Lecture notes on Quantum Chromodynamics

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Version of February 2, 2023

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1 The Quark Model

1.1 Historic overview

Known before 1932:

- Photon γ $m_\gamma = 0$
- Electron e^- $m_e \simeq 0.5$ MeV
- Proton p $m_P \simeq 938$ MeV

Then:

- 1932: Discovery of the neutron (Chadwick) $m_n \simeq 940$ MeV
- 1932: Isospin formalism (Heisenberg)
- 1935: Prediction of the π-meson as carrier (mediator) of strong forces (Yukawa)
- 1938: Extension of the isospin formalism to π-mesons, prediction of the π^0 (Kemmer)
- 1947: Discovery of charged π^+ and π^- (Lattes), $m_{\pi^\pm} \simeq 140$ MeV
- 1950: Discovery of the neutral π^0 $m_{\pi^0} \simeq 135$ MeV

The picture seemed to be converging, however

- 1947(?): Observation of new long-living particles in cosmic rays (V-particles); first signatures of new "strange" particles in accelerator experiments (K-mesons, Λ-hyperons)

1.1.1 Isospin formalism

One observes that p, n and also π^+, π^0, π^- have almost the same masses, why?

Recall the Hydrogen atom: the states $|n, \ell, m\rangle$ with $m = -\ell, \ldots, \ell$ have the same energies (are degenerate) because of the rotational symmetry of the Hamiltonian

? Hidden Symmetry Internal Symmetry

Analogy with spin:

$$|p\rangle, |n\rangle \Leftrightarrow |\uparrow\rangle, |\downarrow\rangle$$

Spin-rotations:

$$\begin{aligned} |\uparrow\rangle &\to \alpha|\uparrow\rangle + \beta|\downarrow\rangle \\ |\downarrow\rangle &\to \gamma|\uparrow\rangle + \delta|\downarrow\rangle \end{aligned} \tag{1.1}$$

Isospin-rotations:

$$\begin{aligned} |p\rangle &\to a|p\rangle + b|n\rangle \\ |n\rangle &\to c|p\rangle + d|n\rangle \end{aligned} \tag{1.2}$$

Symmetry group $SU(2)$:

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{or} \quad U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \qquad UU^\dagger = 1\,, \qquad \det U = 1\,. \tag{1.3}$$

! Isospin: Abstract transformation in Hilbert space of the quantum states

Following this line of reasoning we introduce

$$\textbf{Nucleon} = \begin{pmatrix} p \\ n \end{pmatrix} \tag{1.4}$$

is a particle with *isospin* $1/2$; It has two states with *isospin projection* $+1/2$ and $-1/2$.

For small isospin transformations

$$U = \mathbb{I} + i\sum_{a=1}^{3} \delta\phi_a \frac{\tau_a}{2}\,, \qquad \tau_a = \text{Pauli matrices} \tag{1.5}$$

Isospin operators:

$$\begin{aligned} \hat{I}_a &= \frac{1}{2}\tau_a \qquad (\text{cf.}: \ \hat{S}_a = \frac{1}{2}\sigma_a) \\ \hat{I}^2 &= \hat{I}_1^2 + \hat{I}_2^2 + \hat{I}_3^2 \qquad (\text{cf.}: \ \hat{S}^2 = \hat{S}_1^2 + \hat{S}_2^2 + \hat{S}_3^2) \end{aligned} \tag{1.6}$$

Then

$$\begin{aligned} \hat{I}^2|p(n)\rangle &= I(I+1)|p(n)\rangle\,, \qquad I = 1/2 \\ \hat{I}_3|p\rangle &= +\frac{1}{2}|p\rangle\,, \qquad \hat{I}_3|n\rangle = -\frac{1}{2}|n\rangle \end{aligned} \tag{1.7}$$

The operator $\hat{I}^2$ is fully equivalent (mathematically) to the operator of angular momentum; possible eigenvalues are therefore (follows from group theory)

$$I = 0, 1/2, 1, 3/2, \ldots \tag{1.8}$$

Kemmer postulated that π-mesons form a system with isospin $I = 1$:

$$\pi = \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix} \qquad \begin{aligned} I_3 &= +1 \\ I_3 &= \ \ 0 \\ I_3 &= -1 \end{aligned} \tag{1.9}$$

$\hookleftarrow$ This was a prediction for π^0 !

Isospin summation (cf. spin-summation):
NN-states:

$$\text{isospin}\,1/2 \otimes \text{isospin}\,1/2 = \text{isospin}\,1 + \text{isospin}\,0 \tag{1.10}$$

$$\begin{array}{lll} & I_3 = +1 & pp \\ I = 1: & I_3 = 0 & \frac{1}{\sqrt{2}}(pn + np) \qquad \text{triplet} \\ & I_3 = -1 & nn \\ I = 0: & I_3 = 0 & \frac{1}{\sqrt{2}}(pn - np) \qquad \text{singlet (deuteron)} \end{array} \tag{1.11}$$

πN-states:

$$\text{isospin}\,1 \otimes \text{isospin}\,1/2 = \text{isospin}\,1/2 + \text{isospin}\,3/2 \tag{1.12}$$

$\hookleftarrow$ predictions for $\pi N \to \pi N$ scattering etc.

1.1.2 Strange particles and the "8-fold way"

Before 1953 "strange" V-particles were only seen in cosmic rays but eventually could be observed in also in accelerator experiments that allowed for their detailed study. For example

$$\pi^- + p \to K^0 + \Lambda \tag{1.13}$$

for $E_\pi \sim 1.5$ GeV one has measured, e.g.

$$\sigma(\pi^- p \to K^0 \Lambda) \sim 1\,\text{mb} \equiv 10^{-27}\,\text{cm}^2\,,$$
$$\sigma_{\text{tot}}(\pi^- p \to \text{hadrons}) \sim 40\,\text{mb} \tag{1.14}$$

These cross sections correspond (roughly) to geometric cross sections of hadrons

$$R^2 \sim (1\,\text{fm})^2 = 10^{-26}\,\text{cm}^2 \tag{1.15}$$

hence K and Λ are produced via strong interaction.

"Strange" particles decay, e.g.

$$\Lambda \to p + \pi^-, \quad n + \pi^0 \tag{1.16}$$

A natural life time for decays induced by strong interaction would be

$$\tau_{\text{strong}} \sim R/c \sim 10^{-13}\,\text{cm}/3 \cdot 10^{10}\,\text{cm}\,\text{s}^{-1} \sim 10^{-23}\,\text{s} \tag{1.17}$$

The experimentally measured life time is, however

$$\tau_\Lambda \simeq 2.63 \cdot 10^{-10}\,\text{s} \tag{1.18}$$

hence this is a weak decay, similar to $n \to pe^-\nu$.

It seems that strong and electromagnetic decays of these particles are forbidden, why?

$\hookleftarrow$ **New quantum number — "strangeness"**

$$\begin{array}{rr} p, n, \pi^+, \pi^0, \pi^- & S = 0 \\ \Lambda, \Sigma^+, \Sigma^0, \Sigma^- & S = -1 \\ \Xi^0, \Xi^- & S = -2 \\ K^0, K^+ & S = +1 \\ \bar{K}^0, K^- & S = -1 \end{array} \tag{1.19}$$

assume that strangeness is conserved in strong and electromagnetic interactions, e.g.

$$\pi^- + p \to K^0 + \Lambda \qquad 0 + 0 \to +1 - 1 \quad \text{(allowed)}$$
$$K^- + p \to K^0 + \Xi^0 \qquad -1 + 0 \to +1 - 2 \quad \text{(allowed)} \tag{1.20}$$

In addition we have a conserved electric charge Q and baryon number B:

$$\begin{aligned} Baryons: &\quad B = +1 \\ Antibaryons: &\quad B = -1 \\ Mesons: &\quad B = \ \ 0 \end{aligned} \tag{1.21}$$

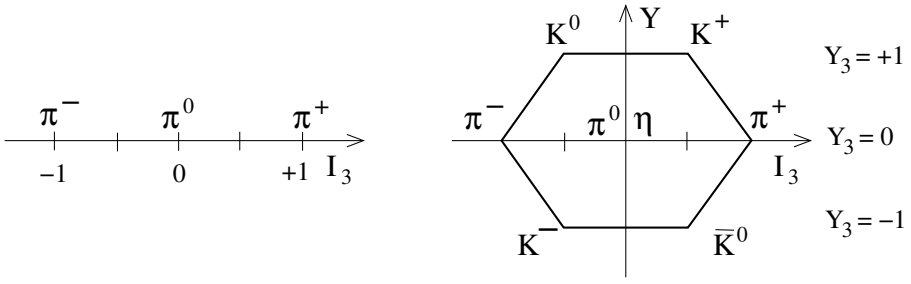
Gell-Mann and Nishijima observed that for all observed particles the following relation holds:

$$\begin{aligned} Q &= I_3 + \frac{1}{2}(S + B)\,, \\ Y &= S + B: \qquad \textbf{Hypercharge} \end{aligned} \tag{1.22}$$

