$ Euclidean) $k_0 \to ik_4$:



$$d^4k = dk_0 d^3\vec{k} = i dk_1 \dots dk_4 = i d^4k_E$$
$$k^2 = k_0^2 - \vec{k}^2 = -(k_1^2 + \dots k_4^2) = -k_E^2 \tag{3.30}$$

so that

$$\int d^4k \frac{1}{[k^2 - a^2 + i\epsilon]^2} = i \int d^4k_E \frac{1}{[k_E^2 + a^2]^2} \tag{3.31}$$

and if $a^2 > 0$ (small momenta) then the $i\epsilon$ is no more needed.

The remaining integral can be calculated Euler's parametrization of the solid angle in generic N-dimensional space:

$$\begin{aligned}
&(k_1 \dots k_n) \mapsto (k, \phi, \theta_1, \dots \theta_{N-2}) \\
&\int d^N k = \int_0^\infty dk\, k^{N-1} \int_0^{2\pi} d\phi \int_0^\pi \sin\theta_1 d\theta_1 \dots \int_0^\pi \sin^{N-2}\theta_{N-2} d\theta_{N-2} \\
&\qquad = \int_0^\infty dk\, k^{N-1} \int d\Omega_N
\end{aligned} \tag{3.32}$$

Then

$$\int d^N k\, F(k^2) = \frac{2\pi^{N/2}}{\Gamma[N/2]} \int_0^\infty dk\, k^{N-1}\, F(k^2) = \frac{\pi^{N/2}}{\Gamma[N/2]} \int_0^\infty \int_0^\infty dk^2\, k^{N-2}\, F(k^2) \tag{3.33}$$

We need $N = 4$ in which case get

$$i \int d^4k_E \frac{1}{[k_E^2 + a^2]^2} = i\pi^2 \int_0^\infty dk^2\, \frac{k^2}{[k^2 + a^2]^2} \tag{3.34}$$

and collecting everything obtain

$$T = \frac{1}{2}\lambda^2 \int_0^1 d\alpha \frac{1}{(2\pi)^4} i\pi^2 \int_0^\infty dk^2\, \frac{k^2}{[k^2 + m^2 - \alpha\bar{\alpha}p^2]^2} \tag{3.35}$$

This would be the end of the story (almost) but the integral diverges. We can try to cut off the k^2-integral at large momenta (so-called cutoff regularization) introducing some large scale

$$M^2 \gg m^2, p^2 \qquad\qquad \int_0^\infty dk_E^2 \mapsto \int_0^{M^2} dk_E^2 \tag{3.36}$$

and obtain in this way

$$\begin{aligned}
\int_0^{M^2} dk^2\, \frac{k^2}{[k^2 + m^2 - \alpha\bar{\alpha}p^2]^2} &= -\frac{M^2}{M^2 + m^2 - \alpha\bar{\alpha}p^2} + \ln \frac{m^2 - \alpha\bar{\alpha}p^2 + M^2}{m^2 - \alpha\bar{\alpha}p^2} \\
&\to -1 + \ln \frac{M^2}{m^2 - \alpha\bar{\alpha}p^2}
\end{aligned} \tag{3.37}$$

leading to

$$T = -\frac{i}{2} \frac{\lambda^2}{16\pi^2} \int_0^1 d\alpha \Big[1 - \ln \frac{M^2}{m^2 - \alpha\bar{\alpha}p^2}\Big] \tag{3.38}$$

This is finite but depends on (unphysical) parameter M.

This situation is generic for (almost) all QFTs. We have to understand what to do with such ill-defined expressions.

3.3 Wilsonian Effective Action

The same problem apparently affects the path integral, so it has to be defined with some restrictions on the allowed paths. In any case some cutoff is necessary. In this section consider Euclidean version of the ϕ^4 theory in generic d dimensions: We will see interesting differences for d=2,3,4; I do not intend using dimensional regularization.

- Define

$$W = \int [\mathcal{D}\phi]^M \exp\left\{ - \int d^d x \left[\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!}\phi^4 \right] \right\} \tag{3.39}$$

where

$$[\mathcal{D}\phi]^M = \prod_{|k|<M} d\phi_k \,, \qquad \phi(x) = \int_{|k|<M} \frac{d^d k}{(2\pi)^d} e^{ikx} \phi(k) \tag{3.40}$$

The parameter M can be thought of as a boundary of our knowledge: Usually we assume that our theory is valid for all distances from large to small (zero), but this may be too simplistic as the theory may change drastically at very small distances where we have no experimental data. If we do not know the “fundamental” theory at very small distances, why can we make any predictions for large distances? Actually we know that we can, and the work by Wilson was motivated by the theory of phase transitions in condensed matter physics where from we know that the properties of matter do not depend very much on the underlying atomic structure.

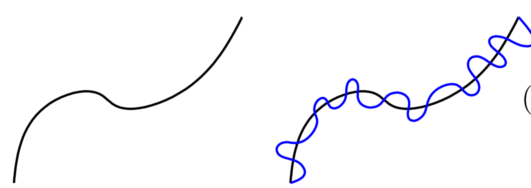
Question asked by Wilson: what happens if we change M? Can it be that changing M and adjusting some parameters in the Lagrangian we end up with the same physical predictions?

- Let

$$\mu < M \tag{3.41}$$

and split the fields in the path integral in “fast” and “slow”

$$\phi(k) = \Phi(k) + \varphi(k) \tag{3.42}$$



so that

$$\begin{aligned} \Phi(k):& \qquad 0 < k < \mu && \text{slow fields} \\ \varphi(k):& \qquad \mu < k < M && \text{fast fields} \end{aligned} \tag{3.43}$$

Then

$$W = \int [\mathcal{D}\Phi]^\mu [\mathcal{D}\varphi]^M_\mu e^{-S[\Phi+\varphi]} \stackrel{!}{=} \int [\mathcal{D}\Phi]^\mu e^{-S_{eff}[\Phi,\mu]} \tag{3.44}$$

or

$$e^{-S_{eff}[\Phi]} \stackrel{!}{=} \int [\mathcal{D}\varphi]_\mu^M e^{-S[\Phi+\varphi]} \tag{3.45}$$

$S_{eff}[\Phi,\mu]$ is called **Wilsonian effective action**

- In our case

$$S[\Phi+\varphi] = S[\Phi] + \cancel{\int d^dx\, \varphi(x) \frac{\delta}{\delta\Phi(x)} S[\Phi]} + S[\varphi] + \int d^dx \Big[\frac{\lambda}{4}\Phi^2(x)\varphi^2(x) + \frac{\lambda}{6}\Phi(x)\varphi^3(x)\Big] \tag{3.46}$$

Linear terms in φ can be neglected because a fast field cannot decay in slow fields (apart from small corrections). One can also argue that calculating the path integral over fast fields φ one can consider $\Phi(x)$ as external classical fields so that they satisfy classical EOM $\frac{\delta}{\delta\Phi(x)}S[\Phi] = 0$. Thus we get

$$e^{-S_{eff}[\Phi]} = e^{S[\Phi]} \int [\mathcal{D}\varphi]_\mu^M \exp\Big\{ - \int d^4x \Big[\frac{1}{2}(\partial_\mu\varphi)^2 + \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4 + \frac{\lambda}{4}\Phi^2\varphi^2 + \frac{\lambda}{6}\Phi\varphi^3\Big]\Big\} \tag{3.47}$$

so that the effective action is given by the usual action plus corrections that we will try to evaluate in perturbation theory.

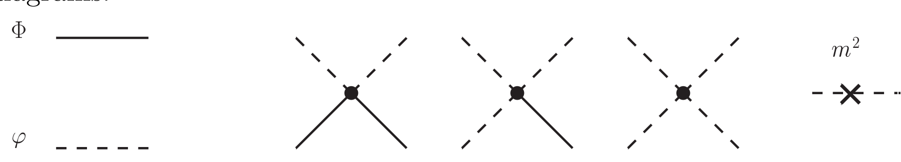
To simplify calculations assume that λ and m are small and use perturbative expansion both in λ and m. Thus choose *Lecture 8*

$$\begin{aligned} S_0[\varphi] &= \int d^dx\, \frac{1}{2}(\partial_\mu\phi)^2\,, \\ S_I[\varphi,\Phi] &= \int d^4x \Big[\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\lambda}{4}\Phi^2\varphi^2 + \frac{\lambda}{6}\Phi\varphi^3\Big] \end{aligned} \tag{3.48}$$

We can write

$$\begin{aligned} \int [\mathcal{D}\varphi]_\mu^M e^{S_0[\varphi]+S_I[\varphi,\Phi]} &= \underbrace{\int [\mathcal{D}\varphi]_\mu^M e^{S_0[\varphi]}}_{\mathcal{N}} \frac{\int [\mathcal{D}\varphi]_\mu^M e^{S_0[\varphi]+S_I[\varphi,\Phi]}}{\int [\mathcal{D}\varphi]_\mu^M e^{S_0[\varphi]}} \\ &= \mathcal{N}\langle 0|\mathrm{Texp}\Big\{ - \int d^dx \,\Big[\frac{1}{2}m^2\hat\phi^2 + \frac{\lambda}{4!}\hat\phi^4 + \frac{\lambda}{4}\Phi^2\hat\varphi^2 + \frac{\lambda}{6}\Phi\hat\varphi^3\Big]\Big\}|0\rangle \end{aligned} \tag{3.49}$$

$\Rightarrow$ Feynman diagrams:



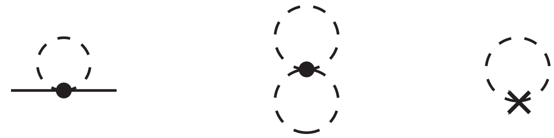
The propagator of the fast field, formally

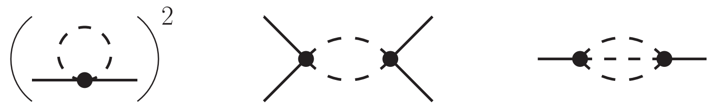
$$= \int\limits_{|p_E|>\mu}^{|p_E|<M} \frac{d^dp_E}{(2\pi)^d} e^{-ip_E(x-y)} \frac{1}{p_E^2} \tag{3.50}$$

but finite interval is often only necessary if one encounters divergent integrals. Otherwise, replacing integration over $\mu < |p_E| < M$ by the integral over the whole space usually produces corrections

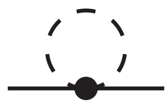
$(\sim P/\mu)^k$ where P is some external momentum such that $|P| \ll \mu$ (because we use μ as a cutoff) that can be neglected

Note that for effective action we need diagrams with arbitrary number of external legs, but to the first few orders in λ, m^2, only a few diagrams contribute:

$O(\lambda)$ 

$O(\lambda^2)$ 

- Example I (simplest)

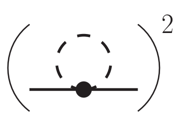
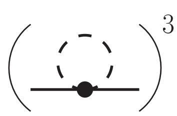


$$
\begin{aligned}
&\leftarrow \langle\Omega|\mathrm{T}\Big\{-\int d^dx\,\frac{\lambda}{4}\Phi^2(x)\varphi^2(x)\Big\}|\Omega\rangle \\
&= -\frac{\lambda}{4}\int d^dx\,\Phi^2(x)\langle\Omega|\mathrm{T}\{\varphi(x)\varphi(x)\}|\Omega\rangle \\
&\overset{!}{=} -\frac{1}{2}\int d^dx\,\Phi^2(x)\delta m^2
\end{aligned}
\tag{3.51}
$$

where

$$
\begin{aligned}
\delta m^2 &= \frac{\lambda}{2}\langle\Omega|\mathrm{T}\{\varphi(x)\varphi(x)\}|\Omega\rangle = \frac{\lambda}{2}\int\limits_{|p_E|>\mu}^{|p_E|<M}\frac{d^dp_E}{(2\pi)^d}\cancel{e^{-ip_E(x-x)}}\frac{1}{p_E^2+\cancel{m^2}} \\
&= \frac{\lambda}{(4\pi)^{d/2}}\frac{1}{\Gamma[d/2]}\frac{1}{d-2}\left[M^{d-2}-\mu^{d-2}\right]
\end{aligned}
\tag{3.52}
$$

— is some number. In higher orders we will get diagrams of the same type

 $+\ \frac{1}{2}$  $+\ \frac{1}{3!}$ 

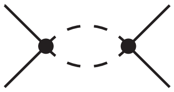
$$
= \exp\Big\{-\frac{1}{2}\int d^dx\,\Phi^2(x)\delta m^2\Big\}
\tag{3.53}
$$

The effective action, therefore, receives a contribution

$$
S_{eff} \ni \overbrace{-\frac{1}{2}\int d^dx\,\Phi^2(x)\Big(m^2}^{S[\Phi]} + \delta m^2\Big)
\tag{3.54}
$$

— mass renormalization (the value of mass parameter changes).

- Example II (less simple)



$$\leftarrow \frac{1}{2}\left(-\frac{\lambda}{4}\right)^2 \int d^dx \int d^dy\, \langle\Omega|\mathrm{T}\Big\{\Phi^2(x)\,\varphi^2(x)\varphi^2(y)\,\Phi^2(y)\Big\}|\Omega\rangle$$

$$= \frac{1}{2}\left(-\frac{\lambda}{4}\right)^2 2\int d^dx \int d^dy\, \Phi^2(x)\Phi^2(y) \int \frac{d^dk_1}{(2\pi)^d} e^{-ik_1(x-y)}\frac{1}{k_1^2}\int \frac{d^dk_2}{(2\pi)^d} e^{ik_2(x-y)}\frac{1}{k_2^2} \tag{3.55}$$

This is a nonlocal contribution: $\Phi^2(x)$ and $\Phi^2(y)$ are at different space-time positions. Let us try to expand and see what comes out:

$$\Phi^2(y) = \Phi^2(x) + (y-x)^\mu\partial_\mu\Phi^2(x) + \frac{1}{2}(y-x)^\mu(y-x)^\nu\partial_\mu\partial_\nu\Phi^2(x) + \dots \tag{3.56}$$

Consider the first term. The y-integration becomes trivial

$$\int d^dy\, e^{iy(k_1-k_2)} = (2\pi)^d\delta(k_1-k_2) \tag{3.57}$$

and we get

$$\begin{aligned} \dots &= \frac{1}{2}\left(-\frac{\lambda}{4}\right)^2 2\int d^dx\, \Phi^4(x) \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^4} \\ &\overset{!}{=} -\frac{1}{4!}\int d^dx\, \Phi^4(x)\,\delta\lambda \end{aligned} \tag{3.58}$$

with

$$\begin{aligned} \delta\lambda &= -4!\left(\frac{\lambda}{4}\right)^2 \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^4} \\ &= -4!\left(\frac{\lambda}{4}\right)^2 \frac{1}{(4\pi)^{d/2}}\frac{1}{\Gamma[d/2]}\int_{\mu^2}^{M^2} dk^2\, k^{d-2}\frac{1}{k^4} \\ &= -\frac{3\lambda^2}{(4\pi)^{d/2}}\frac{1}{\Gamma[d/2]}\frac{1}{d-4}\left(M^{d-4}-\mu^{d-4}\right) \end{aligned} \tag{3.59}$$

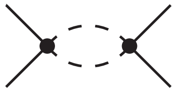
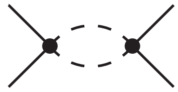
If $d=4$ we can recalculate the integral anew or use that for $d-4\to 0$

$$M^{d-4} = e^{(d-4)\ln M} = 1 + (d-4)\ln M + \mathcal{O}((d-4)^2) \tag{3.60}$$

to get

$$\delta\lambda\Big|_{d=4} = -\frac{3\lambda^2}{16\pi^2}\ln\frac{M}{\mu} \tag{3.61}$$

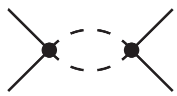
This diagrams will be repeated in higher orders with $1/k!$ coefficients so the the result will exponentiate

$$\sum_k \frac{1}{k!}\Bigg($$  $$\Bigg)^k = \exp\Bigg\{$$  $$\Bigg\} = \exp\Big\{-\frac{1}{4!}\int d^dx\,\Phi^4(x)\,\delta\lambda\Big\} \tag{3.62}$$

This has the same form as the interaction term ϕ^4 in the original action, so that effectively

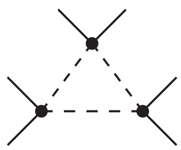
$$\lambda \to \lambda + \delta\lambda\,.$$

• On these two examples we see that the difference of the effective action from the original one amounts to modification of parameters. But we also get new terms:



$$: \quad \Phi^2(x)\Phi^2(y) = \ldots \qquad +\frac{1}{2}(y-x)^\mu(y-x)^\nu\Phi^2(x)\partial_\mu\partial_\nu\Phi^2(x)$$

$$\mapsto \int d^dx \boxed{\Phi^2(x)\partial^2\Phi^2(x)}\,\delta A$$



$$: \qquad \mapsto \int d^dx \boxed{\Phi^6(x)}\,\delta B \tag{3.63}$$

so that the effective Lagrangian is of the form

$$\mathcal{L}_{eff} = \frac{1}{2}\Big(1+\delta Z\Big)(\partial_\mu\Phi)^2 + \frac{1}{2}\Big(m^2+\delta m^2\Big)\Phi^2 + \frac{1}{4!}\Big(\lambda+\delta\lambda\Big)\Phi^4$$
$$+\,\delta A\,\Phi^2\partial^2\Phi^2 + \delta B\,\Phi^6 + \ldots \tag{3.64}$$

i.e. it contains an (infinite) tail of higher-dimension terms. If such terms necessarily appear when one changes the cutoff, it seems there is no reason why they were absent in the original action. So, maybe we have to add them and then an effective Lagrangian becomes

$$\mathcal{L}_{eff} = \frac{1}{2}\Big(1+\delta Z\Big)(\partial_\mu\Phi)^2 + \frac{1}{2}\Big(m^2+\delta m^2\Big)\Phi^2 + \frac{1}{4!}\Big(\lambda+\delta\lambda\Big)\Phi^4$$
$$+\Big(A+\delta A\Big)\,\Phi^2\partial^2\Phi^2 + \Big(B+\delta B\Big)\,\Phi^6 + \ldots \tag{3.65}$$

• Note, however, that we need to compare the resulting Green functions, i.e. results after path integration, but not just the integrands. In order to do this, we have to

(1) Rescale the coordinates/momenta so that the integration regions are the same:

$$k' = k\,\frac{M}{\mu}\,, \qquad x' = x\frac{\mu}{M} \tag{3.66}$$

Let

$$r = \frac{\mu}{M}\,, \qquad \text{rescaling factor} \tag{3.67}$$

then, e.g.,

$$\int_0^\mu d^dk = r^d\int_0^M d^dk'\,, \qquad \int d^dx = r^{-d}\int d^dx'\,, \qquad \frac{\partial}{\partial x^\mu} = r\frac{\partial}{\partial x'^\mu} \tag{3.68}$$

and therefore

$$\int d^dx\, \mathcal{L}_{eff} = \int d^dx'\, r^{-d} \Big\{ \frac{1}{2}\Big(1+\delta Z\Big) r^2 (\partial_\mu \Phi)^2 + \frac{1}{2}\Big(m^2+\delta m^2\Big)\Phi^2 + \frac{1}{4!}\Big(\lambda+\delta\lambda\Big)\Phi^4$$
$$+ \Big(A+\delta A\Big)\, r^2\, \Phi^2 \partial'^2 \Phi^2 + \Big(B+\delta B\Big)\, \Phi^6 + \dots \Big\} \tag{3.69}$$

(2) Rescale the field to get the same free propagator (related to state normalization)

$$\Phi' = \Big[r^{2-d}(1+\delta Z)\Big]^{1/2} \Phi \tag{3.70}$$

Then

$$\int d^dx\, \mathcal{L}_{eff} = \int d^dx' \left\{ \frac{1}{2}(\partial_\mu \Phi')^2 + \frac{1}{2} m'^2 \Phi'^2 + \frac{\lambda'}{4!}\Phi'^4 + A'\Phi'^2\partial'^2\Phi'^2 + B'\,\Phi'^6 + \dots \right\} \tag{3.71}$$

where

$$\begin{aligned} m'^2 &= (m^2+\delta m^2)(1+\delta Z)^{-1} r^{-2} \\ \lambda' &= (\lambda+\delta\lambda)(1+\delta Z)^{-2} r^{d-4} \\ A' &= (A+\delta A)(1+\delta Z)^{-2} r^{d-2} \\ B' &= (B+\delta B)(1+\delta Z)^{-3} r^{2d-6} \end{aligned} \tag{3.72}$$

Thus we get the same action as before (with modified parameters) and can discuss what is the effect.

- Consider the usual quartic interaction $\sim \Phi^4$ for $d = 4$. We get

$$\lambda' = \lambda - \frac{3\lambda^2}{16\pi^2} \ln\frac{M}{\mu} + \mathcal{O}(\lambda^3) \tag{3.73}$$

— the **running coupling**: $\lambda \mapsto \lambda(M)$, $\lambda' \mapsto \lambda(\mu)$

$$\lambda(\mu) = \lambda(M) - \frac{3\lambda^2(M)}{16\pi^2}\ln\frac{M}{\mu} + \mathcal{O}(\lambda^3 \ln^2 M/\mu) \tag{3.74}$$

The expansion becomes ill-behaved if $\lambda(\mu)\ln(M/\mu) = \mathcal{O}(1)$ and has to be resummed. This is done going over to the differential equation (renormalization group equation)

$$\frac{d\lambda^2}{d\ln\mu} = \frac{3}{16\pi^2}\lambda^2(\mu) + \mathcal{O}(\lambda^3) = \beta(\lambda) \qquad \textbf{Beta function} \tag{3.75}$$

Note that in this series there cannot be any large logarithms on the r.h.s. because there is only one scale μ involved. Thus the expansion is well-behaved (so long as the coupling is small).

To solve this equation, write

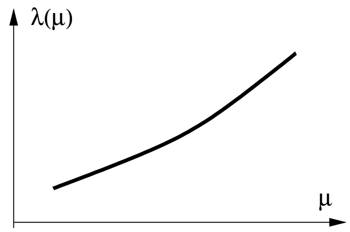
$$\frac{d\lambda^{-1}(\mu)}{d\ln\mu} = -\frac{1}{\lambda^2(\mu)}\frac{d\lambda(\mu)}{d\ln\mu} = -\frac{3}{16\pi^2} \tag{3.76}$$

so that

$$\begin{cases} \dfrac{1}{\lambda(\mu)} = \dfrac{1}{\lambda_0} - \dfrac{3}{16\pi^2}\ln\mu \\ \dfrac{1}{\lambda(M)} = \dfrac{1}{\lambda_0} - \dfrac{3}{16\pi^2}\ln M \end{cases} \quad \Rightarrow \quad \lambda(\mu) = \frac{\lambda(M)}{1 + \frac{3\lambda^2(M)}{16\pi^2}\ln\frac{M}{\mu}} \tag{3.77}$$

or choose

$$\frac{1}{\lambda_0} \stackrel{!}{=} \frac{3}{16\pi^2}\ln\Lambda \quad \Rightarrow \quad \lambda(\mu) = \frac{16\pi^2}{3\ln\dfrac{\Lambda}{\mu}}$$



$$: \tag{3.78}$$

— the coupling increases at small distances alias decreases at large distances (screening by vacuum polarization)

- Now consider contributions of higher dimension. Up to logarithmic corrections from δZ

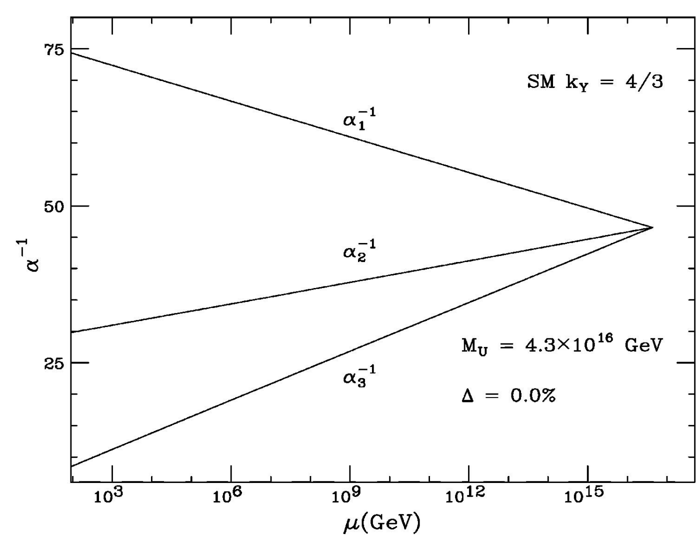
$$A' \simeq A\left(\frac{\mu}{M}\right)^{d-2} \qquad \Rightarrow \qquad A(\mu) \simeq A(M)\left(\frac{\mu}{M}\right)^{d-2} \tag{3.79}$$

and becomes negligibly small at $\mu << M$ if $\underline{d > 2}$ (and $A(M)$ is some "reasonable" number). Similar

$$B' \simeq B\left(\frac{\mu}{M}\right)^{2d-6} \qquad \Rightarrow \qquad B(\mu) \simeq B(M)\left(\frac{\mu}{M}\right)^{2d-6} \tag{3.80}$$

and becomes negligibly small at $\mu << M$ if $\underline{d > 3}$ (and $B(M)$ is some "reasonable" number). Such interactions are called "irrelevant" in condensed matter physics community

- Example: Coupling unification in the Standard Model



from: V.Barger *et al.* PLB **624** (2005) 233

At the scales $10^{15} - 10^{14}$ GeV all three known fundamental interactions (strong, weak and electromagnetic) become equally strong. Expect the theory is modified in some way, also maybe including gravity. If this is the case, one should expect to have a sequence of "irrelevant" interactions suppressed as

$$\left(\frac{\text{momenta probed in our experiments}}{\text{new physics scale}}\right)^{2(?)}$$

There is currently a fashionable direction of research, people try to find traces of such interactions (so far without much success).

Lecture 9

3.4 Nonlinear σ-model in $d = 2$

- Spin-vector field:

$$\vec{n}(x) \qquad \sum_a n^a n^a = 1 \qquad a = 1, 2, \ldots, N \tag{3.81}$$

(N-component vector with unit length, lives in two dimensions $x^\mu = \{x_1, x_2\}$.

Lagrangian density

$$\boxed{\mathcal{L} = \frac{N}{2g}(\partial_\mu n^a(x))(\partial^\mu n^a(x))} \qquad \mu = 1, 2 \tag{3.82}$$

This looks like as free field (with N components), but the condition $|\vec{n}|^2 = 1$ makes the system highly nontrivial.

- We can try to solve this constraint explicitly, writing

$$n^a = \{\pi^1, \ldots, \pi^{N-1}, \sigma\}, \qquad \sigma = (1 - |\vec{\pi}|^2)^{1/2} \tag{3.83}$$

(Historically, sigma-models were first considered as a model for pion interactions;
$\rightarrow$ the notation $\pi(x)$, "pions".) Then

$$\partial_\mu \sigma = \frac{1}{2} \frac{-2\pi^a \partial_\mu \pi^a}{\sqrt{1 - |\vec{\pi}|^2}} \tag{3.84}$$

and

$$[\partial_\mu n^a]^2 = [\partial_\mu \pi^a]^2 + [\partial_\mu \sigma]^2 = [\partial_\mu \pi^a]^2 + \frac{(\vec{\pi} \cdot \partial_\mu \vec{\pi})^2}{1 - \pi^2} \tag{3.85}$$

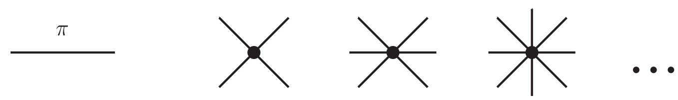
Thus we can define the quantum theory writing the path integral in terms of (independent) π-fields

$$W = \mathcal{N} \int \mathcal{D}\pi \, e^{-\int d^2x \, \mathcal{L}(x)} \tag{3.86}$$

with the Lagrangian density

$$\mathcal{L} = \frac{N}{2g}\left[(\partial_\mu \pi^a)^2 + \frac{(\pi^a \partial_\mu \pi^a)^2}{1 - \pi^2}\right] \tag{3.87}$$

Perturbation theory for this path integral contains a "pion" propagator (massless scalar) and a bunch of vertices corresponding to the expansion $1/(1-\pi^2) = 1 + \pi^2 + \pi^4 + \ldots$



This looks like a theory of $N-1$ massless particles (pions) $\pi^1, \ldots \pi^{N-1}$

- Note that the initial Lagrangian is invariant under rotations of the n-field

$$n_a \mapsto n'_a = R_{ab} n^b \qquad R^T R = \mathbb{1} \qquad O(N) - \text{symmetry} \tag{3.88}$$

(and therefore the choice of N-th component as a sigma-field is just a convention), but the theory in terms of pions seems to have a smaller symmetry

$$\pi_a \mapsto \pi'_a = R_{ab} \pi^b \qquad R^T R = \mathbb{1} \qquad O(N-1) - \text{symmetry} \tag{3.89}$$

How could this happen?
— The Lagrangian in terms of π fields still has $O(N)$ symmetry — this was just a rewriting so the symmetry could not disappear, it is hidden in the relations between different vertices.
— BUT the ground state (vacuum) is not $O(N)$ symmetric:

$$\begin{aligned} &\langle \Omega | \pi(x) | \Omega \rangle = 0 && \text{by construction there are no particles in vacuum} \\ &\langle \Omega | \sigma(x) | \Omega \rangle = 1 && \sigma = 1 \text{ if there is no pion field} \end{aligned} \tag{3.90}$$

This kind of symmetry breaking (Hamiltonian is symmetric but the ground state not) is called spontaneous symmetry breaking; we we study this phenomenon in detail later.

Thus, our expectation seems to be that the Lagrangian in (3.82) describes the system of $N-1$ massless particles, and the $O(N)$ symmetry of the Lagrangian is broken spontaneously to $O(N-1)$. We will see, however, that both statements are wrong. . .

N.B. The theory in terms of π-fields is renormalizable despite having infinite number of vertices: $O(N)$ symmetry ensures that all couplings are related, and these relations will be maintained if the regularization procedure does not break the symmetry explicitly.

- Let us try differently and start with a path integral in terms of original fields inserting the $n^2 = 1$ condition explicitly:

$$W = \mathcal{N} \int \mathcal{D}n(x) \, \exp\Big[- \int d^2x \, \frac{N}{2g} (\partial_\mu n^a)^2(x) \Big] \prod_x \delta(n^2(x) - 1) \tag{3.91}$$

How to rewrite an infinite product of δ-functions in a more manageable way?

$$\delta(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha \, e^{i\alpha y} \qquad \Rightarrow \qquad \delta(n^2(x) - 1) = \mathcal{N} \int d\alpha_x e^{\alpha_x (n^2(x) - 1)} \tag{3.92}$$

The product of such factors for each point x van be viewed as a path integral over a new field $\alpha_x \equiv \alpha(x)$

$$\prod_x \delta(n^2(x)-1) = \mathcal{N}\int \mathcal{D}\alpha(x)\, \exp\left[i\int d^2x\,\alpha(x)(n^2(x)-1)\right] \tag{3.93}$$

Thus we get

$$W = \mathcal{N}\int \mathcal{D}n(x)\,\mathcal{D}\alpha(x)\exp\left\{-\int d^2x\left[\frac{N}{2g}(\partial_\mu n^a)^2(x) + i\alpha(x)(n^2(x)-1)\right]\right\} \tag{3.94}$$

This construction is usually called introduction of an auxiliary field.

For what follows it is convenient to rescale the fields in the following way:

$$n(x) \mapsto n(x)\left(\frac{N}{g}\right)^{-1/2},$$
$$i\alpha(x) \mapsto \alpha(x)\frac{\sqrt{N}}{g}\frac{1}{2} \tag{3.95}$$

Thus we want to calculate the generating functional, in this notation

$$W[J] = \mathcal{N}\int \mathcal{D}n(x)\,\mathcal{D}\alpha(x)\exp\left\{-\int d^2x\left[\frac{1}{2}(\partial_\mu n^a)^2(x) + \frac{1}{2}\frac{\alpha(x)}{\sqrt{N}}\left(n^2-\frac{N}{g}\right) + J^a(x)n^a(x)\right]\right\} \tag{3.96}$$

Advantage of this representation is that the path integral over n-fields can be taken explicitly. Let A_{ij} be a real symmetric $d\times d$ matrix. Then (cf. Sec. 2.3 and 2.5)

$$\int d^dx\, e^{-\frac{1}{2}x_iA_{ij}x_j} = (2\pi)^{d/2}(\det A)^{-1/2} \tag{3.97}$$

and this can be generalized for hermitian operators to give

$$\int \mathcal{D}n(x)\,\exp\left\{-\int d^2x\left[\frac{1}{2}(\partial_\mu n)^2(x) + \frac{1}{2}\frac{\alpha(x)}{\sqrt{N}}n^2(x) + J(x)n(x)\right]\right\}$$
$$= \mathcal{N}\left[\det\left(-\partial^2+\frac{\alpha(x)}{\sqrt{N}}\right)\right]^{-N/2}\exp\left\{\frac{1}{2}\int d^2xd^2y\, J^a(x)\langle x|\frac{1}{-\partial^2+\alpha/\sqrt{N}}|y\rangle J^a(y)\right\} \tag{3.98}$$

Here I used a notation (Schwinger) for the matrix element of the inverse operator $\langle x|\ldots|y\rangle$ that I did not use before. In the same notation the propagator can be written as (here Euclidean space!)

$$\Delta(x-y) = \int \frac{d^dp}{(2\pi)^d}e^{-ip(x-y)}\frac{1}{p^2+m^2} = \langle y|\frac{1}{-\partial^2+m^2}|x\rangle \tag{3.99}$$

Check:

$$\langle y|\frac{1}{-\partial^2+m^2}|x\rangle = \int\frac{d^dp}{(2\pi)^d}\langle y|\frac{1}{-\partial^2+m^2}|p\rangle\langle p|x\rangle = \int\frac{d^dp}{(2\pi)^d}\langle y|p\rangle\frac{1}{p^2+m^2}\langle p|x\rangle \tag{3.100}$$

Finally, use

$$\det M = \exp\left[\mathrm{Tr}\,\ln M\right] \tag{3.101}$$

to get

$$\boxed{\begin{aligned} W[J] &= \mathcal{N}\int \mathcal{D}\alpha(x)\exp\left\{-S_{eff}[\alpha] + \frac{1}{2}\int d^2x d^2y\, J(x)\langle x|(-\partial^2+\alpha/\sqrt{N})^{-1}|y\rangle J(y)\right\} \\ S_{eff}[\alpha] &= \frac{N}{2}\text{Tr}\,\ln\left[-\partial^2 + \frac{\alpha(x)}{\sqrt{N}}\right] - \int d^2x\,\frac{\sqrt{N}}{2g}\alpha(x) \qquad (3.102)\end{aligned}}$$

• So far all manipulations were exact, we did not make any assumptions, but in order to move further we have to make some approximation.

• Main idea: try to consider N as a large number and construct a $1/N$ expansion (instead of the expansion in the coupling g. It turns out that the large-N expansion leads to the semiclassical expansion of the path integral over the auxiliary field $\alpha(x)$ so that the path integral is dominated (for large N) by trajectories that deviate not much from the solution of classical EOM (classical trajectory):

$$\alpha(x) = \underbrace{\alpha_0(x)}_{\text{classical path}} + \underbrace{\alpha_q(x)}_{\text{quantum fluctuations}} \tag{3.103}$$

Since all points in our two-dimensional space are "equal", the classical path (if it exists!) cannot depend on x and can only be a constant in space. For the reasons that will become clear later, let us denote this constant as

$$\alpha_0(x) = \alpha_0 = \sqrt{N}m^2 \tag{3.104}$$

We require

$$\left.\frac{\delta S_{eff}}{\delta\alpha(x)}\right|_{\alpha(x)=\sqrt{N}m^2} = 0 \qquad \text{(Euler-Lagrange)} \tag{3.105}$$

To expand the effective action in α_q use

$$\begin{aligned}\text{Tr}\,\ln\left[-\partial^2 + \frac{\alpha(x)}{\sqrt{N}}\right] &= \text{Tr}\,\ln\left[-\partial^2 + m^2 + \frac{\alpha_q(x)}{\sqrt{N}}\right] \\ &= \text{Tr}\left\{\ln(-\partial^2+m^2) + \ln\left[1 + \frac{1}{-\partial^2+m^2}\frac{\alpha_q(x)}{\sqrt{N}}\right]\right\}\end{aligned} \tag{3.106}$$

and

$$\ln(1+x) = -\sum_{k=1}^{\infty}\frac{(-1)^k}{k}x^k \tag{3.107}$$

so that

$$\begin{aligned}S_{eff}[\sqrt{N}m^2 + \alpha_q] &= \frac{N}{2}\text{tr}\ln(-\partial^2+m^2) - \int d^2x\,\frac{m^2N}{2g} - \frac{\sqrt{N}}{2g}\int d^2x\,\alpha_q(x) \\ &- \frac{N}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\text{Tr}\left[\left(\frac{1}{-\partial^2+m^2}\frac{\alpha_q(x)}{\sqrt{N}}\right)^k\right]\end{aligned} \tag{3.108}$$

The linear term in α_q has to vanish (this is an equation on m^2)

$$\frac{N}{2}\mathrm{Tr}\left[\frac{1}{-\partial^2+m^2}\frac{\alpha_q(x)}{\sqrt{N}}\right]-\frac{\sqrt{N}}{2g}\int d^2x\,\alpha_q(x)\stackrel{!}{=}0 \tag{3.109}$$

Here

$$\begin{aligned}\frac{\sqrt{N}}{2}\mathrm{Tr}\left[\frac{1}{-\partial^2+m^2}\alpha_q(x)\right] &= \frac{\sqrt{N}}{2}\int d^2x\,\langle x|\frac{1}{-\partial^2+m^2}\alpha_q(x)|x\rangle \\ &= \frac{\sqrt{N}}{2}\int d^2x\,\alpha_q(x)\int\frac{d^2p}{(2\pi)^2}\frac{1}{p^2+m^2}\end{aligned} \tag{3.110}$$

The integral is divergent (has to be expected), so let us introduce a cutoff

$$\int\limits_{|p|<M}\frac{d^2p}{(2\pi)^2}\frac{1}{p^2+m^2}=\frac{1}{4\pi}\ln\frac{M^2}{m^2} \tag{3.111}$$

Thus our Euler-Lagrange Eq. (3.109) takes the form

$$\frac{1}{g}=\frac{1}{4\pi}\ln\frac{M^2}{m^2} \qquad\qquad \text{"Gap Equation"} \tag{3.112}$$

This is the standard situation; we have to assume $g\to g(M)$ and introduce the running coupling in such a way that m^2 (which is a physical quantity as we will see) remains constant when we change the cutoff:

$$\frac{1}{g(\mu)}=\frac{1}{4\pi}\ln\frac{\mu^2}{m^2} \tag{3.113}$$

The solution of our Euler-Lagrange Eq. is then

$$m^2=M^2\exp\left(-\frac{4\pi}{g(M)}\right)=\mu^2\exp\left(-\frac{4\pi}{g(\mu)}\right)\qquad\forall\mu \tag{3.114}$$

Note that this result for m^2 is *nonperturbative*, i.e. it cannot be obtained in perturbation theory in the coupling g. Famous mathematical example

$$f(x)=e^{-1/x}=0+0\cdot x+0\cdot x^2+\dots \tag{3.115}$$

has zero Taylor expansion at $x=0$ (all derivatives vanish). This explains the failure of our original guess on the structure of this theory.

• The next step would be to integrate over small fluctuations around the classical solution. To this end one needs to consider the quadratic term in α_q in the effective action which gives rise to the propagator of the α-"particle" and construct a perturbation theory to include higher-order terms. This can be formulated as a set of Feynman rules using which one can construct a systematic $1/N$-expansion (to all orders in $1/N$).

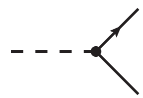
Note that in the generating function (3.102) we have

$$J(x)\langle x|(-\partial^2+\alpha/\sqrt{N})^{-1}|y\rangle J(y)=J(x)\langle x|(-\partial^2+m^2+\alpha_q/\sqrt{N})^{-1}|y\rangle J(y) \tag{3.116}$$

and this describes the propagator of the (original!) n-field and the coupling of a pair of external fields to α_q, which, as we see, is suppressed by $1/\sqrt{N}$. In order to formulate the $1/N$ perturbation theory it is convenient to reintroduce the n-field. In this way one obtains (requires some work)

 $\dfrac{1}{p^2+m^2}$

 $D(p)$

 $-\dfrac{1}{\sqrt{N}}$ (3.117)

where the propagator of the α field ($\to$ exercise) has the form

$$
\begin{aligned}
D(p) &= -\frac{2}{\Gamma(p)}\,,\\
\Gamma(p) &= \int \frac{d^2q}{(2\pi)^2}\frac{1}{[q^2+m^2][(p+q)^2+m^2]}\\
&= \frac{1}{2\pi}\frac{1}{\sqrt{p^2(p^2+4m^2)}}\ln\frac{\sqrt{p^2+4m^2}+\sqrt{p^2}}{\sqrt{p^2+4m^2}-\sqrt{p^2}}
\end{aligned}
\tag{3.118}
$$

Thus we end up with N massive fields and all couplings respect the $O(N)$ symmetry, there is no spontaneous symmetry breaking to $O(N-1)$.

N.B. Note that in there are no vertices with several α-fields coupled together. *Lect.* 10

4 Quantization of gauge fields

4.1 Quantum-mechanical systems with constraints; an example

The following example belongs to R. Jackiw.

- Consider a three-particle (mechanical) system with Lagrangian

$$
\begin{aligned}
L((A(t),x_1(t),x_2(t),\dot{x}_1(t),\dot{x}_2(t)) = \frac{1}{2}\dot{x}_1^2+\frac{1}{2}\dot{x}_2^2+A^2+(\dot{x}_1+\dot{x}_2)A-\frac{1}{2}\omega^2x_{12}^2\,,\\
x_{12}=x_1-x_2
\end{aligned}
\tag{4.1}
$$

Euler-Lagrange equations:

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} &= \frac{\partial L}{\partial x_1} &\Rightarrow\qquad \ddot{x}_1+\dot{A} &= -\omega^2x_{12}\\
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} &= \frac{\partial L}{\partial x_2} &\Rightarrow\qquad \ddot{x}_2+\dot{A} &= +\omega^2x_{12}\\
\frac{d}{dt}\frac{\partial L}{\partial \dot{A}} &= \frac{\partial L}{\partial A} &\Rightarrow\qquad 0 &= \dot{x}_1+\dot{x}_2+2A
\end{aligned}
\tag{4.2}
$$

The Lagrangian is invariant under the following trafo:

$$
\begin{aligned}
x_1(t) &\mapsto x_1(t) + \alpha(t) \\
x_2(t) &\mapsto x_2(t) + \alpha(t) \\
A(t) &\mapsto A(t) - \dot{\alpha}(t)
\end{aligned}
\tag{4.3}
$$

— a toy model for gauge invariance!

This invariance can be used to eliminate $A(t)$ completely imposing a suitable "gauge fixing" condition. Let us choose

$$
\alpha(t) = \int_{t_0}^{t} dt'\, A(t)
\tag{4.4}
$$

Then $A(t) = 0$ and we get equations of motion

$$
\begin{aligned}
\ddot{x}_1 &= -\omega^2 x_{12} \\
\ddot{x}_2 &= +\omega^2 x_{12}
\end{aligned}
\tag{4.5}
$$

which, obviously, describes a system of two particles connected by a spring, and, in addition, a constraint condition (from the third EOM)

$$
\dot{x}_1 + \dot{x}_2 = 0
\tag{4.6}
$$

(the total momentum is zero).

The corresponding Lagrangian formulation: Consider a system with

$$
L(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2 - \frac{1}{2}\omega^2 x_{12}^2
\tag{4.7}
$$

with the constraint

$$
p_1(t) + p_2(t) = 0\,, \qquad\qquad p_k = \frac{\partial L}{\partial \dot{x}_k} \qquad\qquad \text{(canonical momentum)}
\tag{4.8}
$$

This is an example of what is called "constrained canonical system" and the question is how such systems can be quantized (Dirac)

- Without the constraint, everything is very simple. We introduce new coordinates

$$
\begin{aligned}
X &= \frac{1}{2}(x_1 + x_2) \qquad \text{center of mass} \\
x &= x_1 - x_2
\end{aligned}
\tag{4.9}
$$

so that the Lagrangian becomes

$$
L(X, x; \dot{X}, \dot{x}) = \dot{X}^2 + \frac{1}{4}\dot{x}^2 - \frac{1}{2}\omega^2 x^2
\tag{4.10}
$$

and the new canonically conjugated momenta are

$$P = \frac{\partial L}{\partial \dot{X}} = 2\dot{X} = p_1 + p_2$$
$$p = \frac{\partial L}{\partial \dot{x}} = \frac{1}{2}\dot{x} = \frac{1}{2}(p_1 - p_2) \tag{4.11}$$

Canonical quantization:

$$x, X \mapsto \hat{x}, \hat{X}\,, \qquad p, P \mapsto \hat{p}, \hat{P}\,,$$
$$[\hat{p}, \hat{x}] = i\,, \qquad [\hat{P}, \hat{X}] = i \tag{4.12}$$

Hamiltonian

$$\widehat{H} = \frac{1}{2}\widehat{P}^2 + \hat{p}^2 + \frac{1}{2}\omega^2\hat{x}^2 \tag{4.13}$$

gives rise to a Schrödinger equation for the wave function, $\widehat{H}\Psi(X,x) = E\Psi(X,x)$ with solutions, obviously

$$\Psi(X,x) = e^{iPx}\psi_n(x)\,, \qquad E_n = \frac{1}{2}P^2 + \Big(n + \frac{1}{2}\Big)\omega \tag{4.14}$$

where $\psi_n(x)$ are Hermite polynomials (WFs of the harmonic oscillator)

- The question is, how to impose the classical constraint $P = p_1 + p_2 = 0$ in quantum theory?

(1) First thought (wrong):

$\longleftarrow$ Require that the operator P vanishes $\quad \widehat{P} \overset{!}{=} 0$

— contradicts commutation relations...

(2) Second thought (Dirac):

$\longleftarrow$ Require that the Hilbert space of the quantum theory is restricted to a subspace ("physical states") with zero momentum:

$\longleftarrow \quad \forall F(x) \qquad F(\widehat{P})|phys\rangle = 0$

This looks clever, but does not automatically cure all problems. Indeed, let us calculate the commutator

$$? \qquad \langle phys|[\widehat{P}, \widehat{X}]|phys\rangle = 0 \quad \text{or} \quad = i \tag{4.15}$$

The WF of a physical state (that is with zero momentum) does not depend on X, it is a function of x only. Consider ground state as example

$$\Psi_{phys} = \Psi_{vac} \sim e^{-\frac{1}{2}\omega x_{12}^2} \tag{4.16}$$

Then

$$
\begin{aligned}
\ldots &= \frac{i}{2}\int dx_1 dx_2\, e^{-\frac{1}{2}\omega x_{12}^2}\left[\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}, x_1+x_2\right]e^{-\frac{1}{2}\omega x_{12}^2} \\
&= \frac{i}{2}\int dx_1 dx_2\, e^{-\frac{1}{2}\omega x_{12}^2}\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)(x_1+x_2)e^{-\frac{1}{2}\omega x_{12}^2} \\
&= i\int dx_1 dx_2\, e^{-\frac{1}{2}\omega x_{12}^2}e^{-\frac{1}{2}\omega x_{12}^2} \;=\; ?
\end{aligned}
\tag{4.17}
$$

— the integral diverges along x_1+x_2. We will try to formulate a consistent approach using path integral quantization.

4.2 Path-integral quantization of constrained systems

- Our goal is to calculate

$$
\langle phys, f|e^{-i\widehat{H}t}|phys, i\rangle =? \tag{4.18}
$$

For our very simple system we can easily solve the constraint explicitly

$$
P=0 \qquad \Rightarrow \qquad X = \text{ const} \tag{4.19}
$$

and write the path integral in terms of the remaining coordinate:

$$
\ldots = \int dx_f dx_i \Psi_{phys,f}(x_f)\Psi_{phys,i}(x_i)\int\limits_{x(t_i)=x_i}^{x(t_f)=x_f}\mathcal{D}x(t)\,\exp\left\{i\int_{t_i}^{t_f}dt\left(\frac{1}{4}\dot{x}^2-\frac{1}{2}\omega^2x^2\right)\right\} \tag{4.20}
$$

In the general situation this solution can be complicated and not possible to write explicitly. E.g. in QED the constraint was $\nabla\cdot E=0$, how to solve it for $\vec{A}$? And it can be more complicated.

- Thus it makes sense to try to write the path integral in terms of the original coordinates

$$
\int\mathcal{D}x_1\mathcal{D}x_2
$$

- <u>Reminder</u>: path integral in phase space (first for one variable)

We slice the time interval

$$
t_f=t_N>t_{N-1}>\ldots>t_1>t_0=t_i, \tag{4.21}
$$

write the time evolution operator as a product of evolution operators in small slices, and insert unity operators in between:

$$
\langle x_f|e^{-i\widehat{H}(t_f-t_i)}|x_i\rangle = \langle x_f|e^{-i\widehat{H}(t_N-t_{N-1})}\mathbb{1}e^{-i\widehat{H}(t_{N-1}-t_{N-2})}\mathbb{1}\ldots\mathbb{1}e^{-i\widehat{H}(t_1-t_0)}|x_i\rangle \tag{4.22}
$$

where

$$
\mathbb{1}_k=\int dx_k\frac{dp_k}{2\pi}\underbrace{e^{ip_kx_k}}_{\langle x_k|p_k\rangle}|x_k\rangle\langle p_k| \tag{4.23}
$$

Next, we observe that

$$\forall k \qquad \langle p_k|e^{-i\widehat{H}(\hat{p},\hat{x})\Delta t}|x_{k-1}\rangle = e^{-ip_kx_{k-1}-iH(p_k,x_{k-1})\Delta t} \tag{4.24}$$

and collecting everything get

$$\begin{aligned}\langle x_f|e^{-i\widehat{H}(t_f-t_i)}|x_i\rangle &= \int \prod dx_k \prod \frac{dp_k}{2\pi} \exp\Big\{ip_N(x_N - x_{N-1}) + \ldots + ip_1(x_1 - x_0)\Big\}\\ &\qquad\qquad \times \exp\Big\{ - i\Delta t\big[H(p_N, x_{N-1}) + \ldots + H(p_1, x_0)\big]\Big\}\\ &= \int \mathcal{D}x(t)\mathcal{D}p(t)\, \exp\Big\{i\int_{t_i}^{t_f} dt\, \Big[p(t)\dot{x}(t) - H(p(t), x(t)\Big]\Big\}\\ &\to \int\limits_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}x(t)\, \exp\Big\{i\int_{t_i}^{t_f} dt\, L(x(t), \dot{x}(t))\Big\}\end{aligned} \tag{4.25}$$

- Let us try to do the same for our problem. The only difference is that we have two variables:

$\langle 0, phys|e^{-i\widehat{H}(t_f-t_i)}|0, phys\rangle =$

$$\begin{aligned}&= \int dx_1^f dx_2^f\, \Psi_0(x_1^f - x_2^f) \int dx_1^i dx_2^i\, \Psi_0(x_1^i - x_2^i)\langle x_1^f x_2^f|e^{-i\widehat{H}(t_f-t_i)}|x_1^i x_2^i\rangle\\ &= \int dx_1^f dx_2^f\, e^{-\frac{1}{2}\omega(x_{12}^f)^2} \int dx_1^i dx_2^i\, e^{-\frac{1}{2}\omega(x_{12}^i)^2}\\ &\quad\times \int \mathcal{D}x_1\mathcal{D}x_2\mathcal{D}p_1\mathcal{D}p_2 \exp\Big\{i\int dt\, \Big[p_1\dot{x}_1 + p_2\dot{x}_2 - \frac{1}{2}p_1^2 - \frac{1}{2}p_2^2 - \frac{1}{2}\omega^2 x_{12}^2\Big]\Big\}\end{aligned} \tag{4.26}$$

and introducing natural new variables

$$X = \frac{1}{2}(x_1 + x_2)\,, \qquad x = x_1 - x_2\,, \qquad P = p_1 + p_2\,, \qquad p = \frac{1}{2}(p_1 - p_2) \tag{4.27}$$

get

$$\begin{aligned}\ldots &= \int dx_f dX_f e^{-\frac{1}{2}\omega x_f^2} \int dx_i dX_i e^{-\frac{1}{2}\omega x_i^2}\\ &\quad\times \int \mathcal{D}X\, \mathcal{D}x\, \mathcal{D}P\, \mathcal{D}p \exp\Big\{i\int dt\, \Big[P\dot{X} + p\dot{x} - \frac{1}{2}P^2 - p^2 - \frac{1}{2}\omega^2 x^2\Big]\Big\}\end{aligned} \tag{4.28}$$

These are simple integrals, let us try to take them. First do the path integrals over momenta:

$$\begin{aligned}&\int \mathcal{D}P\, e^{i\int dt[P\dot{X} - \frac{1}{2}P^2]} = \mathcal{N}\, e^{i\int dt \frac{1}{2}\dot{X}^2}\\ &\int \mathcal{D}p\, e^{i\int dt[p\dot{x} - p^2]} = \mathcal{N}\, e^{i\int dt \frac{1}{4}\dot{x}^2}\end{aligned} \tag{4.29}$$

Get

$$\begin{aligned}\ldots &= \int dx_f dX_f e^{-\frac{1}{2}\omega x_f^2} \int dx_i dX_i e^{-\frac{1}{2}\omega x_i^2}\\ &\quad\times \int \mathcal{D}X\, \mathcal{D}x\, \exp\Big\{i\int dt\, \Big[\frac{1}{2}\dot{X}^2 + \frac{1}{4}\dot{x}^2 - \frac{1}{2}\omega^2 x^2\Big]\Big\}\end{aligned} \tag{4.30}$$

The remaining path integral over $X(t)$ corresponds to the free motion and the result of this integration is a function of the *difference* between the final and initial "big X" coordinates:

$$\int\limits_{X(t_i)=X_i}^{X(t_f)=X_f} \mathcal{D}X\, e^{i\int dt\, \frac{1}{2}\dot{X}^2} = \langle X_f|e^{-i\frac{1}{2}\widehat{P}^2(t_f-t_i)}|X_i\rangle \;=\; \int \frac{dP}{2\pi}\langle X_f|P\rangle e^{-i\frac{1}{2}P^2 t_{fi}}\langle P|X_i\rangle$$

$$= \int \frac{dP}{2\pi} e^{iP(X_f-X_i)} e^{-i\frac{1}{2}P^2 t_{fi}} = F(X_f - X_i)\,, \tag{4.31}$$

and the remaining factors do not involve any X_f, X_i dependence. Thus if we try to make the remaining integrations $\int dX_f dX_i$ we will find a divergence along the direction of $X_f + X_i$. This is not good ☹.

• The problem arises because we did not define what is a "physical" state properly. First of all, it is not enough to say that the system is in a "physical" state at t_i and t_f, but one has to require that it does not fluctuate into "unphysical" states at intermediate times. Second, we should be careful with counting the number of states: Our two-particle system with c.m. coordinate $X = 0$ and $X = 1$ are not two different states, but two copies of the same state and we have to take into account its contribution only once. In order to ensure this, we have to substitute the unity operator insertions in Eq. (4.22) by projection operators on the physical subspace:

$$\boxed{\mathbb{1} = \sum_{\text{all}} |\Psi\rangle\langle\Psi| \qquad \Longrightarrow \qquad \mathbb{1}_{\text{phys}} = \sum_{\text{phys}} |\Psi\rangle\langle\Psi|} \tag{4.32}$$

For our simple case, this amounts to the modification of Eq. (4.23) to

$$\mathbb{1}_{\text{phys}} = \int \frac{dp_1}{2\pi}\frac{dp_2}{2\pi}(2\pi)\delta(p_1+p_2)\int dx_1 dx_2\, \delta\Big(\tfrac{1}{2}(x_1+x_2)-a\Big) e^{ip_1x_1+ip_2x_2}|x_1x_2\rangle\langle p_1p_2| \tag{4.33}$$

or

$$\mathbb{1}_{\text{phys}} = \int \frac{dp_1}{2\pi}\frac{dp_2}{2\pi}(2\pi)\delta(p_1+p_2)\int dx_1 dx_2\, \delta\Big(\tfrac{1}{2}(x_1+x_2)-a\Big) e^{-ip_1x_1-ip_2x_2}|p_1p_2\rangle\langle x_1x_2| \tag{4.34}$$

(both representations can be useful)

Let us check that $\mathbb{1}_{\text{phys}}$ acts as a unity operator on a physical subspace,

$$\langle y_1y_2|phys\rangle = \Psi(y_1-y_2)$$

We need to show that

$$\langle y_1y_2|\mathbb{1}_{\text{phys}}|phys\rangle \stackrel{?}{=} \langle y_1y_2|phys\rangle \tag{4.35}$$

so let us calculate:

$$\langle y_1y_2|\mathbb{1}_{\text{phys}}|phys\rangle =$$

$$= \langle y_1y_2|\int \frac{dp_1}{2\pi}\frac{dp_2}{2\pi}(2\pi)\delta(p_1+p_2)\int dx_1 dx_2\, \delta\Big(\tfrac{1}{2}(x_1+x_2)-a\Big) e^{-ip_1x_1-ip_2x_2}|p_1p_2\rangle\langle x_1x_2|phys\rangle$$

$$= \int \frac{dp_1}{2\pi}\frac{dp_2}{2\pi}(2\pi)\delta(p_1+p_2)\int dx_1 dx_2\, \delta\Big(\tfrac{1}{2}(x_1+x_2)-a\Big) e^{-ip_1x_1-ip_2x_2}\underbrace{e^{ip_1y_1+ip_2y_2}}_{\langle y_1y_2|p_1p_2\rangle}\Psi(x_{12}) \tag{4.36}$$

where a is an arbitrary constant.
Integrals over momenta are trivial:

$$\int \frac{dp_1}{2\pi}\frac{dp_2}{2\pi}(2\pi)\delta(p_1+p_2)e^{-ip_1x_1-ip_2x_2+ip_1y_1+ip_2y_2}$$
$$= \int \frac{dp_1}{2\pi} e^{ip_1(-x_1+x_2+y_1-y_2)} = \delta(x_{12}-y_{12}) \tag{4.37}$$

so that we get

$$\int dx_1 dx_2\, \delta\big(\tfrac{1}{2}(x_1+x_2)-a\big)\delta(x_{12}-y_{12})\Psi(x_{12})$$
$$= \int d\Big(\tfrac{1}{2}(x_1+x_2)\Big) d(x_{12})\delta\big(\tfrac{1}{2}(x_1+x_2)-a\Big)\delta(x_{12}-y_{12})\Psi(x_{12}) = \Psi(y_{12}) \qquad ☺ \tag{4.38}$$

• Derivation of the path integral can now be repeated inserting these extra δ-functions. Note (will be important later) that it is NOT necessary to use the same constant a at all time slices – they can be different. We get

$$\langle phys, f|e^{-i\widehat{H}(t_f-t_i)}|phys, i\rangle =$$
$$= \int dx_1^f dx_2^f\, \Psi^f_{\rm phys}(x_{12}^f)\delta\big(\tfrac{1}{2}(x_1^f+x_2^f)-a_f\Big) \int dx_1^i dx_2^i\, \Psi^i_{\rm phys}(x_{12}^i)\delta\big(\tfrac{1}{2}(x_1^i+x_2^i)-a_i\Big)$$
$$\times \int \mathcal{D}p_1(t)\mathcal{D}p_2(t)\prod_t \delta(p_1(t)+p_2(t)) \int \mathcal{D}x_1(t)\mathcal{D}x_2(t)\prod_t \delta\big(\tfrac{1}{2}(x_1(t)+x_2(t))-a(t)\Big)$$
$$\times \exp\left\{i\int dt\left[p_1\dot{x}_1+p_2\dot{x}_2-\frac{1}{2}p_1^2-\frac{1}{2}p_2^2-\frac{1}{2}\omega^2x_{12}^2\right]\right\} \tag{4.39}$$

Starting from this expression we can take the integrals over momenta and over x_1+x_2 and will find

$$\ldots = \int dx_f dx_i \Psi^f_{\rm phys}(x_f)\Psi^i_{\rm phys}(x_i)\int \mathcal{D}x(t)\,\exp\left\{i\int dt\left[\frac{1}{4}\dot{x}^2-\frac{1}{2}\omega^2x^2\right]\right\} \tag{4.40}$$

which is the same expression as we found above writing path integral in terms of $x = x_1 - x_2$ directly. So, it seems we are on the right path!

$\boxed{Lect.\ 11}$

• The important point to cancel all divergences and reproduce the correct answer was a simple identity that allowed us to get rid of integrations over non-dynamical coordinates

$$\prod_k \int dX_k\, dx_k \delta(X_k-a_k) = \prod_k dx_k \qquad\qquad x_k = x(t_k) \tag{4.41}$$

or

$$\prod_k \int dX_k\, \delta(X_k-a_k) = 1 \tag{4.42}$$

In order to write this we had to know the expression for X_k in terms of original coordinates:

$$\{x_1, x_2\} \mapsto \{X, x\}, \qquad\qquad X = X(x_1,x_2), \qquad x = x(x_1,x_2) \tag{4.43}$$

What can we do if the explicit expressions for new coordinates $X(x_1, x_2)$, $x(x_1, x_2)$ are not known?

To make the following equations less cumbersome, let us introduce a shorthand notation for the set of coordinates at all time slices

$$\underline{X} = \{X_1 \ldots, X_N\} \qquad\qquad \underline{x} = \{x_1 \ldots, x_N\} \tag{4.44}$$

Note, first of all, that (4.42) can be replaced with the same effect by a more general expression

$$\prod_k \int dX_k \, \delta(f_k(\underline{X}, \underline{x})) \det \left| \frac{df_i(\underline{X}, \underline{x})}{dX_j} \right| = 1 \tag{4.45}$$

which is a multidimensional analogue of

$$\int dX \, f'(X)|\delta[f(X)] = \int d(f(X)) \, \delta[f(X)] = 1 \,, \tag{4.46}$$

assuming that functions $f_k(X_1, \ldots, X_N; x_1, \ldots, x_N)$ have only one solution of $f_k(\underline{X}, \underline{x}) = 0$ for $\underline{X}$ assuming given set of $\underline{x}$.

Now we have to recall some results from theoretical mechanics (in Hamiltonian formulation). Going over to

$$\{\underline{x}_1, \underline{x}_2\} \mapsto \{\underline{X}, \underline{x}\} \,, \qquad\qquad \{\underline{p}_1, \underline{p}_2\} \mapsto \{\underline{P}, \underline{p}\} \tag{4.47}$$

can be viewed as a canonical transformation to new generalized coordinates. An important concept in this formalism are Poisson brackets. One can show that

$$\frac{df_i(\underline{X}, \underline{x})}{dX_j} = \Big\{ P_i, f_j \Big\}_{P,p,X,x} \tag{4.48}$$

where

$$\Big\{ F_i, G_j \Big\}_{P,X,p,x} \stackrel{!}{=} \sum_k \left[\frac{\partial F_i}{\partial P_k} \frac{\partial G_j}{\partial X_k} + \frac{\partial F_i}{\partial p_k} \frac{\partial G_j}{\partial x_k} \right] - (F \leftrightarrow G) \tag{4.49}$$

The most important property: Poisson brackets are invariant under canonical transformations, i.e.

$$\Big\{ F_i, G_j \Big\}_{P,p,X,x} = \Big\{ F_i, G_j \Big\}_{p_1,p_2,x_1,x_2} \tag{4.50}$$

so that we can write (for our case)

$$\frac{df_i(\underline{X}, \underline{x})}{dX_j} = \Big\{ (p_1 + p_2)_i, \tilde{f}_j(\underline{x}_1, \underline{x}_2) \Big\}_{p_1,p_2,x_1,x_2} \tag{4.51}$$

Here we assume that we know $P = p_1 + p_2$ but pretend that we do not know $X(x_1, x_2)$, $x(x_1, x_2)$ Formally

$$\tilde{f}(\underline{x}_1, \underline{x}_2) = f(\underline{X}(x_1, x_2), \underline{x}(x_1, x_2)) \tag{4.52}$$

This relation is of no use if we do not know what is X and x but since the function $f(\underline{X}, \underline{x})$ in (4.45) was (almost) arbitrary, we can use an (almost) arbitrary function of original coordinates $\tilde{f}(\underline{x}_1, \underline{x}_2)$

instead!

• The rest is a technicality, we have to take a continuum limit and replace the sets of coordinates and momenta at given time slices by functions of time. Thus

$$
\begin{aligned}
f_k(\underline{x}_1, \underline{x}_2) &\mapsto f[x_1(t), x_2(t)] \\
\Big\{(p_1+p_2)_i, \tilde{f}_j(\underline{x}_1, \underline{x}_2)\Big\} &\mapsto \Big\{p_1(t)+p_2(t), f[x_1(t'), x_2(t')]\Big\}
\end{aligned}
\tag{4.53}
$$

with

$$
\Big\{F(t), G(t')\Big\} \stackrel{!}{=} \int dt'' \bigg(\frac{\delta F}{\delta p_1(t'')} \frac{\delta G}{\delta x_1(t'')} + \frac{\delta F}{\delta p_2(t'')} \frac{\delta G}{\delta x_2(t'')} - (F \leftrightarrow G) \bigg)
\tag{4.54}
$$

And this is the end, we can write a more general path-integral representation of Green functions for our system

$$
\begin{aligned}
&\langle phys, f| e^{-i\widehat{H}(t_f - t_i)} | phys, i\rangle = \\
&= \int dx_1^f dx_2^f \, \Psi^f_{\text{phys}}(x^f_{12}) \delta\Big(f[x_1^f, x_2^f]\Big) \int dx_1^i dx_2^i \, \Psi^i_{\text{phys}}(x^i_{12}) \delta\Big(f[x_1^i, x_2^i]\Big) \\
&\quad \times \int \mathcal{D}p_1(t) \mathcal{D}p_2(t) \prod_t \delta(p_1(t) + p_2(t)) \\
&\quad \times \int \mathcal{D}x_1(t) \mathcal{D}x_2(t) \, \det\Big\{p_1(t)+p_2(t), f[x_1(t), x_2(t)]\Big\} \prod_t \delta\Big(f[x_1(t), x_2(t)]\Big) \\
&\quad \times \exp\Big\{ i \int dt \, \Big[p_1\dot{x}_1 + p_2\dot{x}_2 - H(x_1, x_2, p_1, p_2)\Big]\Big\}
\end{aligned}
\tag{4.55}
$$

$$
\begin{aligned}
\delta(p_1+p_2) \quad &\leftrightarrow \quad \text{constraint} \\
\delta(f(x_1, x_2)) \quad &\leftrightarrow \quad \text{gauge fixing}
\end{aligned}
\tag{4.56}
$$

• Our goal is of course to develop a technique that can be applied to more complicated systems so let us see how it works for QED. Compare:

	QM example	QED in Coulomb gauge
auxiliary fields	$x_1(t), x_2(t), \cancel{A(t)}$	$A_\mu(t) = \{\cancel{\Phi}, \vec{A}\}$
canon. coord.	$x_1(t), x_2(t)$	$A_k(\vec{x}, t)\,, \quad k = 1,2,3$
canon. momenta	$p_1(t), p_2(t)$	$E_k(\vec{x}, t)\,, \quad k = 1,2,3$
constraint	$p_1(t) + p_2(t) = 0$	$\vec{\nabla} \cdot \vec{E} = 0$
gauge fix.	$x_1 + x_2 - a - 0$	$\vec{\nabla} \cdot \vec{A} = 0$
Poisson brack.	1	?

Let us calculate the Poisson bracket in QED.

$$\Big\{\frac{\partial}{\partial x_k}E^k(x), \frac{\partial}{\partial y_l}A^l(y)\Big\} = \qquad\qquad \partial_x^k = \frac{\partial}{\partial x_k}$$

$$= \int d^4z \Big\{ \frac{\delta \partial_x^k E^k(x)}{\delta E^i(z)} \frac{\delta \partial_y^l A^l(y)}{\delta A^i(z)} - \cancel{\frac{\delta \partial_x^k E^k(x)}{\delta A^i(z)}} \cancel{\frac{\delta \partial_y^l A^l(y)}{\delta E^i(z)}} \Big\}$$

$$= \int d^4z \, \partial_x^k \delta^{(4)}(x-z)\delta^{ki} \; \partial_y^l \delta^{(4)}(y-z)\delta^{li} \tag{4.57}$$

Use

$$\int d^4z \, \delta^{(4)}(x-z)\delta^{(4)}(y-z) = \delta^{(4)}(x-y) \tag{4.58}$$

$$\ldots = \partial_x^k \partial_y^k \delta^{(4)}(x-y) = \vec{\nabla}^2 \delta^{(4)}(x-y) \tag{4.59}$$

(Laplace operator applied to the delta-function). We will need a determinant of this operator which can also be written as

$$\det \vec{\nabla}^2 \delta^{(4)}(x-y) = e^{\mathrm{Tr} \, \ln \vec{\nabla}^2 \delta^{(4)}(x-y)} \tag{4.60}$$

Now all ingredients are in place and we can write the QED path integral (I skip averaging over the states)

$$\int \mathcal{D}A^k(x) \prod_x \delta(\vec{\nabla} \cdot \vec{A}) \int \mathcal{D}E^k \prod_x \delta(\vec{\nabla} \cdot \vec{E}) \det \vec{\nabla}^2 \delta^{(4)}(x-y) \, e^{i\int d^4x \, [\vec{E}\cdot\dot{\vec{A}} - H(E,A)]} \tag{4.61}$$

This is so far a path integral in phase space. We can now rewrite

$$\prod_x \delta(\vec{\nabla} \cdot \vec{E}) = \int \mathcal{D}A_0(x) \, e^{i\int d^4x \, A_0(x) \vec{\nabla}\cdot\vec{E}(x)} \tag{4.62}$$

after which the integral over the electric fields (= generalized momenta) can be done and the result can be written as

$$\ldots = \int \underbrace{\mathcal{D}A^k(x)\mathcal{D}A^0(x)}_{\mathcal{D}A^\mu(x)} \prod_x \delta(\vec{\nabla} \cdot \vec{A}) \underbrace{\det \vec{\nabla}^2 \delta^{(4)}(x-y)}_{\text{constant!}} \exp\Big\{-i\int d^4x \, \frac{1}{4} F_{\mu\nu}F^{\mu\nu}\Big\} \tag{4.63}$$

Finally, the determinant can be thrown out as it is a constant that will cancel in the ratio of path integrals for Green functions (this simplification is specific for QED).

4.3 Faddeev-Popov method, QED in covariant gauges

In 1967 Faddeev and Popov suggested a very elegant formalism to quantize constrained systems with main focus on nonabelian gauge theories. Let us introduce this technique first for QED.

- Leaving aside the question with boundary conditions (which role is mainly to derive the $+i\epsilon$

prescription in Feynman propagator), one can try to write the QED path integral in the "naive" version as

$$\int \mathcal{D}A\, e^{iS[A]} \qquad\qquad \mathcal{D}A \equiv \mathcal{D}A\mathcal{D}A_0\mathcal{D}A_1\dots\mathcal{D}A_3 \tag{4.64}$$

The action is

$$\begin{aligned} S &= \int d^4x \left[-\frac{1}{4}F_{\mu\nu}^2\right] \qquad\qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\ &= \frac{1}{2}\int d^4x\, A_\mu(x)\Big[\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu\Big]A_\nu(x) \end{aligned} \tag{4.65}$$

Going over to momentum space this can be written as

$$\begin{aligned} A_\mu(x) &= \int \frac{d^4k}{(2\pi)^4}e^{-ikx}A_\mu(k) \\ S &= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}A_\mu(k)\Big[-k^2 g_{\mu\nu} + k_\mu k_\nu\Big]A_\nu(k) \end{aligned} \tag{4.66}$$

To formulate the perturbation theory we need to find the propagator which is defined as a Green function (with proper boundary conditions)

$$\Big[\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu\Big]D^{\nu\rho}(x-y) = i\delta_\mu^\rho\delta^{(4)}(x-y) \qquad g_\mu{}^\rho \equiv \delta_\mu^\rho \tag{4.67}$$

which becomes a (matrix) algebraic equation in momentum space:

$$\Big[-k^2 g_{\mu\nu} + k_\mu k_\nu\Big]D^{\nu\rho}(k) = i\delta_\mu^\rho \tag{4.68}$$

This equation, however, does not have any solutions. Indeed, because of Lorentz invariance $D^{\nu\rho}$ must have the structure

$$D^{\nu\rho}(k) = d_1 g^{\nu\rho} + d_2 k^\nu k^\rho \tag{4.69}$$

so that

$$\begin{aligned} \Big[-k^2 g_{\mu\nu} + k_\mu k_\nu\Big]D^{\nu\rho}(k) &= \Big[-k^2 g_{\mu\nu} + k_\mu k_\nu\Big]\Big[d_1 g^{\nu\rho} + d_2 k^\nu k^\rho\Big] \\ &= -d_1 g^{\mu\rho}k^2 + k_\mu k_\rho[\cancel{-d_2k^2} + d_1 - \cancel{d_2k^2}] \end{aligned} \tag{4.70}$$

Thus we are forced to put $d_1 = 0$ to kill the $k_\mu k_\rho$ term and cannot get the $i\delta_\mu^\rho$ required on the r.h.s. In other words, an inverse operator does not exist……

The problem, of course, has its roots in gauge invariance. The action does not change under gauge transformations

$$A_\mu(x) \mapsto A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x) \equiv A_\mu^\alpha(x) \quad \leftarrow \text{ notation} \tag{4.71}$$

and this reflects that fact that physical — electric and magnetic — fields do not change under such trafo. As the result the operator $\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu$ has zero modes — eigenfunctions with zero eigenvalues, and the inverse does not exist. Each physical state has to contribute once to the path integral and writing path integral over all four components in A_μ we overcount - each state comes many times. [We have seen this on the QM example in the previous section].

- To reduce the number of states we can try to impose some gauge condition. For example

$$G(A) = 0 \qquad\qquad \text{where G is some functional} \tag{4.72}$$

The (first) idea by Faddeev and Popov was to insert a unity factor inside the path integral, of the type

$$1 = \int_{-\infty}^{\infty} dx\, \delta(f(x)) \left|\frac{df}{dx}\right| \tag{4.73}$$

which can be generalized to (we used this several times already)

$$1 = \int \mathcal{D}\alpha(x)\delta\Big[G(A^\alpha)\Big] \det\left(\frac{\delta G(A^\alpha)}{\partial\alpha}\right) \tag{4.74}$$

Here

$$\delta\Big[G(A^\alpha)\Big] \equiv \prod_x \delta\Big[G(A^\alpha)\Big] \tag{4.75}$$

and (cf. above)

$$A^\alpha_\mu(x) \equiv A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x) \tag{4.76}$$

Let us take a generalized Lorentz-type gauge condition as example (most important practically):

$$G(A) = \partial_\mu A^\mu - \underbrace{\omega(x)}_{\text{arbitrary function}} \tag{4.77}$$

Then

$$\begin{aligned} \det\left(\frac{\delta G(A^\alpha)}{\partial\alpha}\right) &= \det\left[\frac{\delta}{\delta\alpha(x)}\Big(\partial_\mu A^\mu(y) + \frac{1}{e}\partial^2\alpha(y) - \omega(y)\Big)\right] \\ &= \det\Big[\partial^2\delta^{(4)}(x-y)\Big] \end{aligned} \tag{4.78}$$

Note that this is the same determinant as in previous section. Since it does not depend on A_μ we can pull it out of the path integral to get

$$\int \mathcal{D}A \cdot 1 \cdot e^{iS[A]} = \det\Big[\partial^2\delta^{(4)}(x-y)\Big] \int \mathcal{D}\alpha \int \mathcal{D}A\, e^{iS[A]}\delta\Big[G(A^\alpha)\Big] \tag{4.79}$$

However,

$$\begin{aligned} \int \mathcal{D}A &= \int \mathcal{D}A^\alpha &&\text{shift of variable} \\ S[A] &= S[A^\alpha] &&\text{gauge invariance} \end{aligned} \tag{4.80}$$

Thus

$$\ldots = \det\Big[\partial^2\delta^{(4)}(x-y)\Big]\int\mathcal{D}\alpha\int\mathcal{D}A^\alpha\, e^{iS[A^\alpha]}\delta\Big[G(A^\alpha)\Big]$$

$$= \underbrace{\det\Big[\partial^2\delta^{(4)}(x-y)\Big]\int\mathcal{D}\alpha}_{\text{irrelevant constant}}\int\mathcal{D}A\, e^{iS[A]}\delta\Big[G(A)\Big] \tag{4.81}$$

• We still have a freedom to choose $\omega(x)$ in the gauge condition, and this can be used to bring $\delta\Big[G(A)\Big]$ to smth more manageable. To this end we do not choose a particular function, but integrate over all $\omega(x)$ with the Gaussian weight factor:

$$\int\mathcal{D}\omega(x)\, e^{-i\int d^4x\frac{\omega^2(x)}{2\xi}}\ldots\ldots\delta\Big[\partial_\mu A^\mu-\omega\Big] = \exp\left(-i\int d^4x\,\frac{(\partial_\mu A^\mu)^2}{2\xi}\right) \tag{4.82}$$

This integral is taken trivially using the gauge-fixing delta-function and results in an extra term in the Lagrangian density

$$\mathcal{L}^{(\xi)} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{4.83}$$

(name: gauge-fixed Lagrangian)

We can now calculate all Green functions involving gauge-invariant operators $O(A^\alpha) = O(A)$:

$$\langle\Omega|\mathrm{T}\left\{O_1(A)\ldots O_n(A)\right\}|\Omega\rangle = \frac{\int\mathcal{D}A\,O_1(A)\ldots O_n(A)\,e^{iS^{(\xi)}[A]}}{\int\mathcal{D}A\,e^{iS^{(\xi)}[A]}} \tag{4.84}$$

The Faddeev-Popov determinant $\det\left[\partial^2\delta^{(4)}(x-y)\right]$ and the volume of the gauge group $\int\mathcal{D}\alpha(x)$ cancel out in the ratio!

• Back to the propagator. With the additional term in the action, the equation (4.85) also gets an additional contribution. We get

$$\Big[-k^2g_{\mu\nu}+\Big(1-\frac{1}{\xi}\Big)k_\mu k_\nu\Big]D^{\nu\rho}(k) = i\delta_\mu^\rho \tag{4.85}$$

which is solved by

$$D^{\mu\nu}(k) = \frac{-1}{k^2+i\epsilon}\left(g^{\mu\nu}-(1-\xi)\frac{k^\mu k^\nu}{k^2}\right) \tag{4.86}$$

The gauge parameter ξ can be chosen arbitrary. The most common choices are

$$\begin{aligned} &\xi = 0 && \text{Landau gauge}\\ &\xi = 1 && \text{Feynman gauge}\\ &\xi = 3 && \text{Yennie gauge} \end{aligned} \tag{4.87}$$

Lect. 12

• We expect that all physical results, e.g. the probability to emit a photon in a certain reaction, do not depend on the choice of gauge parameter. How can this happen?

To describe photon emission we can consider a generating functional adding external source to the action:

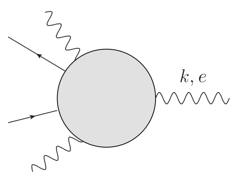
$$S[J] = \int d^4x \left[-\frac{1}{4}F^2_{\mu\nu}(x) - A_\mu(x)J^\mu(x) \right] \tag{4.88}$$

The corresponding Euler-Lagrange equation = Maxwell equation

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{4.89}$$

so that the viervector $J^\nu = \{\rho, \vec{j}\}$ is composed from the charge/current density of external particles (e.g. electrons).

Probability amplitude to emit a photon will then be given by the matrix element of the current sandwiched between the initial and final state:



$$\begin{aligned} iM(k) &= iM^\mu(k)\epsilon^*_\mu(k) \\ M^\mu(k) &= \int d^4x\, e^{ikx} \langle f|\widehat{J}^\mu(x)|i\rangle \end{aligned} \tag{4.90}$$

[We need to write operator $\widehat{J}^\mu(x)$ in real QED where electrons are also quantum particles, but for the present argument we can also take a function $J(x)$ assuming that photons are produced by some classical source.]

Remember that the current J^μ in classical QED is conserved:

$$\partial_\mu J^\mu = 0 \qquad \Leftrightarrow \qquad \frac{d}{dt}\rho + \vec{\nabla}\cdot\vec{j} = 0 \tag{4.91}$$

and therefore

$$0 = i\int d^4x\, e^{ikx}\langle f|\partial_\mu J^\mu(x)|i\rangle \overset{p.I}{=} \int d^4x\, \langle f|J^\mu(x)|i\rangle \left(-i\frac{\partial}{\partial x^\mu}\right) e^{ikx} = k_\mu M^\mu(k) \tag{4.92}$$

which is called a Ward identity. For classical sources, it is therefore a direct consequence of current conservation. In quantum theory (QED) we have to make sure that charged current is conserved as an operator — this deserves a detailed discussion (later).

The $\epsilon^*_\mu(k)$ in above equation is photon polarization vector (complex conjugated because photon in final state). Let us recall how are the polarization vectors defined. The four-vector potential of a real photon, $k^2 = 0$, can be written as

$$A^{(\lambda)}_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2k_0}\left[\epsilon^{(\lambda)}_\mu e^{-ikx} + \epsilon^{*(\lambda)}_\mu e^{ikx}\right], \qquad k_0 = |\vec{k}| \tag{4.93}$$

Let us choose a frame of reference where

$$k_\mu = \{k, 0, 0, k\} \tag{4.94}$$

In this frame we can define four orthogonal vectors

$$\begin{aligned} &\epsilon^{(1)}_\mu = \{0,1,0,0\} \qquad \epsilon^{(2)}_\mu = \{0,0,1,0\} \\ &\epsilon^{(+)}_\mu = \frac{1}{\sqrt{2}}\{1,0,0,1\} \qquad \epsilon^{(-)}_\mu = \frac{1}{\sqrt{2}}\{1,0,0,-1\} \end{aligned} \tag{4.95}$$

as a basis. Only first two possibilities (transverse polarizations) are physical because the other two can be disposed of by the choice of gauge:

- Lorentz gauge

$$\partial^\mu A_\mu(x) = 0 \implies k^\mu A_\mu(k) = 0$$
$$\implies A_\mu(k) = e_\mu^{(-)} A_-(k) \;\; \text{not allowed} \;\; [k^\mu e_\mu^{(-)} = 2] \tag{4.96}$$

- For the special case $k^2 = 0$ Lorentz condition does not specify the gauge uniquely

$$0 = k^\mu A_\mu(k) = k^\mu A'_\mu(k) = k^\mu (A_\mu + k_\mu \lambda(k))$$
$$\implies A_\mu(k) = e_\mu^{(+)} A_+(k) \sim k_\mu \;\; \text{can be gauged away} \tag{4.97}$$

Thus, emission of "plus" or "minus" photons cannot influence any observable quantities.

Imagine we want to calculate the total probability of photon emission:

$$\sigma \sim \sum_{\substack{phys.\\polar.}} |M|^2 = \sum_{\substack{phys.\\polar.}} \epsilon_\mu^{(\lambda),*} \epsilon_\nu^{(\lambda)} M^\mu M^{\nu,*} = |M^1|^2 + |M^2|^2 \tag{4.98}$$

However

$$k_\mu M^\mu = 0 \implies k_0 M^0 - k_3 M^3 = 0 \implies M^0 = M^3 \tag{4.99}$$

Therefore can write also

$$\sum_{\substack{phys.\\polar.}} \epsilon_\mu^{(\lambda),*} \epsilon_\nu^{(\lambda)} M^\mu M^{\nu,*} = |M^1|^2 + |M^2|^2 \;=\; |M^1|^2 + |M^2|^2 + |M^3|^2 - |M^0|^2$$
$$= -g_{\mu\nu} M^\mu M^{\nu,*} \;=\; \sum_{\substack{all\\polar.}} \epsilon_\mu^{(\lambda),*} \epsilon_\nu^{(\lambda)} M^\mu M^{\nu,*} \tag{4.100}$$

i.e. the sum over transverse polarizations is equal to the sum over all polarizations.

On a more technical level, this means that QED interaction vertices have such a form that they effectively annihilate the longitudinal ξ-dependent parts of the propagators in Feynman diagrams (this happens in the sum of all diagrams with photon emission in all points along a given fermion line, not in each diagram separately).

Another formulation — interaction is such that unphysical degrees of freedom can not be produced in collisions of physical particles — the QED scattering matrix (S-matrix) is unitary.

5 Path Integrals for fermions

Path-integral technique allows to build (define) a quantum field theory starting from classical fields. We did it for scalar fields and photons, and now want to do it for fermions.

$$\begin{aligned} &\text{Electric/magnetic fields (class.)} && \xrightarrow{\text{Path.I.}} && \text{Photons (quantum)} \\ &(?) \quad \text{Classical Dirac Field} && \xrightarrow{\text{Path.I.}} && \text{Electrons (quantum)} \end{aligned} \tag{5.1}$$

The problem is, we do not know what is a classical Dirac field — and note that Dirac spinors (solutions of Dirac equations) are not directly observable.

The problem is to incorporate the anti-commutation relation for classical fields that are functions, not operators:

$$\{\Psi(\vec{x},t),\Psi(\vec{y},t)=0 \tag{5.2}$$

and the idea is to write this functions, schematically, as

$$\Psi(x)=\sum_k \theta_k\phi_i(x) \tag{5.3}$$

where $\phi_k(x)$ form a complete set of "normal" functions, and θ_k are new mathematical objects, anticommuting numbers (= Grassman numbers).

- We define Grassman numbers as the set of anticommuting numbers

$$\forall\ \theta,\eta \qquad \theta\,\eta=-\eta\,\theta \tag{5.4}$$

As a consequence

$$\forall\ \theta, \qquad \theta^2=0 \tag{5.5}$$

We assume that Grassman numbers can be added and also multiplied by complex numbers producing objects like $A\theta+B\eta$, $A,B\in\mathbb{C}$, and these operations obeys all natural rules (commutativity, transitivity etc.).

Next, we need to define integrals over Grassman numbers. Note that for path integrals (our goal) we only need analogue of $\int_{-\infty}^{\infty}$:

$$\int d\theta\, f(\theta)=? \tag{5.6}$$

The nice thing is, all functions of Grassman numbers are linear functions (as $\theta^2=0$, so each $f(\eta)$ can be written as

$$f(\theta)=A+B\,\theta \tag{5.7}$$

Working with path integrals we have seen that the invariance w.r.t. the change the variable by a given function $\mathcal{D}\phi(x)=\mathcal{D}(\phi(x)+\phi_0(x)$ is very important and we want to have the same property (translation invariance) for fermions. We therefore require that

$$\int d\theta\, f(\theta)\equiv\int d(\theta+\eta)\, f(\theta+\eta)\ \stackrel{!}{=}\ \int d\theta\, f(\theta+\eta) \tag{5.8}$$

and of course we want to define integration as a linear operation, i.e.

$$\int d\theta\,[Af(\theta)+Bg(\theta)]=A\int d\theta\, f(\theta)+B\int d\theta\, g(\theta)\,, \qquad A,B\in\mathbb{C}$$

Using that $f(\eta)$ is actually a linear function, (5.8) becomes

$$\forall\eta, A, B \qquad \int d\theta\,[A + B\,\theta] \stackrel{!}{=} \int d\theta\,[A + B\,(\theta+\eta)]$$
$$= \int d\theta\,[(A + B\,\eta) + B\theta] = \int d\theta\,[(A + B\,\eta) + B\int d\theta\,\theta \tag{5.9}$$

and the only possibility how it can be satisfied is to require

$$\int d\theta = 0\,,$$
$$\int d\theta\,\theta = 1\,, \qquad \text{(arbitrary constant, can be set to one)} \tag{5.10}$$

In order to deal with multiple integrals we have to add a definition (sign convention)

$$\int d\theta \int d\eta\,\eta\theta \equiv \int d\theta \left(\int d\eta\eta\right)\theta = 1 \tag{5.11}$$

(so that, effectively $d\theta\eta = -\eta d\theta$, etc.).
Last but not least, we need "complex-valued" Grassman numbers, so we introduce an operation of complex conjugation $\theta \mapsto \theta^*$ such that $(\theta^*)^* = \theta$ and

$$(\theta\,\eta)^* \stackrel{!}{=} \eta^*\theta^* = -\theta^*\eta^* \tag{5.12}$$

Note that $\theta^2 = (\theta^*)^2 = 0$ but $\theta\theta^* = -\theta^*\theta \neq 0$.
For the integrals over complex Grassman numbers we assume as above

$$\int d\theta^* d\theta\,\theta\,\theta^* = 1 \tag{5.13}$$

As a consequence

$$\int d\theta^* d\theta\, e^{-\theta^* b\theta} = \int d\theta^* d\theta\,(1 - \theta^* b\theta) = b \tag{5.14}$$

to be compared with a similar integral over "normal" complex variables (Gauss)

$$\int dz^* dz\, e^{-z^* bz} = \frac{2\pi}{b} \tag{5.15}$$

Note that b appears in the numerator in the first case, and in the denominator in the second. This will be important!

Next in complexity, we will need integrals of this type but with constant b replaced by a matrix (or differential operator). To derive these, we need to understand what happens with products of Grassman numbers under unitary transformations.
Assume we have a set of Grassman numbers

$$\theta_i, \qquad i = 1, \ldots, n \tag{5.16}$$

and "rotate" them with a unitary matrix

$$\theta'_i = U_{ij}\theta_j \qquad U^\dagger U = \mathbb{1} \tag{5.17}$$

Then

$$\begin{aligned}\prod_i \theta'_i &= \frac{1}{n!}\epsilon^{ij\ldots\ell}\theta'_i\theta'_j\ldots\theta'_\ell \\ &= \frac{1}{n!}\epsilon^{ij\ldots\ell}U_{ii'}\theta_{i'}U_{jj'}\theta_{j'}\ldots U_{\ell\ell'}\theta_{\ell'} \\ &= \frac{1}{n!}\epsilon^{ij\ldots\ell}U_{ii'}U_{jj'}\ldots U_{\ell\ell'}\epsilon^{i'j'\ldots\ell'}\prod_k\theta_k \\ &= \det U \quad \prod_i \theta_i \end{aligned}\tag{5.18}$$

As a consequence

$$\int\prod_i d\theta_i^* d\theta_i \underbrace{f(\theta,\theta^*)}_{\prod\theta_i\prod\theta_i^*} = \int\prod_i \theta_i^*\theta_i\, f(\theta',\theta'^*) \tag{5.19}$$

because $\det U\cdot(\det U)^* = 1$.

We can use the possibility to make such rotations to bring hermitian matrices to the diagonal form so that

$$\int\prod_k d\theta_k^* d\theta_k\, e^{-\sum_{ij}\theta_i^* B_{ij}\theta_j} = \int\prod_k d\theta_k^* d\theta_k\, e^{-\sum_i\theta_i^* b_i\theta_i} = \prod_i b_i = \det\, B \tag{5.20}$$

For comparison, for "normal" integrals

$$\int\prod_k dz_k^* dz_k\, e^{-z_i^* B_{ij} z_i} = \frac{(2\pi)^n}{\det B} \tag{5.21}$$

Finally, two more integrals that can be derived starting from the above:

$$\begin{aligned}\int\prod_k d\theta_k^* d\theta_k\, e^{\theta_i^* B_{ij}\theta_j}\theta_\ell\theta_m^* &= \det B\,(B^{-1})_{\ell m}\,, \\ \int\prod_k dz_k^* dz_k\, e^{-z_i^* B_{ij} z_i} z_\ell z_m^* &= \frac{(2\pi)^n}{\det B}(B^{-1})_{\ell m}\end{aligned}\tag{5.22}$$

The difference is always in the position of the determinant — numerator vs. denominator!

- Now that the necessary mathematical tools are ready, we can define the classical Dirac field

$$\psi(x) = \sum_i \theta_i\,\phi_i(x) \tag{5.23}$$

where $\phi_i(x)$ are four-component objects (bispinors). We will then write the fermion action as

$$S_F = \int d^4x\, \bar\psi(x)(i\partial\!\!\!/ - m)\psi(x) \tag{5.24}$$

To see that it produces expected results we can calculate the propagator using above integrals (5.20), (5.22):

$$\begin{aligned}\langle 0|\mathrm{T}\{\widehat\psi(x_1)\widehat{\bar\psi}(x_2)\}|0\rangle &= \frac{\int \mathcal{D}\bar\psi\mathcal{D}\psi\, \psi(x_1)\bar\psi(x_2)\, e^{iS_F}}{\int \mathcal{D}\bar\psi\mathcal{D}\psi\, e^{iS_F}} \\ &= \frac{\det(i\partial\!\!\!/ - m)\langle x_1|\frac{i}{i\partial\!\!\!/-m}|x_2\rangle}{\det(i\partial\!\!\!/ - m)}\end{aligned} \tag{5.25}$$

Thus

$$\begin{aligned}S_F(x-y) \;\equiv\; \langle 0|\mathrm{T}\{\widehat\psi(x_1)\widehat{\bar\psi}(x_2)\}|0\rangle &= \int \frac{d^4k}{(2\pi)^4}\langle x_1|k\rangle \frac{i}{k\!\!\!/ - m + i\epsilon}\langle k|x_2\rangle \\ &= \int \frac{d^4k}{(2\pi)^4 i} e^{-ik(x_1-x_2)}\frac{1}{m - k\!\!\!/ - i\epsilon} \\ &= \int \frac{d^4k}{(2\pi)^4 i} e^{-ik(x_1-x_2)}\frac{m + k\!\!\!/}{m^2 - k^2 - i\epsilon}\end{aligned} \tag{5.26}$$

reproducing the well-known expression. Note that for fermion field operators

$$\mathrm{T}\{\widehat\psi(x_1)\widehat{\bar\psi}(x_2)\} = \theta(t_1-t_2)\,\widehat\psi(x_1)\widehat{\bar\psi}(x_2) - \theta(t_2-t_1)\widehat{\bar\psi}(x_2)\widehat\psi(x_1) \tag{5.27}$$

Lect. 13

In order to calculate Green functions with more than two fields we can construct the generating functional

$$\begin{aligned}Z[\bar\eta,\eta] &= \int \mathcal{D}\bar\psi\mathcal{D}\psi\ \exp\Big\{ i\int d^4x\, \Big[\bar\psi(x)(i\partial\!\!\!/ - m)\psi(x) + \bar\eta(x)\psi(x) + \bar\psi(x)\eta(x)\Big]\Big\} \\ &= Z_0 \exp\Big\{-\frac{1}{2}\int d^4x\, d^4y\, \bar\eta(x) S_F(x-y)\eta(y)\Big\}\end{aligned} \tag{5.28}$$

We need one more sign convention to take derivatives over Grassmanian numbers:

$$\frac{d}{d\eta}\theta\eta = -\frac{d}{d\eta}\eta\theta \stackrel{!}{=} -\theta \tag{5.29}$$

Then

$$\langle 0|\mathrm{T}\{\widehat\psi(x_1)\widehat{\bar\psi}(x_2)\}|0\rangle = Z_0^{-1}\left(-i\frac{\delta}{\delta\bar\eta(x_1)}\right)\left(+i\frac{\delta}{\delta\eta(x_2)}\right) Z[\bar\eta,\eta]\Big|_{\eta=\bar\eta=0} \tag{5.30}$$

etc.