The breakthrough: Gell-Mann, Neeman 1961,1964: The "8-fold way" (← Mahajana-Buddhismus)

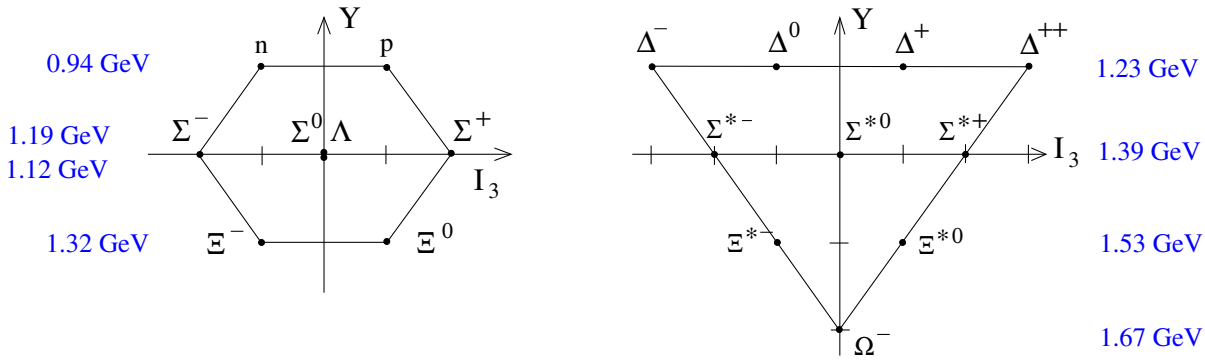
$$\boxed{SU(2)\ \text{(isospin)} \ \Rightarrow\ SU(3)\ \text{(isospin + hypercharge)}} \tag{1.23}$$

Mesons:



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Baryons:



- ↩ Prediction for Ω^- with mass ca. 1670 MeV and $J^P = \frac{3}{2}^+$!

- Symmetry is approximate: different states are not exactly degenerate: splitting appr. 150 MeV

$$\text{Hypercharge symmetry breaking}: \quad \frac{m_\Lambda - m_p}{m_p} \sim 20\% \tag{1.24}$$

It was possible to classify all known hadrons in irreducible representations of the $SU(3)$ group (later) and predict the existence of Ω^-. However, no hadrons could be matched with the representation of the lowest dimension — the fundamental representation of $SU(3)$.

Example: isospin group $SU(2)$ $I = 0, 1/2, 1, 3/2, \ldots$
Fundamenental representation

$$N = \begin{pmatrix} p \\ n \end{pmatrix}, \qquad I = 1/2 \quad I_3 = + \pm 1/2 \tag{1.25}$$

Adjoint representation

$$\pi = \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}, \qquad I = 1 \quad I_3 = -1, 0, 1 \tag{1.26}$$

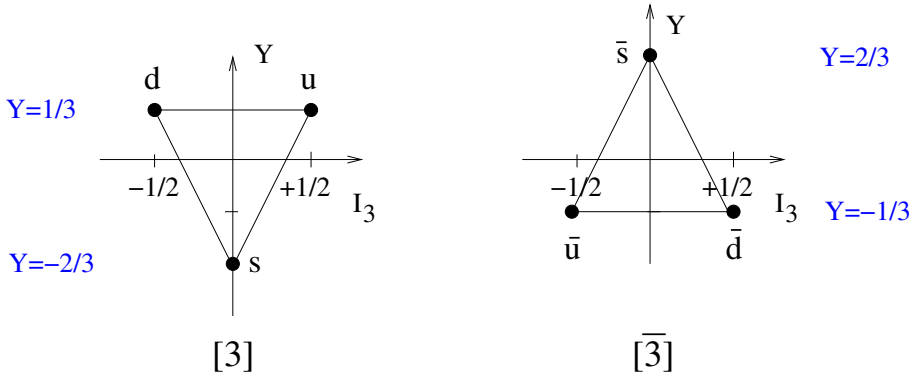
etc.
For the case of the $SU(3)$ classification the analogue of p and n was missing.

Gell-Mann, Zweig 1964 *Quarks*

Quarks	I	I_3	Y	S	B	Q
u	1/2	+1/2	1/3	0	1/3	2/3
d	1/2	-1/2	1/3	0	1/3	-1/3
s	0	0	-2/3	-1	1/3	-1/3

$$Q = I_3 + \frac{1}{2}Y$$

Quarks u, d, s and antiquarks $\bar{u}, \bar{d}, \bar{s}$ transform according the three-dimensional fundamental representations of the $SU(3)$, called [3] and [$\bar{3}$]:

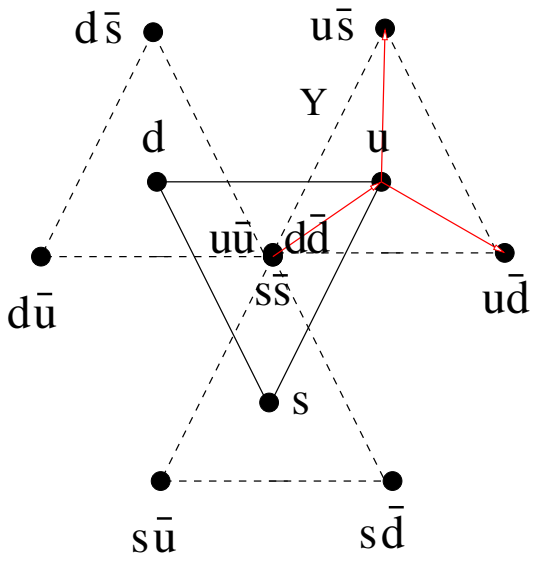


The $SU(3)$ transformations:

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \Rightarrow \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \qquad UU^\dagger = 1\,, \quad \det U = 1 \tag{1.27}$$

Mesons are built from a quark and antiquark:

$$[3] \otimes [\bar{3}] = [1] + [8] \tag{1.28}$$



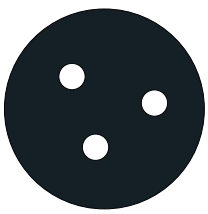
Identify:

$$\begin{aligned} \pi^+ &= u\bar{d}\,, \qquad \pi^- = d\bar{u}\,, \\ K^+ &= u\bar{s}\,, \qquad K^0 = d\bar{s}\,, \qquad \bar{K}^0 = s\bar{d}\,, \qquad K^- = s\bar{u}\,, \\ \pi^0 &= \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})\,, \qquad \eta^0 = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})\,, \\ \eta'^0 &= \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} + s\bar{s}) \qquad \text{(singlet under } SU(3)) \end{aligned} \tag{1.29}$$

Three different quarks u, d, s are usually called "flavors".

1.1.3 The quark model and color

Baryons, in turn, are built of three quarks: **The quark model**

N= 

$m_u \sim m_d \sim 300$ MeV
$m_s \sim 450$ MeV

- **Problem 1:**
  There exist apparently no free quarks in nature — no particles with electric charge $+2/3$ or $-1/3$

$$\hookleftarrow \textbf{Quark confinement}$$

- **Problem 2:**
  $\Omega^-(s = 3/2, s_3 = 3/2)$ is built of three strange quarks:

$$\Omega^- = s^\uparrow s^\uparrow s^\uparrow \tag{1.30}$$

  For the ground state, one expects that the wave function describing space distribution of the three quarks in the nucleon is symmetric, $\Psi(x_1, x_2, x_3) = \Psi(x_2, x_1, x_3)$, etc.

  Hence a totally symmetric wave function for a spin-3/2 particle — contradiction with Pauli principle?