- Everything is now ready to define Quantum Electrodynamics starting from the path integral with

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}^2 + \bar\psi(x)(i\partial\!\!\!/ - m)\psi(x) + e\bar\psi\gamma^\mu\psi\, A_\mu \tag{5.31}$$

Equations of motion (Maxwell+Dirac)

$$(i\not{D} - m)\psi = 0\,, \qquad \bar{\psi}(i\overleftarrow{\not{D}} + m) = 0$$
$$\partial_\mu F^{\mu\nu} = J^\nu\,, \qquad J^\nu = -e\bar{\psi}(x)\gamma^\mu\psi(x) \tag{5.32}$$

Covariant derivative:

$$D_\mu = \partial_\mu - ieA_\mu(x) \tag{5.33}$$

Weil, 1929:
The QED Lagrangian does not change if in addition to the gauge trafo

$$A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x) \tag{5.34}$$

one rotates the fermion field as

$$\psi(x) \to e^{i\alpha(x)}\psi(x) \tag{5.35}$$

— local gauge symmetry.

Noether theorem: Symmetry → conserved current/conserved charge

$$\partial_\nu J^\nu = 0 \qquad \text{on the classsical level} \tag{5.36}$$

Check:

$$\begin{aligned} i\partial_\mu\bar{\psi}(x)\gamma^\mu\psi(x) &= \bar{\psi}(x)[i\overleftarrow{\not{\partial}} + i\overrightarrow{\not{\partial}}]\psi(x) \\ &= \bar{\psi}(x)[(i\overleftarrow{\not{\partial}} - e\not{A} + m) + (i\overrightarrow{\not{\partial}} + e\not{A} - m)]\psi(x) = 0 \end{aligned} \tag{5.37}$$

What happens in quantum theory?

6 Ward identities and quantum anomalies

6.1 Derivation

There exists a simple way to show that current conservation in QED is a direct consequence of local gauge symmetry.

- Let us make an infinitesimal gauge trafo (phase rotation) of the fermion field $\alpha(x) \to 0$

$$\psi(x) \mapsto \psi'(x) = (1 + i\alpha(x))\psi(x)\,, \qquad \bar{\psi}(x) \mapsto \bar{\psi}'(x) = (1 - i\alpha(x))\bar{\psi}(x)\,, \tag{6.1}$$

but do not change the gauge (photon) field. In this case, of course, the Lagrangian density will not be invariant and will change to

$$\begin{aligned} \mathcal{L}(x) \mapsto \mathcal{L}'(x) &= \mathcal{L} - \bar{\psi}\gamma^\mu\psi(x)\partial_\mu\alpha(x) \\ &= \mathcal{L} + \frac{1}{e}J^\mu(x)\partial_\mu\alpha(x) \end{aligned} \tag{6.2}$$

Consider the QED path integral

$$\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}A\,e^{iS[\bar{\psi},\psi,A]}\,\psi(x_1)\bar{\psi}(x_2) = \int \mathcal{D}\bar{\psi}'\,\mathcal{D}\psi'\,\mathcal{D}A\,e^{iS[\bar{\psi}',\psi',A]}\,\psi'(x_1)\bar{\psi}'(x_2) \tag{6.3}$$

Equality holds because this can be viewed as a name change for the integration variables.

However

$$\mathcal{D}\bar{\psi}'\,\mathcal{D}\psi' = \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \tag{6.4}$$

and therefore $\forall\alpha(x)$

$$0 = \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}A\,e^{iS[\bar{\psi},\psi,A]}\Big\{\underbrace{i\int d^4x\,\frac{1}{e}J^\mu(x)\partial_\mu\alpha(x)}_{S'-S} + i\alpha(x_1) - i\alpha(x_2)\Big\}\psi(x_1)\bar{\psi}(x_2) \tag{6.5}$$

Integrating by parts in the first term in braces and writing, e.g., $\alpha(x_1) = \int d^4x\,\alpha(x)\,\delta(x-x_1)$ we obtain

$$\begin{aligned}\langle\Omega|\mathrm{T}\{\partial_\mu\hat{J}^\mu(x)\hat{\psi}(x_1)\hat{\bar{\psi}}(x_2)\}|\Omega\rangle &= e\delta(x-x_1)\langle\Omega|\mathrm{T}\{\hat{\psi}(x_1)\hat{\bar{\psi}}(x_2)\}|\Omega\rangle \\ &\quad - e\delta(x-x_2)\langle\Omega|\mathrm{T}\{\hat{\psi}(x_1)\hat{\bar{\psi}}(x_2)\}|\Omega\rangle\end{aligned} \tag{6.6}$$

which implies that for every matrix element

$$\langle 1|\partial_\mu\hat{J}^\mu(x)|2\rangle = 0 \tag{6.7}$$

This is the statement of current conservation in quantum theory (where from the usual Ward identities follow, as we have seen). As a consequence we have two exact predictions: 1) electric charge is conserved and 2) photon is massless. Both are checked experimentally to a very high accuracy.

• The argumentation seems to be very general and can be applied to any symmetry of the classical Lagrangian. Remember that relativistic fermion fields are four-component objects

$$\psi(x) = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} \tag{6.8}$$

and one can consider rotations between these components. A <u>chiral rotation</u> is defined as

$$\begin{aligned}\psi(x) \mapsto \psi'(x) &= e^{i\alpha\gamma_5}\psi(x) = [1 + i\alpha\gamma_5 + \ldots]\psi(x) \\ \bar{\psi}(x) \mapsto \bar{\psi}'(x) &= \bar{\psi}(x)e^{i\alpha\gamma_5} = \bar{\psi}(x)[1 + i\alpha\gamma_5 + \ldots]\end{aligned} \tag{6.9}$$

where the second line follows from

$$\bar{\psi}(x) = \psi^\dagger\gamma_0 \quad\Rightarrow\quad \bar{\psi}' = (e^{i\alpha\gamma_5}\psi)^\dagger\gamma_0 = \psi^\dagger\underbrace{(e^{i\alpha\gamma_5})^\dagger}_{e^{-i\alpha\gamma_5}}\gamma_0 = \bar{\psi}e^{i\alpha\gamma_5} \tag{6.10}$$

If α does not depend on x (global symmetry transformation) and electron mass can be neglected, $m = 0$, the Lagrangian density becomes invariant under such transformations:

$$\bar{\psi}\not{D}\psi \mapsto \bar{\psi}[1 + i\alpha\gamma_5]D_\mu\gamma^\mu[1 + i\alpha\gamma_5]\psi = \bar{\psi}\not{D}\psi \tag{6.11}$$

thanks to

$$\gamma_\mu\gamma_5 = -\gamma_5\gamma_\mu\,, \qquad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{6.12}$$

— **<u>Chiral Symmetry</u>**

Symmetry of the Lagrangian implies existence of a conserved current (Noether) and this current can be constructed explicitly (following Noether). One obtains the

—**<u>Axial Current</u>**

$$\begin{aligned} J^{\mu 5}(x) &= \bar{\psi}(x)\gamma^\mu\gamma_5\psi \\ \partial_\mu J^{\mu 5} &= 0 \qquad \text{classically} \end{aligned} \tag{6.13}$$

• In order to find out whether axial current is conserved in quantum theory, we can follow the above argumentation:
Let us consider a <u>local</u> chiral rotation instead of the <u>global</u> rotation

$$\alpha \mapsto \alpha(x) \tag{6.14}$$

The action will certainly not remain invariant but change by a term $\sim \partial_\mu\alpha(x)$

$$\bar{\psi}\not{D}\psi \mapsto \bar{\psi}\not{D}\psi - \partial_\mu\alpha(x)J^{\mu 5}(x) \tag{6.15}$$

(similar to what was before). We then proceed in the same way and need to prove that for chiral trafos also

$$\mathcal{D}\bar{\psi}'\,\mathcal{D}\psi' \stackrel{?}{=} \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \tag{6.16}$$

This can be argued in the following way. A Dirac spinor can be decomposed in left-handed and right-handed components as

$$\psi(x) = \frac{1}{2}(1 - \gamma_5)\psi(x) + \frac{1}{2}(1 + \gamma_5)\psi(x) \equiv \psi_L(x) + \psi_R(x) \tag{6.17}$$

The operations

$$P_L \stackrel{!}{=} \frac{1}{2}(1 - \gamma_5)\,, \qquad P_R \stackrel{!}{=} \frac{1}{2}(1 + \gamma_5) \tag{6.18}$$

are indeed projection operators since

$$P_L + P_R = \mathbb{1}\,, \qquad P_L^2 = P_L\,, \qquad P_R^2 = P_R\,, \qquad P_L P_R = P_R P_L = 0 \tag{6.19}$$

The action can also be written as a sum of two terms involving either left or right fermions,

$$\bar{\psi}\not{D}\psi = \bar{\psi}_L\not{D}\psi_L + \bar{\psi}_R\not{D}\psi_R \tag{6.20}$$

so that left-handed and right-handed fields become essentially independent (if $m = 0!$). Also in the path integral the measure splits in

$$\int \mathcal{D}\psi = \int \mathcal{D}\psi_R D\psi_L \tag{6.21}$$

In Weil representation for gamma-matrices γ_5 is diagonal

$$\gamma_5 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} \tag{6.22}$$

and our projection simply selects two upper components of the Dirac spinor as left-handed, and the two lower ones as right-handed. The difference between gauge trafo $e^{i\alpha(x)}$ and chiral trafo $e^{i\alpha(x)\gamma_5}$ is that gauge trafo rotates ψ_L and ψ_R in the same direction, $\psi_{L,R} \mapsto e^{i\alpha(x)}\psi_{L,R}$, whereas chiral trafo rotates ψ_L and ψ_R in opposite directions, $\psi_L \mapsto e^{-i\alpha(x)}\psi_L$, $\psi_R \mapsto e^{+i\alpha(x)}\psi_R$. Since the path integral is taken over equal amount of left-handed and right-handed fields, the extra phases will cancel and the measure will be invariant. Thus (6.16) should hold and repeating the argumentation that we used for gauge trafo, we will conclude that axial current is conserved at quantum level as well.

Unfortunately, this conclusion is wrong (experimental fact). Thus there must have been a loophole in our argumentation, and it turns out (K. Fujikawa, 1979) that (6.16) is actually not true because of the necessity to introduce a regularization. You can find this e.g. in the textbook by Peskin and Schröder, but I will describe instead another method where the problem is seen more clearly.

6.2 Adler-Bardeen anomaly

• Let us try to derive the Ward identity (6.6) in a different way, following the technique in Sec. 3.1. We want to calculate

$$\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\mathcal{D}A\, e^{iS_{QED}}\,\partial_\mu J^\mu(x)\psi(x_1)\bar{\psi}(x_2) = ? \tag{6.23}$$

where $J^\mu = -e\bar{\psi}\gamma^\mu\psi$ is the electromagnetic current. Write

$$\begin{aligned}
\partial_\mu\bar{\psi}(x)\gamma_\mu\psi(x) &= \bar{\psi}(x)\big[\overleftarrow{\partial\!\!\!/} + \overrightarrow{\partial\!\!\!/}\,\big]\psi(x) \\
&= \bar{\psi}(x)\big[\big(\overleftarrow{\partial\!\!\!/} + ie A\!\!\!/(x) - im\big) + \big(\overrightarrow{\partial\!\!\!/} - ie A\!\!\!/(x) + im\big)\big]\psi(x) \\
&= \bar{\psi}(x)\big[\big(\overleftarrow{D\!\!\!\!/} - im\big) + \big(\overrightarrow{D\!\!\!\!/} + im\big]\psi(x) \\
&= -i\bar{\psi}(x)\big[\big(i\overleftarrow{D\!\!\!\!/} + m\big) + \big(i\overrightarrow{D\!\!\!\!/} - m\big)\big]\psi(x)
\end{aligned} \tag{6.24}$$

and observe that

$$\begin{aligned}
\frac{\delta}{\delta\bar{\psi}(y)} e^{i\int d^4x\,\bar{\psi}(x)(iD\!\!\!\!/-m)\psi(x)} &= +i(i\overrightarrow{D\!\!\!\!/} - m)\psi(y)\, e^{iS_F} \\
\frac{\delta}{\delta\psi(y)} e^{i\int d^4x\,\bar{\psi}(x)(iD\!\!\!\!/-m)\psi(x)} &= +i\bar{\psi}(y)(i\overleftarrow{D\!\!\!\!/} + m)\, e^{iS_F}
\end{aligned} \tag{6.25}$$

[taking the derivative, do not forget that fermion operators anticommute — sign in second eq.]

Therefore

$$
\begin{aligned}
\partial_\mu J^\mu(x) e^{iS_F} &= (-e)(-i)\bar{\psi}(x)\big[\big(i \overleftarrow{\not{D}} + m\big) + \big(i \overrightarrow{\not{D}} - m\big)\big]\psi(x)\, e^{iS_F} \\
&= (+e)\left[\left(\frac{\delta}{\delta\psi(x)} e^{iS_F}\right)\psi(x) + \bar{\psi}(x)\left(\frac{\delta}{\delta\bar{\psi}(x)} e^{iS_F}\right)\right]
\end{aligned}
\tag{6.26}
$$

and our object of interest can be written as

$$
\begin{aligned}
&\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\mathcal{D}A\, e^{iS_{QED}}\,\partial_\mu J^\mu(x)\psi(x_1)\bar{\psi}(x_2) = \\
&\quad = e\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\mathcal{D}A\left[\left(\frac{\delta}{\delta\psi(x)} e^{iS_F}\right)\psi(x) + \bar{\psi}(x)\left(\frac{\delta}{\delta\bar{\psi}(x)} e^{iS_F}\right)\right]\psi(x_1)\bar{\psi}(x_2)
\end{aligned}
\tag{6.27}
$$

Here we want to integrate by parts and have two possibilities. E.g. for the first term

$$
\begin{aligned}
\psi(x_1)\bar{\psi}(x_2)\left(\frac{\delta}{\delta\psi(x)} e^{iS_F}\right)\psi(x) \quad &\xrightarrow{p.I} \quad -\, e^{iS_F}\frac{\delta}{\delta\psi(x)}\left[\psi(x_1)\bar{\psi}(x_2)\psi(x)\right] \\
&= -e^{iS_F}\left[\delta(x-x_1)\bar{\psi}(x_2)\psi(x) + \underbrace{\delta(x-x)}_{!?}\psi(x_1)\bar{\psi}(x_2)\right]
\end{aligned}
\tag{6.28}
$$

The term $\sim \delta(x-x_1)$ and the similar term $\sim \delta(x-x_2)$ from $\ldots \frac{\delta}{\delta\bar{\psi}(x)} \ldots$ combine to give

$$
\begin{aligned}
\ldots = -e\delta(x-x_1)&\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\mathcal{D}A\, e^{iS}\bar{\psi}(x_2)\psi(x) \\
- e\delta(x-x_2)&\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\mathcal{D}A\, e^{iS}\psi(x_1)\bar{\psi}(x)
\end{aligned}
\tag{6.29}
$$

which is exactly what we have obtained in Sec. 6.1 by a different method. But in addition we have two unpleasant terms with $\delta^{(4)}(0)$ that signal that we have a potential problem and must be more accurate.

• It is clear that the problem originates from having the fields $\bar{\psi}$ and ψ in the electromagnetic current at the same space-time point. If we separate them a bit,

$$
\bar{\psi}(x)\gamma^\mu\psi(x) \to \bar{\psi}(x+\epsilon)\gamma^\mu\psi(x-\epsilon)\,,
$$

the "bad" terms would not appear. The limit $\epsilon \to$ (here $\epsilon = \epsilon_\nu$ is a four-vector) can, however, turn out to be singular and require some care:

$$
\begin{aligned}
\langle A|\bar{\psi}(x)M\psi(x)|B\rangle &= \lim_{\epsilon^\alpha\to 0}\langle A|\bar{\psi}(x+\epsilon)M\psi(x-\epsilon)|B\rangle \\
&= \int\frac{d^4p}{(2\pi)^4}\langle A|\bar{\psi}(0)M|p\rangle\, e^{-i2p\cdot\epsilon}\langle p|\psi(0)|B\rangle
\end{aligned}
\tag{6.30}
$$

The limit $\epsilon \to 0$ may lead to a divergent integral if the matrix elements do not decrease sufficiently fast with momentum.

Splitting the field positions in local currents is, of course, a way to introduce a regularization. But this is a bad regularization because it breaks Lorentz symmetry and gauge invariance. If the regularization procedure in a quantum theory breaks symmetries of the corresponding classical theory, these symmetries will not be preserved and the classical conservation laws will be broken. Of course, was can simply be stupid to use a bad scheme, so the question is whether a regularization scheme that maintains the wanted symmetry exists. [And a more complicated question: whether a regularization scheme that maintains all classical symmetries]. Lect. 14

In our case:

- We certainly do not want to sacrifice Lorentz symmetry, but can repair it relatively easily if we average over all directions of ϵ_ν in Minkowski space (we will see how this works)
- We do not want to sacrifice gauge symmetry, because it protects conservation of the electric charge = law of nature. It is broken by the "naive" split-point regularization because

$$\begin{array}{ccc} \bar\psi(x)\gamma_\mu(\gamma_5)\psi(x) \mapsto & \bar\psi(x+\epsilon)\gamma_\mu(\gamma_5)\psi(x-\epsilon) & \\ & \downarrow \qquad\qquad \searrow & \\ & \bar\psi(x+\epsilon)e^{-i\alpha(x+\epsilon)} \qquad e^{i\alpha(x-\epsilon)}\psi(x-\epsilon) & \end{array} \tag{6.31}$$

Here $\gamma_\mu(\gamma_5)$ means that the equation is valid both for γ_μ and $\gamma_\mu\gamma_5$.
The gauge-invariant split-point regularization is possible and first suggested by Schwinger:

$$J^{\text{reg}}_{\mu(5)}(x) = \bar\psi(x+\epsilon)\gamma_\mu(\gamma_5)\exp\left\{ie\int_{-1}^{1} du\,\epsilon_\mu A^\mu(x+u\epsilon)\right\}\psi(x-\epsilon) \tag{6.32}$$

Let us verify gauge invariance of this expression.

▶

Under a gauge trafo

$$J^{\text{reg}}_{\mu5}(x) \mapsto \bar\psi(x+\epsilon)e^{-i\alpha(x+\epsilon)}\gamma_\mu\gamma_5\exp\left\{ie\int_{-1}^{1} du\,\epsilon_\mu(A^\mu + \frac{1}{e}\partial^\mu\alpha)(x+u\epsilon)\right\}e^{i\alpha(x-\epsilon)}\psi(x-\epsilon) \tag{6.33}$$

We have to verify that all α-dependent terms in the exponent cancel:

$$-i\alpha(x+\epsilon) + i\alpha(x-\epsilon) + i\int_{-1}^{1} du\,\epsilon_\mu\partial^\mu\alpha(x+u\epsilon) \quad \overset{?}{=} \quad 0 \tag{6.34}$$

Useful identity:

$$\forall f(x) \qquad x^\mu\frac{\partial}{\partial x^\mu}f(ux) = u\frac{d}{du}f(ux) \tag{6.35}$$

Then

$$\begin{aligned} \int_{-1}^{1} du\,\epsilon^\mu\frac{\partial}{\partial x^\mu}\alpha(x+u\epsilon) &= \int_{-1}^{1} du\,\epsilon^\mu\frac{\partial}{\partial(u\epsilon^\mu)}\alpha(x+u\epsilon) \\ &= \int_{-1}^{1} du\,\frac{1}{u}u\frac{d}{du}\alpha(x+u\epsilon) = \alpha(x+\epsilon) - \alpha(x-\epsilon) \end{aligned} \tag{6.36}$$

and indeed cancel the first two terms in (6.34).

◀

• Let us continue with axial vector current (we will also see why there is no problem with vector). We need finite ϵ^ν as a regulator, so can assume it is small and expand where possible. Consider

$$
\begin{aligned}
\partial_\mu J^{\rm reg}_{\mu 5}(x) &= \bar\psi(x+\epsilon)\,[\,\overleftarrow{\not\partial} + \overrightarrow{\not\partial}\,]\gamma_5\Big(1 + ie\int_{-1}^{1} du\,\epsilon_\nu A^\nu(x+u\epsilon) + \dots\Big)\psi(x-\epsilon) \\
&= \bar\psi(x+\epsilon)\Big[\Big(\overleftarrow{\not\partial} + ie\not{A}(x+\epsilon)\Big)\gamma_5 - ie\not{A}(x+\epsilon)\gamma_5 \\
&\qquad + \gamma_\mu\gamma_5\frac{\partial}{\partial x^\mu} ie\int_{-1}^{1} du\,\epsilon_\nu A^\nu(x+u\epsilon) \\
&\qquad + \Big(\overrightarrow{\not\partial} - ie\not{A}(x-\epsilon)\Big)\gamma_5 + ie\not{A}(x-\epsilon)\gamma_5\Big]\psi(x-\epsilon)
\end{aligned} \tag{6.37}
$$

Let us rewrite the term with the integral in another form:

$$
\begin{aligned}
ie\frac{\partial}{\partial x^\mu}\int_{-1}^{1} du\,\epsilon_\nu A^\nu(x+u\epsilon) &= ie\int_{-1}^{1} du\,\epsilon_\nu\Big\{\Big[\partial^\mu A^\nu - \partial^\nu A^\mu\Big](x+u\epsilon) + \partial^\nu A^\mu(x+u\epsilon)\Big\} \\
&= ie\int_{-1}^{1} du\,\Big\{\epsilon_\nu F^{\mu\nu}(x+u\epsilon) + \frac{d}{du}A^\mu(x+u\epsilon)\Big\} \\
&= -2ie\epsilon_\nu F^{\nu\mu}(x) + \mathcal{O}(\epsilon^2) + ieA^\mu(x+\epsilon) - iA^\mu(x-\epsilon)\Big\}
\end{aligned} \tag{6.38}
$$

This expression enters (6.37) multiplied by $\gamma_\mu\gamma_5$ and the terms marked in blue exactly cancel. Here we expanded the integral involving $F^{\mu\nu}(x+u\epsilon)$ in anticipation that only linear terms in ϵ will be important.

Thus we get

$$
\partial_\mu J^{\rm reg}_{\mu 5}(x) = \bar\psi(x+\epsilon)\Big[\overleftarrow{\not{D}} - \underbrace{2ie\epsilon_\nu F^{\nu\mu}(x)\gamma_\mu}_{\text{extra term}} + \overrightarrow{\not{D}}\Big]\gamma_5\psi(x-\epsilon) + +\mathcal{O}(\epsilon^2) \tag{6.39}
$$

• Now we use this expression in the path integral

$$
\begin{aligned}
&\int \mathcal{D}\bar\psi\,\mathcal{D}\psi\mathcal{D}A\,e^{iS}\,\partial_\mu J^{\mu 5}(x)\psi(x_1)\bar\psi(x_2) = \\
&= \int \mathcal{D}\bar\psi\,\mathcal{D}\psi\mathcal{D}A\,e^{iS}\,\bar\psi(x+\epsilon)\Big[\overleftarrow{\not{D}} - 2ie\epsilon_\nu F^{\nu\mu}(x)\gamma_\mu + \overrightarrow{\not{D}}\Big]\gamma_5\psi(x-\epsilon)\psi(x_1)\bar\psi(x_2)
\end{aligned} \tag{6.40}
$$

and proceed exactly as before. The "bad" delta-functions do not appear as the fermions in the current sit at different points, so that we will get the old result plus an extra contribution from the term in $\epsilon_\nu F^{\nu\mu}(x)$:

$$
-2ie\int \mathcal{D}\bar\psi\,\mathcal{D}\psi\mathcal{D}A\,e^{iS}\,\bar\psi(x+\epsilon)\,\epsilon_\nu F^{\nu\mu}(x)\,\gamma_\mu\gamma_5\,\psi(x-\epsilon)\cancel{\psi(x_1)\bar\psi(x_2)} \tag{6.41}
$$

[extra fermion fields at $x_{1,2} \neq x$ do not change anything so we can forget them].

Let us deal with this integral in the following way: We will take the path integral over fermions and leave the integral over photon fields as it stands for now. In this way, taking the integral over fermions we should assume that $F_{\mu\nu}$ is a certain given function — a background electromagnetic field.
Thus we need to evaluate

$$\mathcal{N} \int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi e^{iS_F}\, \bar{\psi}(x+\epsilon)\, \gamma_\mu \gamma_5\, \psi(x-\epsilon) \tag{6.42}$$

[where the external field sits in the covariant derivative in the action] which is the same as

$$\langle \Omega_A | \mathrm{T}\{\bar{\psi}_k(x+\epsilon)(\gamma_\mu\gamma_5)^{ki}\psi_i(x-\epsilon)\}|\Omega_A\rangle = -(\gamma_\mu\gamma_5)^{ki}\langle \Omega_A|\mathrm{T}\{\psi_i(x-\epsilon)\bar{\psi}_k(x+\epsilon)\}|\Omega_A\rangle \tag{6.43}$$

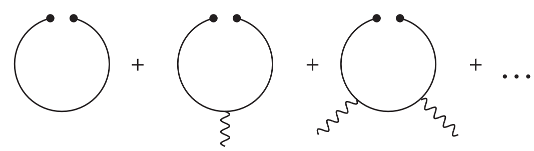
where $|\Omega_A\rangle$ is the state with no fermions, but with classical EM fields present — electron propagator in the background field. It will have to be multiplied by

$$(-2ie)\epsilon_\nu F^{\nu\mu}(x) \tag{6.44}$$

and the limit is taken $\epsilon \to 0$, so that we only need singular terms $1/\epsilon^n$ in the propagator.

- Let us do this calculation. In perturbation theory

$$\langle \Omega_A|\mathrm{T}\{\psi_i(x-\epsilon)\bar{\psi}_k(x+\epsilon)\}|\Omega_A\rangle \quad = \tag{6.45}$$



To leading order in the field (i.e. without field) we have the usual propagator

$$\langle 0|\mathrm{T}\{\psi_i(x-\epsilon)\bar{\psi}_k(x+\epsilon)\}|0\rangle = \langle x-\epsilon| \left(\frac{i}{i\not{\partial} - \not{m}}\right)_{ik} |x+\epsilon\rangle = \int \frac{d^4p}{(2\pi)^4} \langle x-\epsilon|\frac{i}{i\not{\partial}}|p\rangle\langle p|x+\epsilon\rangle$$
$$= \int \frac{d^4p}{(2\pi)^4 i} e^{2ip\cdot\epsilon} \left(\frac{1}{-\not{p}}\right)_{ik} = -\frac{i}{2\pi^2}\frac{2\not{\epsilon}_{ik}}{(2\epsilon)^4} \tag{6.46}$$

<u>*More formulas can be found in the Appendix:*</u>
This and many other calculations will be done using the following basic integrals

$$\int d^dx\, e^{iqx} \frac{\Gamma(\alpha)}{[-x^2+i\epsilon]^\alpha} = -i\pi^{d/2}2^{d-2\alpha}\frac{\Gamma(d/2-\alpha)}{[-q^2-i\epsilon]^{d/2-\alpha}}$$
$$\int d^dq\, e^{-iqx} \frac{\Gamma(\alpha)}{[-q^2-i\epsilon]^\alpha} = +i\pi^{d/2}2^{d-2\alpha}\frac{\Gamma(d/2-\alpha)}{[-x^2+i\epsilon]^{d/2-\alpha}} \tag{6.47}$$

and

$$\int d^d x \, \frac{\Gamma(\alpha)}{(-x^2 - a^2 + i\epsilon)^\alpha} = -i\pi^{d/2} \frac{\Gamma(\alpha - d/2)}{[-a^2 + i\epsilon]^{\alpha - d/2}}$$
$$\int d^d x \, \frac{\Gamma(\alpha)}{(-x^2 - a^2 + i\epsilon)^\alpha} x_\mu x_\nu = -i\pi^{d/2} \left(-\frac{g_{\mu\nu}}{2}\right) \frac{\Gamma(\alpha - d/2 - 1)}{[-a^2 + i\epsilon]^{\alpha - d/2 - 1}} \quad (6.48)$$

At first sight this (simplest) term produces a very strong singularity $\sim 1/\epsilon^3$! However, we have to trace the propagator with $\gamma_\mu \gamma_5$ and get zero:

$$\frac{1}{\epsilon^4} (\gamma_\mu \gamma_5)^{ki} (\not\epsilon)_{ik} = \frac{1}{\epsilon^4} \mathrm{Tr}\,[\gamma_\mu \gamma_5 \not\epsilon] = 0 \quad (6.49)$$

If we consider vector current and not axial-vector, there is no γ_5 and we get a term $\sim \epsilon^\mu/\epsilon^4$. This will have to be multiplied with $\epsilon_\nu F^{\nu\mu}$, see (6.44), and again vanish thanks to $F^{\nu\mu} = -F^{\mu\nu}$. Thus, the term without external field presents no danger.

- Now let us switch on the field:

$$\langle 0|\mathrm{T}\{\psi_i(x-\epsilon)\bar{\psi}_k(x+\epsilon)\}|0\rangle = \langle x-\epsilon|\frac{i}{i\not D}|x+\epsilon\rangle \;=\; \langle x-\epsilon|\frac{1}{\not\partial - ie\not A}|x+\epsilon\rangle$$
$$= \langle x-\epsilon|\frac{1}{\not\partial} + \frac{1}{\not\partial} ie\not A \frac{1}{\not\partial} + \ldots|x+\epsilon\rangle \quad (6.50)$$

This series generates contributions to all orders in the field, but in fact only the first-order term will be necessary (the others are regular at $\epsilon \to 0$). We have

$$\langle x-\epsilon|\frac{1}{\not\partial} ie\not A \frac{1}{\not\partial}|x+\epsilon\rangle = ie\int d^4y \langle x-\epsilon|\frac{1}{\not\partial}|y\rangle \not A(y) \langle y|\frac{1}{\not\partial}|x+\epsilon\rangle$$
$$= ie\left(\frac{i}{2\pi^2}\right)^2 \int d^4y \, \frac{\not x - \not y - \not\epsilon}{(x-y-\epsilon)^4} \not A(y) \frac{\not y - \not x - \not\epsilon}{(y-x-\epsilon)^4} \qquad \boxed{y' = y - x}$$
$$= ie\left(\frac{i}{2\pi^2}\right)^2 \int d^4y \, \frac{-\not y - \not\epsilon}{(y+\epsilon)^4} \not A(x+y) \frac{\not y - \not\epsilon}{(y-\epsilon)^4}$$
$$= \frac{ie}{4\pi^4} 6 \int_0^1 du \, u\bar{u} \int d^4y \frac{(\not y + \not\epsilon)\not A(x+y)(\not y - \not\epsilon)}{[u(y+\epsilon)^2 + \bar{u}(y-\epsilon)^2]^4} \quad (6.51)$$

The usual routine: rewrite the denominator

$$u(y+\epsilon)^2 + \bar{u}(y-\epsilon)^2 = [y + (2u-1)\epsilon]^2 + 4u\bar{u}\epsilon^2 \quad (6.52)$$

and shift the integration variable to get

$$\ldots = \frac{ie}{4\pi^4} 6 \int_0^1 du \, u\bar{u} \int d^4y \frac{(\not y + 2\bar{u}\not\epsilon)\not A(x+y-(2u-1)\epsilon)(\not y - 2u\not\epsilon)}{[y^2 + 4u\bar{u}\epsilon^2]^4} \quad (6.53)$$

Now we can expand the background field around x:

$$A_\xi(x+y-(2u-1)\epsilon) = A_\xi(x) + (y-(2u-1)\epsilon)_\alpha [\partial^\alpha A_\xi](x) + \ldots \quad (6.54)$$

because extra powers of y will result in extra powers of ϵ after integration. After some algebra one obtains

$$\ldots = -\frac{e}{4\pi^2}\left\{\frac{\epsilon^\xi A_\xi}{2\epsilon^4}\not\epsilon + \frac{1}{8\epsilon^2}\Big[\partial^\alpha A^\xi(\gamma_\alpha\gamma_\xi\not\epsilon - \not\epsilon\gamma_\xi\gamma_\alpha) - 4\epsilon^\xi\not\epsilon\,\partial^2 A_\xi\Big] + \ldots\Big]\right\} \tag{6.55}$$

The first term $1/\epsilon^2$ does not contribute for the same reason as the term without photon field above, and the last term $\sim \partial^2 A$ is already non-singular. Thus we only need one term and can use

$$\gamma_\alpha\gamma_\xi\not\epsilon - \not\epsilon\gamma_\xi\gamma_\alpha = 2i\epsilon_{\alpha\xi\rho\sigma}\epsilon^\rho\gamma_5\gamma^\sigma \tag{6.56}$$

to get

$$\begin{aligned}\ldots &= -\frac{e}{32\pi^2}\frac{1}{\epsilon^2}\frac{1}{2}F^{\alpha\xi}2i\epsilon_{\alpha\xi\rho\sigma}\epsilon^\rho\gamma_5\gamma^\sigma \\ &= -\frac{ie}{16\pi^2}\frac{1}{\epsilon^2}\widetilde{F}_{\rho\sigma}(x)\epsilon^\rho\gamma_5\gamma^\sigma \end{aligned}\tag{6.57}$$

where

$$\widetilde{F}_{\rho\sigma} = \frac{1}{2}\epsilon_{\rho\sigma\alpha\beta}F^{\alpha\beta} \tag{6.58}$$

[duality transformation — exchanges electric and magnetic fields; the second pair of Maxwell Eqs. $\partial_\mu\widetilde{F}^{\mu\nu} = 0$ — there exist no magnetic charges]

- It remains to take a trace $\mathrm{tr}[\ldots\gamma_\mu\gamma_5]$, take into account minus sign because we calculate $\psi\ldots\bar\psi$ and need $\bar\psi\ldots\psi$, and multiply by (6.44). Obtain

$$-2ie\epsilon_\mu F^{\alpha\mu}(x)(-1)\frac{-ie}{16\pi^2}\frac{1}{\epsilon^2}\widetilde{F}_{\rho\sigma}(x)\epsilon^\rho\,\underbrace{\mathrm{tr}\big[\gamma_5\gamma^\sigma\gamma^\mu\gamma_5\big]}_{4g^{\mu\sigma}} \tag{6.59}$$

Final step, we have to symmetrize over all directions of ϵ^α in Minkowski space:

$$\langle\,\epsilon_\alpha\epsilon_\beta\rangle = \frac{1}{4}g_{\alpha\beta}\,\epsilon^2 \tag{6.60}$$

and the final result reads

$$\boxed{\partial_\mu J^{\mu5}(x) \;=\; \frac{e^2}{8\pi^2}\,F_{\alpha\beta}\widetilde{F}^{\alpha\beta}}$$

Adler-Bardeen Anomaly (6.61)

This is an exact operator relation, valid for insertions in arbitrary Green functions (up to contact terms)
For the vector current we would zero because of the trace $\mathrm{tr}\big[\gamma_5\gamma^\sigma\gamma^\mu\gamma_5\big] \mapsto \mathrm{tr}\big[\gamma_5\gamma^\sigma\gamma^\mu\big] = 0$.

6.3 Gell-Mann–Oakes–Renner relation and quark masses

We now turn over to QCD, electrons $\mapsto$ quarks and photons $\mapsto$ gluons, but at this point will not need any details on quark-gluon interactions, so we can postpone the consistent construction of QCD as a quantum theory for later.

This section will mostly deal with pions, $m_\pi \simeq 140MeV$, that are build from a light quark and light antiquark, u or d and their somewhat heavier brothers kaons, $m_K \simeq 493MeV$, that include the strange quark (or antiquark).

- As a preliminary step, we have to derive some (simple) relation.

The pion decay constant f_π is defined as

$$\langle\Omega|\bar{u}(0)\gamma_\mu\gamma_5 d(0)|\pi^+(q)\rangle = if_\pi q_\mu \tag{6.62}$$

It can be measured experimentally from the observed decay $\pi^+ \to \mu^+\nu$ which happens when the u-quark and $\bar{d}$ antiquark annihilate via weak interactions and produce a pair of leptons — a muon and a neutrino:

$$\begin{aligned} &f_\pi \simeq 132 \text{ MeV} && (\pi^+ \sim u\bar{d}) \\ &f_K \simeq 155 \text{ MeV} && (K^+ \sim u\bar{s}) \end{aligned} \tag{6.63}$$

Since for any operator

$$\langle p'|\widehat{O}(x)|p\rangle = \langle p'|e^{i\widehat{P}x}\widehat{O}(0)e^{-i\widehat{P}x}|p\rangle = e^{i(p'-p)x}\langle p'|\widehat{O}(0)|p\rangle \tag{6.64}$$

one gets

$$\partial^\mu\langle 0|\bar{u}(x)\gamma_\mu\gamma_5 d(x)|\pi^+(q)\rangle\big|_{x=0} = (-iq^\mu)if_\pi q_\mu = f_\pi m_\pi^2 \tag{6.65}$$

On the other hand

$$\begin{aligned} \partial^\mu\bar{u}(x)\gamma_\mu\gamma_5 d(x) &= (m_u+m_d)\bar{u}(x)i\gamma_5 d(x) + \bar{u}(x)\big[\overleftarrow{\not{D}} - im_u\big]\gamma_5 d(x) - \bar{u}(x)\gamma_5\big[\overrightarrow{\not{D}} + im_d\big]d(x) \\ &= (m_u+m_d)\bar{u}(x)i\gamma_5 d(x) - i\bar{u}(x)\big[i\overleftarrow{\not{D}} + m_u\big]\gamma_5 d(x) + i\bar{u}(x)\gamma_5\big[i\overrightarrow{\not{D}} - m_d\big]d(x) \end{aligned} \tag{6.66}$$

The last two terms do not contribute to the matrix element (Dirac equation) so that we get

$$\partial^\mu\langle\Omega|\bar{u}(x)\gamma_\mu\gamma_5 d(x)|\pi^+(q)\rangle = (m_u+m_d)\langle\Omega|\bar{u}(x)i\gamma_5 d(x)|\pi^+(q)\rangle \tag{6.67}$$

Comparing the two representations, get

$$\boxed{\langle\Omega|\bar{u}(x)i\gamma_5 d(x)|\pi^+(q)\rangle = \frac{f_\pi m_\pi^2}{m_u+m_d}} \tag{6.68}$$

The $J_5(x) = \bar{u}(x)i\gamma_5 d(x)$ is called a pseudoscalar current (flavor-nonsinglet).

Lect. 15

- Now the main part. Consider the correlation function

$$\begin{aligned}
& q_\mu \int d^4x\, e^{iqx} \langle \Omega | \mathrm{T}\,\{\bar{u}(x)\gamma^\mu\gamma_5 d(x)\, \bar{d}(0) i\gamma_5 u(0)\} | \Omega \rangle \\
= \quad & \int d^4x \left(-i\frac{\partial}{\partial x^\mu} e^{iqx}\right) \langle \Omega | \mathrm{T}\,\{\bar{u}(x)\gamma^\mu\gamma_5 d(x)\, \bar{d}(0) i\gamma_5 u(0)\} | \Omega \rangle \\
\overset{p.I.}{=} \quad & i \int d^4x\, e^{iqx} \langle \Omega | \mathrm{T}\,\{\partial_\mu \bar{u}(x)\gamma^\mu\gamma_5 d(x)\, \bar{d}(0) i\gamma_5 u(0)\} | \Omega \rangle \\
\equiv \quad & i \int d^4x\, e^{iqx} \mathcal{N} \int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi \mathcal{D}A\, e^{S_{QCD}}\, \partial_\mu \bar{u}(x)\gamma^\mu\gamma_5 d(x)\, \bar{d}(0) i\gamma_5 u(0)
\end{aligned} \tag{6.69}$$

Here we can use the expression for the divergence of axial current in (6.66) and

$$\begin{aligned}
\partial^\mu \bar{u}(x)\gamma_\mu\gamma_5 d(x)\, e^{iS_F} &= (m_u + m_d)\bar{u}(x) i\gamma_5 d(x) e^{iS_F} \\
&\quad + \bar{u}(x)\gamma_5 \left(\frac{\delta}{\delta \bar{d}(x)} e^{iS_F}\right) - \left(\frac{\delta}{\delta u(x)} e^{iS_F}\right)\gamma_5 d(x)
\end{aligned} \tag{6.70}$$

Note that in this case there is no anomaly since u and d quark are different.
Using this representation and integrating by parts in fermion fields in the path integral obtain

$$\begin{aligned}
\ldots &= (m_u + m_d)\, i \int d^4x\, e^{iqx} \langle \Omega | \mathrm{T}\,\{\bar{u}(x) i\gamma_5 d(x) \bar{d}(0) i\gamma_5 u(0)\} | \Omega \rangle \\
&\quad + i \int d^4x\, e^{iqx} \delta^{(4)}(x) \langle \Omega | \mathrm{T}\,\{-\bar{u}(x)\gamma_5 i\gamma_5 u(0) - \bar{d}(0) i\gamma_5\gamma_5 d(x)\} | \Omega \rangle \\
&= (m_u + m_d)\, i \int d^4x\, e^{iqx} \langle \Omega | \mathrm{T}\,\{\bar{u}(x) i\gamma_5 d(x) \bar{d}(0) i\gamma_5 u(0)\} | \Omega \rangle + \langle \Omega | \bar{u}u + \bar{d}d | \Omega \rangle
\end{aligned} \tag{6.71}$$

This is so far an exact identity. Let us consider it specifically at the point where

$$q^\mu = 0 : \qquad q^\mu = \{0, 0, 0, 0\} \tag{6.72}$$

(all four components are zero). Then obviously

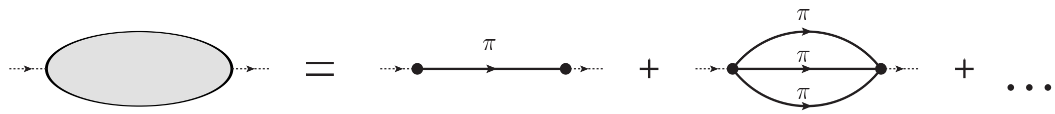
$$0 = (m_u + m_d)\, i \int d^4x\, \langle \Omega | \mathrm{T}\,\{\bar{u}(x) i\gamma_5 d(x) \bar{d}(0) i\gamma_5 u(0)\} | \Omega \rangle + \langle \Omega | \bar{u}u + \bar{d}d | \Omega \rangle \tag{6.73}$$

Again, this is so far exact, but now we want to make an approximation for the correlation function using the fact that pion mass is very small $m_\pi^2 \sim 0.02$ GeV^2 compared to other hadrons.

The correlation function can be written as a sum of contributions of intermediate hadron states with proper quantum numbers inserting

$$\mathbb{1} \;=\; |\pi^+\rangle\langle\pi^+| + \sum_{S\neq\pi} |S\rangle\langle S|$$

or



$$\frac{f_\pi m_\pi^2}{m_u + m_d} \; \frac{1}{m_\pi^2 \underbrace{-q^2}_{\to 0}} \; \frac{f_\pi m_\pi^2}{m_u + m_d} \tag{6.74}$$

If $q^2 \gg m_\pi^2$ the pion contribution is just one of many, but if $q^2 = 0$ it is enhanced by a huge factor $1/m_\pi^2$ compared to all others (multipion states can only produce $\ln m_\pi^2$ but not a pole, higher mass states with pion quantum numbers are rather heavy $m_{\pi'} \sim 1\,\mathrm{GeV}$.)

Taking into account the pion contribution only (PCAC approximation) we obtain

$$-(\langle \bar{u}u \rangle + \langle \bar{d}d \rangle) = (m_u + m_d)\frac{f_\pi^2 m_\pi^4}{(m_u + m_d)^2}\frac{1}{m_\pi^2 \underbrace{-q^2}_{\to 0}} = \frac{f_\pi^2 m_\pi^2}{m_u + m_d} \tag{6.75}$$

or

$$\boxed{-(m_u + m_d)(\langle \bar{u}u \rangle + \langle \bar{d}d \rangle) = f_\pi^2 m_\pi^2 \tag{6.76}}$$

— The Gell-Mann–Oakes–Renner (GMOR) relation.

- What happens with this relation if quark masses become very small?
  - The l.h.s. goes to zero, thus either $m_\pi^2 \to 0$, or $f_\pi^2 \to 0$ (or both).
  - Since $m_\pi^2/m_K^2 \sim 0.08$ but $f_\pi^2/f_K^2 \sim 0.72$, the first option seems much more likely.
  - Massless particles $m_\pi = 0$ can only exist if protected by some symmetry. Since the Lagrangian becomes chiral invariant in massless limit, maybe it is chiral symmetry?
  - Note there is no anomaly in this case and $\partial_\mu J^{\mu 5}_{ud} = 0$ at quantum level.
  - What happens in this limit with

    $$\langle \bar{q}q \rangle \equiv \langle \Omega | \bar{q}(0) q(0) | \Omega \rangle \qquad \textbf{the chiral condensate} \tag{6.77}$$

    Since $\bar{q}q$ is not invariant under chiral trafos, a non-zero vacuum expectation value would mean that chiral symmetry is broken *spontaneously* (because Lagrangian is invariant).
  - ! We will find (next Chapter) that spontaneous breaking of a continuous symmetry necessitates existence of a massless particle (pion) — seems we are on a right track

Leaving a more detailed justification/discussion for later, let us assume that for small quark masses

$$\langle \bar{q}q \rangle \sim \mathcal{O}(1) \tag{6.78}$$

$$f_\pi \sim \mathcal{O}(1) \tag{6.79}$$

$$m_\pi^2 \sim \mathcal{O}(m_q) \tag{6.80}$$

Under this assumption, it turns out to be possible to obtain a very decent estimate for quark masses based on using GMOR relation(s) and some additional arguments (summary in: H. Leutwyler, 1982; the idea goes back to an early (renown) work by Nambu and Iona-Lazinio, 1961).

- To this end we employ GMOR relations for the pion and also for K-mesons:

$$\begin{aligned}(m_u + m_d)(\langle \bar{u}u \rangle + \langle \bar{d}d \rangle) &= -f_\pi^2 m_\pi^2 \\ (m_u + m_s)(\langle \bar{u}u \rangle + \langle \bar{s}s \rangle) &= -f_{K^+}^2 m_{K^+}^2 \\ (m_d + m_s)(\langle \bar{d}d \rangle + \langle \bar{s}s \rangle) &= -f_{K^0}^2 m_{K^0}^2\end{aligned} \tag{6.81}$$

[the last two cannot have very high accuracy because $m_K \sim 490$ MeV is already not very small, but we hope this will be sufficient].

Isospin symmetry:

$$\underbrace{m_{K^+} \simeq m_{K_0}}_{\text{observe}} \qquad \underbrace{f_{K^+} \simeq f_{K_0}}_{\text{observe}} \qquad \underbrace{\langle \bar{u}u \rangle \simeq \langle \bar{d}d \rangle}_{\text{assume}} \tag{6.82}$$

[accuracy $\sim 1 - 3\%$].
It took some time to realize that isospin symmetry does not imply that $m_u \simeq m_d$ but rather is a consequence of both of them being small, $m_u, m_d \ll$ hadron masses.

SU(3) flavor symmetry:
Extension of isospin to include "strange" particles, (Gell-Mann), typical accuracy $\sim 20\%$.

$$\langle \bar{u}u \rangle \, , \langle \bar{d}d \rangle \approx \langle \bar{s}s \rangle \qquad\qquad \text{expect accuracy} \sim 20\%. \tag{6.83}$$

Under these assumptions, forming proper combinations of the relations in (6.81), obtain

$$\frac{m_s + m_u}{m_d + m_u} \approx \frac{f_K^2 m_K^2}{f_\pi^2 m_\pi^2} \qquad\qquad \frac{m_d - m_u}{m_d + m_u} \approx \frac{f_{K^0}^2 m_{K^0}^2 - f_{K^+}^2 m_{K^+}^2}{f_\pi^2 m_\pi^2} \tag{6.84}$$

where from

$$\frac{m_s}{m_d} \approx 18 \pm 5 \qquad\qquad \frac{m_d}{m_u} \approx 2.0 \pm 0.4 \tag{6.85}$$

To fix the overall scale, observe that all hadrons containing s-quark are typically 120-200 MeV heavier that their counterparts with a d-quark, which suggests that $m_s \sim 100 - 200$ MeV. Analyzing all relevant data one comes to an estimate (Leutwyler 1982)

$$m_u \approx 5\,\text{MeV}\,, \qquad m_d \approx 9\,\text{MeV}\,, \qquad m_s \approx 190\,\text{MeV}\,. \tag{6.86}$$

so that u and d quarks are very light as compared to even the pion mass. The error was estimated in 1982 as about 50%. One also obtains

$$\langle \bar{q}q \rangle \approx -(250\,\text{MeV})^3\,, \qquad \text{quark condensate} \tag{6.87}$$

The modern numbers are

$$\begin{aligned} \overline{m}_u(2\,\text{GeV}) &= (2.32 \pm 0.10)\,\text{MeV} \\ \overline{m}_d(2\,\text{GeV}) &= (4.71 \pm 0.09)\,\text{MeV} \qquad \textit{Particle Data Group, updated August 2019} \\ \overline{m}_s(2\,\text{GeV}) &= (92.7 \pm 0.9)\,\text{MeV} \end{aligned} \tag{6.88}$$

(running masses in $\overline{\text{MS}}$ renormalization scheme at the scale $\mu = 2$ GeV).

6.4 Decay $\pi^0 \to \gamma\gamma$

Neutral pions usually end their life by decaying in two photons (probability 98.8%). The measured decay rate

$$\Gamma(\pi^0 \to 2\gamma) = 7.95 \cdot 10^{-6}\,\text{MeV} \tag{6.89}$$

is responsible for a very short life time $\tau \sim 8.5 \cdot 10^{-17}$ sec. compared to $\tau \sim 2.6 \cdot 10^{-8}$ sec. for charged $\pi^{\pm}$. Can we calculate this decay rate in QFT?

$$\langle \gamma(k_1,\lambda_1)\gamma(k_2,\lambda_2)|\pi^0(q = k_1 + k_2)\rangle = \ ? \qquad q^2 = m_\pi^2 \tag{6.90}$$

- Consider

$$\langle \gamma\gamma| \underbrace{\mathbb{1}}_{|\pi^0\rangle\langle\pi^0|+\dots} \partial_\mu J^{\mu 5}|\Omega\rangle \simeq \langle \gamma\gamma|\pi^0\rangle \frac{1}{m_\pi^2 - q^2} \underbrace{\langle \pi^0|\partial_\mu J^{\mu 5}|\Omega\rangle}_{f_\pi m_\pi^2} \tag{6.91}$$

We need the amplitude (6.90) for pions on mass shell, $q^2 = m_\pi^2$, but can consider it for off-shell pions as well. One should expect that

$$\langle \gamma\gamma|\pi^0\rangle_{q^2=m_\pi^2} - \langle \gamma\gamma|\pi^0\rangle_{q^2=0} = \mathcal{O}(m_\pi^2/\Lambda^2) \tag{6.92}$$

where Λ is of order of typical hadron masses, so it is small. (The difference is of the same order as higher-mass terms neglected in (6.91)). We get

$$\langle \gamma\gamma|\pi^0\rangle_{q^2=m_\pi^2} \simeq \langle \gamma\gamma|\pi^0\rangle_{q^2=0} \simeq \frac{1}{f_\pi}\langle \gamma(k_1)\gamma(k_2)|\partial_\mu J^{\mu 5}|\Omega\rangle|_{(k_1+k_2)^2=0} \tag{6.93}$$

Thus the pion can be replaced at small momenta by the divergence of the axial current.

- This current is anomalous! We can use the QED expression (6.61) corrected for
  - pion is an isospin-triplet

$$J^{\mu 5}(x) = \frac{1}{\sqrt{2}}\big[\bar{u}(x)\gamma^\mu\gamma^5 u(x) - \bar{d}(x)\gamma^\mu\gamma^5 d(x)\big] \tag{6.94}$$

- quarks have fractional charges $e \mapsto e_u e$ or $e \mapsto e_d e$, respectively
- quarks have color (the current involves a sum over N_c quark species)

Thus for our case

$$\partial_\mu J^{\mu 5} = \frac{1}{\sqrt{2}}\left[e_u^2 - e_d^2\right] N_c \cdot \frac{e^2}{8\pi^2} F_{\alpha\beta}\widetilde{F}^{\alpha\beta} \tag{6.95}$$

with $e_u = 2/3$, $e_d = -1/3$, $e^2 = 4\pi\alpha_{\text{QED}}$ $\alpha_{\text{QED}} \simeq 1/137$.
Using this expression (exact!) get

$$\langle\gamma\gamma|\pi^0\rangle \simeq \frac{1}{f_\pi}\frac{1}{\sqrt{2}}\left[\frac{4}{9} - \frac{1}{9}\right] N_c \frac{\alpha_{\text{QED}}}{2\pi}\langle\gamma(k_1)\gamma(k_2)|F_{\alpha\beta}\widetilde{F}^{\alpha\beta}|\Omega\rangle \tag{6.96}$$

For a photon in the final state

$$\begin{aligned}
\big(A_\mu(x)\big)_k^{(\lambda)} &= \epsilon_\mu^{*(\lambda)} e^{ikx} \\
\big(F_{\mu\nu}(x)\big)_k^{(\lambda)} &= \big(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\big)_k^{(\lambda)} = i(k_\mu \epsilon_\nu^{*(\lambda)} - k_\nu \epsilon_\mu^{*(\lambda)})e^{ikx}
\end{aligned} \tag{6.97}$$

so that

$$\begin{aligned}
\langle\gamma\gamma|F\widetilde{F}|0\rangle &= \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}\langle\gamma\gamma|F_{\alpha\beta}F_{\mu\nu}|0\rangle \\
&= 2\frac{1}{2}\epsilon^{\alpha\beta\mu\nu}(i)^2[k_{1\alpha}\epsilon_\beta^{*(\lambda_1)} - k_{1\beta}\epsilon_\alpha^{*(\lambda_1)}][k_{2\mu}\epsilon_\nu^{*(\lambda_2)} - k_{2\nu}\epsilon_\mu^{*(\lambda_2)}] \\
&= 4\epsilon_{\alpha\beta\mu\nu}\, k_1^\alpha k_2^\beta\, \epsilon_\mu^{*(\lambda_1)}\epsilon_\nu^{*(\lambda_2)}
\end{aligned} \tag{6.98}$$

and finally

$$\langle\gamma(k_1,\lambda_1)\gamma(k_2,\lambda_2)|\pi^0(q)\rangle = \frac{N_c}{3\sqrt{2}}\frac{\alpha_{\text{QED}}}{2\pi}\frac{1}{f_\pi}4\epsilon_{\alpha\beta\mu\nu}\, k_1^\alpha k_2^\beta\, \epsilon_\mu^{*(\lambda_1)}\epsilon_\nu^{*(\lambda_2)} \tag{6.99}$$

It remains to calculate the decay rate (see, e.g., Peskin-Schröder)

$$\Gamma(\pi^0 \to 2\gamma) = \sum_{\lambda_{1,2}}\frac{1}{2m_\pi}\int\frac{d^3k_1}{(2\pi)^3 2k_{10}}\frac{d^3k_2}{(2\pi)^3 2k_{20}}\delta^{(4)}(k_1+k_2-q)\big|\langle\gamma\gamma|\pi^0(q)\rangle\big|^2 \tag{6.100}$$

[here $q^2 = m_\pi^2$, not zero!].
Obtain

$$\begin{aligned}
\Gamma^{\text{theory}}(\pi^0 \to 2\gamma) &= \left(\frac{N_c\alpha_{\text{QED}}}{3\pi}\right)^2\frac{m_\pi^3}{32\pi f_\pi^2} \;\simeq\; 7.25\,10^{-6}\,\text{MeV} \\
\Gamma^{\text{exp}}(\pi^0 \to 2\gamma) &\simeq 7.95\,10^{-6}\,\text{MeV}
\end{aligned} \tag{6.101}$$

— a very impressive confirmation of $N_c = 3$ (color)!

7 Non-abelian gauge theories

Lect. 16

Two especially important examples (occur in nature):

- $SU(2)$-group (electroweak interactions)

$$2\times 2 \text{ matrices} \qquad UU^\dagger = \mathbb{1}\,, \det U = 1$$

$$\text{generators} \qquad t^a = \frac{1}{2}\sigma^a\,, \qquad a = 1,2,3 \qquad \text{Pauli matrices}$$

$$\text{structure constants} \qquad [t^a, t^b] \overset{!}{=} i f^{abc} t^c$$

$$\Big[\frac{1}{2}\sigma^a, \frac{1}{2}\sigma^b\Big] = i\epsilon^{abc}\frac{1}{2}\sigma_c \qquad \Rightarrow \qquad f^{abc}_{SU(2)} = \epsilon^{abc} \tag{7.1}$$

- $SU(3)$-group (strong interactions, QCD)

$$3\times 3 \text{ matrices} \qquad UU^\dagger = \mathbb{1}\,, \det U = 1$$

$$\text{generators} \qquad t^a = \frac{1}{2}\lambda^a\,, \qquad a = 1,2,\ldots,8 \qquad \text{Gell-Mann matrices}$$

$$\text{structure constants} \qquad [t^a, t^b] \overset{!}{=} i f^{abc} t^c$$

$$f^{abc} = -f^{bac} = -f^{acb} \tag{7.2}$$

[Normalization convention $\mathrm{tr}[t^a t^b] = \frac{1}{2}\delta_{ab}$]

- Gauge transformations

$$\psi(x) \Leftrightarrow \begin{pmatrix} \psi_1 \\ \ldots \\ \psi_{N_c} \end{pmatrix} \qquad \psi(x) \mapsto e^{i\alpha^a t^a}\Psi(x)$$

$$A^a_\mu \mapsto A^a_\mu + \frac{1}{g}\partial_\mu \alpha^a + f^{abc} A^b_\mu \alpha^c \qquad \text{for infinitesimal } \alpha^a \to 0 \tag{7.3}$$

In what follows I sometimes use a matrix notation

$$A_\mu \equiv A^a_\mu t^a \tag{7.4}$$

- Covariant derivative and nonabelian strength tensor*

$$\begin{aligned} D_\mu &= \partial_\mu \mathbb{1} - igA^a_\mu t^a \quad \equiv \quad \partial_\mu - igA_\mu \\ [D_\mu, D_\nu] &\overset{!}{=} -igF^a_{\mu\nu}t^a \quad \equiv \quad -igF_{\mu\nu} \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \end{aligned} \tag{7.5}$$

- Lagrangian

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar\psi(i\not{D} - m)\psi \tag{7.6}$$

*In adjoint rep. $(T^c)_{ab} = -if_{abc}$, covariant derivative $D^{ab}_\mu = \partial_\mu \delta^{ab} - igA^c_\mu (T^c)_{ab}$, where $a,b,c = 1,\ldots,N_c^2-1$

Note that the interactions have become much more complicated:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$$

$$\mathcal{L}_I = g\bar{\psi} A^a_\mu t^a \gamma^\mu \psi - g f^{abc} (\partial_\alpha A^a_\mu) A^{\alpha,b} A^{\mu,c} - \frac{1}{4} g^2 (f^{eab} A^a_\mu A^b_\nu)(f^{ecd} A^{\mu,c} A^{\nu,d}) \tag{7.7}$$

Assuming that this theory can be quantized in the same way as QED one obtains the set of

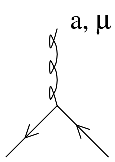
- Feynman rules

Propagators:

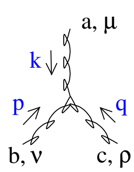
$$\langle 0|T\{\psi^i_\alpha(x)\bar{\psi}^j_\beta(y)\}|0\rangle = \delta^{ij} \int \frac{d^4p}{(2\pi)^4 i} e^{-ip(x-y)} \left(\frac{1}{m - \not{p} - i\epsilon} \right)_{\alpha\beta}$$

$$\langle 0|T\{A^a_\mu(x) A^b_\nu(y)\}|0\rangle = \delta^{ab} \int \frac{d^4k}{(2\pi)^4 i} e^{-ik(x-y)} \frac{1}{k^2 + i\epsilon} \Big[g_{\mu\nu} - (1-\xi) \frac{k_\mu k_\nu}{k^2} \Big] \tag{7.8}$$

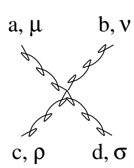
Vertices:



$$= ig\gamma_\mu t^a$$



$$= g f^{abc} \big[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu \big]$$



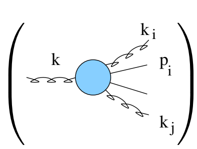
$$\begin{aligned} = -ig^2 \big[& f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\ & + f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\ & + f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) \big] \end{aligned} \tag{7.9}$$

Is this a valid quantum theory?

7.1 QCD Ward identity at tree level

I will consider QCD as an example of a non-abelian QFT but the choice is mainly about terminology (quarks, gluons, ...). The results are valid for any gauge group.

We have just discussed how important is to maintain gauge symmetry in QED at quantum level. Symmetry results in existence of a conserved current and at quantum level is expressed by the Ward identity. At the amplitude level in QED

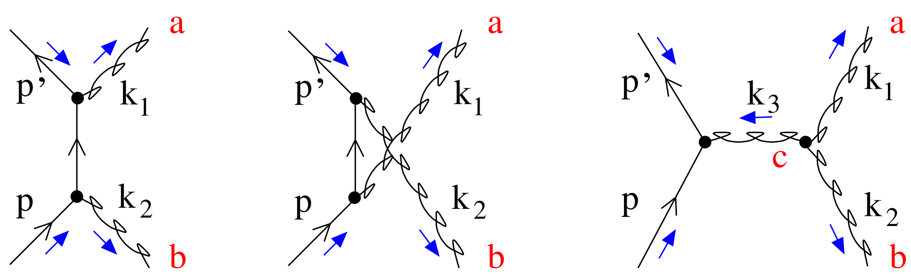
$k^\mu \cdot$ 

$= 0$ (7.10)

This holds for *arbitrary* photon and electron virtualities up to the δ-function contributions that we found above (contact terms). We want this identity to hold in QCD as well (otherwise the vacuum will be "colored" etc.)

- Let us check the QCD Ward identity on a simple example: quark-antiquark annihilation in a pair of gluons†

$$q(p) + \bar{q}(p') \to g^a(k_1) + g^b(k_2) \tag{7.11}$$



(1) (2) (3)

- The first two diagrams together:

$$\begin{aligned} iM^{\mu\nu}_{1,2}\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2) &= \epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2) \\ &\times (ig)^2\bar{v}(p')\left\{\gamma^\mu t^a \frac{i}{\not p - \not k_2 - m}\gamma^\nu t^b + \gamma^\nu t^b \frac{i}{\not k_2 - \not p' - m}\gamma^\mu t^a\right\}u(p) \end{aligned} \tag{7.12}$$

Replace

$$\epsilon^*_\nu(k_2) \longrightarrow k_{2\nu} \tag{7.13}$$

Obtain

$$iM^{\mu\nu}_{1,2}\epsilon^*_\mu(k_1)k_{2\nu} = \epsilon^*_\mu(k_1)(ig)^2\bar{v}(p')\left\{\gamma^\mu t^a \frac{i}{\not p - \not k_2 - m}\not k_2 t^b + \not k_2 t^b \frac{i}{\not k_2 - \not p' - m}\gamma^\mu t^a\right\}u(p) \tag{7.14}$$

Thanks to Dirac equation can replace

$$\begin{aligned} (\not p - m)u(p) = 0, &\qquad \text{in the first term} \qquad \not k_2 u(p) = (\not k_2 - \not p + m)u(p) \\ \bar{v}(p')(\not p' + m) = 0, &\qquad \text{in the second term} \qquad \bar{v}(p')\not k_2 = \bar{v}(p')(\not k_2 - \not p' - m) \end{aligned} \tag{7.15}$$

The propagators cancel and we get

$$\begin{aligned} iM^{\mu\nu}_{1,2}\epsilon^*_\mu(k_1)k_{2\nu} &= \epsilon^*_\mu(k_1)(ig)^2\bar{v}(p')\Big\{-i\gamma^\mu[t^a,t^b]\Big\}u(p) \\ &= -g^2\epsilon^*_\mu(k_1)\bar{v}(p')\gamma^\mu f^{abc}t^c u(p) \end{aligned} \tag{7.16}$$

†copypasted from my QCD lecture notes; example from Peskin&Schröder

- The third diagram:

$$iM_3^{\mu\nu}\epsilon_\mu^*(k_1)\epsilon_\nu^*(k_2) = \epsilon_\mu^*(k_1)\epsilon_\nu^*(k_2) \times (ig)\bar{v}(p')\gamma_\rho t^c u(p)\frac{-i}{k_3^2} g f^{abc}\Big[g^{\mu\nu}(k_2-k_1)^\rho + g^{\nu\rho}(k_3-k_2)^\mu + g^{\rho\mu}(k_1-k_3)^\nu\Big] \tag{7.17}$$

$$\begin{aligned}&\text{now replace} && \epsilon_\nu^*(k_2) \longrightarrow k_{2\nu} \\ &\text{and use} && k_1+k_2+k_3 = 0 \longrightarrow k_2 = -k_1-k_3\end{aligned} \tag{7.18}$$

Then

$$\begin{aligned}\epsilon_\nu^*(k_2)[***] &\longrightarrow k_{2\nu}[***] \\ &= k_2^\mu(k_2-k_1)^\rho + k_2^\rho(k_3-k_2)^\mu + g^{\rho\mu}(k_1-k_3)\cdot k_2 \\ &= (-k_1-k_3)^\mu(-2k_1-k_3)^\rho + (-k_1-k_3)^\rho(2k_3+k_1)^\mu + g^{\rho\mu}(k_1-k_3)\cdot(-k_1-k_3) \\ &= g^{\rho\mu}k_3^2 - k_3^\rho k_3^\mu - g^{\rho\mu}k_1^2 + k_1^\rho k_1^\mu\end{aligned} \tag{7.19}$$

Hence

$$iM_3^{\mu\nu}\epsilon_\mu^*(k_1)k_{2\nu} = \epsilon_\mu^*(k_1)(ig)\bar{v}(p')\gamma_\rho t^c u(p)\frac{-i}{k_3^2} g f^{abc}\Big[g^{\rho\mu}k_3^2 - k_3^\rho k_3^\mu - g^{\rho\mu}k_1^2 + k_1^\rho k_1^\mu\Big]$$

Assume $k_1^2 = 0$ (on-shell) <u>and</u> $\epsilon_\mu^*(k_1)k_1^\mu = 0$ (physical polarization). Then:

- the last two terms vanish
- the second term vanishes as well:

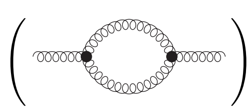
$$(-k_3^\rho)\bar{v}(p')\gamma_\rho u(p) = \bar{v}(p')[(\not{p}'+m)+(\not{p}-m)]u(p) = 0 \tag{7.20}$$

- the first term gives:

$$iM_3^{\mu\nu}\epsilon_\mu^*(k_1)k_{2\nu} = \epsilon_\mu^*(k_1)g^2\bar{v}(p')\gamma^\mu f^{abc}t^c u(p) \tag{7.21}$$

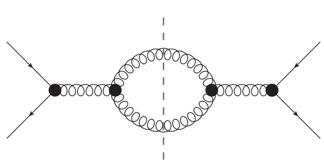
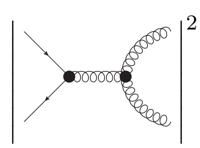
  and exactly cancels the contribution of the first two diagrams!

Happy end? — **No!** — a disaster in loop diagrams (true quantum effects)

$q_\nu \cdot$  $\neq 0$

$\searrow$ all polarizations in intermediate state (7.22)

Gluons with unphysical polarizations can be produced $\longrightarrow$ unitarity is broken:

2 Im  $\neq \int d\,(\text{phase space})$  (7.23)

The moral is, we have to go back and define the quantum theory properly.

7.2 Path integral quantization for non-abelian gauge theories

Let us try to go the same road as in QED and see whether we will have any new issues. We want to define a path-integral over gauge fields in a non-abelian theory (say, QCD)

$$\int \mathcal{D}A \exp\left[i\int d^4x\left(-\frac{1}{4}F^a_{\mu\nu}F^{a;\mu\nu}\right)\right] \;=\; ? \tag{7.24}$$

The gauge trafo is now

$$\begin{aligned} &\psi(x)\mapsto e^{i\alpha^a t^a}\psi(x)\,, \qquad\qquad \alpha^a=\alpha^a(x)\,,\\ &A^a_\mu t^a\mapsto e^{i\alpha^a t^a}\left[A^b_\mu t^b+\frac{i}{g}\partial_\mu\right]e^{-i\alpha^a t^a}. \end{aligned} \tag{7.25}$$

This is valid for arbitrary $\alpha(x)$. If $\alpha(x)\to 0$ this can be simplified to

$$\begin{aligned} A^a_\mu\mapsto (A^\alpha)^a_\mu &= A^a_\mu+\frac{1}{g}\partial_\mu\alpha^a+f^{abc}A^b_\mu\alpha^c\\ &= A^a_\mu+\frac{1}{g}\Big(\partial_\mu\delta^{ac}+gf^{abc}A^b_\mu\Big)\alpha^c\\ &= A^a_\mu+\frac{1}{g}\Big(\partial_\mu\delta^{ac}-ig(T^b)_{ac}A^b_\mu\Big)\alpha^c \qquad\qquad (T^b)_{ac}=-if_{acb},\\ &= A^a_\mu+\frac{1}{g}D^{ac}_\mu\alpha^c \end{aligned} \tag{7.26}$$

— covariant derivative in adjoint rep., a 8×8 matrix (in QCD)

We impose the gauge condition

$$G(A)=0 \tag{7.27}$$

and insert the Faddeev-Popov "1" inside the path integral:

$$1=\int\mathcal{D}\alpha(x)\,\delta\Big[G(A^\alpha)\Big]\;\det\left(\frac{\delta G(A^\alpha)}{\partial\alpha}\right) \tag{7.28}$$

[Note that if G is a linear operator then $\det(\ldots)$ does not depend on α.]

Then

$$\ldots=\int\mathcal{D}A\int\mathcal{D}\alpha\,\delta\Big[G(A^\alpha)\Big]\;\det\left(\frac{\delta G(A^\alpha)}{\partial\alpha}\right)\;e^{iS[A]} \tag{7.29}$$

and using

$$\begin{aligned} &\int\mathcal{D}A=\int\mathcal{D}A^\alpha &&\text{shift of variable}\\ &S[A]=S[A^\alpha] &&\text{gauge invariance} \end{aligned} \tag{7.30}$$

arrive at

$$\ldots=\cancel{\left(\int\mathcal{D}\alpha\right)}\int\mathcal{D}A\,\delta\Big[G(A)\Big]\;\det\left(\frac{\delta G(A^\alpha)}{\partial\alpha}\right)\;e^{iS[A]} \tag{7.31}$$

Let us take the same generalized Lorentz-type gauge condition as in QED:

$$G(A) = \partial^\mu A^a_\mu - \underbrace{\omega^a(x)}_{\text{arbitrary function}} \tag{7.32}$$

and integrate over all functions $\omega^a(x)$ with a Gaussian weight (as in QED). In this way we will get a new (gauge-fixing) term in the action and obtain the (expected) gluon propagator

$$\langle 0|T\{A^a_\mu(x)A^b_\nu(y)\}|0\rangle = \delta^{ab}\int \frac{d^4k}{(2\pi)^4 i}e^{-ik(x-y)}\frac{1}{k^2+i\epsilon}\Big[g_{\mu\nu} - \xi\frac{k_\mu k_\nu}{k^2}\Big] \tag{7.33}$$

- The difference is the determinant:

in QED:
$$\det\left(\frac{\delta G(A^\alpha)}{\partial\alpha}\right) = \det\left[\frac{\delta}{\delta\alpha(x)}\Big(\partial_\mu A^\mu(y) + \frac{1}{e}\partial_\mu\partial^\mu\alpha(y) - \omega(y)\Big)\right] = \det\left[\partial^2\delta^{(4)}(x-y)\right]$$

in QCD:
$$\det\left(\frac{\delta G(A^\alpha)}{\partial\alpha}\right) = \det\left[\frac{\delta}{\delta\alpha(x)}\Big(\partial_\mu A^\mu(y) + \frac{1}{g}\partial_\mu D^\mu\alpha(y) - \omega(y)\Big)\right] = \det\left[\partial_\mu D^\mu\delta^{(4)}(x-y)\right] \tag{7.34}$$

so that the determinant depends now on the gauge field in a nontrivial way and cannot be discarded.

- Faddeev&Popov's (second) trick: Write the det. as a path integral over auxiliary grassmanian fields

$$\det\left[\partial_\mu D^\mu\delta^{(4)}(x-y)\right] = \int \mathcal{D}c\,\mathcal{D}\bar{c}\,\exp\Big\{i\int d^4x\,\bar{c}^a(x)\Big(-\partial^\mu D^{ac}_\mu\Big)c^c(x)\Big\} \tag{7.35}$$

  - $c^a(x), a = 1,\ldots,8$ are complex Grassman fields (anticommuting) in adjoint rep. of the gauge group (as gluons). Note $\bar{c}^a \equiv (c^*)^a$.
  - $c^a(x)$ are <u>spin-zero</u> (scalar) fields (do not obey the spin-statistics theorem), thus they cannot be physical
  - $c^a(x)$ are called Faddeev-Popov <u>**ghost fields**</u>

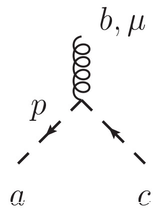
$$\boxed{\mathcal{L}_{\text{ghost}}(x) = \bar{c}^a(x)\Big(-\partial^2\delta^{ac} - g\partial^\mu f^{abc}A^b_\mu(x)\Big)c^c(x)} \tag{7.36}$$

Propagator:

$$\overline{c^a(x)\bar{c}^b}(y) \;=\; \delta^{ab}\int\frac{d^4p}{(2\pi)^4 i}e^{-ik(x-y)}\frac{1}{-k^2-i\epsilon}$$

$$= \frac{i\delta^{ab}}{p^2+i\epsilon} \tag{7.37}$$

Ghost-gluon vertex:



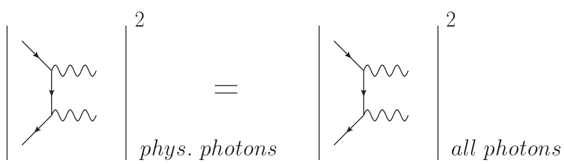
$$= -g f^{abc} p_\mu \tag{7.38}$$

The complete nonabelian (QCD) Lagrangian is then

$$\boxed{\mathcal{L}_{\rm QCD} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \bar{\psi}(i\not{D} - m)\psi + \bar{c}^a(-\partial^\mu D^{ac}_\mu)c^c} \tag{7.39}$$

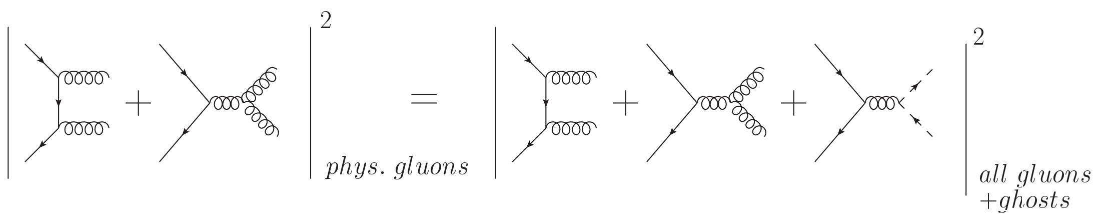
• The role of the ghost fields is to subtract contributions of unphysical polarizations of gluon fields. How it works:

In QED:



(7.40)

In QCD:

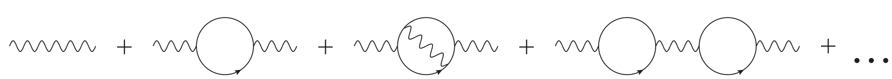


Lect. 17

• Why the gauge fields remain massless? Consider QED first:
Free photon propagator in Feynman gauge

$$D^{(0)}_{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4 i}\frac{g_{\mu\nu}}{k^2 + i\epsilon}e^{-ikx}, \qquad D^{(0)}_{\mu\nu}(k) = \frac{g_{\mu\nu}}{k^2 + i\epsilon} \tag{7.41}$$

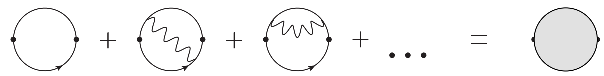
Exact photon propagator

$D_{\mu\nu}(k) =$  (7.42)

The last pictured contribution is the repetition of the second one; can happen separated by large time interval

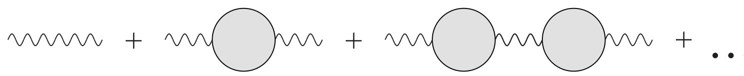
Such contributions are called "one-particle reducible", they are simple and can be summed up

One defines *photon self energy* as the sum of all 1PI diagrams (amputated):

$\Pi_{\mu\nu}(k) =$ 

$$= i\int d^4x\, e^{ikx}\, \langle\Omega|J_\mu(x)J_\nu(0)|\Omega\rangle \tag{7.43}$$

Then

$D_{\mu\nu}(k) =$  (7.44)

or

$$D_{\mu\nu}(k) \;=\; D^{(0)}_{\mu\nu}(k) + D^{(0)}_{\mu\mu_1}(k)\Pi^{\mu_1\mu_2}(k)D^{(0)}_{\mu_2\nu}(k) + \ldots \tag{7.45}$$

$\Rightarrow$ Dyson equation:

$$D_{\mu\nu}(k) \;=\; D^{(0)}_{\mu\nu}(k) + D^{(0)}_{\mu\mu_1}(k)\Pi^{\mu_1\mu_2}(k)\overset{\swarrow\ \text{exact!}}{D_{\mu_2\nu}(k)} \tag{7.46}$$

Using $D^{(0)}_{\mu\nu}(k) = g_{\mu\nu}/k^2$ this yields an equation

$$k^2 D_{\mu\nu}(k) \;=\; g_{\mu\nu} + \Pi_\mu{}^{\mu_2}(k)D_{\mu_2\nu}(k) \qquad \Longrightarrow \qquad \Big[k^2 g_{\mu\mu_1} - \Pi_{\mu\mu_1}\Big]D_{\mu_1\nu} \;=\; g_{\mu\nu} \tag{7.47}$$

Let (Lorentz invariance)

$$\begin{aligned} \Pi_{\mu\nu}(k) &= g_{\mu\nu}\, a_1(k^2) + k_\mu k_\nu\, a_2(k^2) \\ D_{\mu\nu}(k) &= g_{\mu\nu}\, d_1(k^2) + k_\mu k_\nu\, d_2(k^2) \end{aligned} \tag{7.48}$$

Then

$$g_{\mu\nu} \;=\; \big[k^2 - a_1\big]d_1\, g_{\mu\nu} + \big[k^2 - a_1\big]k_\mu k_\nu\, d_2 - k_\mu k_\nu\, d_1 a_1 - k^2 d_2 a_2\, k_\mu k_\nu \tag{7.49}$$

Collecting the terms $\propto g_{\mu\nu}$:

$$1 \;=\; \big[k^2 - a_1\big]d_1 \qquad \Longrightarrow \qquad d_1(k^2) \;=\; \frac{1}{k^2 - a_1(k^2)} \tag{7.50}$$

Therefore

$$D_{\mu\nu}(k) = \frac{g_{\mu\nu}}{k^2 - a_1(k^2) + i\epsilon} + \mathcal{O}(k_\mu k_\nu) \tag{7.51}$$

and in a general situation $a_1(k^2 = 0) \neq 0$ so that the photon would get a mass if not the Ward identity:

$$\begin{aligned} k^\mu \Pi_{\mu\nu} = 0 \quad &\Rightarrow \quad \Pi_{\mu\nu}(k) = \big(g_{\mu\nu}k^2 - k_\mu k_\nu\big)\,\Pi(k^2)\,, \\ &\Rightarrow \quad a_1(k^2) = k^2\Pi(k^2)\,, \qquad a_2(k^2) = -\Pi(k^2) \end{aligned} \tag{7.52}$$

$$k^\mu \Pi_{\mu\nu} = 0 \Rightarrow k^\mu \big[g_{\mu\nu}\, a_1(k^2) + k_\mu k_\nu\, a_2(k^2)\big] = 0 \quad \Rightarrow \quad a_1(k^2) = -k^2 a_2(k^2) \tag{7.53}$$

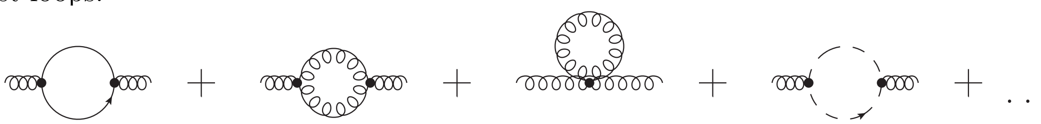
Thus finally

$$\boxed{D_{\mu\nu}(k) \;=\; \frac{g_{\mu\nu}}{k^2[1-\Pi(k^2)]}} \tag{7.54}$$

— the QED photon remains massless (to all orders in perturbation theory)

• The derivation in QCD is the same, but in order that the Ward identity is satisfied we need to include ghost loops:

$\Pi^{\mathrm{QCD}}_{\mu\nu}(p) =$



$$= (g_{\mu\nu}p^2 - p_\mu p_\nu)\Pi^{\mathrm{QCD}}(p^2) \tag{7.55}$$

Thus, as a consequence of gauge invariance (= Ward id.), gluon has no mass.

8 Spontaneous symmetry breaking and the Higgs mechanism

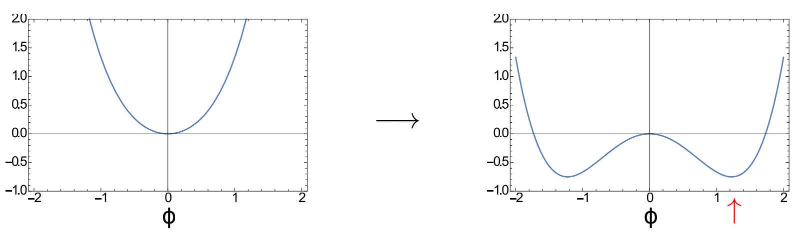
• Example I: Breaking of a discrete symmetry

Consider as simple scalar theory with a sign change for the mass term:

$$\begin{aligned} \mathcal{L}(x) &= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \\ \longrightarrow \quad & \\ \mathcal{L}(x) &= \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4 \end{aligned} \tag{8.1}$$

Potential

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{8.2}$$



$$\phi_0 = \pm v = \sqrt{\frac{6}{\lambda}}\mu \tag{8.3}$$

Since the lowest energy state is by definition the vacuum state, v" corresponds to the vacuum expectation value (VEV) of $\phi(x)$: $v = \langle\Omega|\phi|\Omega\rangle$. Note that $\langle\Omega|\phi|\Omega\rangle$ is obviously not invariant under $\phi \to -\phi$ so that choosing the particular sign $\langle\Omega|\phi|\Omega\rangle = \pm v$ we break the symmetry.

Let

$$\phi(x) = v + \sigma(x) \tag{8.4}$$

This makes sense if we consider small fluctuations near the bottom of the potential. In this case $v = \langle\Omega$

Let us rewrite

$$\mathcal{L}(\phi) = \mathcal{L}(v+\sigma) \longrightarrow \mathcal{L}'(\sigma) \tag{8.5}$$

- The linear term $\sim \sigma$ will not appear (because $\phi = v$ is a minimum)
- Can omit constants like v^4, v^2 as they do not influence EOM

Obtain after a short calculation

$$\mathcal{L}' = \frac{1}{2}(\partial_\mu\sigma)^2 - \underbrace{\frac{1}{2}(2\mu^2)\sigma^2}_{\text{mass term}} \quad \underbrace{-\sqrt{\frac{\lambda}{6}}\mu\sigma^3 - \frac{\lambda}{4!}\sigma^4}_{\text{interactions}} \tag{8.6}$$

- a scalar field σ with mass $m_\sigma = \sqrt{2}\mu$, with $\sim \sigma^3$ and $\sim \sigma^4$ interactions.
- the symmetry $\phi \to -\phi$ is not manifest, it is hidden in the relations between the couplings

$$m_\sigma\sigma^2\,, \quad g_1\sigma^3\,, \quad g_2\sigma^4$$

- these relations must hold also for running couplings (if the symmetry is not broken by renormalization, so it persists in quantum theory)

— This is all what happens with discrete symmetries, hier $\mathbb{Z}(2)$.

- Example II: Breaking of a continuous symmetry

Let us now assume that ϕ has several components — "Linear σ-model"

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi^a)^2 + \frac{1}{2}\mu^2(\phi^a)^2 - \frac{\lambda}{4}[(\phi^a)^2]^2$$

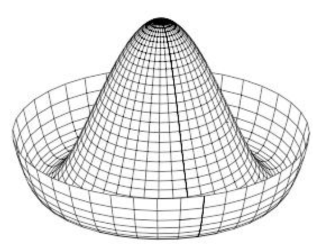
$$i = 1\,\ldots, N\,, \qquad \searrow \; 4, \text{ not } 4! \text{ to avoid ugly } \sqrt{6} \tag{8.7}$$

This model has a $O(N)$ symmetry (cf. Sec. 3.4)

$$\phi^a \mapsto R_{ab}\phi^b\,, \qquad R^T R = \mathbb{1} \tag{8.8}$$

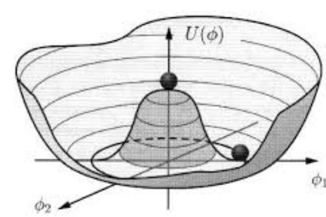
The potential has a "mexican hat" shape

$$V(\phi^a) = V(\phi^2) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4$$



minima: $\quad (\phi^a)^2 = v^2 = \dfrac{\mu^2}{\lambda}$

choose: $\quad \phi_0^a = \{0, 0, \ldots, v\} \qquad (8.9)$



In order to describe the system close to the chosen minimum, redefine (cf. Sect. 3.4)

$$\phi^a(x) = \{\underbrace{\pi^1(x), \dots \pi^{N-1}(x)}_{N-1}, v + \sigma(x)\} \tag{8.10}$$

and rewrite the Lagrangian in terms of new fields omitting unessential constants. Get

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu \pi^k)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 \\ &- \sqrt{\lambda}\mu\sigma^3 - \sqrt{\lambda}\mu(\pi^k)^2\sigma - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}\pi^2\sigma^2 - \frac{\lambda}{4}\pi^4 \end{aligned} \tag{8.11}$$

- massive field σ, $m_\sigma = \sqrt{2}\mu$ — exactly as before
- $N-1$ massless "pions"

- Why $N-1$ — a group theory interpretation.

In N dimensions there are $N(N-1)/2$ independent rotations (number of generators/symmetries): Indeed, a rotation happens in a plain, and a plain is defined by two basis vectors; hence there exist $N(N-1)/2$ different plains.

After the symmetry breaking $O(N) \mapsto O(N-1)$ the $(N-1)(N-2)/2$ symmetry transformations remain (corresponding symmetries are not broken) and the number of broken symmetries is

$$\frac{1}{2}N(N-1) - \frac{1}{2}(N-1)(N-2) = N-1 \tag{8.12}$$

This is exactly the number of massless pions that we have found.

- This situation is very general and can be formulated as

For every spontaneously broken continuous symmetry there must be a massless particle	**Goldstone theorem**	(8.13)

Proof:

►

Classically

$$\mathcal{L} = \text{ (terms with derivatives) } - V(\phi) \tag{8.14}$$

Let $\phi_0^a = \text{const}$ be a minimum of $V(\phi)$, i.e.

$$\frac{\partial}{\partial \phi^a} V\Big|_{\phi^a(x)=\phi_0^a} = 0 \tag{8.15}$$

and

$$V(\phi) = V(\phi_0) + \frac{1}{2}(\phi - \phi_0)^a(\phi - \phi_0)^b \left(\frac{\partial^2}{\partial\phi_a \partial\phi_b} V\right)_{\phi_0} + \dots \tag{8.16}$$

Let

$$m^2_{ab} \stackrel{!}{=} \left(\frac{\partial^2}{\partial \phi_a \partial \phi_b} V \right)_{\phi_0} \tag{8.17}$$

— It is a symmetric matrix $m^2_{ab} = m^2_{ba}$ with non-negative eigenvalues (because ϕ_0 = minimum). Let

$$\phi^a \mapsto \phi^a + \Delta^a(\phi) \tag{8.18}$$

be an infinitesimal symmetry transformation of the Lagrangian:

$$\mathcal{L}(\phi^a) = \mathcal{L}(\phi^a + \Delta^a(\phi)) \tag{8.19}$$

This is valid for arbitrary $\phi(x)$. For constant fields $\phi(x) = \phi$ the terms with derivatives vanish, so that the potential has to be invariant under such trafos as well:

$$V(\phi^a) = V(\phi^a + \Delta^a(\phi)) \tag{8.20}$$

and therefore

$$\Delta^a(\phi) \frac{\partial}{\partial \phi^a} V(\phi) = 0 \tag{8.21}$$

Let us take another derivative of this equation, $\partial/\partial\phi_b$, and put $\phi \to \phi_0$ at the end. Obtain

$$0 = \left(\frac{\partial \Delta^a}{\partial \phi^b} \right)_{\phi_0} \underbrace{\left(\frac{\partial V}{\partial \phi^a} \right)_{\phi_0}}_{=0} + \Delta^a(\phi_0) \underbrace{\left(\frac{\partial^2}{\partial \phi_a \partial \phi_b} V \right)_{\phi_0}}_{=m^2_{ab}} \tag{8.22}$$

Therefore

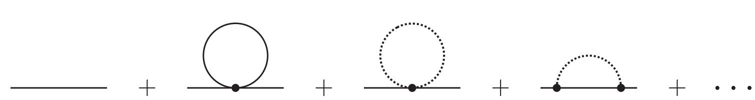
$$\Delta^a(\phi_0)\, m^2_{ab} = 0 \tag{8.23}$$

Thus, there are two possibilities:

(1) $\Delta^a(\phi_0) = 0$ — this means that the corresponding symmetry transformation is not broken

(2) $\Delta^a(\phi_0) \neq 0$ — the symmetry is broken and $\Delta^a(\phi_0)$ is the (left) eigenvector of m^2_{ab} (the mass matrix) with zero eigenvalue — a massless particle.

• A full-fledged proof in quantum theory is more complicated. E.g. in the above example of the linear σ-model is has to happen that exact "pion" propagator still contains a pole at $p^2 = 0$, alias that the self-energy vanishes at zero momentum in the sum of all diagrams in every order of pert. theory:



[here solid line is a pion, short dashes a sigma-meson]. The idea of the proof is that taking into account quantum corrections leads to a renormalized Lagrangian with an effective potential $V(\phi) \to V_{eff}(\phi)$ and the above argumentation remains valid if $V_{eff}(\phi)$ retains the same symmetries.

8.1 The abelian Higgs model

- Consider the following theory (a version of scalar QED):

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \left|D_\mu\phi\right|^2 - V(\phi)\,, \qquad D_\mu = \partial_\mu + ieA_\mu \tag{8.24}$$

where

$A_\mu(x)$: a photon
$\phi(x)$: a complex scalar field

This theory is gauge-invariant [$U(1)$-symmetry]

$$\begin{aligned} \phi(x) &\mapsto e^{i\alpha(x)}\phi(x) \\ A_\mu(x) &\mapsto A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x) \end{aligned} \tag{8.25}$$

Let us take the potential for the scalar field with a "wrong" sign of the mass term (as in above examples)

$$V(\phi) = -\mu^2\phi^*\phi + \frac{\lambda}{2}(\phi^*\phi)^2,, \qquad \mu^2 > 0 \tag{8.26}$$

In this case the $U(1)$ symmetry will be broken spontaneously with

$$|\phi_0|^2 = v^2 = \frac{\mu^2}{\lambda} \tag{8.27}$$

We can choose ϕ_0 to have arbitrary phase

$$\phi_0 = ve^{i\alpha} \tag{8.28}$$

All choices are equivalent, let us take a real ϕ_0 for definiteness

$$\phi_0 = v = \sqrt{\frac{\mu^2}{\lambda}} \tag{8.29}$$

Expanding around the chosen minimum, we can write

$$\phi(x) = \underbrace{\phi_0}_{=v} + \frac{1}{\sqrt{2}}\big(\phi_1(x) + i\,\phi_2(x)\big) \tag{8.30}$$

and re-write the Lagrangian in terms of real fields $\phi_1(x)$ and $\phi_2(x)$.