Gell-Mann (1972), Fritzsch (1973): a new degree of freedom:

— Each quark exists in three versions (states), called "colors"

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \qquad d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}, \qquad s = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}, \tag{1.31}$$

! A totally antisymmetric wave function can be built as

$$\Omega^-(3/2, 3/2) = s^\uparrow_\alpha s^\uparrow_\beta s^\uparrow_\gamma \epsilon^{\alpha\beta\gamma} \Psi(x_1, x_2, x_3) \tag{1.32}$$

! This state is invariant under rotations in the color space:

$$q_\alpha \to \sum_{\beta=1}^{3} U_{\alpha\beta} q_\beta \,; \quad \alpha = 1, 2, 3 \qquad q = (u, d, s)]$$

$$UU^\dagger = 1\,, \quad \det U = 1 \tag{1.33}$$

— again a $SU(3)$-group.

$$\underline{\textbf{color}\text{–}SU(3) \quad \neq \quad \textbf{flavor}\text{–}SU(3)}$$

The color–$SU(3)$ symmetry plays a fundamental role in QCD; the flavor–$SU(3)$ is (as we know now) rather accidental.

Postulate:
Only $SU(3)$-invariant ("colorless") states exist in nature.

- Baryons:

$$q_\alpha q_\beta q_\gamma \epsilon^{\alpha\beta\gamma} \to U_{\alpha\alpha'} U_{\beta\beta'} U_{\gamma\gamma'} \epsilon^{\alpha\beta\gamma} q_{\alpha'} q_{\beta'} q_{\gamma'} = \det U \, \epsilon^{\alpha'\beta'\gamma'} q_{\alpha'} q_{\beta'} q_{\gamma'} \tag{1.34}$$

- Mesons:

$$\bar{q}_\alpha q_\alpha \to \bar{q}_{\alpha'} U^\dagger_{\alpha'\alpha} U_{\alpha\beta'} q_{\beta'} = \bar{q}_{\alpha'} q_{\alpha'} \tag{1.35}$$

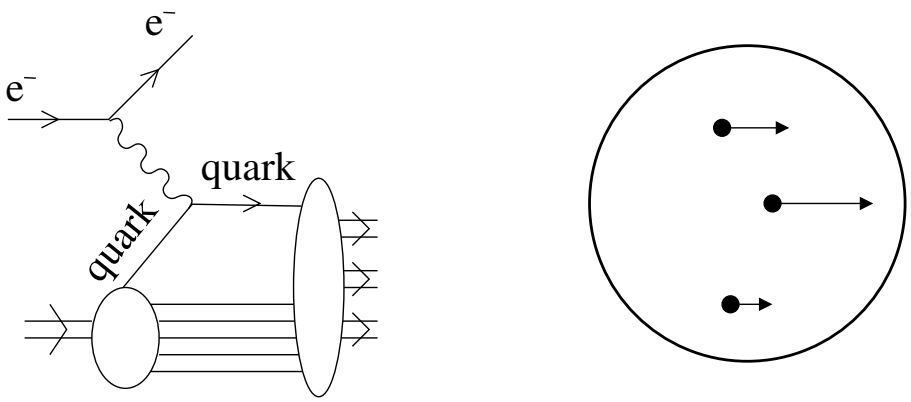
Confinement: only colorless particles exist.

A big question: are quarks mathematical constructs only, or they indeed exist materially inside hadrons?

- 1969: Crucial evidence:
  Scattering of electrons from protons with large momentum transfer (at large angle) (Bjorken)

$$e^-(k_1) + N \to e^-(k_2) + X(\leftarrow \text{ any hadron state}) \tag{1.36}$$

  ! Nucleons contain quasi-free point-like particles inside them ("partons")



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- 1973: Gross, Wilczek, Politzer: **"Asymptotic freedom"**
  — Theory of quark-gluon interactions

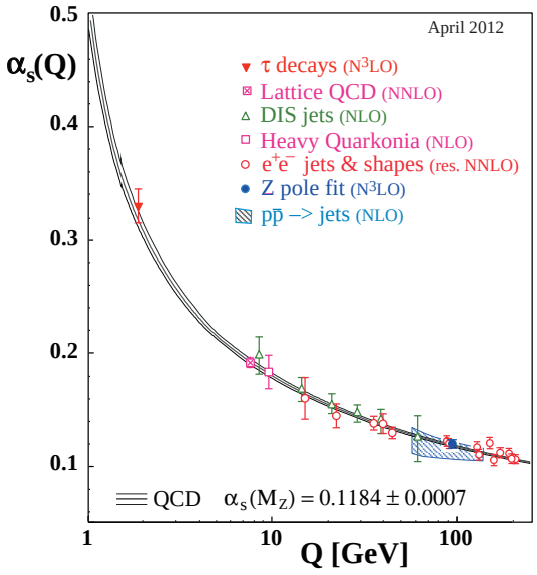
Quantum Chromodynamics

— a "nonabelian gauge theory":

QED	gauge transformations	$U(1)$	$e \to e^{i\phi(x)} e$	$\leftarrow$ photons
QCD	gauge transformations	$SU(3)$	$q \to Uq$	$\leftarrow$ gluons

QCD (color) charge (coupling constant) is small at small distances and becomes large at hadronic scales.

World summary, see [S. Bethke, arXiv:1210.0325]



1.2 Elements of group theory: The $SU(3)$ group

Consider a three-dimensional abstract Hilbert space with orthonormal basis vectors

$$|1\rangle, \quad |2\rangle, \quad |3\rangle; \qquad\qquad \langle i|k\rangle = \delta_{ik} \tag{1.37}$$

these could be e.g. $|1,2,3\rangle = |u\rangle, |d\rangle, |d\rangle$ or $|1,2,3\rangle = |u_1\rangle, |u_2\rangle, |u_3\rangle$

The $SU(3)$ group:

$$\boxed{U|i\rangle = |j\rangle U_{ji} \qquad UU^\dagger = 1\,, \quad \det U = 1} \tag{1.38}$$

Infinitesimal transformations

$$U = \mathbb{I} + i\delta\phi\, H\,, \qquad \delta\phi \in \mathbb{R} \tag{1.39}$$

Then

$$\begin{aligned} &(\mathbb{I} + i\delta\phi\, H)(\mathbb{I} - i\delta\phi\, H^\dagger) \;=\; 1 \\ &\det(\mathbb{I} + i\delta\phi\, H) = e^{\mathrm{Tr}\ln(\mathbb{I}+i\delta\phi\, H)} = 1 + i\delta\phi \mathrm{Tr}\, H \;=\; 1 \end{aligned} \tag{1.40}$$

yields

$$H = H^\dagger\,, \qquad \mathrm{Tr}\, H = 0 \tag{1.41}$$

i.e. H is a hermitian 3×3 matrix with Tr$=0$.

A suitable basis (Gell-Mann matrices)

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{1.42}$$

Normalization convention

$$\mathrm{Tr}\,(\lambda_a\lambda_b) = 2\delta_{ab}\,, \qquad\qquad a,b = 1,2,\ldots,8 \tag{1.43}$$

Each infinitesimal $SU(3)$ transformation can be written as

$$U = \mathbb{I} + i\delta\phi_a\frac{\lambda_a}{2} \tag{1.44}$$

The matrices

$$t^a = \frac{1}{2}\lambda^a \tag{1.45}$$

are called *generators* of the $SU(3)$ transformations.

They satisfy the following (canonical) commutation relations:

$$\boxed{[t_a, t_b] = if_{abc}t^c} \tag{1.46}$$

The f_{abc} symbols are called *structure constants* of the $SU(3)$.

Multiply by t^d and take the trace; since $\mathrm{Tr}\,(t_c t_d) = (1/2)\delta_{cd}$ obtain

$$if_{abc} = 2\,\mathrm{Tr}\,([t_a, t_b]t_c) \tag{1.47}$$

From this representation one can easily see that f_{abc} are totally antisymmetric and real.

An anticommutator of two t^a matrices:

$$\{t_a, t_b\} = c_{ab}\,\mathbb{I} + d_{abc}t_c \tag{1.48}$$

Taking the trace:

$$\mathrm{Tr}\,(\{t_a, t_b\}) = 2\cdot\frac{1}{2}\delta_{ab} = c_{ab}\mathrm{Tr}\,(\mathbb{I}) = 3c_{ab} \quad \Rightarrow \quad c_{ab} = \frac{1}{3}\delta_{ab} \tag{1.49}$$

also

$$d_{abc} = 2\mathrm{Tr}\,(\{t_a, t_b\}t_c) \tag{1.50}$$

Hence d_{abc} symbols are real and totally symmetric in indices.