We will get:
(1) From the potential

$$V(\phi) = \cancel{-\frac{1}{2\lambda}\mu^4} + \frac{1}{2}(2\mu^2)\phi_1^2 + \text{ interactions} \tag{8.31}$$

Thus

$$\left.\begin{array}{l} \text{— } \phi_1 \text{ has mass } \sqrt{2}\mu \\ \text{— } \phi_2 \text{ is massless} \end{array}\right\} \text{ as expected (Goldstone)}$$

(2) From the interactions with the gauge field

$$
\begin{aligned}
\left|D_\mu\phi\right|^2 &= \Big[(\partial_\mu - ieA_\mu)\phi^*\Big]\Big[(\partial^\mu + ieA^\mu)\phi\Big] \\
&= \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 + \sqrt{2}evA_\mu\partial^\mu\phi_2 + e^2v^2A_\mu A^\mu \\
&\quad + \mathcal{O}(\phi^3, \phi^2 A, \ldots)
\end{aligned} \tag{8.32}
$$

(?!) A mass term for the photon field

$$
\mathcal{L} = \ldots + \frac{1}{2}m_A^2 A^2\,, \qquad m_A^2 = 2e^2\phi_0^2 = 2e^2v^2 \tag{8.33}
$$

The sign is correct because

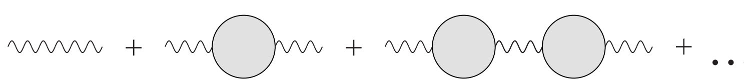
$$
\frac{1}{2}m_A^2A^2 = \frac{1}{2}m_A^2\Big(A_0^2 - A_z^2 - \underbrace{\vec{A}_\perp^2}_{\text{phys. photons}}\Big) \tag{8.34}
$$

How could it happen that photon acquires a mass?

- Remind the usual argument: $\boxed{\textit{Lect. 18}}$

Photon propagator

$$
D_{\mu\nu}(k) = \tag{8.35}
$$



where the shaded blob is

$$
\Pi_{\mu\nu}(k) = i\int d^4x\, e^{ikx}\,\langle\Omega|J_\mu(x)J_\nu(0)|\Omega\rangle = (g_{\mu\nu}k^2 - k_\mu k_\nu)\Pi(k^2) \tag{8.36}
$$

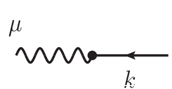
so that

$$
\begin{aligned}
D_{\mu\nu}(k) &= \frac{g_{\mu\nu}}{k^2} + \frac{g_{\mu\mu'}}{k^2}(g_{\mu'\nu'}k^2 - k_{\mu'}k_{\nu'})\Pi(k^2)\frac{g_{\nu'\nu}}{k^2} + \ldots \\
&= \frac{g_{\mu\nu}}{k^2(1-\Pi(k^2))} + \mathcal{O}(k_\mu k_\nu)
\end{aligned} \tag{8.37}
$$

Thus photon has no mass, but under implicit assumption that $\Pi(k^2)$ does not have a pole at $k^2 \to 0$. A pole means a contribution of a massless particle. Since one-particle reducible diagrams with a photon in the intermediate state are excluded by construction, the lowest mass state contributing to $\Pi(k^2)$ in QED is an electron-positron pair, thus no pole.

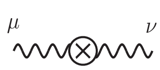
- What happens in our case?

$\mathcal{L} \ni \sqrt{2}evA_\mu\partial^\mu\phi_2$



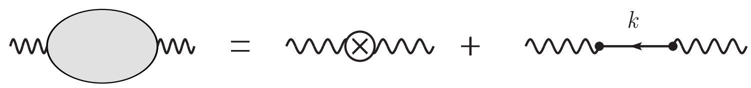
$= \sqrt{2}ve(-ik^\mu) \;=\; -\,im_Ak^\mu$

$\mathcal{L} \ni \frac{1}{2}m_A^2A_\mu A^\mu$



$= m_A^2 g_{\mu\nu}$ (8.38)

Note that $m_A = \sqrt{2}ve = \mathcal{O}(e)$ so that to $\mathcal{O}(e^2)$ accuracy



$$
\begin{aligned}
&= m_A^2 g_{\mu\nu} + (-im_A k^\mu)\frac{1}{-k^2}(+im_A k^\nu) \\
&= m_A^2(g_{\mu\nu} - \underbrace{\frac{k_\mu k_\nu}{k^2}}_{\text{from Goldstone}}) \equiv (g_{\mu\nu}k^2 - k_\mu k_\nu)\Pi(k^2)
\end{aligned}
\tag{8.39}
$$

We conclude that

- $k^\mu \Pi_{\mu\nu} = 0$ — Ward Id. is satisfied
- $\Pi(k^2) = \dfrac{m_A^2}{k^2}$ has a pole at $k^2 = 0$, thus

$$
D_{\mu\nu}(k) = \frac{g_{\mu\nu}}{k^2(1-\Pi(k^2))} = \frac{g_{\mu\nu}}{k^2(1-\frac{m_A^2}{k^2})} = \frac{g_{\mu\nu}}{k^2 - m_A^2} \;\Rightarrow\; \text{mass} \tag{8.40}
$$

- **Unitary gauge** [the fate of the Goldstone boson...]

Our theory is invariant under (local) phase transformations

$$
\phi(x) \mapsto e^{i\alpha(x)}\phi(x) \qquad\qquad A_\mu(x) \mapsto A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x) \tag{8.41}
$$

We can choose the phase (fix the gauge) in such a way that $\forall x, \phi(x) =$ real number. In this case $\phi(x) \equiv \phi_1(x)$ in our notation, and the $\phi_2(x)$ does not exist! (It basically means that our Goldstone particle can be gauged away). What happens in this case?

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}^2 + \Big[(\partial_\mu - ieA_\mu)\phi^*\Big]\Big[(\partial^\mu + ieA^\mu)\phi\Big] - V(\phi) \\
&\overset{\phi=\phi^*}{\mapsto} -\frac{1}{4}F_{\mu\nu}^2 + (\partial_\mu\phi)(\partial^\mu\phi) + e^2\phi^2 A_\mu A^\mu - V(\phi)
\end{aligned}
\tag{8.42}
$$

If $V(\phi)$ has a minimum at $\phi = v = \langle\Omega|\phi|\Omega\rangle$, then we can write $\phi(x) = v + \phi_1(x)$ and get

$$
e^2\phi^2 A_\mu A^\mu \mapsto e^2(v+\phi_1(x))^2 A_\mu^2 = m_A^2 A_\mu^2 + \text{ interactions} \tag{8.43}
$$

Thus we get

— a massive vector field A_μ
— a massive scalar field ϕ_1 from $V(v+\phi_1)$
— a massless (Goldstone) particle disappeared?

Let us count the number of degrees of freedom:

- At the Lagrangian level: 2 real functions to describe a complex-valued $\phi(x)$, 2 real functions to describe photons with physical polarizations. In total 2+2=4.

- In the particle spectra: 1 real function to describe a real field $\phi_1(x)$, 3 real functions to describe a massive vector field A_μ (A reduction from 3 to two polarizations is specific for massless vector particles as for $k^2 = 0$ Lorentz gauge condition does not fix the potential uniquely). In total 1+3=4.

Thus, the degree of freedom corresponding to the Goldstone boson is effectively "eaten up" by the vector field which can have not only two usual transverse polarizations, but also a longitudinal polarization as well.

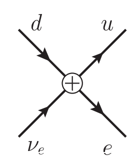
9 The Standard Model: Theory

Weak interactions:
First example: a β-decay
- Pauli (1930): Neutrino needed to carry away energy

$$n \to pe\bar{\nu}_e \qquad \leftrightarrow \qquad d \to ue\bar{\nu}_e \tag{9.1}$$

- Fermi (1933): A four-fermion Lagrangian
- Yang&Lee (1954-56). Wu (1957, exp): parity violation
- Marshak&Sudarshan, Feynman&Gell-Mann (1957): A $V - A$ structure (vector minus axial)

$$\begin{aligned} \delta\mathcal{L}_F &= 2\sqrt{2}G_F(\bar{e}_L\gamma^\mu\nu_L)(\bar{u}_L\gamma_\mu d_L) \\ d_L &= \frac{1-\gamma_5}{2}d = P_L d \end{aligned} \tag{9.2}$$



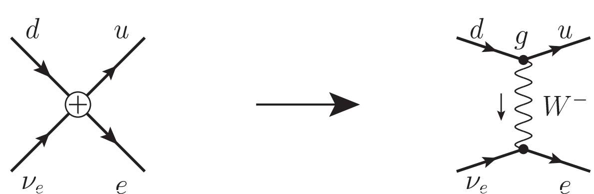
!! Only left-handed particles take part in weak interactions

Problem: a theory involving the four-fermion interactions is not renormalizable. Within the Wilson effective action approach this means that G_F has to be of the order of inverse UV cutoff squared:

$$G_F \sim \frac{1}{M_{UV}^2} \tag{9.3}$$

so it can only be an effective description which must be substituted by a more fundamental theory at large momenta (small distances).

An old idea: split the four-fermion interaction in two parts mediated by some vector particle exchange (W-boson):



$$D_{\mu\nu}^{(W)} \sim \frac{g_{\mu\nu}}{m_W^2 - q^2} \qquad \overset{q^2\to 0}{\Rightarrow} \qquad \frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2} \tag{9.4}$$

In order to realize this scenario one needs a massive vector particle with electric charge.

9.1 The electroweak (Glashow-Weinberg-Salam) Lagrangian

The SM Lagrangian is rather complicated, so let us start building it piece-by-piece.

- Assume there exists a complex scalar field $\phi(x)$
- Assume gauge symmetry $SU(2) \times U(1)$ such that

$$\phi(x) \mapsto \phi'(x) = e^{i\alpha^a(x)t^a} e^{i\frac{1}{2}\beta(x)} \phi(x) \tag{9.5}$$

so that $\phi(x)$ is a two-component spinor w.r.t. $SU(2)$:

$$\phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} \qquad t^a = \frac{1}{2}\sigma^a\,, \qquad a = 1,2,3 \tag{9.6}$$

The factor 1/2 in $e^{i\frac{1}{2}\beta(x)}$ is the $U(1)$ charge of the scalar field.
— convenient to introduce this way instead of writing (later) gauge-field trafo with $\ldots \frac{1}{e_\phi}\partial_\mu \beta(x)$

- Assume the Lagrangian involves a Mexican-hat-type potential $V(\phi$ such that $\phi(x)$ develops a nonzero VEV. The standard choice is

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{9.7}$$

- There exist in total 4 symmetry transformations: $\alpha^1, \alpha^2, \alpha^3, \beta$. If we choose

$$\alpha_1 = 0 \qquad \alpha_2 = 0 \qquad \alpha_3 = \beta \tag{9.8}$$

the VEV does not change (for our specific choice), $\langle \phi' \rangle = \langle \phi \rangle$, so that

— one symmetry is not broken
— three symmetries are broken spontaneously
— will get three massless Goldstone bosons

- We will need four gauge fields (3 for $SU(2)$ and 1 for $U(1)$) to construct a gauge-invariant Lagrangian

— Three Goldstone bosons will be used to produce three massive gauge fields
— One gauge field will remain massless. This will be the photon.

! We will get a unified theory of electromagnetic and weak interactions

- The Lagrangian will have a kinetic energy term for the scalar field in which we wile need to promote ∂_μ to a covariant derivative:

U(1) charge

$$\mathcal{L} = \ldots + \left| D_\mu \phi \right|^2 + \ldots$$
$$D_\mu \phi(x) = \big(\partial_\mu \underbrace{-igA^a_\mu t^a}_{\text{like in QCD}} \underbrace{-i\frac{1}{2}g' B_\mu}_{\text{like in QED}} \big)\phi(x) \tag{9.9}$$

Then

$$\left|D_\mu\phi\right|^2_{\phi=\langle\phi\rangle} = \frac{1}{2}(0,v)\left(gA^a_\mu t^a + \frac{1}{2}g'B_\mu\mathbb{1}\right)\left(gA^{b\mu}t^b + \frac{1}{2}g'B^\mu\mathbb{1}\right)\begin{pmatrix}0\\v\end{pmatrix}$$
$$= \frac{1}{2}\frac{v^2}{4}\left[g^2(A^1_\mu)^2 + g^2(A^2_\mu)^2 - (-gA^3_\mu + g'B_\mu)^2\right] \tag{9.10}$$

Define:

$$W^\pm_\mu = \frac{1}{\sqrt{2}}\left(A^1_\mu \mp iA^2_\mu\right) \qquad \text{W bosons} \tag{9.11}$$

$$Z^0_\mu = \frac{1}{\sqrt{g^2+g'^2}}\left(gA^3_\mu - g'B_\mu\right) \qquad \text{Z boson} \tag{9.12}$$

$$A_\mu = \frac{1}{\sqrt{g^2+g'^2}}\left(g'A^3_\mu + gB_\mu\right) \qquad \text{Photon} \tag{9.13}$$

$$\tag{9.14}$$

Or, define (Weinberg mixing angle)

$$\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}\,, \qquad \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}} \tag{9.15}$$

and

$$\begin{pmatrix}Z^0\\A\end{pmatrix} = \begin{pmatrix}\cos\theta_W & -\sin\theta_W\\ \sin\theta_W & \cos\theta_W\end{pmatrix}\begin{pmatrix}A^3\\B\end{pmatrix} \tag{9.16}$$

Then

$$m_W = g\frac{v}{2}$$
$$m_Z = (g^2+g'^2)^{1/2}\frac{v}{2} = \frac{m_W}{\cos\theta_W}$$
$$m_A = 0 \tag{9.17}$$

! A prediction: a new particle: Z^0 boson. (We only need $W^\pm$ for β-decay)

! A prediction: Higgs particle:
The scalar field will fluctuate around its VEV:

$$\phi(x) = \underbrace{U(x)}_{\to 1,\text{unitary gauge}} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}0\\v+h(x)\end{pmatrix} \qquad \boxed{h(x):\quad \text{Higgs field}} \tag{9.18}$$

An example of a (renormalizable) scalar potential that leads to the nonzero VEV is

$$V(\phi) = -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 \tag{9.19}$$

The minimum of the potential is at

$$v = \left(\frac{\mu^2}{\lambda}\right)^{1/2} \tag{9.20}$$

and in unitary gauge the potential takes the form

$$V(\phi) = \mu^2 h^2 + \lambda v h^3 + \frac{1}{4}\lambda\, h^4 \equiv \frac{1}{2}m_h^2 + \sqrt{\frac{\lambda}{2}} m_h h^3 + \frac{1}{4}\lambda\, h^4 \tag{9.21}$$

with

$$m_h = \sqrt{2}\mu^2 = \sqrt{\frac{\lambda}{2}} v \tag{9.22}$$

The term $\ldots \left|D_\mu \phi\right|^2 \ldots$ will then produce the kinetic energy contribution to Higgs and also Higgs couplings to $W^\pm$ and Z-bosons:

$$\left|D_\mu \phi\right|^2 \to \frac{1}{2}(\partial_\mu h)^2 + \left[m_W^2 W^{\mu +} W_\mu^- + \frac{1}{2} m_Z^2 Z^\mu Z_\mu\right]\left(1 + \frac{h}{v}\right)^2 \tag{9.23}$$

$\boxed{\textit{Lect. } 19}$

- The next step: we want to add fermions — quarks and leptons.

Problem:
Only left-handed particles take part in weak interactions. For free particles

$$\bar{\psi} i \partial\!\!\!/ \psi = \bar{\psi}_L i \partial\!\!\!/ \psi_L + \bar{\psi}_R i \partial\!\!\!/ \psi_R \tag{9.24}$$

Usually we introduce the interaction by promoting $i\partial\!\!\!/$ to a covariant derivative, but how to do this in such a way that electromagnetism interacts equally with left-handed and right-handed fermions, but weak interaction only affects left-handed fermions?

Solution:
Left-handed ψ_L and right-handed ϕ_R fermions transform according to different reps. of the $SU(2)$ group:

$$\begin{aligned} \psi_L &= \begin{pmatrix} \psi_L^1 \\ \psi_L^2 \end{pmatrix} && \text{doublet (spinor)} \qquad \text{(like } \phi\text{)} \\ \psi_R &= \psi_R && \text{singlet (scalar)} \end{aligned} \tag{9.25}$$

This means that from the viewpoint of SU(2) interactions, e.g., the left-handed electron and right-handed electron are completely different particles!
The covariant derivatives are in this case also different:

$$\begin{aligned} (D_\mu)_{\text{spinor}} &= \partial_\mu \mathbb{1} - ig A_\mu^a t^a - ig' Y B_\mu \mathbb{1} && \leftarrow \text{a 2x2 matrix} \\ (D_\mu)_{\text{scalar}} &= \partial_\mu - ig' \tilde{Y} B_\mu \end{aligned} \tag{9.26}$$

where Y and $\tilde{Y}$ are numbers which can (will) be chosen in such a way that our fermions have correct electric charges. They are called **Hypercharges**. For the scalar field we have chosen

$$Y_\phi = \frac{1}{2} \tag{9.27}$$

In terms of the physical states this becomes

$$D_\mu = \partial_\mu - \frac{ig}{\sqrt{2}}\Big(W_\mu^+ T^+ + (W_\mu^- T^-\Big) - \frac{i}{\sqrt{g^2+g'^2}} Z_\mu (g^2 T^3 - g'^2 Y) - \frac{i}{gg'}\sqrt{g^2+g'^2} A_\mu \underbrace{(T_3+Y)}_{Q} \tag{9.28}$$

where $T^\pm = (T_1 \pm iT_2)$.

• The following table summarizes the list of fermion fields that exist in Nature:

Quarks

$$\begin{array}{ccccccc}
Q_L & = & \begin{pmatrix} U \\ D \end{pmatrix}_L & = & \overbrace{\begin{pmatrix} u \\ d \end{pmatrix}_L, \quad \begin{pmatrix} c \\ s \end{pmatrix}_L, \quad \begin{pmatrix} t \\ b \end{pmatrix}_L,}^{\text{families (generations)}} & & Y = \frac{1}{6} \\
 & & U_R & = & u_R, \quad c_R, \quad t_R & & \widetilde{Y} = \frac{2}{3} \\
 & & D_R & = & d_R, \quad s_R, \quad b_R & & \widetilde{Y} = -\frac{1}{3}
\end{array} \tag{9.29}$$

Leptons

$$\begin{array}{ccccc}
L_L & = & \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L, & & Y = -\frac{1}{2} \\
 & & (?)(\nu_e)_R, \quad (?)(\nu_\mu)_R, \quad (?)(\nu_\tau)_R, & & \\
L_R = & & e_R, \quad \mu_R, \quad \tau_R, & & \widetilde{Y} = -1
\end{array} \tag{9.30}$$

(Here the electric charge $Q = T_3 + Y$, T_3 is zero for r.h. quarks as they are singlets to $SU(2)$). At the time when the SM was formulated, only left-handed neutrinos were known to exist. Today we know that right-handed neutrinos exist as well, but we are still not sure if they can be included in the SM in the "standard" way.

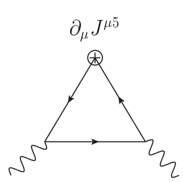
Now we have all particles build in, but there are still two problems:

1. There is a danger that $SU(2) \times U(1)$ symmetry can be broken by quantum anomalies [we have axial-vector interactions]

2. Our fermions are so far massless

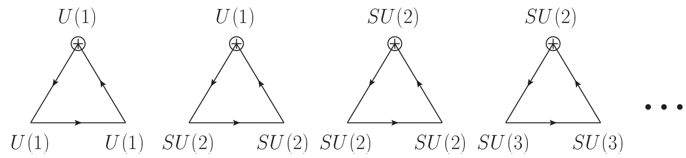
• (1) First, anomaly problem.

The Adler-Bardeen anomaly that we considered in detail in Sect. 6.2 can also be derived from the calculation of the triangle diagram



$$\partial_\mu J^{\mu 5}(x) \;=\; \frac{e^2}{8\pi^2} F_{\alpha\beta}\widetilde{F}^{\alpha\beta} \tag{9.31}$$

Naively, this diagram has to vanish but in fact it does not because of the necessity to introduce a regularization. In SM model there are potentially many anomalous "triangles" — one for every (classically) conserved current and for every pair of gauge fields



(9.32)

If at least one of these anomalies is non-zero, gauge symmetry will be broken and our construction will be worthless. However, a miracle happens (sometimes!):

All anomalies cancel in the sum of quark and lepton triangles if

— the number of lepton families is equal to the number of quark families
— the hypercharges are chosen to reproduce the electric charges of observed particles

Under these two conditions the SM is anomaly-free!

- (1) Second, fermion masses (quarks and leptons)

The usual way to introduce masses for Dirac fermions

$$\bar{\psi} i \not{D} \psi \mapsto \bar{\psi}(i\not{D} - m)\psi \tag{9.33}$$

However

$$m\bar{\psi}\psi = m\bar{\psi}\Big[\frac{1+\gamma_5}{2} + \frac{1-\gamma_5}{2}\Big]\psi = m\bar{\psi}_L\psi_R + m\bar{\psi}_R\psi_L \tag{9.34}$$

and we cannot write such terms because $\psi_L \equiv \psi_L^a$ is a spinor whereas ψ_R is a scalar (of SU(2)).

Solution:
add a term to the Lagrangian (electron as example)

$$\begin{aligned}\delta\mathcal{L}_e &= -\lambda_e\big(\bar{E}_L \cdot \Phi\big)e_R + \text{h.c.} = -\lambda_e\big(\bar{E}_L^a \Phi^a\big)e_R + \text{h.c.}\\ &= -\lambda_e\big(\bar{\nu}_e^L, e_L\big)\begin{pmatrix}\phi^1\\ \phi^2\end{pmatrix} e_R + \text{h.c.}\end{aligned} \tag{9.35}$$

["h.c." = hermitian conjugate]
After the spontaneous symmetry breaking

$$\Phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v + h(x)\end{pmatrix}, \qquad v = \langle\Phi\rangle \tag{9.36}$$

we get

$$\delta\mathcal{L}_e = \underbrace{-\frac{1}{\sqrt{2}}\lambda_e v(\bar{e}_L e_R)}_{\text{mass term}} \underbrace{-\frac{1}{\sqrt{2}}\lambda_e h(\bar{e}_L e_R)}_{\text{electron-higgs coupling}} + \text{h.c.} \tag{9.37}$$

so

$$m_e = \frac{1}{\sqrt{2}}\lambda_e v\,, \qquad\qquad \lambda_e:\ \text{"Yukawa coupling" (parameter)} \tag{9.38}$$

Down-type quarks can be given mass in precisely the same way, but for up-type quarks we have to do it slightly differently: Observe that there exist two possibilities to make a SU(2) scalar from two spinors:

$$\begin{aligned}
&\Phi \mapsto U\Phi\,, \qquad UU^\dagger = \mathbb{1}, \qquad \det U = 1 \\
&(1) \qquad \Phi^\dagger\Phi \mapsto \Phi^\dagger U^\dagger U\Phi = \Phi^\dagger\Phi\,, \\
&(2) \qquad \Phi^a\epsilon_{ab}\Phi^b \mapsto \Phi^{a'}U_{a'a}\epsilon_{ab}U_{bb'}\Phi = \det U\,\Phi^a\epsilon_{ab}\Phi^b
\end{aligned} \tag{9.39}$$

Hence we can add (e.g. for quarks in the first family)

$$\delta\mathcal{L}_q = -\lambda_d\big(\bar{Q}_L\cdot\Phi\big)d_R - \lambda_u\big(\bar{Q}^a_L\epsilon_{ab}\Phi^{b\dagger}\big)u_R + \text{h.c.} \tag{9.40}$$

to get

$$m_d = \frac{1}{\sqrt{2}}\lambda_d v\,, \qquad\qquad m_u = \frac{1}{\sqrt{2}}\lambda_u v \tag{9.41}$$

☹ A new parameter for every massive fermion...

☺ A prediction: Higgs coupling to quarks/leptons is proportional to their mass! Higgs couples most strongly to top-quarks and very weakly to u, d quarks and electrons.

N.B. One can check that the hypercharges sum to zero in both terms so that they are $U(1)$ invariant.

9.2 The CKM matrix

• With three families of quarks and leptons one has more possibilities to add Yukawa interactions, e.g.,

$$\delta\mathcal{L}_Q = -\lambda\big(\bar{Q}^{(1)}_L\cdot\Phi\big)Q^{(2)}_R + \text{h.c.} \tag{9.42}$$

where the superscript (1), (2) labels the family. In this way after symmetry breaking one also obtains nondiagonal in flavor terms like

$$-\frac{1}{\sqrt{2}}\lambda_{ds}(\bar{d}_L s_R) + \text{h.c.} \tag{9.43}$$

— a mass matrix. Physical quark/lepton states Q' correspond to eigenstates of the mass matrix, and their masses correspond to the eigenvalues, e.g.

$$\begin{aligned}
U^i_L &= \mathbb{U}^{ik}_u\,U'^j_L\,, & D^i_L &= \mathbb{U}^{ik}_d\,D'^j_L\,, \\
U^i_L &= \big(u_L, c_L, t_L\big)\,, & D^i_L &= \big(d_L, s_L, b_L\big)\,.
\end{aligned} \tag{9.44}$$

Note that $\mathbb{U}^{ik}_u$ and $\mathbb{U}^{ik}_d$ are in general different unitary matrices (do not need to be the same), $\mathbb{U}\mathbb{U}^\dagger = \mathbb{1}$, $\det\mathbb{U} = 1$.

• The "primed" fields are those ones that we identify as quarks which participate, e.g., in strong interactions. It makes sense, therefore, to rewrite also the weak interaction terms in terms of physical fields. For example, the $u \to d$ transition due to emission of the W-boson (β-decay)

$$W^+_\mu \bar{u}^i_L \gamma^\mu d^i_L \qquad \mapsto \qquad W^+_\mu \bar{u}'^i_L \left(\mathbb{U}^\dagger_u \mathbb{U}_d\right)_{ij} \gamma^\mu d'^j_L \tag{9.45}$$

<u>The CKM matrix:</u>

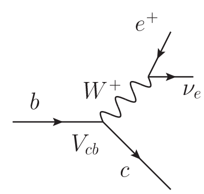
$$V_{ij} = \left(\mathbb{U}^\dagger_u \mathbb{U}_d\right)_{ij} \qquad \Leftarrow \quad 3 \times 3 \text{ unitary matrix} \tag{9.46}$$

and as the result the weak interaction becomes non-diagonal — we obtain mixing between different families:

$$V_{ij} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{9.47}$$

Thus in Nature we observe transitions between quarks belonging to different families, e.g.

$$b \to c e^+ \nu \tag{9.48}$$



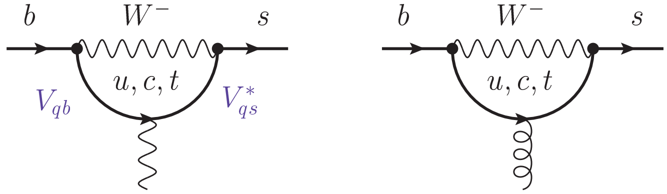
— always between "up"-type and "down"-type quarks
— always with a change of electric charge by one unit

• Why only with an electric charge change? We also have weak interactions generated by a neutral Z^-0 boson exchange?
— Yes, but:

$$Z^0_\mu \bar{u}^i_L \gamma^\mu u^i_L \qquad \mapsto \qquad Z^0_\mu \bar{u}'^i_L \underbrace{\left(\mathbb{U}^\dagger_u \mathbb{U}_u\right)_{ij}}_{\mathbb{1}} \gamma^\mu u'^j_L \tag{9.49}$$

— The SM Lagrangian does not contain terms corresponding to transitions between quarks of different flavor mediated by electrically neutral Z^0 boson. Standard abbreviation: FCNC (Flavor Changing Neutral Currents)

Such processes are allowed, however, due to quantum corrections



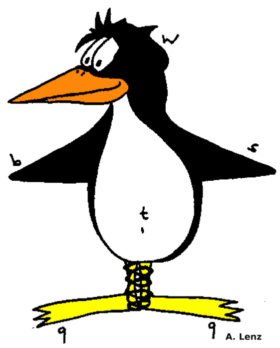
(9.50)

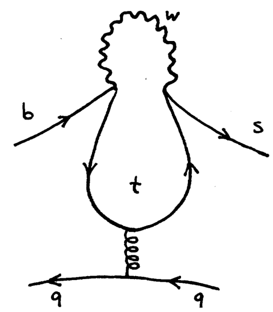
Note that the UV-divergence in these diagrams cancels in the sum of u, c, t quarks since

$$V_{ub}V_{us}^* + V_{cb}V_{cs}^* + V_{tb}V_{ts}^* = 0 \qquad \Leftarrow \qquad V\,V^\dagger = \mathbb{1} \tag{9.51}$$

This also means that a nonzero answer only appears because of different quark masses.

Such contributions are usually referred to as "penguin diagrams":





courtesy of A.Lenz (9.52)

- How many physical parameters are contained in the CKM matrix?

◦ A real orthogonal 3×3 matrix has three parameters (e.g. Euler angles)

◦ A complex unitary 3×3 matrix has 9 real parameters, but:
— Phases of quark fields are unphysical, one can rotate quark fields in
$\bar{q}^{(i)}V_{ij}q^{(j)} \to \bar{q}^{(i)}e^{-i\phi_i}V_{ij}e^{i\phi_j}q^{(j)}$
without causing observable effects.

◦ We have 6 quark fields, thus 6 arbitrary phases, but if we rotate all 6 quarks in the same way, there will be no effect on the CKM matrix. Thus in this way one can get rid of 6-1=5 phases.

◦ Thus one phase is physical, the CKM matrix has 4 parameters — 3 angles and one phase.

The presence of a complex parameter in the Lagrangian has very important physics consequences — a breakdown of the CP symmetry (parity combined with charge conjugation) [more later]. *Lect.* 20

9.3 Quantization of spontaneously broken gauge theories

Our discussion was so far essentially classical. We used unitary gauge in order to isolate physical degrees of freedom. It is not clear, however, how/whether this gauge condition is maintained in perturbation theory, and it is in fact not clear how to formulate perturbation theory in terms of physical particles. A naive formula for the $W^\pm$ propagator

$$\overline{W_\mu(x)W_\nu(y)} \stackrel{?}{=} \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)}\, \frac{g_{\mu\nu}}{k^2 - m_W^2} \tag{9.53}$$

cannot be the full story: This propagator (inspired by usual Feynman gauge) contains contribution of all four polarizations and for massless gauge bosons we have seen that contributions of two unphysical polarizations cancel each other, so that only physical transversely polarized states contribute. For a massive vector particle, the longitudinal polarization is physical, we do not want it to cancel, so smth is not OK.

Let us consider an an abelian Higgs model, Sect. 8.1 as an example, where we will slightly change the notations to be closer to the SM.

- Thus, consider a theory

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \left|D_\mu\Phi\right|^2 - V(\Phi)\,, \qquad D_\mu = \partial_\mu + ieA_\mu \tag{9.54}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2}}\big(\phi_1(x) + i\phi_2(x)\big) \tag{9.55}$$

with real $\phi_{1,2}$. Expanding usual $\Phi(x) \to e^{i\alpha(x)}\Phi(x)$ in real and imaginary parts for $\alpha(x) \to 0$ one obtains that this theory is invariant under infinitesimal abelian trafos

$$\delta\phi_1 = -\alpha(x)\phi_2\,, \qquad \delta\phi_2 = +\alpha(x)\phi_1\,, \qquad \delta A_\mu = -\frac{1}{e}\partial_\mu\alpha(x) \tag{9.56}$$

Assume that $V(\Phi)$ has a nontrivial minimum such that ϕ_1 acquires a nonzero VEV, $\langle\phi_1\rangle = v$, and in order to construct perturbation theory around this minimum we change variables by a shift:

$$\phi_1(x) = v + h(x)\,, \qquad \phi_2(x) = \phi(x) \tag{9.57}$$

so $\{\phi_1, \phi_2\} \mapsto \{h(x), \phi(x)\}$. Then $h(x)$ is our "Higgs" and ϕ is the Goldstone boson. The Lagrangian becomes in these variables

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}(\partial_\mu h - eA_\mu\phi)^2 + \frac{1}{2}(\partial_\mu\phi + eA_\mu(v+h))^2 - V(\phi, h) \tag{9.58}$$

This Lagrangian is still invariant under local symmetry trafos

$$\delta h = -\alpha(x)\phi\,, \qquad \delta\phi = +\alpha(x)(v+h)\,, \qquad \delta A_\mu = -\frac{1}{e}\partial_\mu\alpha(x) \tag{9.59}$$

so that if we want to build a quantum theory as a path integral over $h(x), \phi(x), A_\mu(x)$ we have to follow the Faddeev-Popov construction.

- Start with a "naive" path integral

$$W = \int \mathcal{D}A(x)\mathcal{D}h(x)\mathcal{D}\phi(x)\, e^{i\int d^4x\,\mathcal{L}[A,h,\phi]} \tag{9.60}$$

and introduce some gauge-fixing constraint similar as we did for QED (and QCD)

$$G(A, h, \phi) = \omega(x) \tag{9.61}$$

Following the familiar road, insert the Faddeev -Popov "1" in the path integral and use gauge invariance to separate an overall integral over the gauge group. Obtain

$$W = \cancel{\left(\int \mathcal{D}\alpha(x)\right)} \int \mathcal{D}A(x)\mathcal{D}h(x)\mathcal{D}\phi(x)\, e^{i\int d^4x\,\mathcal{L}[A,h,\phi]}\delta(G(A,h,\phi) - \omega(x))\det\left(\frac{\delta G}{\delta\alpha}\right) \tag{9.62}$$

Next (path-)integrate over $\omega(x)$ with a Gaussian weight to get rid of the $\delta(G(A,h,\phi)-\omega(x))$ at the cost of having an additional (gauge-fixing) term in the Lagrangian:

$$\mathcal{L}(A,h,\phi) \mapsto \mathcal{L}[A,h,\phi] - \frac{1}{2}(G)^2 \tag{9.63}$$

G is, in principle, an arbitrary linear operator, but a suitable choice brings advantages. Let

$$G = \frac{1}{\sqrt{\xi}}\Big(\partial_\mu A^\mu - \xi e v \phi\Big) \qquad \boxed{R_\xi \text{ gauge}} \tag{9.64}$$

Then in the Lagrangian we get

$$\begin{aligned}\frac{1}{2}G^2 &= \frac{1}{2\xi}\Big(\partial_\mu A^\mu - \xi e v\phi\Big)^2 \\ &= \underbrace{\frac{1}{2\xi}(\partial_\mu A^\mu)^2}_{(1)} \underbrace{- ev\phi\partial_\mu A^\mu}_{(2)} \underbrace{+\frac{\xi}{2}e^2v^2\phi^2}_{(3)}\end{aligned} \tag{9.65}$$

(1): The usual gauge fixing term for A_μ
(2): Cancels the term $\sim \partial_\mu\phi A^\mu$ from the expansion of $\ldots \frac{1}{2}(\partial_\mu\phi + eA_\mu(v+h))^2 \ldots$ in (9.58).
(3): Produces a gauge-dependent mass term for the Golstone boson

$$m_\phi^2 = \xi e^2 v^2 \equiv \xi m_A^2 \qquad m_A^2 = (ev)^2 \tag{9.66}$$

Collecting all quadratic terms in the gauge-fixed Lagrangian:

$$\begin{aligned}\mathcal{L} = -\frac{1}{2}A_\mu\Big(-g^{\mu\nu}\partial^2 + \Big(1-\frac{1}{\xi}\Big)\partial^\mu\partial^\nu - m_A^2 g^{\mu\nu}\Big)A_\nu + \frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}m_h^2h^2 + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\xi}{2}m_A^2\phi^2 \\ + \text{ interactions}\end{aligned} \tag{9.67}$$

[The mass term for h comes from $V(\phi)$].

- We still have the determinant of the gauge variation of G which can be computed using (9.59):

$$\begin{aligned}\frac{\delta G}{\delta\alpha} &= \frac{1}{\sqrt{\xi}}\frac{\delta}{\delta\alpha}\Big\{\partial_\mu\Big[A^\mu - \frac{1}{e}\partial^\mu\alpha\Big] - \xi ev\Big[\phi + (v+h)\alpha\Big]\Big\} \\ &= \frac{1}{\sqrt{\xi}}\Big[-\frac{1}{e}\partial^2 - \xi ev(v+h)\Big] = \frac{1}{\sqrt{\xi}e}\Big[-\partial^2 - \xi m_A^2\Big(1+\frac{h}{v}\Big)\Big]\end{aligned} \tag{9.68}$$

The determinant of this operator can be computed as a path integral over Faddeev-Popov ghost fields which add an extra term to the Lagrangian

$$\mathcal{L}_{\text{ghost}} = \bar{c}(x)\Big[-\partial^2 - \xi m_A^2\Big(1+\frac{h}{v}\Big)\Big]c(x) \tag{9.69}$$

Note that the ghosts do not couple to the gauge field (because we have an abelian theory) but they couple to the higgs and cannot be ignored.

- From the collection of all quadratic terms in the Lagrangian we can find the propagators. E.g. for the gauge field

$$-\frac{1}{2}A_\mu\Big(-g^{\mu\nu}\partial^2 + \Big(1-\frac{1}{\xi}\Big)\partial^\mu\partial^\nu - m_A^2 g^{\mu\nu}\Big)A_\nu \tag{9.70}$$

goes over in momentum space to

$$g^{\mu\nu}k^2 - \Big(1-\frac{1}{\xi}\Big)k^\mu k^\nu - m_A^2 g^{\mu\nu} = \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)(k^2 - m_A^2) + \left(\frac{k^\mu k^\nu}{k^2}\right)\frac{1}{\xi}(k^2 - \xi m_A^2) \tag{9.71}$$

and inverting this matrix one obtains the propagator

$$\overline{A^\mu(x)A^\nu}(y) = \int \frac{d^4k}{(2\pi)^4 i}\, e^{-ik(x-y)} \frac{1}{k^2 - m_A^2}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2 - \xi m_A^2}(1-\xi)\right) \tag{9.72}$$

Note that the transverse component of A indeed acquires the mass m_A, and the higgs field indeed acquires the mass m_h (easy to see from $\mathcal{L}$), whereas unphysical components of A, the Goldstone boson and the ghost all acquire the same gauge-dependent mass $\sqrt{\xi}m_A$.

Here is the full list:

A_μ :  $\dfrac{-i}{k^2 - m_A^2}\left(g^{\mu\nu} - \dfrac{k^\mu k^\nu}{k^2 - \xi m_A^2}(1-\xi)\right)$

h : $\dfrac{i}{k^2 - m_h^2}$

ϕ : $\dfrac{i}{k^2 - \xi m_A^2}$

c : $\dfrac{i}{k^2 - \xi m_A^2}$ (9.73)

The dependence on gauge parameter seems to be rather involved, let us check that this dependence indeed cancels on a simple example. [taken from Peskin&Schröder]

- To this end, let us add fermions to our model, in a way that resembles the SM:

$$\mathcal{L} = \bar{\psi}_L(i\not{D})\psi_L + \bar{\psi}_R(i\not{\partial})\psi_R - \lambda_f\big(\bar{\psi}_L\Phi\psi_R + \bar{\psi}_R\Phi^*\psi_L\big) \tag{9.74}$$

with $D_\mu = \partial_\mu + ieA_\mu$ as usual. Inserting in the last term

$$\Phi = \frac{1}{\sqrt{2}}\Big(v + h(x) + i\phi(x)\Big) \tag{9.75}$$

get

$$\begin{aligned} & -\frac{\lambda_f}{\sqrt{2}}\left[v\left(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L\right)\Big(1+\frac{h}{v}\Big) + i\phi\Big(\bar{\psi}_L\psi_R - \bar{\psi}_R\psi_L\Big)\right] \\ = & -\frac{\lambda_f}{\sqrt{2}}\left[v\,\bar{\psi}\Big(\frac{1+\gamma_5}{2} + \frac{1-\gamma_5}{2}\Big)\psi\Big(1+\frac{h}{v}\Big) + i\phi\bar{\psi}\Big(\frac{1+\gamma_5}{2} - \frac{1-\gamma_5}{2}\Big)\psi\right] \\ = & -\frac{\lambda_f}{\sqrt{2}}\left[v\,\bar{\psi}\,\psi\Big(1+\frac{h}{v}\Big) + i\phi\bar{\psi}\gamma_5\psi\right] \;=\; -\,m_f\bar{\psi}\,\psi\Big(1+\frac{h}{v}\Big) - \frac{i\lambda_f}{\sqrt{2}}\phi\,\bar{\psi}\gamma_5\psi \end{aligned} \tag{9.76}$$

Thus the fermion receives a mass as intended:

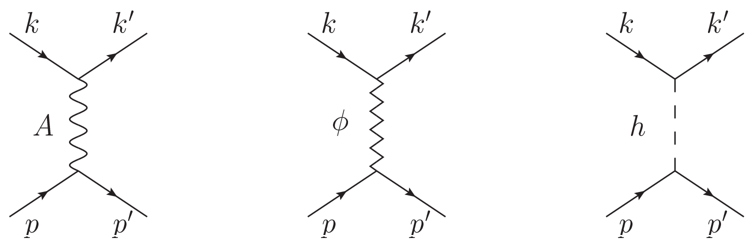
$$m_f = \lambda_f\frac{v}{\sqrt{2}} \tag{9.77}$$

and it interacts with the higgs and with the Goldstone boson.
[This theory is actually not good as it suffers from an axial anomaly, but we will not go that far]

- Let us calculate the fermion-fermion scattering cross section in this model to lowest order:

$$f(p) + f(k) \;\to\; f(p') + f(k') \tag{9.78}$$

We will have three Feynman diagrams — D1,D2,D3 —(the ghost does not contribute at leading order)



(9.79)

(D3): Since higgs propagator does not depend on ξ, D3 is not involved.

(D2): Goldstone boson:

$$i\mathcal{M}_\phi = \left(\frac{\lambda_f}{\sqrt{2}}\right)^2 \bar{u}(p')\gamma_5 u(p) \frac{i}{q^2 - \xi m_A^2} \bar{u}(k')\gamma_5 u(k) \tag{9.80}$$

(D1): Gauge boson:

$$i\mathcal{M}_A = (-ie)^2 \left(\bar{u}(p')\gamma_\mu \frac{1-\gamma_5}{2} u(p)\right) \frac{-i}{q^2 - m_A^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2 - \xi m_A^2}(1-\xi)\right)\left(\bar{u}(k')\gamma_\nu \frac{1-\gamma_5}{2} u(k)\right) \tag{9.81}$$

To see cancellation of the gauge dependence, rewrite

$$\begin{aligned}
&\frac{-i}{q^2 - m_A^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2 - \xi m_A^2}(1-\xi)\right) \\
&\qquad = \frac{-i}{q^2 - m_A^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{m_A^2} + \frac{q^\mu q^\nu}{m_A^2} - \frac{q^\mu q^\nu}{q^2 - \xi m_A^2}(1-\xi)\right) \\
&\qquad = \frac{-i}{q^2 - m_A^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{m_A^2}\right) + \frac{-i}{q^2 - \xi m_A^2}\left(\frac{q^\mu q^\nu}{m_A^2}\right)
\end{aligned} \tag{9.82}$$

The first term does not depend on ξ. In the second term the momenta $q_\mu q_\mu$ will be multiplying spinor products:

$$\begin{aligned}
q^\mu \bar{u}(p')\gamma_\mu \frac{1-\gamma_5}{2} u(p) &= (p-p')^\mu \bar{u}(p')\gamma_\mu \frac{1-\gamma_5}{2} u(p) \qquad \boxed{(\not{p} - m)u(p) = 0} \\
&= \frac{1}{2}\bar{u}(p')\Big[\underbrace{(\not{p} - \not{p}')}_{(\not{p}-m_f)-(\not{p}'-m_f)\to 0} - (\not{p} - \not{p}')\gamma_5\Big] u(p) \\
&= \frac{1}{2}\bar{u}(p')\left[\not{p}'\gamma_5 + \gamma_5\not{p}\right] u(p) \;=\; m_f \bar{v}(p')\gamma_5 u(p)\,, \\
q^\nu \bar{u}(k')\gamma_\nu \frac{1-\gamma_5}{2} u(k) = -(k-k')^\mu \bar{u}(k')&\gamma_\nu \frac{1-\gamma_5}{2} u(k) = -m_f \bar{u}(k')\gamma_5 u(k)
\end{aligned} \tag{9.83}$$

Then

$$
\begin{aligned}
i\mathcal{M}_A &= (-ie)^2 \left(\bar{u}(p')\gamma_\mu \frac{1-\gamma_5}{2} u(p)\right) \frac{-i}{q^2 - m_A^2} \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{m_A^2}\right) \left(\bar{u}(k')\gamma_\nu \frac{1-\gamma_5}{2} u(k)\right) \\
&\quad + (-ie)^2 \left(\bar{u}(p')\gamma_\mu \frac{1-\gamma_5}{2} u(p)\right) \frac{-i}{q^2 - \xi m_A^2} \left(\frac{q^\mu q^\nu}{m_A^2}\right) \left(\bar{u}(k')\gamma_\nu \frac{1-\gamma_5}{2} u(k)\right) \\
&= (-ie)^2 \left(\bar{u}(p')\gamma_\mu \frac{1-\gamma_5}{2} u(p)\right) \frac{-i}{q^2 - m_A^2} \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{m_A^2}\right) \left(\bar{u}(k')\gamma_\nu \frac{1-\gamma_5}{2} u(k)\right) \\
&\quad + (-ie)^2 (-m_f^2) \frac{1}{m_A^2} \left(\bar{u}(p')\gamma_5 u(p)\right) \frac{-i}{q^2 - \xi m_A^2} \left(\bar{u}(k')\gamma_5 u(k)\right)
\end{aligned}
\tag{9.84}
$$

and using $m_f = \lambda_f v/\sqrt{2}$, $m_A = ev$

$$
(-ie)^2 (-m_f^2) \frac{1}{m_A^2} = \left(\frac{\lambda_f}{\sqrt{2}}\right)^2 \tag{9.85}
$$

so that the expression in the second line exactly cancels the Goldstone boson contribution in (9.80). Thus the dependence on gauge parameter disappears, as it should.

• Note that the final result for the gauge-invariant sum of gauge-boson and Goldstone-boson contributions is just as if we would forget Goldstone altogether and calculate the gauge boson diagram using the propagator

$$
\overline{A^\mu(x) A}^\nu(y) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)} \frac{-i}{k^2 - m_A^2} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m_A^2}\right) \tag{9.86}
$$

The tensor structure is the polarization sum. In the rest frame of the gauge boson this becomes a projection on the space-like directions, these are precisely the three possible polarizations of a massive vector particle:

$$
\sum_{e_\mu q^\mu = 0} e_\mu^{(\lambda)} e_\nu^{(\lambda)*} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{m_A^2}\right) \tag{9.87}
$$

Thus the role of the Goldstone boson contribution is to cancel the contribution of the fourth, unphysical, polarization.

• Perturbation theory looks very different for different values of ξ:

∘ Landau (Lorentz) gauge $\xi = 0$:

 $= \dfrac{-i}{k^2 - m_A^2} \left(g^{\mu\nu} - \dfrac{k^\mu k^\nu}{k^2}\right)$ $= \dfrac{i}{k^2}$ (9.88)

∘ Feynman–'t Hooft gauge $\xi = 1$:

 $= \dfrac{-ig^{\mu\nu}}{k^2 - m_A^2}$ $= \dfrac{i}{k^2 - m_A^2}$ (9.89)

○ Unitary gauge $\xi \to \infty$:

$$= \frac{-i}{k^2 - m_A^2}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m_A^2}\right) \tag{9.90}$$

$$= 0$$

(only physical degrees of freedom remain)

• Non-abelian theories with spontaneous symmetry breaking — similar but much more cumbersome. A complete set of Feynman rules for the SM can be found e.g. in: J. C. Romao and J. P. Silva, Int. J. Mod. Phys. A **27** (2012), 1230025.

Lect. 21

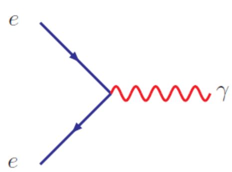
10 The Standard Model: Phenomenology

10.1 W, Z and Higgs

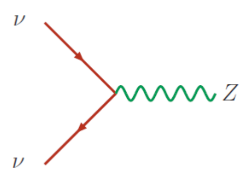
follows Harlander's 2008 Maria Lach lectures

- SM Couplings to leptons:

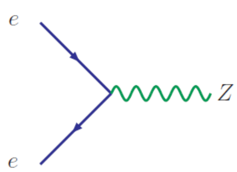
$$\boxed{L_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \qquad \nu_R \qquad e_R} \tag{10.1}$$



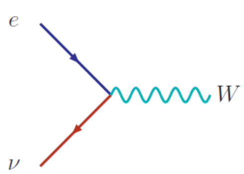
$ieQ_e\gamma_\mu$



$\frac{ie}{4s_W c_W}\gamma_\mu(1-\gamma_5)$



$\frac{ie}{4s_W c_W}\gamma^\mu(g_v^e + g_a^e\gamma_5)$ (10.2)



$\frac{ie}{2\sqrt{2}s_W}\gamma^\mu(1-\gamma_5)$

$$g_v^e = -1 + 4\sin^2\theta_W$$
$$g_a^e = -1\,,$$
$$s_W \equiv \sin\theta_W$$

$$\text{Parameter:} \qquad g, g' \quad \longrightarrow \quad e,\ \sin\theta_W$$

Here and below

$$s_W \equiv \sin\theta_W \qquad c_W \equiv \cos\theta_W \qquad e = \sqrt{4\pi\alpha_{\text{QED}}} > 0 \tag{10.3}$$

- SM Couplings to quarks:

$$\boxed{Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \qquad u_R \qquad d_R} \tag{10.4}$$

— the same with different hypercharges:

$$\begin{array}{llll} \text{leptons:} & Y_L^L = -1 & Y_R^\nu = 0 & Y_R^e = -2 \\ \text{quarks:} & Y_L^Q = \frac{1}{3} & Y_R^u = \frac{4}{3} & Y_R^d = -\frac{2}{3} \\ & \overset{Q=I_3+\frac{Y}{2}}{\Rightarrow} & Q_u = \frac{2}{3} & Q_d = -\frac{1}{3} \end{array} \tag{10.5}$$

[$I_3 = +\frac{1}{2}$ for upper components of the SU(2) doublets and $I_3 = -\frac{1}{2}$ for lower components]

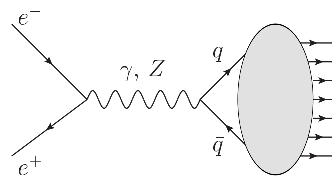
- Electric charge is known very precisely:

$$\frac{1}{\alpha} = 137.035999084(21) \qquad \text{value from PDT 2020} \tag{10.6}$$

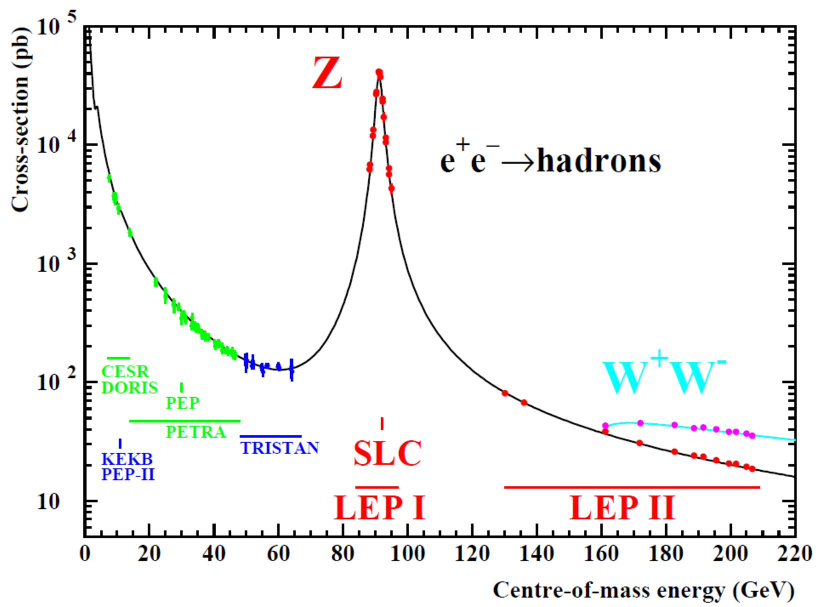
but how can one determine the Weinberg's mixing angle and also W, Z masses?

- Z-boson contribution is seen very clearly in the total cross section of electron-positron annihilation

$$e^+ + e^- \to \text{hadrons} \tag{10.7}$$



LEP at CERN, 1989-2000:



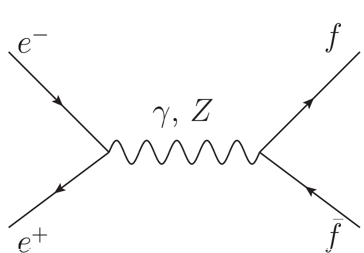
(10.8)

From the position of the peak (now more methods)

$$m_Z = 91.1876(21)\text{GeV/c}^2 \qquad \text{value from PDT 2020} \tag{10.9}$$

- Weinberg's mixing angle

The classical method: Forward-Backward (FB) asymmetry in fermion pair production in e^+e^- annihilation.



$$\frac{d\sigma(s)}{d\cos\theta} = \sigma(s)\left[\frac{3}{8}(1+\cos^2\theta) + A_{FB}^f(s)\cos\theta\right] \tag{10.10}$$

At $s = m_Z^2$ (at the Z-boson peak) at tree level in SM

$$A_{FB}^f = \frac{3}{4} A_e A_f \,, \qquad\qquad A_f \equiv 2 \frac{g_V^f g_A^f}{(g_V^f)^2 + (g_A^f)^2} \tag{10.11}$$

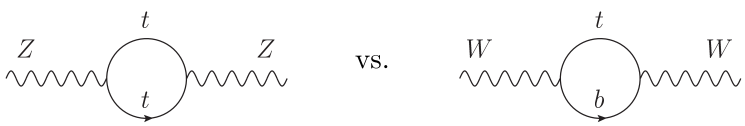
where $g_{A,V}^f$ are defined in (10.2).

$$\frac{g_V^f}{g_A^f} = 1 - 4|Q_f| \sin^2\theta_W \qquad \Rightarrow \qquad \sin^2\theta_W \simeq 0.23 \tag{10.12}$$

With quantum corrections, the analysis becomes more complicated. By definition

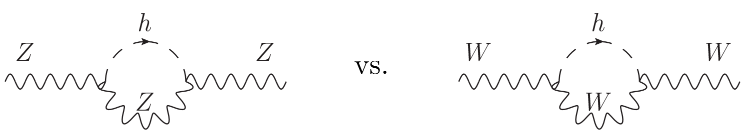
$$\sin^2\theta_W \stackrel{!}{=} 1 - \frac{m_W^2}{m_Z^2} \tag{10.13}$$

but m_W^2 and m_Z^2 are renormalized slightly differently, so $\sin^2\theta_W$ is also affected.
One has to compare



$$\Delta\rho \propto m_t^2 \tag{10.14}$$

and also



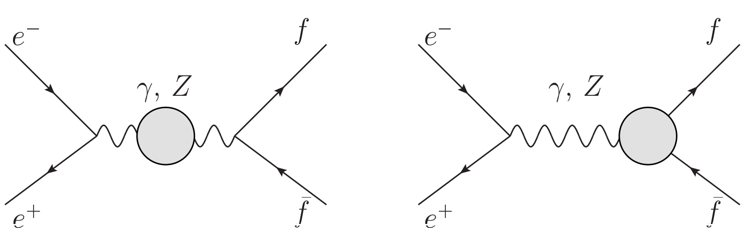
$$\Delta\rho \propto \ln m_h^2 \tag{10.15}$$

Difference in self-energies: $\Delta\rho$ — "ρ-parameter".

Then

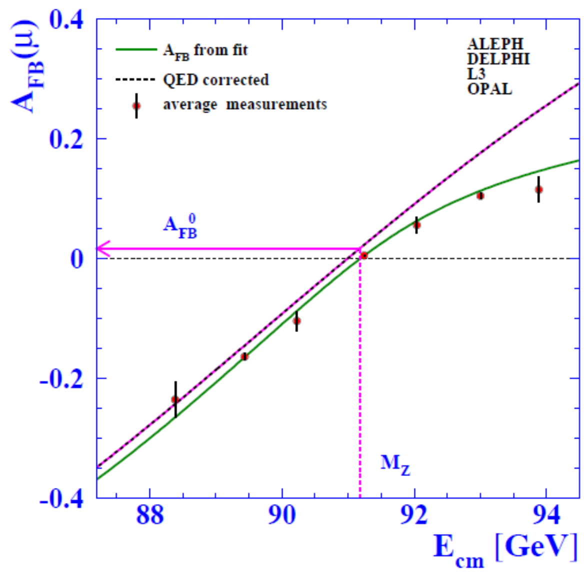
$$\sin^2\theta_W \stackrel{!}{=} 1 - \frac{m_W^2}{m_Z^2} = \frac{g'^2}{g^2 + g'^2}\left(1 - \frac{c_W^2}{s_W^2}\Delta\rho\right) \tag{10.16}$$

Similar contributions one has to take into account in the amplitude



$$\Delta A \propto \ln m_h^2 \quad \text{or} \quad \propto m_t^2 \tag{10.17}$$

and also take into account interference of γ and Z (important off-peak). From the fit to the energy dependence



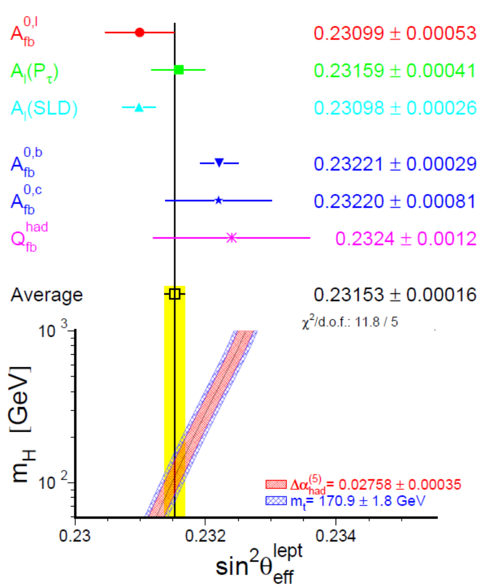
one obtained (LEP)

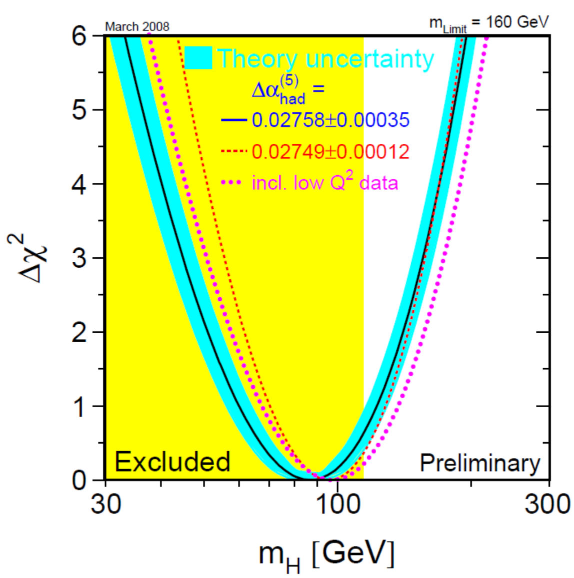
$$\sin^2\theta_W^{\text{eff}} = 0.23153 \pm 0.00016 \qquad \leftarrow \text{ defined through tree relation to } A_{FB} \tag{10.18}$$

In more modern studies one usually prefers to give the value referring of $\overline{\text{MS}}$ renormalization scheme at the scale $\mu = m_Z$:

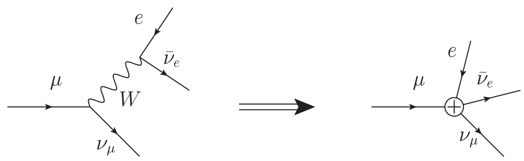
$$\sin^2\theta_W = 0.23122(4) \qquad \text{PDT 2020} \tag{10.19}$$

• Note that quantum corrections to FB asymmetry turn out to be $\propto m_t^2$ or $\propto \ln m_h^2$, so that they depend strongly on the top quark mass and only weakly on the higgs mass. This feature is general. As the result, top quark mass $m_t \sim 175$ GeV was known (from self-consistency of different observables) long before its discovery, but for higgs mass there were only estimates, e.g.,





- Muon decay



$$\tau_\mu^{-1} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left(1 - \frac{8m_e^2}{m_\mu^2}\right) \qquad \Rightarrow \qquad G_F = 1.1663787(6) \times 10^{-5}\ \text{GeV}^{-2} \qquad \text{PDT 2020} \tag{10.20}$$

Comparing to the calculation in SM

$$\frac{1}{m_W^2 - p^2} \mapsto \frac{1}{m_W^2} \qquad \Rightarrow \qquad G_F = \frac{\pi}{\sqrt{2}m_W^2}\frac{\alpha}{\sin^2\theta_W} \qquad \Leftarrow \qquad e^2 = 4\pi\alpha = g^2\sin^2\theta_W \tag{10.21}$$

The latest value (global fit for many measurements)

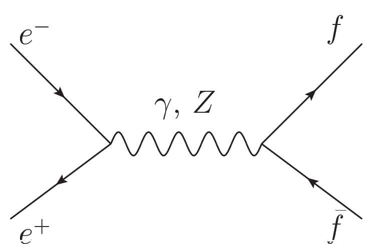
$$m_W = 80.379(12)\ \text{GeV} \qquad\qquad \text{PDT 2020} \tag{10.22}$$

- Also

$$\begin{aligned} & m_W = \frac{1}{2}gv \\ \Rightarrow \quad & G_F = \frac{1}{4\sqrt{2}m_W^2}\frac{4\pi\alpha}{\sin^2\theta_W} = \frac{1}{4\sqrt{2}m_W^2}\frac{g^2\sin^2\theta_W}{\sin^2\theta_W} = \frac{1}{\sqrt{2}v^2} \\ \Rightarrow \quad & v \simeq 246.22\ \text{GeV} \end{aligned} \tag{10.23}$$

- Z-width and the number of neutrinos

Z-boson is not a stable particle. The height of the peak in the $e^+e^- \to$ hadrons is determined by the sum of the decay rates in different channels. Close to the peak

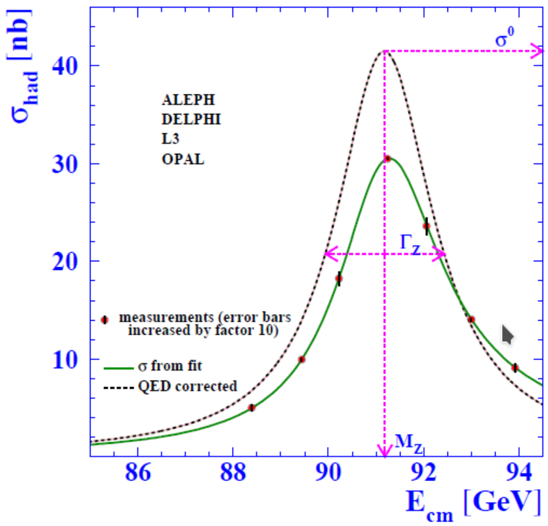


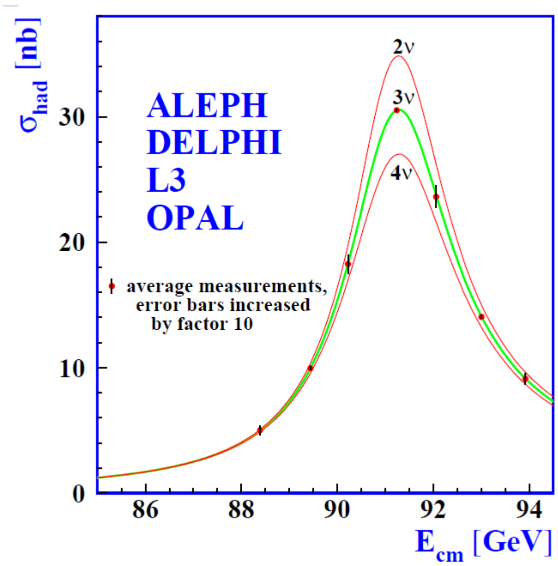
$$\sigma(s) = \frac{12\pi}{m_Z^2}\frac{s\,\Gamma_{ee}\,\Gamma_{ff}}{(s - m_Z)^2 + s^2\Gamma_Z^2/m_Z^2} \tag{10.24}$$

(here only Z-boson contribution, no photon), where Γ_Z is the total decay rate

$$\Gamma_Z = \underbrace{3\Gamma_{\ell\ell}}_{\text{leptons}} + \underbrace{\Gamma_{\text{had}}}_{\text{quarks}} + \underbrace{\Gamma_{\text{inv}}}_{\text{neutrinos}} \tag{10.25}$$

Neutrinos cannot be detected but Γ_{inv} is proportional to their number. One observes





with the fit on the number of neutrinos giving

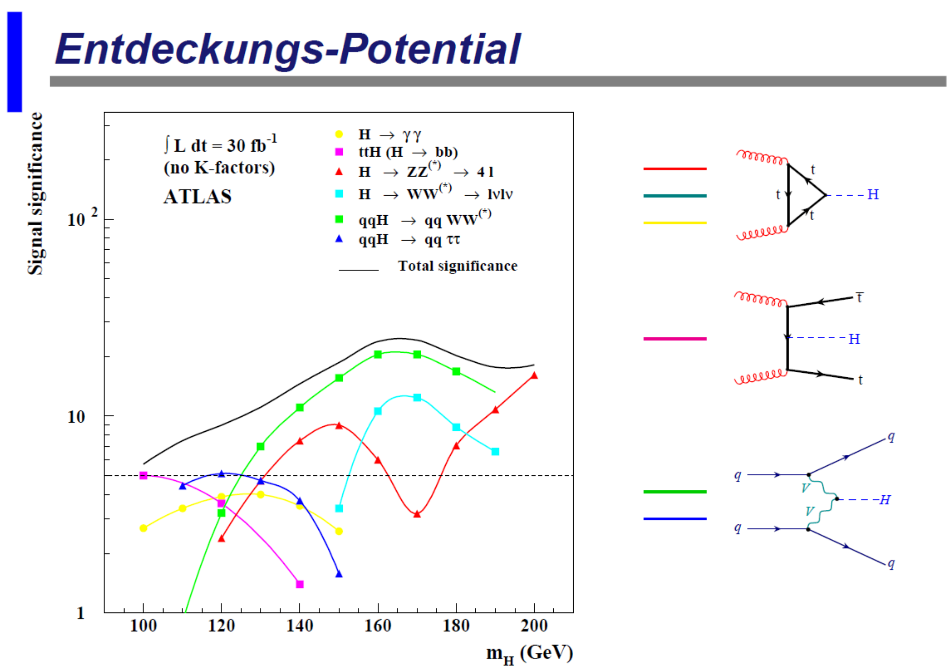
$$N_\nu = 2.9840 \pm 0.0082 \tag{10.26}$$

Thus there exist three lepton families and for self-consistence of the SM also three quark families. There is no room left for another, fourth, family!

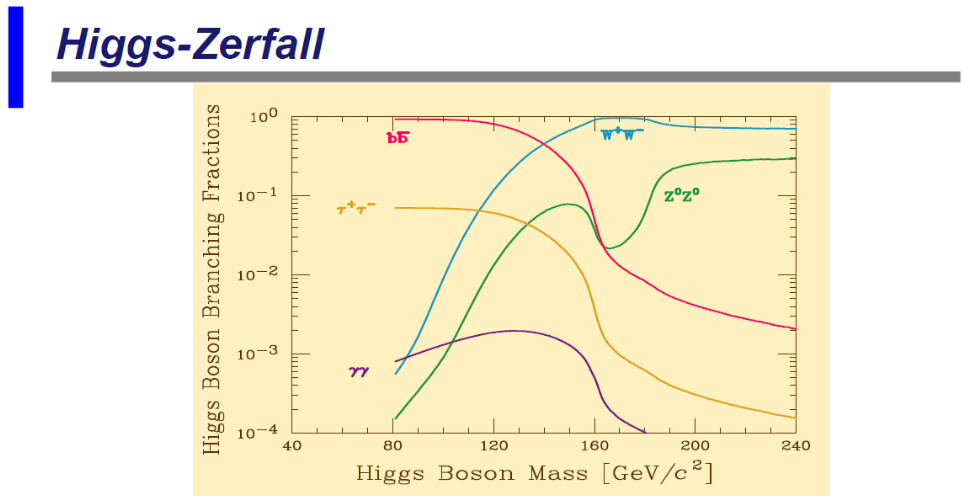
- Higgs particle

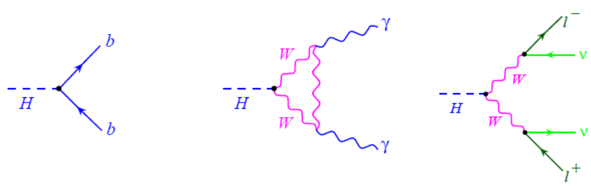
Higgs particle is very elusive and in order to find it one had build a new accelerator (LHC) and look very carefully at all possibilities Higgs could be produced, and all possible decay channels.

— **Higgs production at LHC:** $pp \to h + X$

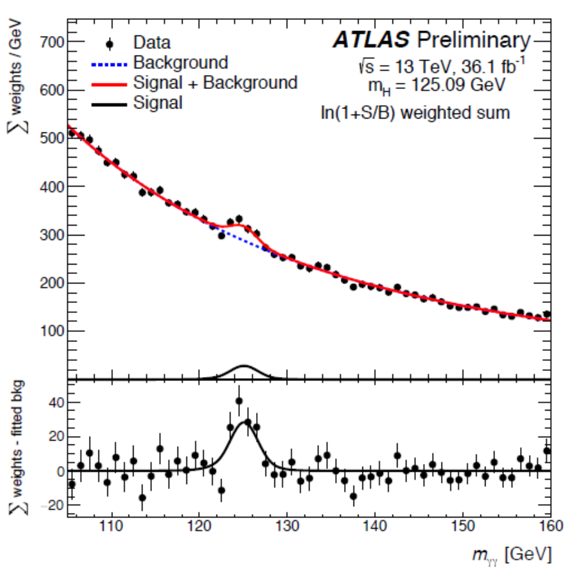


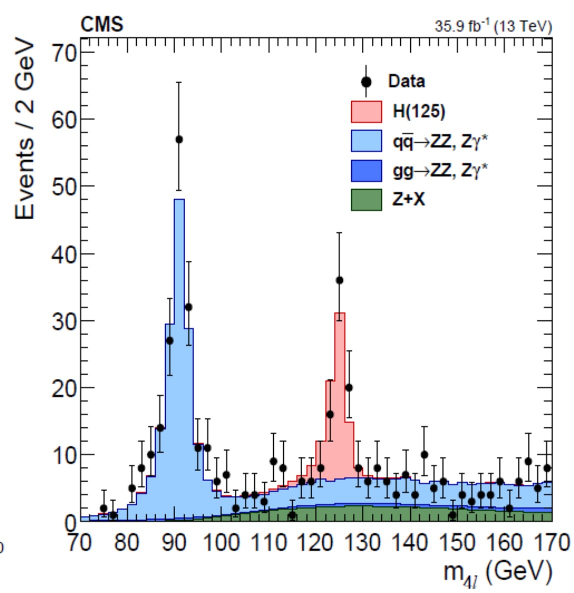
— **Higgs decays:**



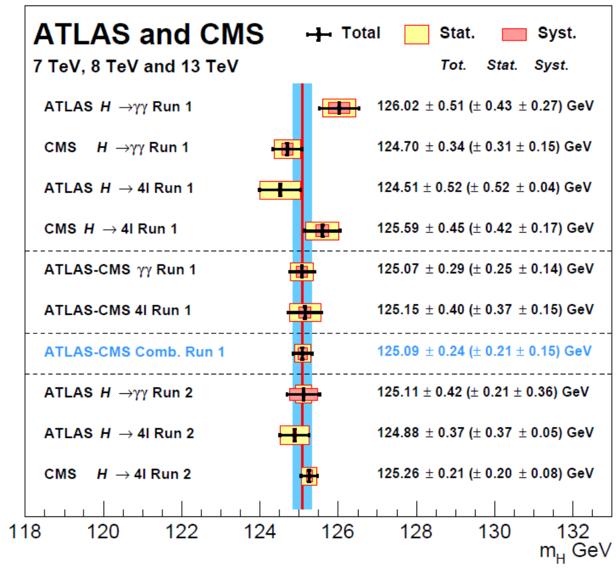


— **Higgs discovery 2012 and eight years after ...**





— **Higgs mass summary, PDT 2019**



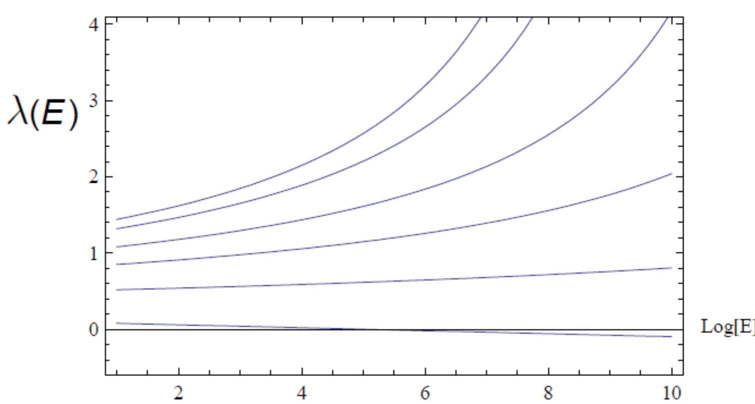
$$\boxed{m_h = 125.10 \pm 0.14 \text{ GeV}} \tag{10.27}$$

What does it tell us? Is m_h too large or too small or just right?
Remind scalar potential

$$V(\phi) = -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 \;\Rightarrow\; \mu^2 h^2 + \lambda v h^3 + \frac{1}{4}\lambda\, h^4 \equiv \frac{1}{2}m_h^2 h^2 + \sqrt{\frac{\lambda}{2}}m_h h^3 + \frac{1}{4}\lambda\, h^4 \tag{10.28}$$

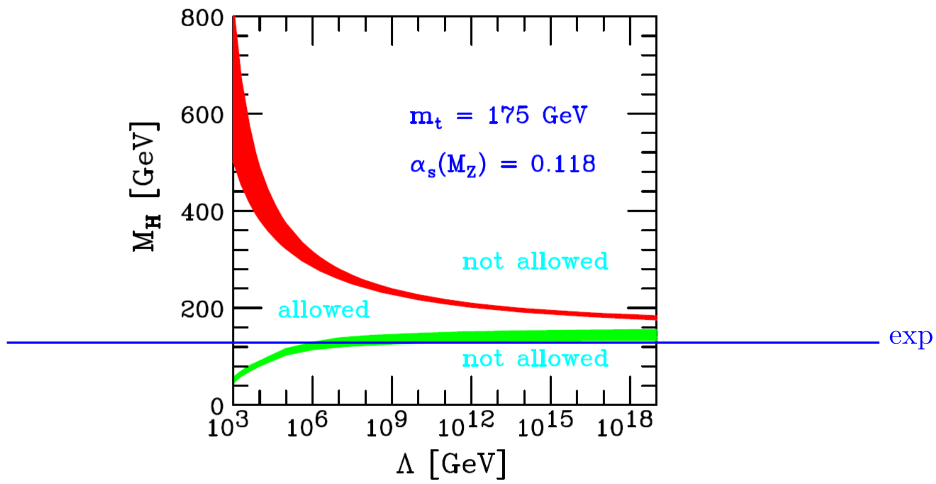
$$v = \left(\frac{\mu^2}{\lambda}\right)^{1/2} \qquad\qquad m_h = \sqrt{2}\mu^2 = \sqrt{\frac{\lambda}{2}}v \tag{10.29}$$

Taking into account quantum corrections $\lambda \to \lambda(\mu)$ (running coupling) [coupling to top quark is also important] and many things can happen:



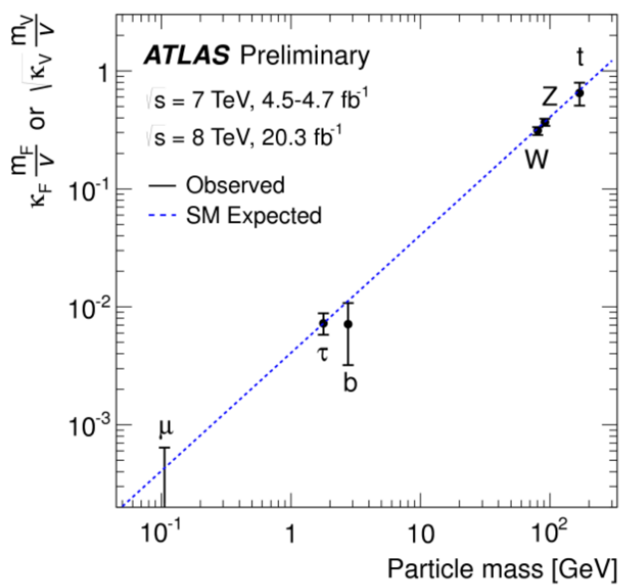
Depending on the initial value at low scale (experiment), the coupling can
— rise slowly
— become infinite at some point ("zero charge" problem)
— become negative (vacuum stability problem)

If a problem occurs, it signals that the SM **must** be modified and replaced by a new more fundamental theory at higher energies. What we know now:



so that it seems that SM could survive as a self-consistent QFT to very large energies.
[This does not mean that it necessarily remains the correct theory].

- SM predicts that higgs couplings to fermions are proportional to the mass:



— OK for now, but low accuracy

Lect. 22

10.2 CKM mixing matrix

- Standard parametrization

$$V_{ij} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{10.30}$$

where

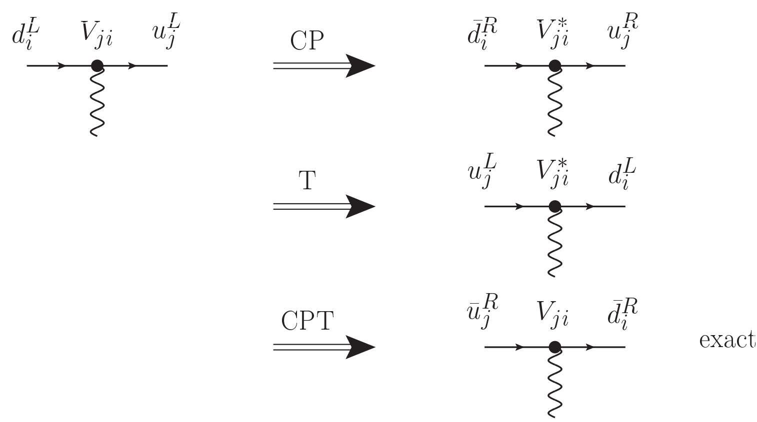
$$c_{ij} = \cos\theta_{ij}\,, \qquad s_{ij} = \sin\theta_{ij}\,, \qquad i,j = 1,2,3 \quad \text{(families)} \tag{10.31}$$

— three Euler angles and one phase.

- ○ The phase is attributed to mixing between 1-st and 3-rd generations (convention).
- ○ Experimentally (we do not know why)

$$s_{13} \ll s_{23} \ll s_{12} \ll 1 \tag{10.32}$$

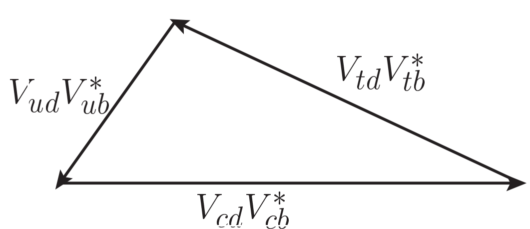
• Quarks from different families get mixed by V, antiquarks by $V^\dagger$. If $V \neq V^\dagger$ (complexity) antiquarks are different from quarks with respect to CP and T trafos.
How these transformations operate:



• Unitarity triangle

$$
\begin{aligned}
&V\,V^\dagger = \mathbb{1} \qquad \Rightarrow \qquad \text{six "triangle" relations}\\
&V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0\\
&\ldots
\end{aligned} \tag{10.33}
$$

— a sum of three complex numbers equals zero:

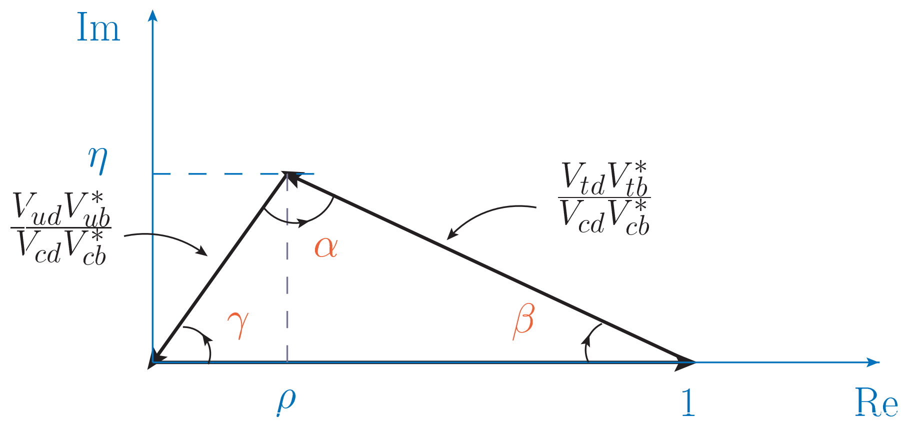


area $= \frac{1}{2}J$

$$\text{Jarlskog invariant:} \qquad J = \ \mathrm{Im}\big(V_{ij}V_{kl}V_{il}^*V_{kj}^*\big) \tag{10.34}$$

Rescaled unitarity triangle:

$$\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} + 1 + \frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*} = 0 \tag{10.35}$$



(10.36)

Determination and over-determination of the CKM parameters — a huge experiment/theory effort in the past 30 years; a Nobel prize for Kobayashi & Maskawa 2008

Prize motivation: "for the discovery of the origin of the broken symmetry which predicts the existence of at least three families of quarks in nature."

A lion share of the info about CKM comes from weak decays of B-mesons:

- LHCB experiment at CERN: $\sim 10^{11}$ detected $b\bar{b}$ pairs

$$pp \to b + \bar{b} + X$$

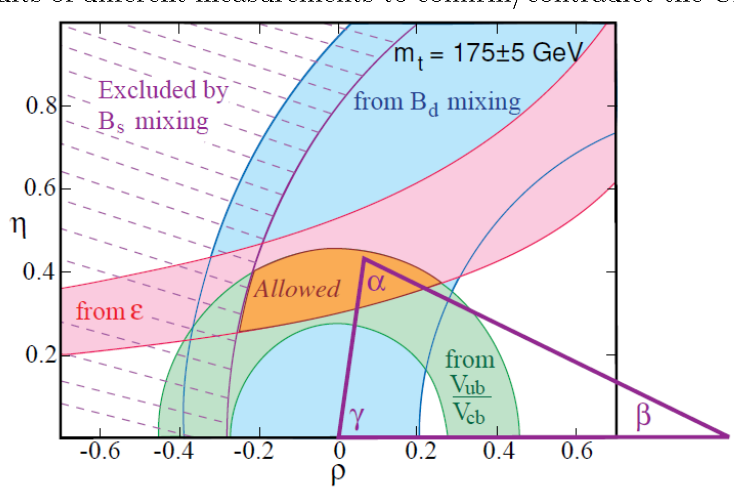
- dedicated "B-factories": BABAR (SLAC) and BELLE (KEK)

$$e^+e^- \to \Upsilon \ (= b\bar{b} \text{ resonances}) \to B_q + \bar{B}_q \qquad\qquad B_q = \bar{b}q\,, q = u, d, s$$

$\sim 10^9$ detected $b\bar{b}$ pairs at BELLE

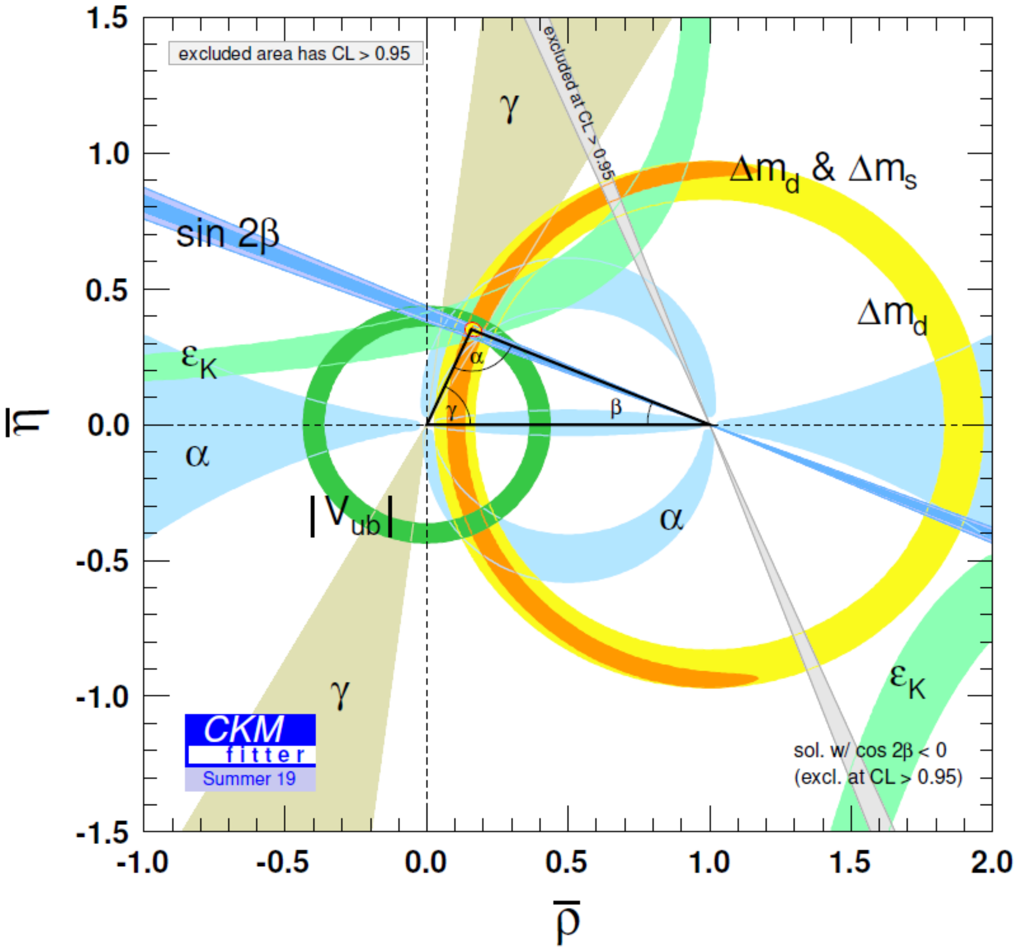
Basic idea: confront results of different measurements to confirm/contradict the CKM mixing idea,

e.g.



— this was the situation about 20 years ago...

. . . and this is how it looks today:

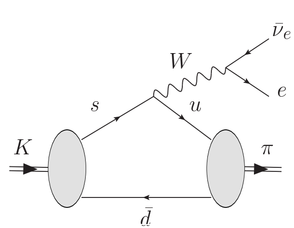


(CKM fitter, May 2020)

Some highlights:

- Sides of the triangle (magnitude of CKM matrix elements)

$$
\begin{array}{llll}
(a) & |V_{ud}| = 0.97425(22) & \Leftarrow & \text{nuclear beta-decays} \\
(b) & |V_{us}| = 0.2252(9) & \Leftarrow & K_L^0 \to \pi e \bar{\nu}_e
\end{array}
$$



$$\propto |V_{us}| f_+(0)$$
$$f_+(0) = 0.9644(49)[\text{lattice QCD}]$$

$$(c) \quad |V_{cd}| = 0.229(6)(24) \quad \Leftarrow \quad D \to K \ell \bar{\nu}_\ell, \pi \ell \bar{\nu}_\ell \tag{10.37}$$

For the mixing angles involving b-quark there is some discrepancy between different methods:

$$\begin{array}{lll} \circ & B \to X_u \ell \bar{\nu}_\ell & \text{"inclusive", sum over all final states} \\ \circ & B \to \pi \ell \bar{\nu}_\ell & \text{"exclusive", a particular final state} \end{array} \tag{10.38}$$

[theory treatment is very different]

$$\begin{array}{lll} (d) & |V_{cb}|^{incl} = (42.2 \pm 0.8) \times 10^{-3} & |V_{cb}|^{excl} = (38.5 \pm 0.9) \times 10^{-3} \\ (e) & |V_{ub}|^{incl} = (4.25 \pm 0.12\,^{+0.15}_{-0.14} \pm 0.23) \times 10^{3} & |V_{ub}|^{excl} = (3.70 \pm 0.10 \pm 0.12) \times 10^{-3} \end{array} \tag{10.39}$$

This discrepancy, most likely, is not a defect of the CKM construction but rather an indication that we do not understand effects of strong interactions in B-decays as well as we would like to.

- CKM phases — from $\cancel{CP}$ observables
- Discovery of CP-violation (1967): $K_0 \leftrightarrow \bar{K}_0$-oscillations

Strong interaction eigenstates:

$$K^0 = \bar{s}\, d \quad (S = +1) \qquad\qquad \bar{K}^0 = \bar{d}\, s \quad (S = -1) \tag{10.40}$$

CP trafo:

$$\mathrm{CP}\,|K^0\rangle = -|\bar{K}^0\rangle \qquad\qquad \mathrm{CP}\,|\bar{K}^0\rangle = -|K^0\rangle \tag{10.41}$$

Therefore, CP-eigenstates:

$$\begin{aligned} CP = +1: &\qquad |K_1\rangle = \frac{1}{\sqrt{2}}\Big(|K^0\rangle - |\bar{K}^0\rangle\Big) \\ CP = -1: &\qquad |K_2\rangle = \frac{1}{\sqrt{2}}\Big(|K^0\rangle + |\bar{K}^0\rangle\Big) \end{aligned} \tag{10.42}$$

If CP is conserved, K_1^0 can decay in two pions, $K_1^0 \to \pi\pi$ $(CP = +1)$ but K_2^0 can only decay in three pions, $K_2^0 \to \pi\pi\pi$ $(CP = -1)$, which is much slower because of a small phase space. Hence different life times:

$$\begin{aligned} \text{"long living" } K_2: &\qquad \tau_L = (5.116 \pm 0.020) \times 10^{-8}\mathrm{s} \\ \text{"short living" } K_1: &\qquad \tau_S = (8.953 \pm 0.005) \times 10^{-11}\mathrm{s} \end{aligned} \tag{10.43}$$

Experimentally (1967), the decay $K_2 \to \pi\pi$ was, however, observed, with a small branching fraction Br$\sim 10^{-3}$. This proves that CP symmetry is broken and the "true" physical eigenstates of neutral kaons include a (small) admixture of the state with opposite CP:

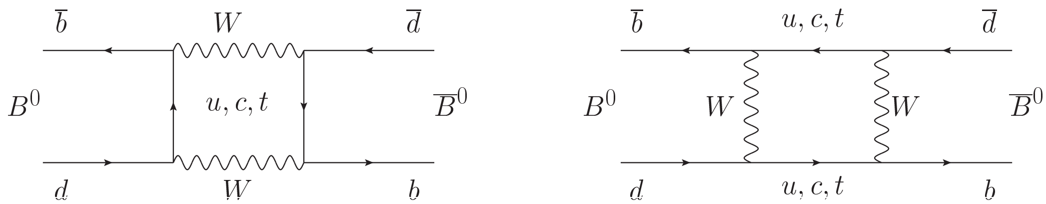
$$|K_S^0\rangle = \frac{1}{\sqrt{1+|\epsilon|^2}}\Big(|K_1\rangle + \epsilon|K_2\rangle\Big) \qquad |K_L^0\rangle = \frac{1}{\sqrt{1+|\epsilon|^2}}\Big(|K_2\rangle + \epsilon|K_1\rangle\Big) \tag{10.44}$$

with $\epsilon \ll 1$. The value of ϵ is a measure of CP-violation (one of many). The experimental value is

$$|\epsilon_K| = (2.23 \pm 0.01) \times 10^{-3} \tag{10.45}$$

- Analogous effect was found in 2000 in the $B_d^0 \leftrightarrow \bar{B}_d^0$ system.
- What is the origin of $B_d^0 \leftrightarrow \bar{B}_d^0$ in the SM?