Using trace folmulas and explicit expressions for λ_a matrices one can calculate f_{abc} and d_{abc} explicitly; these expressions are, however, rarely needed (cf.: Dirac matrices)

Useful identities:

$$\begin{aligned}
&f_{abr}f_{rcs} + f_{bcr}f_{ras} + f_{car}f_{rbs} = 0\\
&f_{abr}d_{rcs} + f_{cbr}d_{ras} = d_{acr}f_{rbs}\\
&f_{ars}f_{brs} = 3\delta_{ab}\\
&d_{aab} = 0 \qquad \leftarrow \text{ summation over "a" implied}\\
&d_{ars}d_{brs} = \frac{5}{3}\delta_{ab}
\end{aligned} \tag{1.51}$$

Very useful identities:

$$f_{abc}t^a t^b = \frac{3}{2} i t^c$$

$$t^a t^a = \frac{4}{3}\mathbb{I}$$

$$t^a t^b t^a = -\frac{1}{6} t^b \tag{1.52}$$

A very powerful identity:

$$(t^a)_{\alpha\beta}(t^a)_{\alpha'\beta'} = \frac{1}{2}\delta_{\alpha\beta'}\delta_{\alpha'\beta} - \frac{1}{6}\delta_{\alpha\beta}\delta_{\alpha'\beta'} \tag{1.53}$$

1.2.1 Representations of the $SU(3)$ group

- A unitary representation of the SU(3) group is a homomorphism

  $$U \to D(U)$$

  of the 3×3 matrices U onto unitary $n \times n$ matrices D,

  $$D(U)D^\dagger(U) = 1$$

  (in general of other dimension), which respects the group multiplication:

  $$D(U)D(V) = D(UV) \tag{1.54}$$

  $D(U)$ can also be viewed (more generally) as linear operators acting on the representation space

  $$R \xrightarrow{D(U)} R \qquad R = \mathbb{C}^n \qquad |\alpha\rangle \in R \quad \text{n-dimensional vectors} \tag{1.55}$$

- A representation is called <u>reducible</u> if it is block-diagonal in a certain basis

  $$D(u) = \begin{pmatrix} D_1(U) & 0 \\ 0 & D_2(U) \end{pmatrix} \quad \begin{matrix} \leftarrow \{ n_1 \\ \leftarrow \{ n_2 \end{matrix} \qquad n_1 + n_2 = n \tag{1.56}$$

  Otherwise it is called <u>irreducible</u>

  A necessary and sufficient condition:
  Represention $D(U)$ is irreducible if and only if

  $$\forall |\alpha\rangle \in R \quad \text{linear combinations of} \quad D(U)|\alpha\rangle \quad \text{span the whole space} \tag{1.57}$$

- Two representations D_1 and D_2 are called equivalent if

  $$\exists S \quad \forall U \qquad S^{-1}D_1(U)S = D_2(U) \tag{1.58}$$

- Simplest representations:
  - $U \to \mathbb{I}$ [1] $\longleftarrow$ trivial representation
  - $U \to U$ [3] $\longleftarrow$ fundamental representation

– $U \to U^*$ $\quad [\bar{3}]$

- Adjoint representation [8]:

  Representation space:

$$R = \mathbb{C}^8 \qquad c = \{c_1, \dots c_8\} \qquad c_a \in \mathbb{C} \quad a = 1, \dots, 8 \tag{1.59}$$

  Let

$$C = c_a t^a \qquad \longleftarrow \quad \text{a complex } 3 \times 3 \text{ matrix with Tr } = 0 \tag{1.60}$$

  Define

$$C \quad \stackrel{D(U)}{\longrightarrow} \quad UCU^\dagger \tag{1.61}$$

  or, equivalently

$$\mathrm{Tr}\,(Ct^a) \to \mathrm{Tr}\,(UCU^\dagger t^a) \quad \Rightarrow \quad \frac{1}{2}c_b \to c_a \mathrm{Tr}\,(Ut^a U^\dagger t^b)$$

L4

  Infinitesimal transformations:

$$U = \mathbb{I} + i\delta\phi_a t^a \quad \longrightarrow \quad D(U) = \mathbb{I} + i\delta\phi_a T^a \tag{1.62}$$

  Lie algebra:

$$[t_a, t_b] = i f_{abc} t^c \quad \longrightarrow \quad [T_a, T_b] = i f_{abc} T^c \tag{1.63}$$

  (The generators in all representations obey the same commutation relations)
  In our case (adjoint representation)

$$\begin{aligned} c_b \longrightarrow\ & 2\mathrm{Tr}\left[(1 + i\delta\phi_c t^c) t^a (1 - i\delta\phi_{c'} t^{c'}) t^b\right] c_a \\ =\ & 2\mathrm{Tr}\,[t^a t^b] c_a + 2i\delta\phi_c \big\{\mathrm{Tr}\,(t^c t^a t^b) - \mathrm{Tr}\,(t^a t^c t^b)\big\} \\ =\ & \big[\delta_{ab} + i\delta\phi_c i f_{cab}\big] c_a \\ \equiv\ & \big[(\mathbb{I})_{ba} + i\delta\phi_c (T^c)_{ba}\big] c_a \end{aligned} \tag{1.64}$$

  It follows

$$\boxed{(T^c)_{ba} = -i f_{cba}} \tag{1.65}$$

  $\hookleftarrow$ generators in the adjoint representation

Example: Classification of the $\bar{q}q$ states under $SU(3)$-flavor

Let

$$q_1 \equiv u \qquad q_2 \equiv d \qquad q_3 \equiv s$$

and assume

$$\begin{aligned} [3] \qquad & |q_i\rangle \quad \longrightarrow \quad |q_j\rangle U_{ji}\,, \qquad U \in SU(3) \\ [\bar{3}] \qquad & |\bar{q}_i\rangle \quad \longrightarrow \quad |\bar{q}_j\rangle U^*_{ji} \end{aligned} \tag{1.66}$$

Now consider quark-antiquark states. Representation space is 9-dimensional R : $\quad |q_i\rangle|\bar{q}_j\rangle$

$$D(U)(|q_i\rangle|\bar{q}_j\rangle) = |q_{i'}\rangle|\bar{q}_{j'}\rangle U_{i'i}U^*_{j'j} \tag{1.67}$$

This representation is reducible:

- $SU(3)$-invariant state:

$$|1\rangle = \frac{1}{\sqrt{3}}|q_i\rangle|\bar{q}_i\rangle = \frac{1}{\sqrt{3}}\big(|u\rangle|\bar{u}\rangle + |d\rangle|\bar{d}\rangle + |s\rangle|\bar{s}\rangle\big) \tag{1.68}$$

  Check

$$|1'\rangle = D(U)|1\rangle = \frac{1}{\sqrt{3}}|q_{i'}\rangle|\bar{q}_{j'}\rangle U_{i'i}U^*_{j'i} = \frac{1}{\sqrt{3}}|q_{i'}\rangle|\bar{q}_{j'}\rangle U_{i'i}U^\dagger_{ij'}$$

$$= \frac{1}{\sqrt{3}}|q_{i'}\rangle|\bar{q}_{j'}\rangle\delta_{i'j'} = |1\rangle \tag{1.69}$$

- An arbitrary orthogonal state

$$|C\rangle = C_{ij}|q_i\rangle|\bar{q}_j\rangle \qquad\qquad C_{ij} = 3\times 3 \text{ matrix with } \operatorname{Tr} C = 0 \tag{1.70}$$

  Check $SU(3)$ transformation:

$$|C'\rangle = D(U)|C\rangle = C_{ij}|q_{i'}\rangle|\bar{q}_{j'}\rangle U_{i'i}U^*_{j'j} = (UCU^\dagger)_{i'j'}|q_{i'}\rangle|\bar{q}_{j'} \tag{1.71}$$

  Thus

$$C' = UCU^\dagger \tag{1.72}$$

  ! This is precisely the transformation rule of the adjoint representation

Result:

$$\boxed{[3]\otimes[\bar{3}] = [1]+[8]} \tag{1.73}$$

Example II: Classification of three-quark states:

$$\boxed{[3]\otimes[3]\otimes[3] = [1]+[8]+[8]+[10]} \tag{1.74}$$

Start with the first pair:

$$[3]\otimes[3] = [\bar{3}]+[6]$$

$$\swarrow \qquad \searrow$$

$$\epsilon_{ijk}|q_i\rangle|q_k\rangle \qquad |q_i\rangle|q_k\rangle + |q_k\rangle|q_i\rangle$$

Add the third quark:

$$[\bar{3}]\otimes[3] = [1]+[8]$$

$$[6]\otimes[3] = [8]+[10]$$

$$\searrow$$

$$|q_i\rangle|q_j\rangle|q_k\rangle + \text{ permutations}$$

Symmetry in quantum mechanics:

$$[\hat{H}, T^a] = 0 \tag{1.75}$$

Example: Angular momentum

$$[\hat{H}, \vec{L}] = 0\,, \qquad [\hat{L}_i, \hat{L}_j] = \frac{1}{2}\underbrace{\epsilon_{ijk}}_{\downarrow}\hat{L}_k \tag{1.76}$$

structure constants of $SU(2) \sim SO(3)$

Hence one additive quantum number:

$$L_z|\Psi\rangle = m|\Psi\rangle\,, \qquad [L_x, L_z] \neq 0 \qquad [L_y, L_z] \neq 0 \tag{1.77}$$

$\hookleftarrow$ $SO(3)$ group has rank one.
For $SU(3)$:

$$[T_3, T_8] = 0 \qquad \Leftarrow \quad \text{group has rank two} \tag{1.78}$$

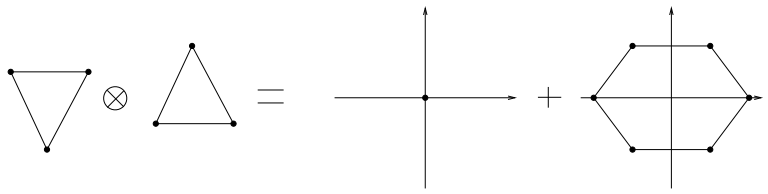
therefore can require that simultaneously

$$\hat{T}_3|\Psi\rangle = t_3|\Psi\rangle\,, \qquad \hat{T}_8|\Psi\rangle = t_8|\Psi\rangle \tag{1.79}$$

$\hookleftarrow$ two quantum numbers, $\Psi = \Psi(t_3, t_8)$.
Following Gell-Mann we identify (for $SU(3)$-flavor)

$$\begin{aligned} &I_1 = t_1, \qquad I_2 = t_2, \qquad I_3 = t_3, \qquad \text{(isospin)} \\ &Y = \frac{2}{\sqrt{3}}t_8 \qquad\qquad\qquad\qquad \text{(hypercharge)} \end{aligned} \tag{1.80}$$

The eigenvalues of I_3 and Y that occur in a given representation can be shown as points in the isospin-hypercharge plane, for example



2 Nonabelian quantum field theories

2.1 Geometry of gauge invariance

Weil (1923): gauge invariance

$$\left.\begin{aligned} \psi(x) \longrightarrow \psi'(x) &= e^{i\alpha(x)}\psi(x) \\ A_\mu(x) \longrightarrow A'_\mu(x) &= A_\mu(x) - \tfrac{1}{e}\partial_\mu\alpha(x) \end{aligned}\right\} \Longrightarrow \mathcal{L}_{\text{QED}}(\psi, A) = \mathcal{L}_{\text{QED}}(\psi', A') \tag{2.1}$$

[A_μ transformations introduced by Maxwell; Weil added ψ]
Modern interpretation:

Let $\psi(x) \longrightarrow \psi'(x) = e^{i\alpha(x)}\psi(x)$ and require $\mathcal{L}(\psi) = \mathcal{L}(\psi')$

What is the most general form of Lagrange density consistent with this symmetry?

- Simple:

$$m\bar{\psi}\psi\,, \qquad g^2(\bar{\psi}\psi)^2\,, \ldots \qquad \text{all allowed} \tag{2.2}$$

- Complicated: derivatives

$$\begin{aligned} n^\mu\partial_\mu\psi(x) &:= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big[\psi(x+\epsilon n) - \psi(x)\Big] \\ &\longrightarrow \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big[e^{i\alpha(x+\epsilon n)}\psi(x+\epsilon n) - e^{i\alpha(x)}\psi(x)\Big] \qquad \text{bad} \end{aligned} \tag{2.3}$$

General solution (differential geometry)

In addition to $\psi(x)$, consider a function of two variables $U(y,x)$ with transformation property

$$\begin{aligned} U(y,x) &\longrightarrow U'(y,x) = e^{i\alpha(y)}U(y,x)e^{-i\alpha(x)} \\ U(x,x) &= 1 \end{aligned} \tag{2.4}$$

Its utility is that $\psi(y)$ and $U(y,x)\psi(x)$ have the same transformation laws.

Define **covariant derivative**:

$$n^\mu D_\mu\psi(x) := \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big[\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)\Big] \tag{2.5}$$

Simplest choice:

$$UU^* = 1\,, \qquad U(y,x) = e^{i\phi(y,x)} \tag{2.6}$$

Then

$$\begin{aligned} U(x+\epsilon n, x) &= 1 + i\epsilon n^\mu \boxed{\frac{\partial}{\partial y^\mu}\phi(y,x)\Big|_{y=x}} + \ldots \\ &:= 1 - i\epsilon\epsilon n^\mu \boxed{eA_\mu(x)} \qquad \text{e = arbitrary constant} \end{aligned} \tag{2.7}$$

! A new vector function $A_\mu(x)$ — **a vector field**

Math termonology: U is called a comparator of local symmetry transformations
A_μ is called a connection, it appears in a local limit of U *

*Remark: One can choose

$$U(y,x) = \text{Pexp}\Big\{\int_0^1 du\,(y-x)^\mu A_\mu(ux+(1-u)y)\Big\} + \text{additional vector fields} \tag{2.8}$$

Thus

$$\boxed{D_\mu\psi(x) := (\partial_\mu + ieA_\mu)\psi(x)} \tag{2.9}$$

From the transformation law

$$1 - i\epsilon n^\mu eA_\mu(x) \longrightarrow 1 - i\epsilon n^\mu eA'_\mu(x) = e^{i\alpha(x+\epsilon n)}\big[1 - i\epsilon n^\mu eA_\mu(x)\big]e^{-i\alpha(x)} \tag{2.10}$$

follows

$$A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x) \tag{2.11}$$

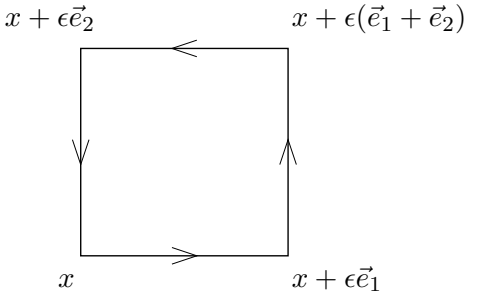
so that

$$\begin{aligned} D_\mu\psi(x) &\longrightarrow D'_\mu\psi'(x) \\ &= \Big[\partial_\mu + ie\Big(A_\mu - \frac{1}{e}\partial_\mu\alpha\Big)\Big]e^{i\alpha(x)}\psi(x) \\ &= e^{i\alpha(x)}D_\mu\psi(x) \end{aligned} \tag{2.12}$$

Thus we are allowed to have in $\mathcal{L}$ terms like

$$\bar\psi \not{D}\psi\,, \qquad \bar\psi D_\mu D^\mu\psi\,, \quad \text{etc.} \tag{2.13}$$

What else? L5



$$W(x) = U(x, x+\epsilon\vec{e}_2)U(x+\epsilon\vec{e}_2, x+\epsilon\vec{e}_1+\epsilon\vec{e}_2)U(x+\epsilon\vec{e}_1+\epsilon\vec{e}_2, x+\epsilon\vec{e}_1)U(x+\epsilon\vec{e}_1, x) \tag{2.14}$$

A straightforward calculation for $\epsilon \to 0$

$$\begin{aligned} W(x) &= 1 - i\epsilon^2 e\Big[\partial_1 A_2(x) - \partial_2 A_1(x)\Big] + \mathcal{O}(\epsilon^3) \\ W(x) &= \text{ invariant} \quad \Longrightarrow \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \text{ invariant} \\ &[W'(x) = W(x) \Rightarrow F'_{\mu\nu} = F_{\mu\nu}] \end{aligned} \tag{2.15}$$

Alternatively, consider

$$[D_\mu, D_\nu]\psi = [\partial_\mu, \partial_\nu]\psi + ie\Big([\partial_\mu, A_\nu] - [\partial_\nu, A_\mu]\Big)\psi - e^2[A_\mu, A_\nu]\psi \tag{2.16}$$

The first and the last terms obviously vanish. The other two:

$$[\partial_\mu, A_\nu]\psi = \partial_\mu(A_\nu\psi) - A_\nu\partial_\mu\psi = \underbrace{(\partial_\mu A_\nu(x))}_{\text{derivative only acts on A!}}\cdot\psi(x) \tag{2.17}$$

Hence

$$[D_\mu, D_\nu]\psi = ie\Big(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\Big)\psi(x) \qquad \boxed{[D_\mu, D_\nu] = ieF_{\mu\nu}} \tag{2.18}$$

Summing up, the QED Lagrangian

$$\mathcal{L}_{\text{QED}} = \bar\psi(i\not{D})\psi - m\bar\psi\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.19}$$

- need a real function with mass dimension four
- could add

$$\ldots - c\,\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} \qquad \text{parity conservation } \Rightarrow c = 0 \tag{2.20}$$

- could add

$$\ldots + c_5\bar\psi D^2\psi + c_6(\bar\psi\psi)^2 + \ldots \tag{2.21}$$

  - not renormalizable
  - $c_5 \sim \dfrac{1}{\Lambda_{UV}}\,, c_5 \sim \dfrac{1}{\Lambda_{UV}^2}$

QED Lagrangian is defined (almost) uniquely by the requirements

- relativistic (Lorentz) invariance
- local gauge symmetry $U(1)$

! Photon exists "because" we require local gauge symmetry !