—second-order effect in weak interaction:



(10.46)

The transition can be described by an effective local operator insertion

$$\langle \overline{B}_d^0 | H_W | B_d^0 \rangle \neq 0 \tag{10.47}$$

Schrödinger Eq. for unstable particle (at rest):

$$i\frac{d}{dt}|\Psi\rangle = \left(m - \frac{i}{2}\Gamma\right)|\Psi\rangle \qquad \Rightarrow \qquad |\Psi(t)\rangle = e^{imt}e^{-\frac{1}{2}\Gamma t}|\Psi(t=0)\rangle \tag{10.48}$$

Generalization for a two-component system:

$$\begin{aligned} i\frac{d}{dt}\begin{pmatrix} B^0 \\ \bar{B}^0 \end{pmatrix} &= \left(M - \frac{i}{2}\Gamma\right)\begin{pmatrix} B^0 \\ \bar{B}^0 \end{pmatrix} \qquad\qquad M = M^\dagger, \quad \Gamma = \Gamma^\dagger, \\ &= \begin{pmatrix} H_{11} & H_{21} \\ H_{12} & H_{22} \end{pmatrix}\begin{pmatrix} B^0 \\ \bar{B}^0 \end{pmatrix} = \begin{pmatrix} m_{11} - \frac{i}{2}\Gamma_{11} & m_{12} - \frac{i}{2}\Gamma_{12} \\ m_{21} - \frac{i}{2}\Gamma_{21} & m_{22} - \frac{i}{2}\Gamma_{22} \end{pmatrix}\begin{pmatrix} B^0 \\ \bar{B}^0 \end{pmatrix} \end{aligned} \tag{10.49}$$

Hermiticity:

$$M = M^\dagger, \quad \Gamma = \Gamma^\dagger \qquad \Rightarrow \qquad m_{21} = m_{12}^*\,, \quad \Gamma_{21} = \Gamma_{12}^* \tag{10.50}$$

CPT symmetry:

$$m_{11} = m_{22} \equiv m\,, \qquad\qquad \Gamma_{22} = \Gamma_{11} \equiv \Gamma \tag{10.51}$$

CP-breaking allows for

$$\cancel{CP} \quad \Rightarrow \quad m_{21} = m_{12}^* \neq m_{12}\,, \qquad \Gamma_{21} = \Gamma_{12}^* \neq \Gamma_{12} \tag{10.52}$$

Diagonalizing this matrix one obtains physical mass eigenstates ("light" and "heavy") with the corresponding decay widths

$$\begin{aligned} |B_L\rangle &= p|B^0\rangle + q|\bar{B}^0\rangle\,, && \text{with } \; m_L,\; \Gamma_L \\ |B_H\rangle &= p|B^0\rangle - q|\bar{B}^0\rangle\,, && \text{with } \; m_H,\; \Gamma_J \\ p, q &\in \mathbb{C}\,, \qquad |p|^2 + |q|^2 = 1 && \end{aligned} \tag{10.53}$$

During the time evolution, therefore, B^0 slowly goes over in $\bar{B}^0$ and back. The transition probability as a function of time:

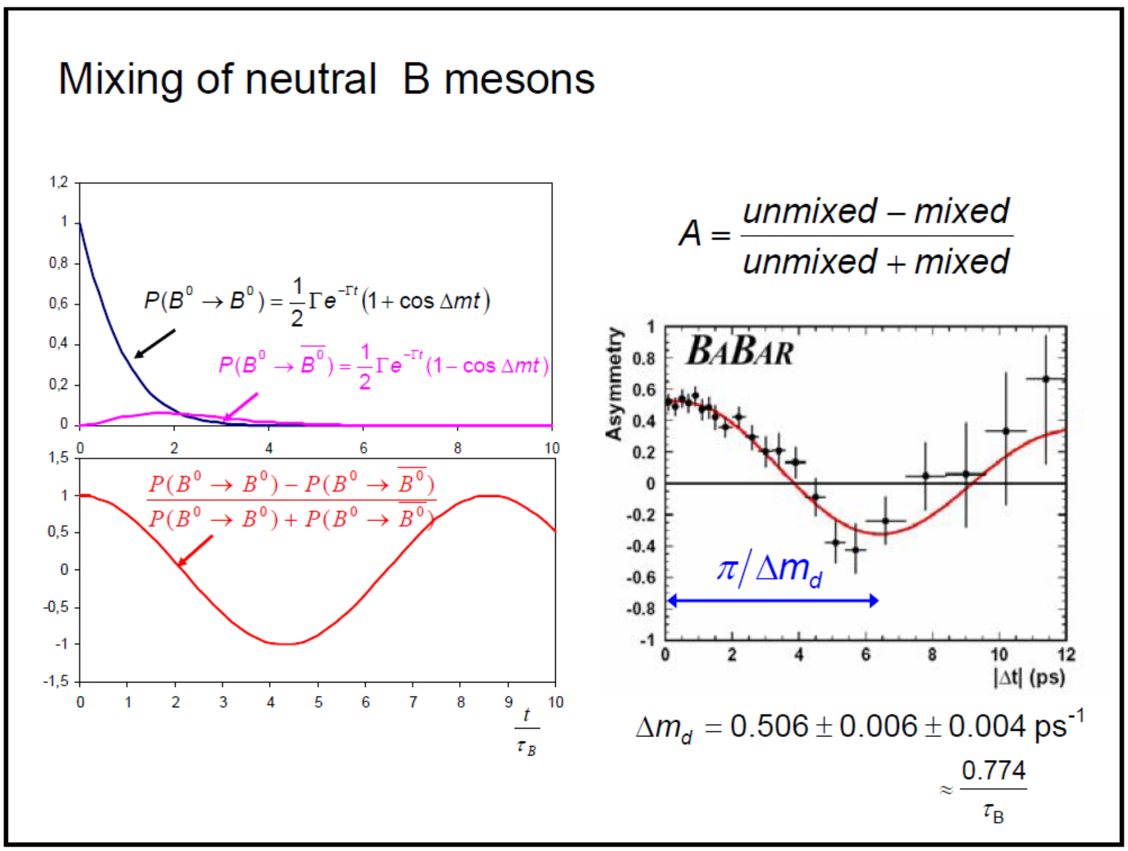
$$P_{B^0\to\bar{B}^0}(t) = |\langle\bar{B}^0(t)|B^0(0)\rangle|^2 = \frac{1}{4}\left|\frac{q}{p}\right|^2 \left|e^{-im_H t}e^{-\frac{1}{2}\Gamma_H t} - e^{-im_L t}e^{-\frac{1}{2}\Gamma_L t}\right|^2$$
$$= \frac{1}{4}\left|\frac{q}{p}\right|^2 \left[e^{-\Gamma_L t} + e^{-\Gamma_H t} - 2e^{-\frac{1}{2}(\Gamma_H+\Gamma_L)t}\cos(\Delta m t)\right] \qquad (10.54)$$

with

$$\Delta m = m_H - m_L = 0.502(7)\ \text{ps}^{-1} \qquad (\sim 3.2\cdot 10^{-4}\ \text{eV} \qquad (10.55)$$

The standard model prediction (from box diagrams above) is

$$\left|\frac{q}{p}\right| - 1 \simeq 4\pi\frac{m_c^2}{m_t^2}\sin\beta \qquad (\sim 5\cdot 10^{-4}) \qquad (10.56)$$



from: P. Uwer, "Quark mixing and CKM matrix"

- Studies of $\not{CP}$ are very broad and go in three main directions:

 1. $\not{CP}$ in mixing of neutral mesons

$$P(B^0\to\bar{B}^0) \overset{\not{CP}}{\neq} P(\bar{B}^0\to B^0) \qquad \Rightarrow \qquad \left|\frac{q}{p}\right| \neq 1 \qquad (10.57)$$

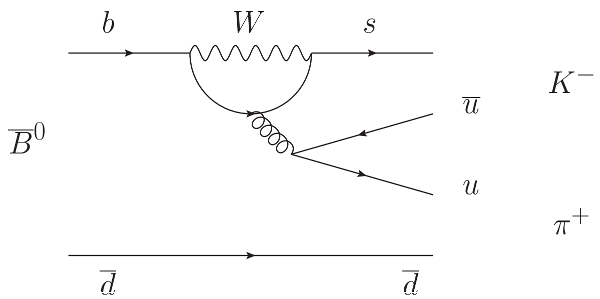
2. "Direct" $\not{CP}$ in meson decays:

$$P(\bar{B}^0 \to \overline{\text{given final state}}) \neq P(B^0 \to \text{ given final state}) \tag{10.58}$$

Example: BABAR 2004

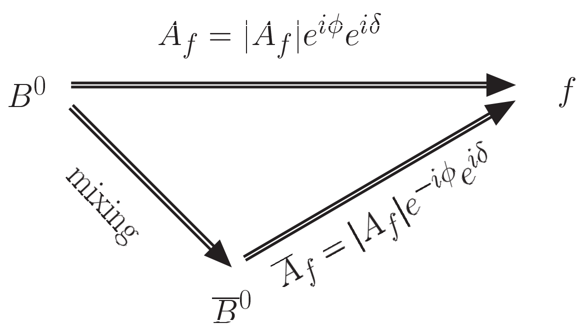
$$A_{\rm CP} = \frac{N(B^0 \to K^+\pi^-) - N(\bar{B}^0 \to K^-\pi^+)}{N(B^0 \to K^+\pi^-) + N(\bar{B}^0 \to K^-\pi^+)} \tag{10.59}$$

This asymmetry is generated by "penguin" diagrams like



$$A_{\rm CP} \sim \underbrace{\sin(\phi_1 - \phi_2)}_{\text{weak phases}}\ \underbrace{\sin(\delta_1 - \delta_2)}_{\text{strong phases}} \tag{10.60}$$

3. Interference between mixing and decay



f : final state with fixed CP (10.61)

$\rightsquigarrow$ $\sin 2\beta$ from $B^0 \to J/\Psi K_S$ ("golden mode")
$\sin 2\alpha$ from $B^0 \to \pi^+\pi^-$ etc.

Lect. 23

11 Some open issues

The SM works very well. Why look beyond?

- Neutrino mixing/masses
- Small deviations are (maybe) appearing...
- Theory issues (fine tuning...)
- "Big" questions — baryon asymmetry, dark matter, etc ($\Leftarrow$ Cosmology)

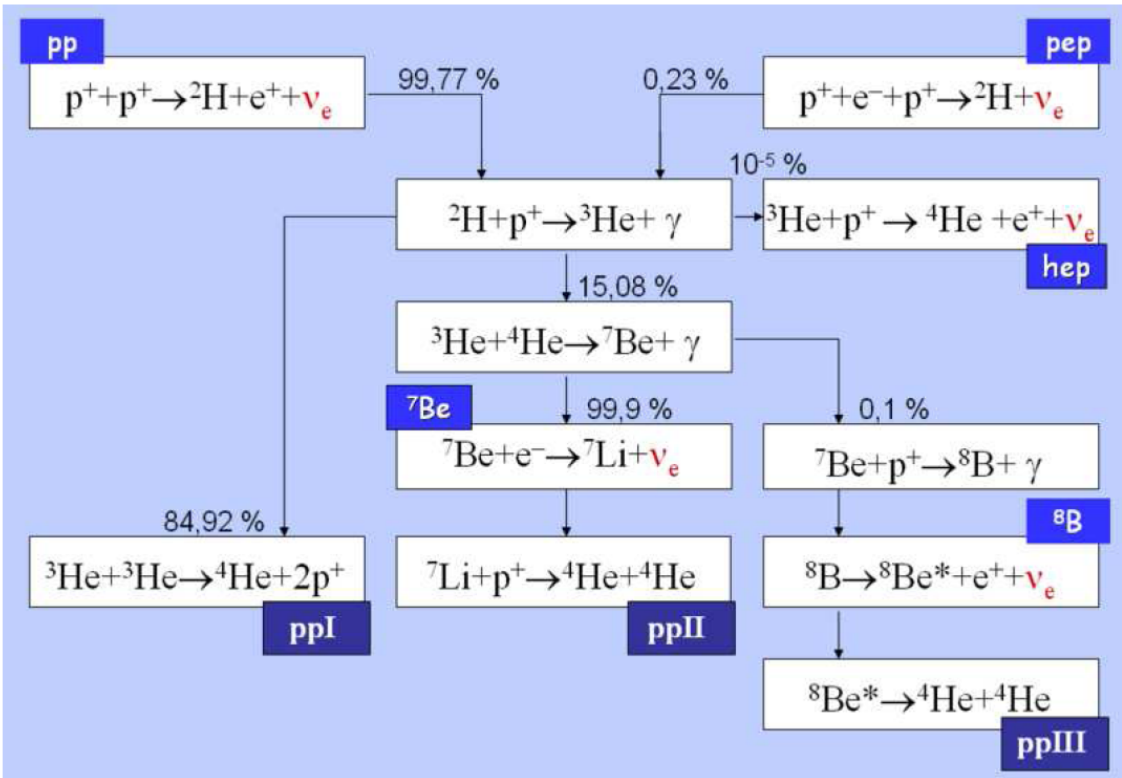
I will give a quick tour.

11.1 Neutrino mixing

• A major source of neutrinos on Earth is the Sun. Neutrinos are produced in the Sun via a complicated chain of reactions, roughly

$$4p \longrightarrow {}^4\mathrm{He} + 2e^+ + 2\nu_e \tag{11.1}$$

The fundamental process is an inverse β-decay: $p \to ne^+\nu_e$ or, on the quark level, $u \to de^+\nu_e$. The actual production process is shown on the following diagram:



This process is complicated but well understood. The amount of neutrinos that the Sun sends to us and their energy spectrum are known rather well. The problem is (was) that the amount of neutrinos detected on the Earth turns out to be much less than expected. Thus, either smth is happening with neutrinos on the way to us, or our sun models are badly wrong...
This discrepancy has become known as the solar neutrino problem.

• The produced neutrino ν_e is the weak interaction eigenstate but not necessarily a mass eigenstate (B. Pontekorvo, 1957) if there some mixing in neutrino sector between ν_e, ν_μ, ν_τ of the type that we have seen in the quark sector. Denote neutrino mass eigenstates by ν_1, ν_2, ν_3 and assume that (for two flavors as example)

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{11.2}$$

The electron neutrino produced in the sun is a linear combination of the mass eigenstates

$$\nu_e = cos\theta\ \nu_1 + \sin\theta\ \nu_2 \tag{11.3}$$

On the way to the Earth the wave functions describing mass eigenstates acquire a phase

$$\nu_1(t) = \nu_1(0)e^{iE_1 t} \qquad\qquad \nu_2(t) = \nu_2(0)e^{iE_2 t}$$

$$E_1 = \sqrt{m_1^2 + |\vec{p}|^2} \qquad\qquad E_2 = \sqrt{m_2^2 + |\vec{p}|^2} \tag{11.4}$$

and if $E_1 \neq E_2$ ($m_1 \neq m_2$) the content of ν_e in terms of ν_1 and ν_2 will change with time:

$$\begin{aligned}\nu_e(t) &= cos\theta\ \nu_1(t) + \sin\theta\ \nu_2(t)\\ &= cos\theta\ \nu_1(0)e^{iE_1 t} + \sin\theta\ \nu_2(0)e^{iE_2 t}\end{aligned} \tag{11.5}$$

This can be expressed in terms of ν_e and ν_μ using

$$\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix} \tag{11.6}$$

so that

$$\begin{aligned}\nu_e(t) &= \cos\theta\big(\cos\theta\,\nu_e(0) - \sin\theta\,\nu_\mu(0)\big)e^{iE_1 t} + \sin\theta\big(\sin\theta\,\nu_e(0) + \cos\theta\,\nu_\mu(0)\big)e^{iE_2 t}\\ &= \Big(\cos^2\theta\, e^{iE_1 t} + \sin^2\theta e^{iE_2 t}\Big)\nu_e(0) + \cos\theta\sin\theta\Big(e^{iE_2 t} - e^{iE_1 t}\Big)\nu_\mu(0)\end{aligned} \tag{11.7}$$

Thus (Pontecorvo):
the electron neutrino can oscillate in a muon neutrino if (1) $m_1 \neq m_2$ and (2) $\theta \neq 0$.

The probability for the change of flavor a to flavor b is then

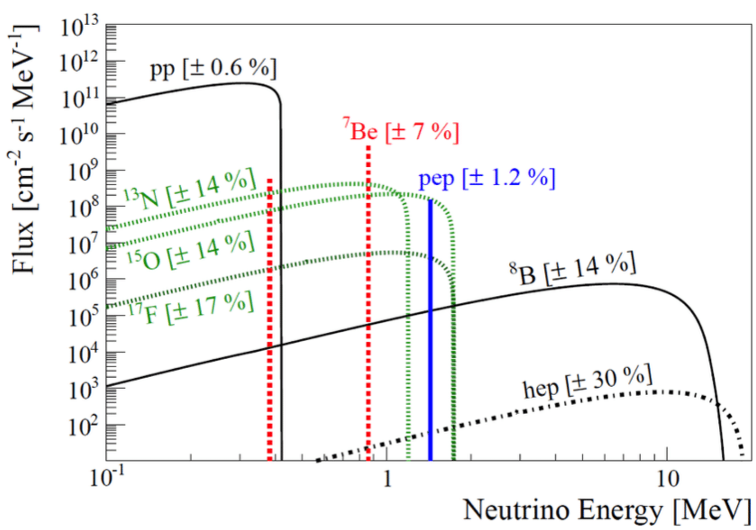
$$\begin{aligned}P(\nu_a \to \nu_b) &= |\langle\nu_b(t)|\nu_a(0)\rangle|^2\\ &= |\cos\theta\sin\theta\Big(e^{iE_2 t} - e^{iE_1 t}\Big)|^2 = \Big(\frac{1}{2}\sin 2\theta\Big)^2\, 2\,[1 - \cos\Delta E t]\end{aligned} \tag{11.8}$$

Here

$$t \simeq \frac{L}{c} \qquad \text{(distance from the sun)}$$

$$\Delta E = \sqrt{m_1^2 + |\vec{p}|^2} - \sqrt{m_2^2 + |\vec{p}|^2} \simeq \frac{1}{2E}(m_1^2 - m_2^2) \tag{11.9}$$

The energy of the neutrinos depends on the production process but generally is in the MeV range:



- <u>Detection on the Earth:</u>

Neutrinos can only be detected via weak interaction, and only process involving electrons but not muons can only detect electron neutrinos. The existing experiments use

$$
\begin{aligned}
\mathrm{CI}^{37} + \nu_e &\to \mathrm{Ar}^{37} + e^- && \text{Davies,} \quad \text{Homestoke} \\
n + \nu_e &\to p + e^- \\
\mathrm{Ga}^{71} + \nu_e &\to \mathrm{Ge}^{71} + e^- && \text{Gallex,} \quad \text{Sage,} \quad \text{GNO} \\
n + \nu_e &\to p + e^- && (11.10)
\end{aligned}
$$

and all find much less ν_e compared to what us produced in the sun. In addition, the SNO experiment had different detection channels sensitive to all three neutrino flavors.

- Other neutrino sources

◦ Atmospheric neutrinos from collisions of cosmic ray particles with the atmosphere

◦ Accelerator neutrinos, e.g., from $pp \to \pi + X \to \mu + \ldots$

◦ Reactor anti-neutrinos (from specially tuned nuclear reactions)

- Our current knowledge about neutrino mixing (PDT 2020)

□ *Atmospheric ν_μ and $\bar{\nu}_\mu$ disappear most likely converting to ν_τ and $\bar{\nu}_\tau$. The results show an energy and distance dependence perfectly described by mass-induced oscillations.*

□ *Accelerator ν_μ and $\bar{\nu}_\mu$ disappear over distances of $\sim$ 200 to 800 km. The energy spectrum of the results show a clear oscillatory behavior also in accordance with mass-induced oscillations with wavelength in agreement with the effect observed in atmospheric neutrinos.*

□ *Accelerator ν_μ and $\bar{\nu}_\mu$ appear as ν_e and $\bar{\nu}_e$ at distances $\sim$ 200 to 800 km.*

□ *Solar ν_e convert to ν_μ and/or ν_τ. The observed energy dependence of the effect is well described by massive neutrino conversion in the Sun matter according to the MSW effect.*

□ *Reactor $\bar{\nu}_e$ disappear over distances of $\sim$ 200 km and $\sim$ 1.5 km with different probabilities. The observed energy spectra show two different mass-induced oscillation wavelengths: at short distances in agreement with the one observed in accelerator ν_μ disappearance, and a long distance compatible with the required parameters for MSW conversion in the Sun.*

MSW: Mihheev-Smirnov-Wolfenstein Effect —neutrino propagation in the sun taking into account the matter density

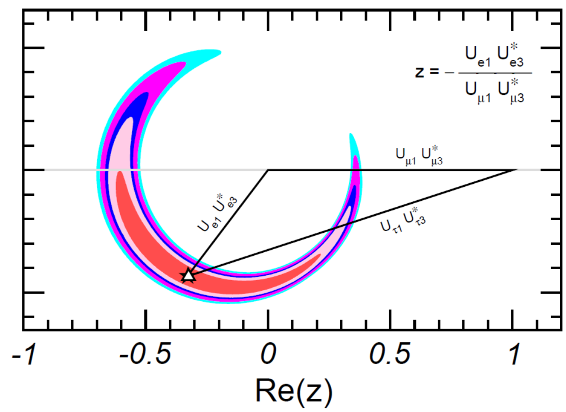
...and here are the current numbers: (PDT 2020)

$$
\begin{aligned}
\Delta m_{21}^2 &= (7.39^{+0.21}_{-0.20}) \cdot 10^{-5}\ \mathrm{eV}^2 \\
\Delta m_{32}^2 &= (2.449^{+0.032}_{-0.030}) \cdot 10^{-3}\ \mathrm{eV}^2 \\
\theta_{12} &= (33.82^{+0.78}_{-0.76})^\circ \\
\theta_{23} &= (48.3^{+1.2}_{-1.9})^\circ \\
\theta_{13} &= (8.61^{+0.13}_{-0.13})^\circ && (11.11)
\end{aligned}
$$

- Constraint on neutrino masses (from kinematics of weak decays)

$$m_{\nu_e} < 1.1\,\text{eV} \qquad\qquad {}^3\text{H} \to {}^3\text{He} + e^- + \bar{\nu}_e \tag{11.12}$$

- Lepton unitarity triangle (assuming three families) ‡



⇝ CP-violation likely also in leptonic sector

- Could neutrinos be Mayorana fermions?

Dirac fermion in Weil rep:

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{11.13}$$

ψ_L and ψ_R have different properties under Lorentz trafos. However, one can show that $\sigma^2\psi_R^*$ transform in the same way as ψ_L so that instead of defining a Dirac bispinor as a pair of left-handed and right-handed fields it is possible to introduce two left-handed fields instead:

$$\Psi = \begin{pmatrix} \psi_L^{(1)} \\ \sigma^2\psi_L^{(2)*} \end{pmatrix} \tag{11.14}$$

The point is that we actually do not need two different fields and can choose

$$\psi_L^1 = \psi_L^2 = \chi \tag{11.15}$$

reducing the number of degrees of freedom. The Dirac mass term then becomes

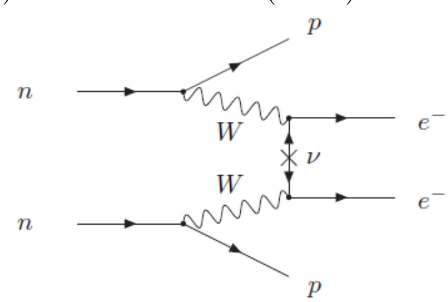
$$m\bar{\Psi}\Psi = m(\psi_R^\dagger\psi_L + \psi_L^\dagger\psi_R) = -m(\chi^T\sigma^2\chi + \text{h.c.}) \tag{11.16}$$

— a Mayorana mass term — and one does not need right-handed particles at all!

If neutrino is a Mayorana particle, neutrino and antineutrino are described by the same field and therefore $\langle\nu(x)\nu(0)\rangle \neq 0$.

‡a major difference to CKM is that one can have three physical phases, not just one, if neutrinos are Mayorana particles (see below).

This can be decided by observation (Mayorana) or non-observation (Dirac) of the neutrinoless double beta decay:



— very hard to measure...

11.2 Anomalous magnetic moment of the muon

- Energy shift of a charged particle in magnetic field

$$E = -\vec{\mu} \cdot \vec{B} \tag{11.17}$$

Classically

$$\vec{\mu} = \frac{q}{2mc}\vec{L} \tag{11.18}$$

In QM

$$\vec{\mu} = g\,\frac{q}{2mc}\vec{S} \qquad\qquad g: \text{ gyromagnetic ratio} \tag{11.19}$$

- For Dirac fermions

$$g = 2 \tag{11.20}$$

▶

Dirac eq. in external field

$$(i\gamma^\mu D_\mu - m)\Psi = 0 \qquad D_\mu = \partial_\mu - ieA_\mu \qquad \vec{\pi} = \vec{p} - e\vec{A} \tag{11.21}$$

To study the nonrelativistic limit $|\vec{p}| \ll m$ and weak fields $eA_0 \ll m$ it is convenient to use Dirac rep. for γ-matrices. In this case the lower two components in the bispinor are v/c suppressed compared to the upper two ones. To see this, make an ansatz

$$\Psi(x) = \begin{pmatrix} \chi(x)\, e^{-ipx} \\ \phi(x)\, e^{-ipx} \end{pmatrix} \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \tag{11.22}$$

and inserting this into the Dirac Eq.

$$\begin{aligned} (1) \quad & i\frac{\partial}{\partial t}\chi = \vec{\sigma} \cdot \vec{\pi}\,\phi + eA_0\chi \\ (2) \quad & i\frac{\partial}{\partial t}\phi = \vec{\sigma} \cdot \vec{\pi}\,\chi + (eA_0 - 2m)\phi \end{aligned} \tag{11.23}$$

From (2) get $\phi = (\vec{\sigma} \cdot \vec{\pi}/2m)\chi$ and inserting in (1):

$$i\frac{\partial}{\partial t}\chi = \left[\frac{(\vec{\sigma} \cdot \vec{\pi})^2}{2m} + eA_0\right]\chi \tag{11.24}$$

Since

$$(\vec{\sigma} \cdot \vec{\pi})^2 = \pi^2 + e\vec{\sigma} \cdot \vec{B} \tag{11.25}$$

get

$$i\frac{\partial}{\partial t}\chi = \left[\frac{(\vec{p} - e\vec{A})^2}{2m} + \frac{e}{2m}\vec{\sigma} \cdot \vec{B} + eA_0\right]\chi \tag{11.26}$$

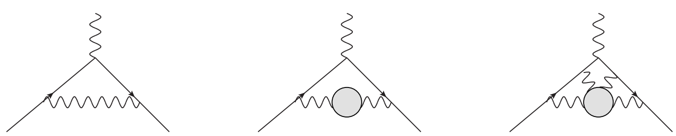
and finally rewrite

$$\frac{e}{2m}\vec{\sigma} \cdot \vec{B} = g\frac{e}{2m}\frac{\vec{\sigma}}{2} \cdot \vec{B} = g\frac{e}{2m}\vec{S} \cdot \vec{B} \qquad \text{with} \quad g = 2 \tag{11.27}$$

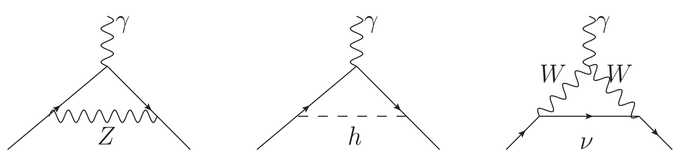
- Deviation from $g = 2$ are due to QED corrections:

$$a_e = \frac{1}{2}(g_e - 2) = \underbrace{\frac{\alpha}{2\pi}}_{\text{J.Schwinger}} - 0.328\ldots\left(\frac{\alpha}{\pi}\right)^2 + 1.182\ldots\left(\frac{\alpha}{\pi}\right)^3 - 1.9144\ldots\left(\frac{\alpha}{\pi}\right)^4 \qquad \text{Kinoshita 2007} \tag{11.28}$$

but not only, as other particles (e.g. quarks) can contribute through loop diagrams:



In addition there are contributions due to weak interactions:



Luckily, all these non-QED corrections are extremely small so that measurements of $(g_e - 2)$ allow one for a very precise determination of electron charge (fine structure constant) that I quoted already:

$$\frac{1}{\alpha} = 137.035999084(21) \qquad\qquad \text{value from PDT 2020} \tag{11.29}$$

This value is (was until very recently . . .) in perfect agreement with other measurements that have comparable accuracy (current champion: Cesium-133 atoms [Parker et al, 1812.04130]).

- The muon is just a heavier copy of the electron, so that the anomalous magnetic moment of the muon can be calculated in precisely the same way. A difference of principle is, however, that contributions of heavy particles (inside loops) are in general proportional to

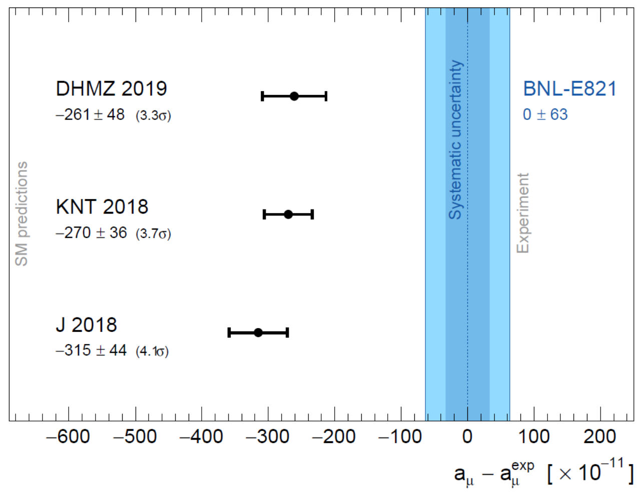
$$\propto \frac{m_{e,\mu}^2}{M^2}$$

(can be shown) and going over from an electron to a muon are enhanced by a factor

$$\frac{m_\mu^2/M^2}{m_e^2/M^2} = \left(\frac{m_\mu}{m_e}\right)^2 \sim 40000 \tag{11.30}$$

$\rightsquigarrow$ weak interaction effects important (calculable)
$\rightsquigarrow$ sensitive to possible "new physics" particles inside loops

Physicists are, therefore, very much intrigued by the disagreement of the calculations with the measurements:



(plot taken from PDT 2020) that persists for many years already. The points show some of the most recent theory calculations, the blue band the recent measurements.

Main theory uncertainty comes from hadronic (quark) contributions inside loops. Improving the current accuracy is a hot research topic.

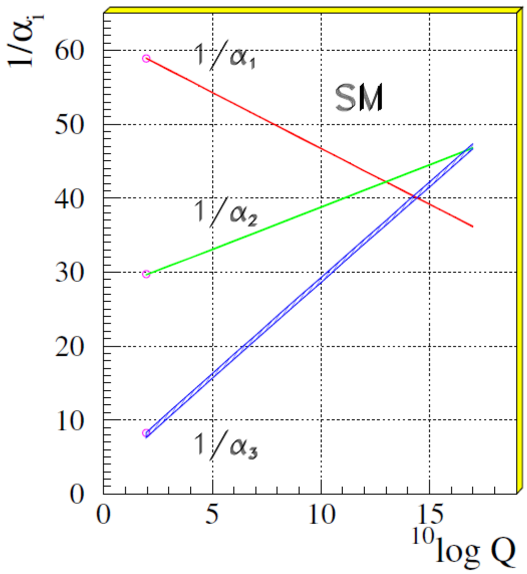
• Deviations from the SM at the 3σ level also show up in some rare B meson decays, but they are not so well established (may disappear with time). *Lect. 24*

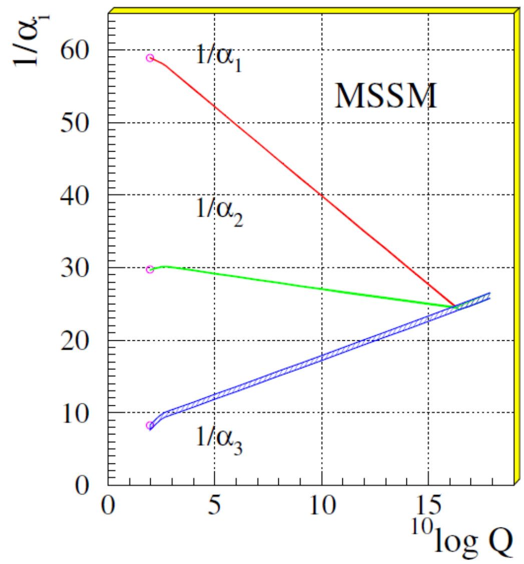
11.3 Theory issues

The SM is a self-consistent QFT, although maybe not as elegant as we would like it to be. If we assume that SM will be substituted by a more fundamental theory at very small distances, there are some issues concerning how this transition can look like. I sketch a few of them.

11.3.1 Unification of couplings

The three coupling constants of known interactions — strong, weak and electrodynamics — become very close in size at energies around $10^{14} - 10^{15}$ GeV, suggesting that at such energies all three interactions become part of a single theory — "Grand unification". If the SM survives without modifications up to such scales, the couplings do not coincide exactly. This inspired speculations that some new particles should show up at scales around 1000 GeV; their contributions modify the β-functions that govern scale dependence of the couplings and one can enforce all three to coincide at one point. The best studied example - the Minimum Supersymmetric extension of the Standard Model (MSSM):





This looks impressive, but, unfortunately, MSSM is practically excluded by the new LHC data — the predicted particles most likely do not exist...

11.3.2 The hierarchy problem

Mass renormalization in a field theory:

$$m_{\rm exp} = \underbrace{m_0}_{\text{divergent}} + \underbrace{\delta m}_{\text{divergent}} \tag{11.31}$$

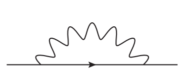
Nobody is worried because infinities parametrize our ignorance of the theory at very large momenta.

However, IF we assume that our theory is correct up to the scale Λ, and calculate contributions up to this scale (i.e., with a rigid cutoff at momenta $> \Lambda$)

$$m_{\rm exp} = \underbrace{m_0(\Lambda)}_{\text{finite}} + \underbrace{\delta m(\Lambda)}_{\text{finite}} \tag{11.32}$$

we would prefer to have $\delta m \lesssim m_0$. What happens in "real life"?

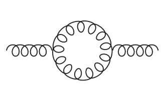
- Fermions

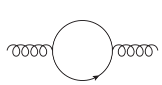


$$\delta m_F \simeq g^2 \int^{\Lambda} d^4k \, \frac{\{k\}}{k^2(k^2 - m_F^2)} \sim g^2 m_F \ln \frac{\Lambda}{m_F} \tag{11.33}$$

Naively one should expect $\delta m_F \sim \Lambda$ from power counting, however, the term $\propto k$ in the numerator vanishes upon integration if the regulator does not break Lorentz symmetry.

- Gauge bosons



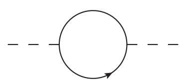


$$\delta m_G \simeq g^2 \int^{\Lambda} d^4k \, \frac{\{k\, k\}}{k^2\, k^2} \sim g^2 m_{\rm loop}^2 \ln \frac{\Lambda}{m_L} \tag{11.34}$$

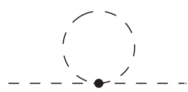
(two powers of k in the numerator either from vertices or from fermion propagators)
Naively one would expect $\delta m_G^2 \sim \Lambda^2$. However, quadratic divergences cancel thanks to gauge symmetry (Ward identity).

☐ unbroken $\longrightarrow$ complete cancellation, $\delta m_G^2 = 0$ (photon)
☐ spont. broken $\longrightarrow$ logarithmic div. allowed, $\delta m_G^2 \sim m^2 \ln \frac{\Lambda}{m}$ (W,Z-bosons)

- Scalars (Higgs in SM)



$$\delta m_h \simeq g^2 \int^{\Lambda} d^4k \, \frac{\{k\,k\}}{(k^2 - m_t^2)\,(k^2 - m_t^2)} \sim g^2\Lambda^2$$



$$\delta m_h \simeq g^2 \int^{\Lambda} d^4k \, \frac{\{1\}}{(k^2 - m_h^2)} \sim g^2\Lambda^2 \tag{11.35}$$

No cancellation of quadratic divergences.

Then for $m_h \sim 100$ GeV requiring $\delta m_h < m_h$ implies $\Lambda < 1000$ GeV (roughly).

Another way to state the problem: If we assume that the SM is correct up to scales $\sim 10^{14}$ GeV (unification of couplings), then (using realistic values of the couplings)

$$\begin{aligned} \text{spin} - \frac{1}{2}: \qquad & \underbrace{m_{\exp}}_{O(100)\,\text{GeV}} = \underbrace{m_0}_{O(100)\,\text{GeV}} + \underbrace{\delta m}_{O(1)\,\text{GeV}} \\ \text{spin} - 0: \qquad & \underbrace{m_{\exp}}_{O(100)\,\text{GeV}} = \underbrace{m_0}_{O(10^{33})\,\text{GeV}} + \underbrace{\delta m}_{O(10^{33})\,\text{GeV}} \end{aligned} \tag{11.36}$$

— a fine tuning or naturalness problem.

't Hooft: existence of light fundamental scalar particles requires additional symmetries

Proposed solutions:
— Supersymmetry
— Extra space-time dimensions, e.g. Randall-Sundrum models
— 'Conformal SM" — spont. symmetry breaking with $m_0 = 0$ and Coleman-Weinberg potential

Nothing convincing so far...

11.4 Baryon asymmetry of the universe

used: V. Rubakov. CERN Yellow Reports: School Proceedings, Vol. 2/2017, CERN-2017-005-SP

- Observe matter but little antimatter in present Universe
- Baryon-to-photon ratio

$$\eta_B = \frac{n_B}{n_\gamma} = 6 \cdot 10^{-10} \qquad \text{(almost constant in time)} \tag{11.37}$$

In early Universe ($T > 10^{12}\,\text{K} = 100\,\text{MeV}$) creation and annihilation of $\bar{q} - q$ pairs should have resulted in

$$n_q \simeq n_{\bar{q}} \simeq n_\gamma$$

Hence

$$\frac{n_q - n_{\bar{q}}}{n_q + n_{\bar{q}}} \sim 10^{-9} \tag{11.38}$$

(In early Universe for 10^9 antiquarks there had been $10^9 + 1$ quarks. Most of them annihilated producing photons but a tiny amount of quarks survived.)

- How did it happen and when?

1. The Universe just started this way (unlikely, e.g., contradicts inflation)

2. Occurred around the GUT scale $\sim 10^{15}$ GeV
   (GUT=Grand Unified Theories usually have baryon number (B) violating interactions; e.g. proton decay predicted but not observed — one of generic problems of such theories)

3. Occur-ed at the electroweak phase transition. This is the era when Higgs field acquired nonzero VEV and also other particles become massive.

Sakharov 1967: Three necessary conditions for baryogenesis

1. B-violation (obvious)

2. C and CP violation:
   If not, without a preference of matter over antimatter the B-violation will take place at the same rate in both directions.

3. Thermodynamic non-equilibrium:
   CPT guarantees equal masses for baryons and antibaryons; Chemical reactions would drive the necessary reactions to correct for any developing asymmetry.
   [In a stationary system (no time dependence at all), if the baryon number was zero initially, it will stay zero forever]

Remarkably,

SM has all necessary ingredients!

- <u>Baryon number is violated nonperturbatively</u>

Baryon current:

$$B^\mu_{(q)} = \bar{q}\gamma^\mu q \;\equiv\; \{\rho, \vec{j}\} = \{\underbrace{q^\dagger q}_{\text{quark number density}}, q^\dagger\gamma_0\vec{\gamma}q\} \tag{11.39}$$

is not conserved because of the anomaly

$$\partial_\mu B^\mu = \left(\frac{1}{3}\right)_{B_q} \cdot 3_{\text{colors}} \cdot 3_{\text{generations}} \cdot \frac{g^2}{32\pi^2} \underbrace{F^a_{\mu\nu}\widetilde{F}^{a,\mu\nu}}_{SU(2)\,\text{field strength}} \tag{11.40}$$

Similarly for lepton current

$$\partial_\mu L^\mu = 3_{\text{generations}} \cdot \frac{g^2}{32\pi^2} F^a_{\mu\nu}\widetilde{F}^{a,\mu\nu} \tag{11.41}$$

Anomaly cancellation implies that

$$\partial_\mu \Big(B^\mu - L^\mu \Big) = 0 \tag{11.42}$$

but B and L are not conserved separately.
If large gauge field fluctuations occur, such that

$$Q \equiv \int_{t_{\rm init}}^{t_{\rm fin}} dt \int d^3x \, \frac{g^2}{32\pi^2} F^a_{\mu\nu} \widetilde{F}^{a,\mu\nu} \neq 0 \tag{11.43}$$

one obtains[§]

$$B_{\rm fin} - B_{\rm init} = \int_{t_{\rm init}}^{t_{\rm fin}} dt \int d^3x \, \partial_\mu B^\mu = \int d^3x \, B^0(\vec{x}, t_{\rm fin}) - \int d^3x \, B^0(\vec{x}, t_{\rm init}) = 3Q \tag{11.44}$$

and similar for three lepton families

$$L_{\rm fin} - L_{\rm init} = 3Q \tag{11.45}$$

In other words,

$$\begin{aligned} &B \;-\; L \text{ conserved} \\ &B \text{ and } L \text{ violated} \end{aligned} \tag{11.46}$$

- How can B be violated if all terms in the Lagrangian conserve this symmetry?

◦ Consider massless fermions in a background field $\vec{A}(\vec{x}, t)$ ($A_0 = 0$, Coulomb gauge) such that

$$\vec{A}(\vec{x}, t \to -\infty) \to \text{ vacuum} \qquad\qquad \vec{A}(\vec{x}, t \to +\infty) \to \text{ vacuum} \tag{11.47}$$

(This could be a quantum fluctuation)

Dirac equation

$$i \frac{\partial}{\partial t} \Psi = i\gamma_0 \vec{\gamma} \cdot (\vec{\partial} - ig\vec{A}) \Psi \equiv H_{\rm Dirac}(t) \Psi \tag{11.48}$$

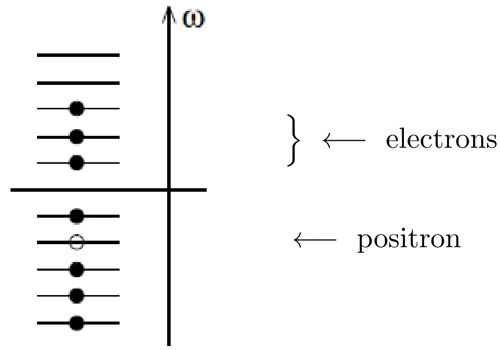
◦ Assume that $\vec{A}$ varies slowly in time; then fermions "sit" on energy levels of the Hamiltonian at the given time

$$H_{\rm Dirac}(t) \Psi_n = E_n(t) \Psi_n \tag{11.49}$$

◦ How do the eigenvalues behave in time?

[§] one can show that Q is integer (winding number = topological charge)

Dirac picture (Dirac sea) at $\vec{A} = 0$: $(t \to \pm\infty)$

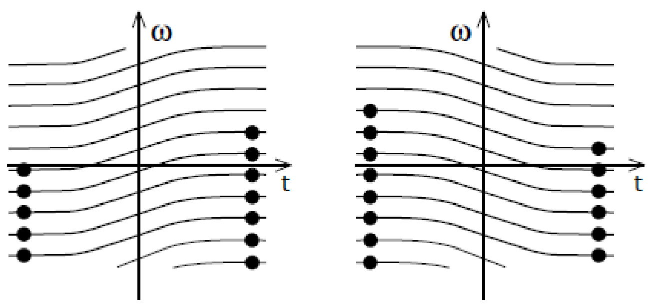


(11.50)

Note that all eigenvalues come in pairs: $+E_n$ and $-E_n$ because of γ_5 symmetry of the D. equation: If Ψ_n is a solution of $H\Psi_n = E_n\Psi_n$, then $\Psi_n' = \gamma_5\Psi$ is a solution of $H\Psi_n' = -E_n\Psi_n'$.

Time evolution of levels in special (topologically nontrivial) gauge fields $Q \neq 0$:

Example: $Q = 2$



left-handed fermions right-handed fermions.

The number of levels crossing zero equals Q. The motion of levels shown in the left-hand panel above corresponds to the case in which the initial state of the fermionic system is vacuum (no real fermions or antifermions) whereas the final state contains $Q = 2$ real (left-handed) fermions.

Note that left-handed and right-handed levels move in opposite directions. In QCD (QED) left-handed and right-handed fermions interact with gluons (photons) in the same way. As the result

$$\begin{aligned} B = N_L + N_R \quad & \text{is conserved} \qquad \partial_\mu J^\mu = 0 \\ Q^5 = N_L - N_R \quad & \text{is violated} \qquad \partial_\mu J^{\mu 5} \neq 0 \end{aligned} \tag{11.51}$$

If, however, only left-handed fermions interact (the case in SM for the $SU(2)$ gauge field, then the number of fermions of a given type is not conserved $\rightsquigarrow$ baryon number not conserved.

∘ In order that this mechanism works, one needs large gauge field fluctuations with nonzero Q (topological charge). They are called instantons. However, at zero temperature a probability of such a quantum fluctuation is suppressed by a factor

$$e^{-16\pi^2/g^2} \sim 10^{-165} \tag{11.52}$$

so you should not worry, we are not going to decay any time soon. (Explanation goes beyond these lectures).

At high temperatures, however, such fields can result from a thermal fluctuation (sphalerons) and their probability becomes sufficiently large such that B-violation is rapid compared to cosmological expansion. This happens at temperatures

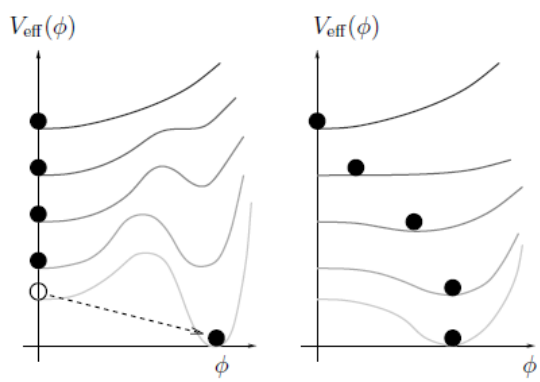
$$T > \underbrace{v_T \equiv \langle\phi\rangle_T}_{\text{Higgs VEV at temp. T}} \tag{11.53}$$

Thus, excitingly, there is a possibility to generate baryon asymmetry at electroweak epoch $T_{EW} \sim$ 100 GeV. This scenario is known as electroweak baryogenesis.

Unfortunately, this does not seem to work. . .

1. CP-violation too weak

2. Universe expands too slowly — expansion time too large to have considerable deviations from equilibrium. The only chance: 1-st order phase transition from the unbroken to the broken phase. [EW symmetry is restored at hight T, $\langle\phi\rangle = 0$, just like a superconducting state becomes normal at high T]



first order second order

— A 1-st order phase transition occurs from a supercooled state via spontaneous creation of bubbles of a new (broken) phase — a "boiling Universe", strongly out of equilibrium.

Unfortunately this does not happen in SM (would require higgs mass $<$ 50 GeV).

3. B-violating processes do not switch off fast enough in broken phase $\rightsquigarrow$ B-asymmetry will be washed out even if generated.

The (generally accepted) conclusion is that electroweak baryogenesis requires considerable extension of the SM, and existence of relatively light new scalar particles (a supersymmetric partner of the top quark, "stop", would be a good candidate). This looks increasingly unlikely in view of LHC results (do not see anything of this kind).

- A popular alternative: leptogenesis (Fukugita & Yanagida, 1986)

Basic idea: baryon asymmetry is generated not directly, but first lepton-antilepton asymmetry occurs and then it is converted to baryon asymmetry via anomaly and sphaleron transitions (as explained above).

Typical (original) scenario: assume existence of a (very) heavy right-handed Mayorana neutrino
— Why right-handed: no SM gauge interactions (both U(1) and SU(2))
— Why Mayorana: do not need a left-handed counterpart
— "Seasaw" mechanism to explain very small (usual) neutrino mass pattern
that decays in leptons and a higgs boson. This decay breaks CP so one also gets an additional source of $\not{C}\not{P}$, which is also welcome.

Typical Lagrangian ($e^i = e, \mu, \tau$)

$$\mathcal{L} = \bar{E}_L^i i\not{\partial} E_L^i + \bar{e}_R^i i\not{\partial} e_R^i + \bar{\nu}_R^i i\not{\partial} \nu_R^i + f_{ij}\bar{e}_R^i E_L^j H^\dagger + g_{ij}\bar{\nu}_R^i E_L^i H - \frac{1}{2} M_i \nu_R^i \nu_R^i + \text{ h.c.} \tag{11.54}$$

Note that the Mayorana mass term explicitly violates the lepton number. The vacuum expectation value of the Higgs field generates charged lepton Dirac mass m_ℓ matrix and Dirac neutrino mass matrix $m_\nu^{(D)}$ in a usual way. In addition there is a Mayorana mass matrix M that can contain large numbers, e.g., $\sim 10^{15}$ GeV.

The mass matrices $m_\nu^{(D)}$ and M contain altogether 6 physical CP phases, which lead to CP violating decays and scatterings. Diagonalizing the 6×6 neutrino mass matrix one obtains three heavy and three light neutrino mass eigenstates.

Currently, there exist many concrete models based on these ideas. *Lect.* 25

12 Supersymmetry

Supersymmetry (SUSY) is a major new idea in comparison to "conventional" symmetries and an important ingredient in practically all modern approaches to particle physics beyond SM. It also has applications to condensed matter physics. SUSY has grown up to a very broad field and a description pretending to be at least half-complete would take several semesters of dedicated lectures. Here I will give an elementary introduction based on the QM example first discussed by Witten in 1981. My presentation follows: M. Shifman, "Beginning Supersymmetry", in: ITEP Lectures on Particle Physics and Field Theory.