A very powerful idea: construct physical theories starting with geometric symmetry principles.

2.2 The Yang-Mills Lagrangian

Let

$$\psi = \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} \tag{2.22}$$

Global $SU(2)$ transformations:

$$\psi \longrightarrow \psi' = e^{i\alpha^k\frac{\sigma_k}{2}}\psi; \qquad \sigma_k = \text{ Pauli matrices} \tag{2.23}$$

for example isospin — original motivation for YM

Local $SU(2)$ transformations (YM):

$$\psi(x) \longrightarrow \psi'(x) = e^{i\alpha^k(x)\frac{\sigma_k}{2}}\psi(x); \qquad \sigma_k = \text{ Pauli matrices} \tag{2.24}$$

Y&M asked: how to built a theory (Lagrangian) invariant under these trafos?

Main difference to QED: the symmetry is nonabelian:

$$e^{i\alpha}\cdot e^{i\beta} = e^{i\beta}\cdot e^{i\alpha} \qquad \text{but} \qquad e^{i\alpha^k\frac{\sigma_k}{2}}\cdot e^{i\beta^k\frac{\sigma_k}{2}} \neq e^{i\beta^k\frac{\sigma_k}{2}}\cdot e^{i\alpha^k\frac{\sigma_k}{2}} \tag{2.25}$$

Comparator of local $SU(2)$ transformations

$$\begin{aligned} &U(y,x) = 2\times 2 \text{ matrix}\,, \qquad UU^\dagger = 1 \\ &U(y,x) \longrightarrow V(y)U(y,x)V^\dagger(x) \end{aligned} \tag{2.26}$$

with

$$V(x) = e^{i\alpha^k(x)\frac{\sigma_k}{2}}\,, \qquad V(x)V^\dagger(x) = \mathbb{I} \tag{2.27}$$

It follows

$$U(x+\epsilon n, x) = \mathbb{I} + ig\epsilon n^\mu A^k_\mu \frac{\sigma^k}{2} + \mathcal{O}(\epsilon^2) \tag{2.28}$$

where g (arbitrary constant) will be called "coupling".
Covariant derivative

$$\boxed{D_\mu = \partial_\mu - igA^k_\mu\frac{\sigma^k}{2} \;\equiv\; \partial_\mu\cdot\mathbb{I} - igA^k_\mu\frac{\sigma^k}{2}} \tag{2.29}$$

Transformation rule for A^k_μ follows from

$$\mathbb{I} + ig\epsilon n^\mu A^k_\mu\frac{\sigma^k}{2} \longrightarrow \mathbb{I} + ig\epsilon n^\mu (A^k_\mu)'\frac{\sigma^k}{2} = V(x+\epsilon n)\Big(\mathbb{I} + ig\epsilon n^\mu A^k_\mu\frac{\sigma^k}{2}\Big)V^\dagger(x) \tag{2.30}$$

where we have to expand everything to $\mathcal{O}(\epsilon)$.

First term:

$$\begin{aligned} V(x+\epsilon n)V^\dagger(x) &= \left(\Big[1 + \epsilon n^\mu\frac{\partial}{\partial x^\mu} + \mathcal{O}(\epsilon^2)\Big]V(x)\right)V^\dagger(x) \\ &= \mathbb{I} + \epsilon n^\mu\left(\frac{\partial}{\partial x^\mu}V(x)\right)V^\dagger(x) + \mathcal{O}(\epsilon^2) \\ &= \mathbb{I} - \epsilon n^\mu V(x)\left(\frac{\partial}{\partial x^\mu}V^\dagger(x)\right) + \mathcal{O}(\epsilon^2) \end{aligned} \tag{2.31}$$

Therefore, comparing terms in $\sim ig\epsilon n^\mu$

$$\boxed{A^k_\mu\frac{\sigma^k}{2} \to V(x)\left(A^k_\mu\frac{\sigma^k}{2} + \frac{i}{g}\partial_\mu\right)V^\dagger(x)} \tag{2.32}$$

For small transformations

$$V(x) = \mathbb{I} + i\alpha^k(x)\frac{\sigma^k}{2} + \dots \tag{2.33}$$

this becomes

$$A^k_\mu \frac{\sigma^k}{2} \to A^k_\mu \frac{\sigma^k}{2} + \frac{1}{g}(\partial_\mu \alpha^k)\frac{\sigma^k}{2} + \underbrace{i\left[\alpha^k \frac{\sigma^k}{2}, A^j_\mu \frac{\sigma^j}{2}\right]}_{\text{new !}} \tag{2.34}$$

Check transformation for the covariant derivative:

$$\begin{aligned} D_\mu\psi \longrightarrow D'_\mu\psi' &= \left(\partial_\mu - ig(A')^k_\mu \frac{\sigma^k}{2}\right)\psi' \\ &= \left(\partial_\mu - igA^k_\mu\frac{\sigma^k}{2} - i(\partial_\mu\alpha^k)\frac{\sigma^k}{2} + g\left[\alpha^k\frac{\sigma^k}{2}, A^j_\mu\frac{\sigma^j}{2}\right]\right)\left(1 + i\alpha^k\frac{\sigma^k}{2}\right)\psi \\ &= \left(1 + i\alpha^k\frac{\sigma^k}{2}\right)\left(\partial_\mu - igA^k_\mu\frac{\sigma^k}{2}\right)\psi + \mathcal{O}(\alpha^2) = V(x)D_\mu\psi + \mathcal{O}(\alpha^2) \qquad \boxed{OK} \end{aligned} \tag{2.35}$$

As a consequence

$$[D_\mu, D_\nu]\psi(x) \longrightarrow V(x)[D_\mu, D_\nu]\psi(x) = \underbrace{V(x)[D_\mu, D_\nu]V^\dagger(x)}\,\underbrace{V(x)\psi(x)} \tag{2.36}$$

Define a nonabelian field strength tensor (Feldstärke) as

$$\boxed{[D_\mu, D_\nu] = -igF^k_{\mu\nu}\frac{\sigma^k}{2}, \qquad F^k_{\mu\nu}\frac{\sigma^k}{2} \longrightarrow V(x)F^k_{\mu\nu}\frac{\sigma^k}{2}V^\dagger(x)} \tag{2.37}$$

Inserting explicit expression for the covariant derivative this becomes

$$F^k_{\mu\nu}\frac{\sigma^k}{2} = \partial_\mu A^k_\nu\frac{\sigma^k}{2} - \partial_\nu A^k_\mu\frac{\sigma^k}{2} - ig\left[A^k_\mu\frac{\sigma^k}{2}, A^j_\nu\frac{\sigma^j}{2}\right] \tag{2.38}$$

Use

$$\left[\frac{\sigma^k}{2}, \frac{\sigma^j}{2}\right] = i\epsilon^{kjl}\frac{\sigma^l}{2} \tag{2.39}$$

multiply by σ^p and take the trace:

$$\boxed{F^k_{\mu\nu} = \partial_\mu A^k_\nu - \partial_\nu A^k_\mu + g\epsilon^{kjl}A^j_\mu A^l_\nu} \tag{2.40}$$

! $F^k_{\mu\nu}$ is not yet $SU(2)$ invariant, but this is easy to repair:

$$\mathrm{Tr}\left[(F^k_{\mu\nu}\frac{\sigma^k}{2})^2\right] = \frac{1}{2}F^k_{\mu\nu}F^{\mu\nu,k} = \text{ invariant} \tag{2.41}$$

Thus, a possible Lagrangian invariant under local $SU(2)$ is (Yang-Mills)

$$\boxed{\mathcal{L}_{\rm YM} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}(F^k_{\mu\nu})^2} \tag{2.42}$$

! Very simple and very similar to QED

Euler-Lagrange equations:

- "Dirac":

$$(i\not{D} - m)\psi = 0 \qquad \text{! field hidden inside D} \tag{2.43}$$

- "Maxwell":

$$\partial^\mu F^k_{\mu\nu} + \underbrace{g\epsilon^{kjl}A^{j,\mu}F^l_{\mu\nu}} = -g\bar{\psi}\gamma_\nu\frac{\sigma^k}{2} \equiv j^k_\nu\,, \quad [\leftarrow \text{ the } SU(2)\text{-charge curent}]$$

$$\text{! non-linear equation: terms} \sim A^2, A^3 \tag{2.44}$$

Generalization to $SU(3)$, [and $SU(4)$, $\dots SU(N)$] is trivial:

$$\psi = \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\\ \psi_3(x)\end{pmatrix} \tag{2.45}$$

Local $SU(3)$ transformations:

$$\begin{aligned}&\psi(x) \longrightarrow V(x)\psi(x)\\ &V(x) = 1 + i\alpha^a(x)t^a + \mathcal{O}(\alpha^2)\,, \qquad a = 1, 2, \dots, 8\end{aligned} \tag{2.46}$$

so that

$$\begin{aligned}&\frac{\sigma^a}{2} \;\longrightarrow\; t^a = \frac{\lambda^a}{2} \quad \text{Gell-Mann matrices}\\ &\big[\frac{\sigma^k}{2}, \frac{\sigma^j}{2}\big] = i\epsilon^{kjl}\frac{\sigma^l}{2} \;\longrightarrow\; [t^a, t^b] = if^{abc}t^c\end{aligned} \tag{2.47}$$

It follows

$$\begin{aligned}\psi \;&\longrightarrow\; (1 + i\alpha^a t^a)\psi\\ A^a_\mu \;&\longrightarrow\; A^a_\mu + \frac{1}{g}\partial_\mu\alpha^a + f^{abc}A^b_\mu\alpha^c\end{aligned} \tag{2.48}$$

and further

$$\begin{aligned}[D_\mu, D_\nu] &= -igF^a_{\mu\nu}t^a\\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu\\ D_\mu &= \partial_\mu - igA^a_\mu t^a\end{aligned} \tag{2.49}$$

Remark: Sign of g is a convention, differs in various textbooks

and finally

$$\boxed{\mathcal{L}_{\rm QCD} = \mathcal{L}_{\rm SU(3)} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \sum_{\psi=u,d,s,\ldots} \bar\psi(i\not{D} - m_\psi)\psi} \tag{2.50}$$

L6

2.3 Quantization and Feynman rules

Assume that QCD can be quantized in the same way as QED (will have some surprises).

Generic Green functions:

$$\langle\Omega|T\{\hat{\bar\psi}^i_\alpha(x_1)\hat A^a_\mu(x_2)\hat\psi^j_\beta(x_3)\ldots\}|\Omega\rangle = \frac{\langle 0|T\{(\hat{\bar\psi}_I)^i_\alpha(x_1)(\hat A_I)^a_\mu(x_2)(\hat\psi_I)^j_\beta(x_3)\ldots e^{i\int d^4x\,\mathcal{L}_I(x)}\}|0\rangle}{\langle 0|T\{e^{i\int d^4x\,\mathcal{L}_I(x)}\}|0\rangle} \tag{2.51}$$

Here

- $\hat\psi\ldots$ on the l.h.s. are Heisenberg operators
- $|\Omega\rangle$ on the l.h.s. is the exact vacuum (ground state)
- All operators on the r.h.s. are written in the interaction representation
- $|0\rangle$ on the r.h.s. is the "perturbative" vacuum (ground state if interactions are switched off)

Then:

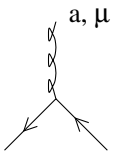
Propagators:

$$\begin{aligned}\langle 0|T\{\psi^i_\alpha(x)\bar\psi^j_\beta(y)\}|0\rangle &= \delta^{ij}\int\frac{d^4p}{(2\pi)^4 i}e^{-ip(x-y)}\left(\frac{1}{m-\not{p}-i\epsilon}\right)_{\alpha\beta}\\ \langle 0|T\{A^a_\mu(x)A^b_\nu(y)\}|0\rangle &= \delta^{ab}\int\frac{d^4k}{(2\pi)^4 i}e^{-ik(x-y)}\frac{1}{k^2+i\epsilon}\Big[g_{\mu\nu}-\xi\frac{k_\mu k_\nu}{k^2}\Big]\end{aligned} \tag{2.52}$$

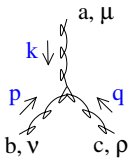
Interaction:

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_I\\ \mathcal{L}_I &= g\bar\psi A^a_\mu t^a\gamma^\mu\psi - gf^{abc}(\partial_\alpha A^a_\mu)A^{\alpha,b}A^{\mu,c} - \frac{1}{4}g^2(f^{eab}A^a_\mu A^b_\nu)(f^{ecd}A^{\mu,c}A^{\nu,d})\end{aligned} \tag{2.53}$$

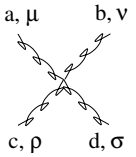
Vertices:



$$= ig\gamma_\mu t^a$$



$$= gf^{abc}\big[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\big]$$



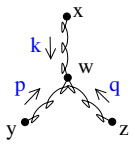
$$\begin{aligned} = -ig^2\big[&f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ &+ f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ &+ f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\big] \end{aligned} \tag{2.54}$$

For example:

$$\langle\Omega|T\{A^d_\alpha(x)A^e_\beta(y)A^f_\lambda(z)\}|\Omega =$$

$$= i\int d^4w\langle 0|T\{T\{A^d_\alpha(x)A^e_\beta(y)A^f_\lambda(z)\mathcal{L}_I(w)\}|0\rangle$$

$$= i\int d^4w\langle 0|T\{A^d_\alpha(x)A^e_\beta(y)A^f_\lambda(z)\big[-gf^{abc}(\partial_\rho A^a_\mu(w))A^{\rho,b}(w)A^{\mu,c}(w)\big]\}|0\rangle$$



$$:= \int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}e^{-ikx-ipy-iqz}(2\pi)^4\delta^{(4)}(p+q+k)$$

$$\times\frac{g_{\alpha\alpha'}}{k^2}\frac{g_{\beta\beta'}}{p^2}\frac{g_{\lambda\lambda'}}{q^2}\cdot(\text{Vertex})^{\alpha'\beta'\gamma'}_{def} \tag{2.55}$$

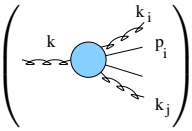
Note $\partial_\rho \longrightarrow -ik_\rho$ if the momentum points into the diagram (Einlaufender Impuls).

Set of Feynman rules defines a quantum theory, but is it selfconsistent?

2.4 Faddeev-Popov ghost fields

- In QED, selfconsistency requires that the amplitudes satisfy Ward identities

$$k^\mu \cdot \left(\quad \right) = 0 \tag{2.56}$$



for arbitrary photon momenta and for electrons on mass shell $p^2 = m^2$.

$\Longrightarrow$ electromagnetic charge is conserved in quantum theory
$\Longrightarrow$ photon remains massless
$\Longrightarrow$ results of calculations do not depend on gauge parameter, etc.

- In QCD, gluons are charged, so maybe we have to impose a condition $k^2 = 0$ but otherwise we expect that Ward identities must hold true.