12.1 Supersymmetric Quantum Mechanics

• Consider a one-dimensional quantum system described by the Hamiltonian

$$H = \frac{1}{2}\Big(\frac{p^2}{2m} + W^2(x) + \sigma_3 \frac{1}{\sqrt{m}}\frac{dW}{dx}\Big) \tag{12.1}$$

Here:

σ^3 = third Pauli matrix
$W(x)$ = arbitrary function

• The Hamiltonian looks peculiar; let us show, however, that it actually describes a one-dimensional motion of an electron in a magnetic field of special type.

Start with (cf. Sec. 11.2)

$$\begin{aligned} H &= \frac{1}{2m}(p-eA)^2 + \frac{e}{2m}\vec{\sigma}\cdot\vec{B} \\ &= \frac{1}{2m}p^2 - \frac{ie}{2m}\vec{\nabla}\cdot\vec{A} - \frac{e}{m}\vec{A}\cdot\vec{p} + \frac{e^2}{2m}\vec{A}^2 + \frac{e}{2m}\vec{\sigma}\cdot\vec{B} \end{aligned} \tag{12.2}$$

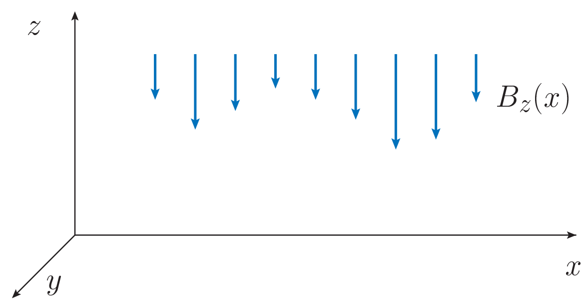
Choose

$$A_x = A_z = 0\,, \qquad A_y = \frac{\sqrt{m}}{e}\underbrace{W(x)}_{\text{only depends on x}} \tag{12.3}$$

Then

$$\partial_x A_y \sim B_z(x)\,, \qquad B_x = B_z = 0 \tag{12.4}$$

— only z-component of $\vec{B}$ is nonzero, and it only depends on x.



It is easy to see that in this way one arrives at the Hamiltonian in (12.1).

- A (nonrelativistic) electron wave function is a two-component spinor

$$\Psi(x) = \psi_1(x)\underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{|\uparrow\rangle} + \psi_2(x)\underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{|\downarrow\rangle} \tag{12.5}$$

- Note that

$$[H, \sigma_3] = 0 \tag{12.6}$$

so that spin projection on the z-axis is conserved with time. We have two independent sets of solutions with “spin-up” and “spin-down”.

In what follows let us give them new names (convention)

$$\begin{aligned} |\uparrow\rangle : &\qquad \text{"Fermion state"} \\ |\downarrow\rangle : &\qquad \text{"Boson state"} \end{aligned} \tag{12.7}$$

- The most important special property of the Hamiltonian in Eq. (12.1) is that it can be written as a square of an operator called the supercharge.

Supercharges:

$$Q_1 \stackrel{!}{=} \frac{1}{2}\left(\sigma_1 \frac{p}{\sqrt{m}} + \sigma_2 W\right) \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Q_2 \stackrel{!}{=} \frac{1}{2}\left(\sigma_2 \frac{p}{\sqrt{m}} - \sigma_1 W\right) \tag{12.8}$$

Using these definitions it is easy to show that

$$\circ \qquad Q_1 = Q_1^\dagger\,, \qquad Q_2 = Q_2^\dagger\,, \qquad Q_2 = -i\sigma_3 Q_1$$

$$\circ \qquad H = 2Q_1^2 = 2Q_2^2 \qquad (= 2Q_1Q_1^\dagger = 2Q_2Q_2^\dagger)$$

$$\circ \qquad \frac{d}{dt}Q_{1,2} = -i[Q_{1,2}, H] = 0 \qquad \text{(conserved charges)} \tag{12.9}$$

We can also summarize these relations as

$$\{Q_i, Q_j\} = \delta_{ij} H \qquad\qquad i,j = 1,2$$

$$[Q_i, H] = 0 \tag{12.10}$$

so that we have three operators: Q_1, Q_2 and H, which form the so-called *graded algebra* (involves commutators and anticommutators

$Q_{1,2}$ are called odd elements of the algebra (fermion type)
H is called even element of the algebra (boson type)

Rationale for the names: Easy to check

$$Q_i|\uparrow\rangle \sim |\downarrow\rangle$$

$$Q_i|\downarrow\rangle \sim |\uparrow\rangle \tag{12.11}$$

so that Q_i generically convert “fermions” in “bosons” and v.v. (unless they annihilate the state)

- **Properties of this system**

In what follows for simplicity assume

$$|W(x)| \to \infty \quad \text{for} \quad x \to \pm\infty \tag{12.12}$$

[only discrete energy levels, no continuous spectrum]

1. All energy eigenvalues are non-negative, $E \geq 0$
   Indeed

$$\forall |a\rangle \qquad E_a = \langle a|H|a\rangle = 2\langle a|Q_1^\dagger Q_1|a\rangle = 2\langle b|b\rangle > 0 \tag{12.13}$$

where $|b\rangle = Q_1|a\rangle$.

2. <u>Ground state energy is exactly zero, unless SUSY is spontaneously broken</u>
   Let $|0\rangle$ be the ground state. A symmetry is not broken (in general) if the generator of the corresponding symmetry transformation annihilates the state (cf. derivation of Goldstone theorem). In our case "unbroken SUSY" means that

$$Q_1|0\rangle = Q_2|0\rangle = 0 \tag{12.14}$$

   In this case, obviously $E_0 = 2\langle 0|Q_1^\dagger Q_1|0\rangle = 0$.
   Alternatively, if

$$|b\rangle = Q_1|0\rangle \neq 0 \tag{12.15}$$

   then

$$H|b\rangle = HQ_1|0\rangle = Q_1 H|0\rangle = E_0 Q_1|0\rangle = E_0|b\rangle \tag{12.16}$$

   so that there exist two degenerate vacuum (lowest energy) states (with equal positive energy), as typical for spontaneous symmetry breaking.

3. <u>Ground state wave function for unbroken SUSY</u>
   We have to solve

$$Q_1|0\rangle = 0 \Rightarrow \frac{d}{dx}\Psi_0(x) = \sqrt{m}W(x)\sigma_3\Psi_0(x) \tag{12.17}$$

   Therefore

$$\Psi_0(x) = C \cdot \exp\left[\int_0^x dy\,\sqrt{m}W(y)\sigma_3\right] \times \Big(|\uparrow\rangle \quad \text{or} |\downarrow\rangle\Big) \tag{12.18}$$

   Thus, formally we have <u>two</u> solutions but in order to have a probability interpretation of the wave function we must require that the solution is normalized to

$$\int\limits_{-\infty}^{\infty} dx|\Psi_0(x)|^2 = 1 \tag{12.19}$$

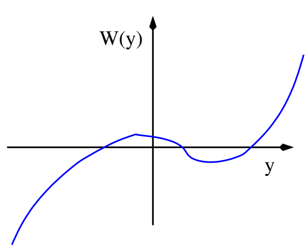
   This can be achieved by changing the normalization of the WF, but only if the above integral exists, i.e,, if it is finite. This is a certain condition on $W(x)$.

   It is possible (easy) to show that:

   A normalizable solution exists if and only if

$$\text{Sign}\,W(y)|_{y\to+\infty} = -\text{Sign}\,W(y)|_{y\to-\infty} \tag{12.20}$$

   so that $W(y)$ has to look like



   For example

   $W(y) = \text{const} \cdot y$

   $W(y) = \text{const} \cdot y(y^2 - a^2)$

   etc. (12.21)

The sign of the constant does not matter as we will see, it can be positive or negative.

Take $W(y) = ay$ for illustration. Then

$$\Psi_0(x) = C \cdot \exp\left[\sqrt{m}a\frac{x^2}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right] \times \left[\begin{pmatrix}1\\0\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}0\\1\end{pmatrix}\right] \tag{12.22}$$

$\rightsquigarrow$ always ONE normalizable solution:

$$|\downarrow\rangle \text{ for } a > 0 \text{ and } |\uparrow\rangle \text{ for } a < 0.$$

4. <u>All excited states come in pairs with equal energy</u>
   Let

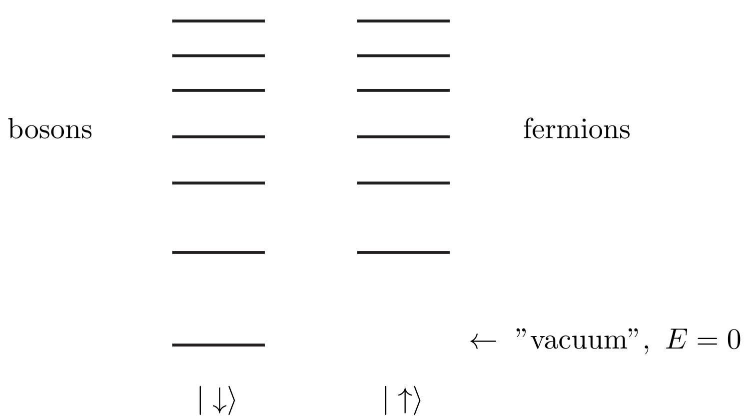
$$\Psi_{\text{fermi}} = \sqrt{\frac{2}{E}} Q_1 \Psi_{\text{boson}}\,, \qquad \text{where} \quad H\,\Psi_{\text{boson}} = E\Psi_{\text{boson}} \tag{12.23}$$

Then

$$H\,\Psi_{\text{fermi}} = E\Psi_{\text{fermi}} \tag{12.24}$$

Follows from $[H, Q_1] = 0$ (as above) and Eq. (12.11). The factor $\sqrt{2/E}$ inserted to have the same normalization of states.

Thus the energy spectrum of our system looks as follows:



(12.25)

To summarize, basic properties of SUSY systems

- Energy of the ground state is exactly zero
  ← in particular all corrections to $E_0 = 0$ in pert. theory vanish!
- All particles (excites states) come in pairs: Boson + Fermion with equal masses
  (if SUSY is not broken spontaneously)

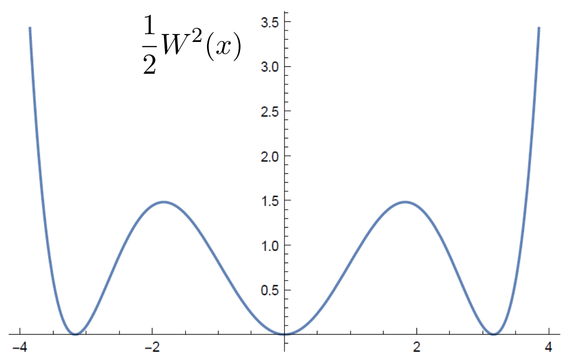
How all corrections to $E_0 = 0$ cancel?

<u>Example:</u>
Take

$$W(x) = \sqrt{m}\omega x\,(1 - \lambda x^2)\,, \quad \lambda \ll 1 \tag{12.26}$$

Remind

$$H = \frac{1}{2}\frac{p^2}{2m} + \underbrace{\frac{1}{2}W^2(x)}_{\substack{\downarrow\\ \text{bosonic part}}} + \underbrace{\sigma_3 \frac{1}{2\sqrt{m}}\frac{dW}{dx}}_{\substack{\downarrow\\ \text{fermionic part of the potential}}} \tag{12.27}$$



bosonic part of the potential

Classically, the particle sits at one of the minima; all three with $E = 0$. Let us consider small oscillations around $x = 0$. Then

$$\text{Bosonic part} \qquad \frac{1}{2}W^2(x) = \frac{1}{2}m\omega^2x^2(1-\lambda x^2)^2 = \underbrace{\frac{1}{2}m\omega^2x^2}_{\text{oscillator}} - \lambda\, m\omega^2x^4 + \mathcal{O}(\lambda^2)$$

$$\text{Fermionic part} \qquad \frac{1}{2}\sigma_3\frac{1}{\sqrt{m}}\frac{dW}{dx} = \frac{1}{2}\sigma_3\omega(1-3\lambda x^2) \tag{12.28}$$

Then for the ground state $|\downarrow\rangle$ in leading order $\lambda \to 0$

$$E_0 = \frac{\omega}{2} - \frac{\omega}{2} = 0 \tag{12.29}$$

First order in pert. theory

$$\Delta E = \langle 0|\delta H|0\rangle = \lambda \int_{-\infty}^{\infty} dx\, \underbrace{\left(\frac{m\omega}{\pi}\right)^{1/2} e^{-m\omega x^2}}_{|\Psi_0|^2(x)} \left\{-m\omega^2x^4 + \frac{3}{2}\omega x^2\right\} = 0 \tag{12.30}$$

and this cancellation of bosonic and fermionic contributions will continue to all orders in λ (becomes quite nontrivial in high orders).

12.2 Superspace and superfields

Questions:

- Are there other systems with similar properties?
- How can one construct them in a systematic way?

• Symmetries are usually related to invariance of the action under certain trafos. In standard QM

$$S = \int dt\, L(t)\,, \qquad\qquad L = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 - V(\phi) \tag{12.31}$$

invariance under time translations

$$t \to t + \tau \tag{12.32}$$

results in energy conservation. The generator of these transformations is the Hamiltonian

$$\phi(t+\delta t) - \phi(t) = i[H,\phi](t)\,\delta t \tag{12.33}$$

The major idea in SUSY is to extend

$$\text{time} \quad t \quad \longrightarrow \quad \text{supertime} \quad (t,\theta,\bar\theta) \tag{12.34}$$

where θ, $\bar\theta$ are (complex) Grassman variables, or, for a field theory

$$\text{space time} \quad (t,\vec{x}) \quad \longrightarrow \quad \text{superspace(-time)} \quad (x^\mu,\theta_\alpha,\bar\theta_{\dot\alpha}) \tag{12.35}$$

where θ_α, $\bar\theta_{\dot\alpha}$ are two-component Grassman spinors.

SUSY transformations are defined as

$$\boxed{\begin{aligned} &t \mapsto t+\tau \\ &\theta \mapsto \theta + \zeta\,, \qquad \bar\theta \mapsto \bar\theta + \bar\zeta\,, \qquad t \mapsto t + i\theta\bar\zeta - i\zeta\bar\theta \end{aligned}} \tag{12.36}$$

• The first thing to check is that these trafos form a group, i.e. a combination of two such trafos is a trafo of the same type. Consider

$$\theta \xrightarrow{\zeta_1} \theta_1 \xrightarrow{\zeta_2} \theta_2 \tag{12.37}$$

Then

$$\begin{aligned} \square \qquad & \theta_1 = \theta + \zeta_1\,, \qquad \theta_2 = \theta_1 + \zeta_2 = \theta + (\zeta_1+\zeta_2) \qquad \checkmark \\ & \qquad \text{similarly} \qquad \bar\theta_2 = \bar\theta + (\bar\zeta_1 + \bar\zeta_2) \\ \square \qquad & t_2 = t_1 + i\theta_1\bar\zeta_2 - i\zeta_2\bar\theta_1 \\ & \quad = (t + i\theta\bar\zeta_1 - i\zeta_1\bar\theta) + i(\theta+\zeta_1)\bar\zeta_2 - i\zeta_2(\bar\theta+\bar\zeta_1) \\ & \quad = t + i\theta(\bar\zeta_1+\bar\zeta_2) - i(\zeta_1+\zeta_2)\bar\theta \underbrace{+\, i\zeta_1\bar\zeta_2 - i\zeta_2\bar\zeta_1}_{\tau,\ \text{time translation}} \qquad \checkmark \end{aligned} \tag{12.38}$$

so that a superposition of two SUSY trafos is a SUSY trafo accompanied by a time translation. This looks good.

- The next step is to introduce functions on the superspace

$$\text{variable} \quad \phi(t) \quad \longrightarrow \quad \text{supervariable} \quad \Phi(t,\theta,\bar{\theta})$$
$$\Big[\text{field} \quad \phi(\vec{x},t) \quad \longrightarrow \quad \text{superfield} \quad \Phi(x^{\mu},\theta_{\alpha},\bar{\theta}_{\dot{\alpha}})\Big]$$

Define the **Real superfield**

$$\Phi = \Phi^*$$
$$\Phi(t,\theta,\bar{\theta}) = \phi(t) + \theta\bar{\psi}(t) + \psi(t)\bar{\theta} + D(t)\theta\bar{\theta} \tag{12.39}$$

It contains

$$\begin{aligned} &\phi(t)\,, \quad D(t): && \text{real boson fields} \\ &\psi(t)\,, \quad \bar{\psi}(t): && \text{real fermion fields} \end{aligned} \tag{12.40}$$

We require that under SUSY transformations

$$\Phi(t,\theta,\bar{\theta}) \mapsto \Phi(t',\theta',\bar{\theta}') \tag{12.41}$$

What does this imply for component fields?

$$\begin{aligned} \Phi = \phi(t) + \theta\bar{\psi}(t) + \psi(t)\bar{\theta} + D(t)\theta\bar{\theta} \mapsto\ & \phi(t + i\theta\bar{\zeta} - i\zeta\bar{\theta}) + (\theta+\zeta)\bar{\psi}(t + i\theta\bar{\zeta} - i\zeta\bar{\theta}) \\ & + \psi(t + i\theta\bar{\zeta} - i\zeta\bar{\theta})(\bar{\theta}+\bar{\zeta}) + D(t + i\theta\bar{\zeta} - i\zeta\bar{\theta})(\theta+\zeta)(\bar{\theta}+\bar{\zeta}) \end{aligned} \tag{12.42}$$

Keeping linear terms in ζ, $\bar{\zeta}$ only (quadratic terms give rise to extra time translations)

$$\begin{aligned} \ldots &= \phi(t) + i\theta\bar{\zeta}\dot{\phi}(t) - i\zeta\bar{\theta}\dot{\phi}(t) \\ &\quad + \zeta\bar{\psi}(t) + \theta\bar{\psi}(t) - i\theta\zeta\bar{\theta}\dot{\bar{\psi}} + \psi(t)\bar{\theta} - \bar{\zeta}\psi(t) - i\theta\bar{\zeta}\bar{\theta}\dot{\psi} \\ &\quad + \zeta\bar{\theta}D(t) + \theta\bar{\zeta}D(t) + \theta\bar{\theta}D(t) \\ &= \Big[\phi(t) + \zeta\bar{\psi}(t) + \psi(t)\bar{\zeta}\Big] + \theta\Big[\bar{\psi}(t) + i\bar{\zeta}\dot{\phi}(t) + \bar{\zeta}D(t)\Big] \\ &\quad + \Big[\psi(t) - i\zeta\dot{\phi}(t) + \zeta D\Big]\bar{\theta} + \Big[D(t) - i\dot{\psi}(t)\bar{\zeta} + i\zeta\dot{\bar{\psi}}\Big]\bar{\theta}\theta \end{aligned} \tag{12.43}$$

Thus

$$\boxed{\begin{aligned} \phi(t) &\mapsto \phi(t) + \zeta\bar{\psi}(t) + \psi(t)\bar{\zeta} && = \phi + \delta\phi \\ \psi(t) &\mapsto \psi(t) - i\zeta\dot{\phi}(t) + \zeta D && = \psi + \delta\psi \\ \bar{\psi}(t) &\mapsto \bar{\psi}(t) + i\bar{\zeta}\dot{\phi}(t) + \bar{\zeta}D(t) && = \bar{\psi} + \delta\bar{\psi} \\ D(t) &\mapsto D(t) - i\dot{\psi}(t)\bar{\zeta} + i\zeta\dot{\bar{\psi}} && = D + \delta D \end{aligned}} \tag{12.44}$$

Lect. 26

(!) Note that δD is a total (time) derivative (important later)

A field that transforms in this way under SUSY trafos is called a superfield.

- Take two superfields, Φ_1 and Φ_2. Then

$$\begin{aligned} &\circ \quad \Phi = \Phi_1 + \Phi_2 \quad \text{is a superfield} \\ &\circ \quad \Phi = \Phi_1 \cdot \Phi_2 \quad \text{is a superfield} \end{aligned} \tag{12.45}$$

First statement is trivial, for the second one

$$\begin{aligned} \phi &= \phi_1\,\phi_2 \\ \bar\psi &= \bar\psi_1\,\phi_2 + \bar\psi_2\,\phi_1 \\ \psi &= \psi_1\,\phi_2 + \psi_2\,\phi_1 \\ D &= D_1\,\phi_2 + D_2\,\phi_1 + \bar\psi_1\,\psi_2 + \bar\psi_2\,\psi_1 \end{aligned} \tag{12.46}$$

and, e.g.,

$$\begin{aligned} \delta\phi &= \phi_1\delta\phi_2 + \phi_2\delta\phi_1 \\ &= \phi_1(\zeta\bar\psi_2 + \psi_2\bar\zeta) + \phi_2(\zeta\bar\psi_1 + \psi_1\bar\zeta) \\ &= \zeta(\phi_1\bar\psi_2 + \phi_2\bar\psi_1) + (\psi_1\phi_2 + \psi_2\phi_1)\bar\zeta = \zeta\bar\psi + \psi\bar\zeta \qquad \checkmark \end{aligned} \tag{12.47}$$

and similar for the other components.

- Further

$$\begin{aligned} &\circ \quad \frac{\partial}{\partial t}\Phi(t,\theta,\bar\theta) \quad \text{is a superfield} \\ &\circ \quad \frac{\partial}{\partial \theta}\Phi(t,\theta,\bar\theta) \quad \underbrace{\text{is \underline{NOT} a superfield}}_{\text{does not transform properly}} \end{aligned} \tag{12.48}$$

- **<u>Covariant derivatives</u>**

Definition:

$$\begin{aligned} D &\overset{!}{=} \frac{\partial}{\partial\theta} + i\bar\theta\frac{\partial}{\partial t} \\ \bar D &\overset{!}{=} -\frac{\partial}{\partial\bar\theta} - i\theta\frac{\partial}{\partial t} \end{aligned} \tag{12.49}$$

Exercise: show that

$$\begin{aligned} \left\{D,\bar D\right\} &= -2i\frac{\partial}{\partial t} \\ D^2 = \bar D^2 &= 0 \end{aligned} \tag{12.50}$$

- **Chiral (antichiral) time**

A covariant derivative of Φ can be written in a compact form introducing an (auxiliary) time shift:

$$\begin{aligned} D\Phi &= \left(\frac{\partial}{\partial\theta} + i\bar\theta\frac{\partial}{\partial t}\right)\left[\phi(t) + \theta\bar\psi(t) + \psi(t)\bar\theta + D(t)\theta\bar\theta\right] \\ &= \bar\psi(t) + D(t)\bar\theta + i\bar\theta\frac{\partial\phi}{\partial t} + i\bar\theta\theta\frac{\partial\bar\psi}{\partial t} \\ &= \left[\bar\psi + \left(i\frac{\partial\phi}{\partial t} + D\right)\bar\theta\right]\Bigg|_{t-i\theta\bar\theta} \end{aligned} \tag{12.51}$$

Define

$$t_{ach} \overset{!}{=} t - i\theta\bar\theta \qquad \textbf{antichiral time}$$

$$t_{ch} \overset{!}{=} t + i\theta\bar\theta \qquad \textbf{chiral time} \tag{12.52}$$

Useful properties:

1. .
$$D\,t_{ach} = 0 \qquad \bar D\,t_{ch} = 0 \tag{12.53}$$
Indeed, e.g.
$$\left(-\frac{\partial}{\partial\bar\theta} - i\theta\frac{\partial}{\partial t}\right)(t + i\theta\bar\theta) = i\theta - i\theta = 0 \tag{12.54}$$

2. Under SUSY transformations (12.36)
$$(t - i\theta\bar\theta) \mapsto (t - i\theta\bar\theta) - 2i\zeta\bar\theta - i\zeta\bar\zeta \tag{12.55}$$
so that SUSY variation of t_{ach} only contains $\bar\theta$, not θ. Similarly, SUSY trafo of t_{ch} only contains θ.

Look now at the SUSY trafo for $D\Phi$:

$$\begin{aligned} D\Phi &= \bar\psi(t_{ach}) + \left(i\frac{\partial\phi}{\partial t} + D\right)(t_{ach})\bar\theta \\ &\mapsto \bar\psi(t_{ach} - 2i\zeta\bar\theta) + \left(i\frac{\partial\phi}{\partial t} + D\right)(t_{ach} - 2i\zeta\bar\theta)(\bar\theta + \bar\zeta) \\ &= \bar\psi(t_{ach}) - 2i\zeta\bar\theta\frac{\partial\bar\psi}{\partial t} + \left(i\frac{\partial\phi}{\partial t} + D\right)(t_{ach})\bar\theta + \left(i\frac{\partial\phi}{\partial t} + D\right)(t_{ach})\bar\zeta \\ &= \left[\bar\psi(t_{ach}) + \left(i\frac{\partial\phi}{\partial t} + D\right)(t_{ach})\bar\zeta\right] + \left[2i\zeta\frac{\partial\bar\psi}{\partial t} + \left(i\frac{\partial\phi}{\partial t} + D\right)(t_{ach})\right]\bar\theta \\ &\quad (+\text{ quadratic terms} \sim \zeta\bar\zeta) \end{aligned} \tag{12.56}$$

where from

$$\delta\bar\psi = \Big(i\frac{\partial\phi}{\partial t} + D\Big)\bar\zeta \qquad\qquad \delta\Big(i\frac{\partial\phi}{\partial t} + D\Big) = 2i\zeta\frac{\partial\bar\psi}{\partial t} \tag{12.57}$$

This agrees with (12.44) because, e.g.

$$\left.\begin{aligned} \delta\phi(t) &= \zeta\bar\psi(t) \ + \cancel{\psi(t)\bar\zeta} \\ \delta D(t) &= \cancel{-i\dot\psi(t)\bar\zeta} \ + i\zeta\dot{\bar\psi} \end{aligned}\right\} \quad\Rightarrow\quad \delta\Big(i\frac{\partial\phi}{\partial t} + D\Big) = i\zeta\dot{\bar\psi} + i\zeta\dot{\bar\psi} \qquad \checkmark \tag{12.58}$$

Thus

- $D\Phi$ is a superfield (SF), as desired.
- $D\Phi$ contains less components compared to Φ (effectively a superfield of a new type)

- **Chiral superfields**

Definition:

$$\begin{aligned} A &= \phi(t_{ch}) + \theta\bar\psi(t_{ch}) \qquad \longleftarrow \quad \text{chiral superfield} \\ \bar A &= \phi(t_{ach}) + \psi(t_{ach})\bar\theta \qquad \longleftarrow \quad \text{antichiral superfield} \end{aligned} \tag{12.59}$$

Equivalent definition:

$$\begin{aligned} &\bar A \quad \text{is called a antichiral SF} \quad \text{if} \quad D\,\bar A = 0 \\ &A \quad \text{is called a chiral SF} \quad \text{if} \quad \bar D\,A = 0 \end{aligned} \tag{12.60}$$

A covariant derivative of a real SF is an (anti)chiral SF because $\bar A = D\Phi \Rightarrow D\bar A = D^2\Phi = 0$ and similar $A = \bar D\Phi \Rightarrow \bar D A = \bar D^2\Phi = 0$.

12.3 SUSY action and Lagrangian

Now we have all necessary ingredients to define a SUSY theory:

$$\boxed{S_{\rm SUSY} = \int dt\, d\bar\theta d\theta\Big[\frac{1}{2}\bar D\Phi\, D\Phi - F(\Phi)\Big]} \tag{12.61}$$

$F(\Phi)$ (arbitrary function) is called a superpotential

- $S_{\rm SUSY}$ is invariant under SUSY transformations.

▶ $\int d\bar\theta\, d\theta$ singles out the D-term of $[\ldots]$. Under SUSY trafos

$$D \mapsto D - i\dot\psi(t)\bar\zeta + i\zeta\dot{\bar\psi}$$

so that δD is a time derivative. Thus the action is invariant ! ◀

- In terms of the component fields:
First term:

$$\int dt\, d\bar\theta d\theta\, \frac{1}{2} \bar D\Phi\, D\Phi$$

$$\overset{t\to t_{ach}}{=} \frac{1}{2}\int dt\, d\bar\theta d\theta \Big[\psi + \Big(-i\frac{\partial\phi}{\partial t} + D\Big)\theta\Big]_{t+2i\theta\bar\theta}\Big[\bar\psi + \Big(i\frac{\partial\phi}{\partial t} + D\Big)\bar\theta\Big]_t$$

$$= \int dt \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}D^2 - i\bar\psi\frac{\partial\psi}{\partial t}\right]_t \tag{12.62}$$

Second term:

$$\int dt\, d\bar\theta d\theta\, F(\Phi) = \int dt \left[F'(\phi)\, D + F''(\phi)\bar\psi\,\psi\right]_t \tag{12.63}$$

Thus we end up with the SUSY Lagrangian

$$\boxed{L_{\text{SUSY}} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}D^2 - i\bar\psi\frac{\partial\psi}{\partial t} - F'(\phi)\, D - F''(\phi)\bar\psi\,\psi} \tag{12.64}$$

- Note that the D-term enters without derivatives so that it can be eliminated using EOM:

$$\frac{\partial L}{\partial D} = 0 \qquad \Rightarrow \qquad D = F'(\phi) \tag{12.65}$$

so that we can rewrite our Lagrangian as

$$\boxed{L_{\text{SUSY}} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\big(F'(\phi)\big)^2 - i\bar\psi\frac{\partial\psi}{\partial t} - F''(\phi)\bar\psi\,\psi} \tag{12.66}$$

- Equations of motion:

$$\begin{aligned} \frac{\partial L}{\partial\bar\psi} = 0 \quad &\Rightarrow \frac{d\psi}{dt} = iF''\psi \\ \frac{d}{dt}\frac{\partial L}{\partial\dot\psi} - \frac{\partial L}{\partial\psi} = 0 \quad &\Rightarrow \frac{d\bar\psi}{dt} = -iF''\bar\psi \end{aligned} \tag{12.67}$$

This implies

$$\frac{d}{dt}\bar\psi\psi = \big(\frac{d}{dt}\bar\psi\big)\psi + \bar\psi\big(\frac{d}{dt}\bar\psi\big) = 0 \tag{12.68}$$

so that

$$q = \bar\psi\psi$$

is a conserved charge (fermion number).

- To bring our discussion to a close, we want to recover our original QM model expression from what we have now.

$$\begin{aligned}\phi(t) \mapsto x(t) \quad &= \text{coordinate} \\ \psi(t) \mapsto ? & \end{aligned} \tag{12.69}$$

The inverse transformation from Schrödinger QM for fermions to anticommuting operators is what is called second quantization. [cf. Bjorken, Drell, Ch. 13.2]
One introduces field operators that satisfy anticomm. relations

$$\{\hat{\psi}(t), \hat{\psi}(t)\} = 0\,, \qquad \{\hat{\bar{\psi}}(t), \hat{\bar{\psi}}(t)\} = 0\,, \qquad \{\hat{\psi}(t), \hat{\bar{\psi}}(t)\} = 1 \tag{12.70}$$

as some time-dependent functions times creation or annihilation operators for which one can take

$$\hat{a}^\dagger = \bar{\psi}(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \sigma_+ \qquad \hat{a} = \psi(0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \sigma_- \tag{12.71}$$

where

$$\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)\,, \qquad [\sigma_+, \sigma_-] = \sigma_3 \tag{12.72}$$

Note that with this definition

$$\hat{a}^\dagger \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{12.73}$$

so that if we interpret $|\downarrow\rangle$ as a state without fermions (vacuum), applying $\hat{a}^\dagger$ we obtain a state with one fermion, just as we expect for the creation operator to do.
Taking into account the Grassmanian nature of ψ, $\bar{\psi}$ we can rewrite the last term as

$$\bar{\psi}\psi = \frac{1}{2}[\bar{\psi}, \psi] + \cancel{\frac{1}{2}\{\bar{\psi}, \psi\}} \tag{12.74}$$

and proceed from Lagrangian to the Hamiltonian:

$$p = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi}\,, \qquad \pi = \frac{\partial L}{\partial \dot{\psi}} = i\bar{\psi}\,, \qquad H = \dot{\phi}\, p + \dot{\psi}\pi - L \tag{12.75}$$

One gets

$$H = \frac{1}{2}p^2 + \frac{1}{2}(F')^2 + \frac{1}{2}\sigma_3 F'' \tag{12.76}$$

(the term in $dot\psi$ cancels out)
Now we can identify

$$F'(x) = W(x) \qquad \text{(and } m = 1\text{)} \tag{12.77}$$

and get our starting expression.

And the very last point. In this representation

$$q = \bar{\psi}\psi = \frac{1}{2}(1 + \sigma_3) \tag{12.78}$$

so that

$$q \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \qquad\qquad q \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \tag{12.79}$$

which justifies the names “boson state” and “fermion state” that we have chosen at the beginning.

Appendices

A Collection of formulas

Dirac algebra in 4 Dimensions

Traces with even number of γ-matrices

$$\mathrm{Tr}\{\mathbf{1}\} = 4 \tag{A.1}$$

$$\mathrm{Tr}\{\gamma_\mu\gamma_\nu\} = 4g_{\mu\nu} \tag{A.2}$$

$$\mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta\} = 4[g_{\mu\nu}g_{\alpha\beta} + g_{\mu\beta}g_{\nu\alpha} - g_{\mu\alpha}g_{\nu\beta}] \tag{A.3}$$

Traces with odd number of γ-matrices

$$\mathrm{Tr}\{\gamma_{\mu_1}\ldots\gamma_{\mu_{2k+1}}\} = 0\,, \qquad k = 0,1,2,\ldots \tag{A.4}$$

Traces including a γ_5-matrix

$$\mathrm{Tr}\{\gamma_5\} = 0 \tag{A.5}$$

$$\mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_5\} = 0 \tag{A.6}$$

$$\mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta\gamma_5\} = 4i\epsilon_{\mu\nu\alpha\beta} \tag{A.7}$$

$$\mathrm{Tr}\{\gamma_{\mu_1}\ldots\gamma_{\mu_{2k+1}}\gamma_5\} = 0\,, \qquad k = 0,1,2,\ldots \tag{A.8}$$

Useful identities for products of γ-matrices

$$\gamma_\mu\gamma^\mu = 4 \tag{A.9}$$

$$\gamma_\mu\gamma_\alpha\gamma^\mu = -2\gamma_\alpha \tag{A.10}$$

$$\gamma_\mu\gamma_\alpha\gamma_\beta\gamma^\mu = 4g_{\alpha\beta} \tag{A.11}$$

$$\gamma_\mu\gamma_\alpha\gamma_\beta\gamma_\rho\gamma^\mu = -2\gamma_\rho\gamma_\beta\gamma_\alpha \tag{A.12}$$

$$\gamma_\mu\gamma_\alpha\gamma_\nu = g_{\alpha\mu}\gamma_\nu + g_{\alpha\nu}\gamma_\mu - g_{\mu\nu}\gamma_\alpha + i\epsilon_{\mu\alpha\nu\beta}\gamma_5\gamma_\beta \tag{A.13}$$

Useful identities for products of ϵ-tensors

$$\epsilon_{\alpha\beta\mu\nu}\epsilon^{\alpha\beta\mu\nu} = -24 \tag{A.14}$$

$$\epsilon_{\alpha\beta\mu\nu}\epsilon^{\rho\beta\mu\nu} = -6g^\rho_\alpha \tag{A.15}$$

$$\epsilon_{\alpha\beta\mu\nu}\epsilon^{\rho\sigma\mu\nu} = -2[g^\rho_\alpha g^\sigma_\beta - g^\sigma_\alpha g^\rho_\beta] \tag{A.16}$$

$$\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}\epsilon^{\beta_1\beta_2\beta_3\beta_4} = -\det\left(g^{\beta_k}_{\alpha_i}\right) \tag{A.17}$$

$$\frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\sigma^{\mu\nu} = i\sigma_{\alpha\beta}\gamma_5 \tag{A.18}$$

!!! We use definitions from Bjorken and Drell:

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3\,, \qquad \epsilon_{0123} = +1 \tag{A.19}$$

Be careful, some other (equally famous) books use different definitions:

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3\,, \qquad \epsilon^{0123} = -\epsilon_{0123} = +1 \qquad \text{Itzykson, Zuber} \tag{A.20}$$

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = -i\gamma^0\gamma^1\gamma^2\gamma^3\,, \qquad \epsilon^{0123} = -\epsilon_{0123} = +1 \qquad \text{Okun} \tag{A.21}$$

This ambiguity is a standard source of sign errors!

Identities involving Dirac spinors

$$\begin{aligned} \bar{u}^\lambda(p)\, u^{\lambda'}(p) &= 2m\delta_{\lambda\lambda'} \\ \bar{v}^\lambda(p)\, v^{\lambda'}(p) &= -2m\delta_{\lambda\lambda'} \\ \bar{u}^\lambda(p)\, v^{\lambda'}(p) &= \bar{v}^\lambda(p)\, u^{\lambda'}(p) \;=\; 0 \end{aligned} \tag{A.22}$$

$$\bar{u}^\lambda(p)\gamma_\mu u^{\lambda'}(p) = \bar{v}^\lambda(p)\gamma_\mu v^{\lambda'}(p) \;=\; 2p_\mu\delta_{\lambda\lambda'} \tag{A.23}$$

$$\sum_{\lambda=\pm 1/2}\left[u^\lambda_\alpha(p)\bar{u}^\lambda_\beta(p) - v^\lambda_\alpha(p)\bar{v}^\lambda_\beta(p)\right] = 2m\delta_{\alpha\beta} = 2m(\mathbb{I})_{\alpha\beta} \tag{A.24}$$

$$\begin{aligned} \sum_{\lambda=\pm 1/2} u^\lambda_\alpha(p)\bar{u}^\lambda_\beta(p) &= (\not{p} + m)_{\alpha\beta} \\ \sum_{\lambda=\pm 1/2} v^\lambda_\alpha(p)\bar{v}^\lambda_\beta(p) &= (\not{p} - m)_{\alpha\beta} \end{aligned} \tag{A.25}$$

Hermitian and Charge conjugation

$$\gamma^0\gamma^\dagger_\nu\gamma^0 = \gamma_\nu \tag{A.26}$$

$$C = i\gamma^2\gamma^0\,, \qquad C^{-1}\gamma_\mu C = -\gamma^T_\mu\,, \qquad C = -C^{-1} = -C^\dagger = -C^T \tag{A.27}$$

Integration in the 4 dimensional Euclidean space

Definitions:

$$k_o \to ik_4 \tag{A.28}$$

$$d^4k \;=\; dk_o d^3\vec{k} = id^4k_E \tag{A.29}$$

$$k^2 \;=\; k_0^2 - \vec{k}^2 = -(k_1^2 + k_2^2 + k_3^2 + k_4^2) = -k_E^2 \tag{A.30}$$

Integration:

$$\int d^D k_E f(k_E^2) = \int d\Omega_D \int_0^\infty dk_E k_E^{D-1} f(k_E^2) \tag{A.31}$$

$$= \frac{\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \int_0^\infty dk_E^2 \left(k_E^2\right)^{\frac{D}{2}-1} f(k_E^2) \tag{A.32}$$

Dimensional Regularization ($D = 4 - 2\epsilon$)

Definitions:

$$\int d^4 k \to \int d^D k \tag{A.33}$$

$$e_0 \to e_0 \mu^{2-\frac{D}{2}} \tag{A.34}$$

Dirac algebra in D Dimensions

Defining the ϵ-tensor and γ_5 in D dimensions involves subtleties that would require a detailed explanation; we will leave out the corresponding formulas.

$$\gamma_\mu \gamma^\mu = D \tag{A.35}$$

$$\gamma_\mu \gamma_\alpha \gamma^\mu = (2-D)\gamma_\alpha \tag{A.36}$$

$$\gamma_\mu \gamma_\alpha \gamma_\beta \gamma^\mu = 4 g_{\alpha\beta} + (D-4)\gamma_\beta \gamma_\alpha \tag{A.37}$$

$$\gamma_\mu \gamma_\alpha \gamma_\beta \gamma_\rho \gamma^\mu = -2\gamma_\rho \gamma_\beta \gamma_\alpha + (4-D)\gamma_\alpha \gamma_\beta \gamma_\rho \tag{A.38}$$

Feynman parameter integrals for products of propagators:

$$\frac{1}{A \cdot B} = \int_0^1 dx \frac{1}{\left[xA + (1-x)B\right]^2} \tag{A.39}$$

$$\frac{\Gamma(a)\Gamma(b)}{A^a \cdot B^b} = \int_0^1 dx\, dy\, x^{a-1} y^{b-1}\, \delta(1-x-y) \frac{\Gamma(a+b)}{\left[xA + yB\right]^{a+b}}$$

$$= \int_0^1 dx\, x^{a-1}(1-x)^{b-1} \frac{\Gamma(a+b)}{\left[xA + (1-x)B\right]^{a+b}} \tag{A.40}$$

This representation can be generalized to an arbitrary number of the denominators, e.g.,

$$\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{A^a \cdot B^b \cdot C^c} = \int_0^1 dx\, dy\, dz\, x^{a-1} y^{b-1} z^{c-1}\, \delta(1-x-y-z) \frac{\Gamma(a+b+c)}{\left[xA + yB + zC\right]^{a+b+c}} \tag{A.41}$$

Loop integrals in D Dimensions

$$\int d^D k \, \frac{\Gamma(a)}{[-k^2 - A - i\epsilon]^a} = i\pi^{\frac{D}{2}} \frac{\Gamma\left(a - \frac{D}{2}\right)}{[-A - i\epsilon]^{a-\frac{D}{2}}} \tag{A.42}$$

$$\int d^D k \, \frac{\Gamma(a)}{[-k^2 - A - i\epsilon]^a} k_\mu k_\nu = i\pi^{\frac{D}{2}} \left(-\frac{g_{\mu\nu}}{2}\right) \frac{\Gamma\left(a - 1 - \frac{D}{2}\right)}{[-A - i\epsilon]^{a-1-\frac{D}{2}}} \tag{A.43}$$

$$\int d^D x \, \frac{\Gamma(\alpha)}{(-x^2 - a^2 + i\epsilon)^\alpha} = -i\pi^{D/2} \frac{\Gamma(\alpha - D/2)}{[-a^2 + i\epsilon]^{\alpha - D/2}}$$

$$\int d^D x \, \frac{\Gamma(\alpha)}{(-x^2 - a^2 + i\epsilon)^\alpha} x_\mu x_\nu = -i\pi^{D/2} \left(-\frac{g_{\mu\nu}}{2}\right) \frac{\Gamma(\alpha - D/2 - 1)}{[-a^2 + i\epsilon]^{\alpha - D/2 - 1}} \tag{A.44}$$

Fourier integrals in D Dimensions

$$\int d^D x \, e^{iqx} \frac{\Gamma(\alpha)}{[-x^2 + i\epsilon]^\alpha} = -i\pi^{D/2} 2^{D-2\alpha} \frac{\Gamma(D/2 - \alpha)}{[-q^2 - i\epsilon]^{D/2 - \alpha}}$$

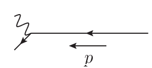
$$\int d^D q \, e^{-iqx} \frac{\Gamma(\alpha)}{[-q^2 - i\epsilon]^\alpha} = +i\pi^{D/2} 2^{D-2\alpha} \frac{\Gamma(D/2 - \alpha)}{[-x^2 + i\epsilon]^{D/2 - \alpha}} \tag{A.45}$$

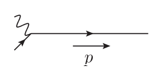
B Feynman rules for QED

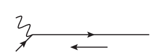
B.0.1 "standard" version

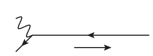
- Lines with arrows:
  Charged particles are shown by solid lines with an arrow. The direction of the arrow shows the flow of the *negative* charge. For example $\longrightarrow$ means that the negative charge is flowing from left to right; it can be an electron moving to the right, or positron to the left.

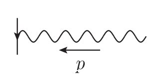
- External lines:

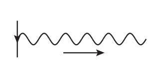
Electron in the initial state:  $= u^\lambda(p)$

Electron in the final state:  $= \bar{u}^\lambda(p)$

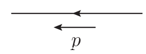
Positron in the initial state:  $= \bar{v}^\lambda(p)$

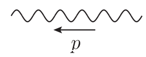
Positron in the final state:  $= v^\lambda(p)$

Photon in the initial state:  $= e_\mu(p)$

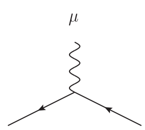
Photon in the final state:  $= e^*_\mu(p)$ (B.1)

- Propagators:

Dirac propagator  $= \dfrac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$

Photon propagator (Feynman gauge):  $= \dfrac{-ig_{\mu\nu}}{p^2+i\epsilon}$ (B.2)

- Interaction vertex:



$$= -ie\gamma^\mu\,, \qquad e = \sqrt{4\pi\alpha_{\mathrm{QED}}} > 0 \qquad \alpha_{\mathrm{QED}} \simeq \frac{1}{137} \tag{B.3}$$

- Integration over loop momenta:

$$\int \frac{d^4p}{(2\pi)^4} \quad \text{for each loop momentum} \tag{B.4}$$

- Energy-momentum conservation

$$(2\pi)^4\delta^{(4)}(\text{sum of all ingoing minus outgoing momenta}) \tag{B.5}$$

- Sign factors

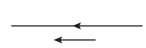
a) Extra (-1) factor for each closed fermion loop

b) Extra (-1) between the diagrams that differ only by the exchange of two identical external fermion lines. This applies not only to the exchange of identical particles in the final state but also, for example, the exchange of initial particle and final antiparticle. (B.6)

B.0.2 Alternative possibility (used e.g. in my QED lectures)

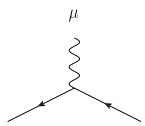
The differences are the following:

- Propagators:

Dirac propagator  $= \dfrac{(\not{p}+m)}{m^2-p^2-i\epsilon}$

Photon propagator (Feynman gauge):  $= \dfrac{g_{\mu\nu}}{p^2+i\epsilon}$ (B.7)

- Interaction vertex:



$$= -e\gamma^\mu\,, \qquad e = \sqrt{4\pi\alpha_{\mathrm{QED}}} > 0 \qquad \alpha_{\mathrm{QED}} \simeq \frac{1}{137} \tag{B.8}$$

- Integration over loop momenta:

$$\int \frac{d^4p}{(2\pi)^4 i} \qquad \text{for each loop momentum} \tag{B.9}$$

that is, each propagator has an extra factor i, each momentum integral $1/i$ and each vertex $1/i$. This set of rules is equivalent to the standard one because of the Euler's formula: For each Feynman diagram the following relation holds:

$$L = I - V + 1$$

where

L = the number of loops

I = the number of internal lines (number of propagators)

V = the number of interaction vertices (B.10)

Thus, for an arbitrary Feynman diagram, replacing the "standard" expressions by the "nonstandard" ones one obtains an extra factor

$$\left(\frac{1}{i}\right)^L \left(i\right)^I \left(\frac{1}{i}\right)^V = \left(i\right)^{I-V-L} = -i\,. \tag{B.11}$$

Since this factor is the same for all diagrams, it can be absorbed in the definition of the Green functions and is at the end irrelevant (because physical observables are written in terms of $|\text{amplitude}|^2$)