2.4.1 Reminder: Photon polarization vectors

Consider a real photon $k^2 = 0$. Let

$$k_\mu = \{k, 0, 0, k\} \tag{2.57}$$

The four-vector $A_\mu(k)$ of a real photon can be decomposed in the basis

$$\begin{aligned} e_\mu^{(1)} &= \{0, 1, 0, 0\} \qquad e_\mu^{(2)} = \{0, 0, 1, 0\} \\ e_\mu^{(+)} &= \frac{1}{\sqrt{2}}\{1, 0, 0, 1\} \qquad e_\mu^{(-)} = \frac{1}{\sqrt{2}}\{1, 0, 0, -1\} \end{aligned} \tag{2.58}$$

Only first two possibilities (transverse polarizations) are physical because the other two can be disposed of by the choice of gauge:

- Lorentz gauge

$$\begin{aligned} \partial^\mu A_\mu(x) = 0 \implies & k^\mu A_\mu(k) = 0 \\ \implies & A_\mu(k) = e_\mu^{(-)} A_-(k) \quad \text{not allowed} \quad [k^\mu e_\mu^{(-)} = 2] \end{aligned} \tag{2.59}$$

- For the special case $k^2 = 0$ Lorentz condition does not specify the gauge uniquely

$$\begin{aligned} 0 = k^\mu A_\mu(k) = k^\mu A'_\mu(k) = k^\mu (A_\mu + k_\mu \lambda(k)) \\ \implies A_\mu(k) = e_\mu^{(+)} A_+(k) \sim k_\mu \quad \text{can be gauged away} \end{aligned} \tag{2.60}$$

Thus, emission of "plus" or "minus" photons cannot influence any observable quantities
Note: it does not mean that we always take Lorentz gauge.

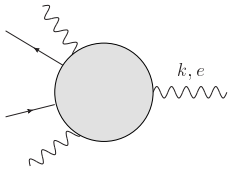
2.4.2 Reminder: Current conservation and Ward Identity in QED

- $U(1)$ gauge symmetry $\Rightarrow$ conserved current (Noether theorem)

$$j^\mu(x) = -e\bar{\psi}(x)\gamma^\mu\psi(x)\,, \qquad \partial_\mu j^\mu(x) = 0$$

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi - j_\mu(x)A^\mu(x) \tag{2.61}$$

- Photon emission in QED:



$$iM(k) = iM^\mu(k)\epsilon^*_\mu(k)$$

$$M^\mu(k) = \int d^4x\, e^{ikx}\langle f|\hat{j}^\mu(x)|i\rangle \tag{2.62}$$

- Ward Identity = Current conservation in quantum theory:

$$0 = k_\mu M^\mu(k) = \int d^4x\, \langle f|\hat{j}^\mu(x)|i\rangle \left(-i\frac{\partial}{\partial x^\mu}\right) e^{ikx} = i\int d^4x\, e^{ikx}\langle f|\partial_\mu\hat{j}^\mu(x)|i\rangle \tag{2.63}$$

- Unitarity (conservation of probability) in quantum mechanics:

$$\frac{d}{dt}\int d^3x\, |\Psi(x,t)|^2 = 0 \qquad \Longleftarrow \text{ Hamiltonian is a hermitian operator} \tag{2.64}$$

- Unitarity in QED: unphysical photons cannot be produced in collisions of "physical" particles

Total cross section for photon emission:

$$\sigma \sim \sum_{\substack{phys.\\ polar.}} |M|^2 = \sum_{\substack{phys.\\ polar.}} \epsilon^{(\lambda),*}_\mu \epsilon^{(\lambda)}_\nu M^\mu M^{\nu,*} = |M^1|^2 + |M^2|^2 \tag{2.65}$$

However

$$k_\mu M^\mu = 0 \implies k_0 M^0 - k_3 M^3 = 0 \implies M^0 = M^3 \tag{2.66}$$

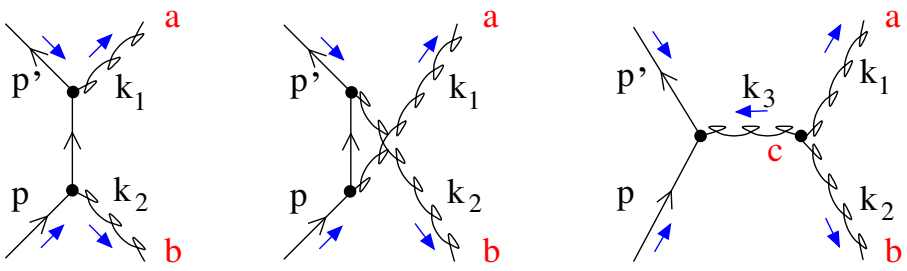
Therefore can write also

$$\sum_{\substack{phys.\\ polar.}} \epsilon^{(\lambda),*}_\mu \epsilon^{(\lambda)}_\nu M^\mu M^{\nu,*} = |M^1|^2 + |M^2|^2 + |M^3|^2 - |M^0|^2 = -g_{\mu\nu}M^\mu M^{\nu,*} = \sum_{\substack{all\\ polar.}} \epsilon^{(\lambda),*}_\mu \epsilon^{(\lambda)}_\nu M^\mu M^{\nu,*} \tag{2.67}$$

i.e. the sum over transverse polarizations is equal to the sum over all polarizations.
Accepted terminology: The QED S-matrix is unitary L7

2.4.3 Quark-antiquark annihilation into a pair of gluons

Let us check what happens in QCD on a simple example:



$$q(p) + \bar{q}(p') \to g^a(k_1) + g^b(k_2) \tag{2.68}$$

• The first two diagrams together:

$$iM^{\mu\nu}_{1,2}\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2) = \epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2) \times (ig)^2\bar{v}(p')\left\{\gamma^\mu t^a \frac{i}{\not p - \not k_2 - m}\gamma^\nu t^b + \gamma^\nu t^b \frac{i}{\not k_2 - \not p' - m}\gamma^\mu t^a\right\}u(p) \tag{2.69}$$

Replace

$$\epsilon^*_\nu(k_2) \longrightarrow k_{2\nu} \tag{2.70}$$

Obtain

$$iM^{\mu\nu}_{1,2}\epsilon^*_\mu(k_1)k_{2\nu} = \epsilon^*_\mu(k_1)(ig)^2\bar{v}(p')\left\{\gamma^\mu t^a \frac{i}{\not p - \not k_2 - m}\not k_2 t^b + \not k_2 t^b \frac{i}{\not k_2 - \not p' - m}\gamma^\mu t^a\right\}u(p) \tag{2.71}$$

Thanks to Dirac equation can replace

$$\begin{aligned} (\not p - m)u(p) = 0, &\qquad \text{in the first term} &\qquad \not k_2 u(p) = (\not k_2 - \not p + m)u(p) \\ \bar{v}(p')(\not p' + m) = 0, &\qquad \text{in the second term} &\qquad \bar{v}(p')\not k_2 = \bar{v}(p')(\not k_2 - \not p' - m) \end{aligned} \tag{2.72}$$

The propagators cancel and we get

$$\begin{aligned} iM^{\mu\nu}_{1,2}\epsilon^*_\mu(k_1)k_{2\nu} &= \epsilon^*_\mu(k_1)(ig)^2\bar{v}(p')\Big\{-i\gamma^\mu[t^a,t^b]\Big\}u(p) \\ &= -g^2\epsilon^*_\mu(k_1)\bar{v}(p')\gamma^\mu u(p) f^{abc}t^c \end{aligned} \tag{2.73}$$

• The third diagram:

$$iM^{\mu\nu}_3\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2) = \epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2) \tag{2.74}$$
$$\times (ig)\bar{v}(p')\gamma_\rho t^c u(p)\frac{-i}{k_3^2}gf^{abc}\Big[g^{\mu\nu}(k_2-k_1)^\rho + g^{\nu\rho}(k_3-k_2)^\mu + g^{\rho\mu}(k_1-k_3)\Big]$$

$$\begin{aligned} &\text{now replace} && \epsilon^*_\nu(k_2) \longrightarrow k_{2\nu} \\ &\text{and use} && k_1 + k_2 + k_3 = 0 \longrightarrow k_2 = -k_1 - k_3 \end{aligned} \tag{2.75}$$

Then

$$\begin{aligned} \epsilon^*_\nu(k_2)[***] &\longrightarrow k_{2\nu}[***] \\ &= k_2^\mu (k_2 - k_1)^\rho + k_2^\rho (k_3 - k_2)^\mu + g^{\rho\mu}(k_1 - k_3)\cdot k_2 \\ &= (-k_1 - k_3)^\mu (-2k_1 - k_3)^\rho + (-k_1 - k_3)^\rho (2k_3 + k_1)^\mu + g^{\rho\mu}(k_1 - k_3)\cdot(-k_1 - k_3) \\ &= g^{\rho\mu} k_3^2 - k_3^\rho k_3^\mu - g^{\rho\mu} k_1^2 + k_1^\rho k_1^\mu \end{aligned} \tag{2.76}$$

Hence

$$iM_3^{\mu\nu}\epsilon^*_\mu(k_1)k_{2\nu} = \epsilon^*_\mu(k_1)(ig)\bar v(p')\gamma_\rho t^c u(p)\frac{-i}{k_3^2} g f^{abc}\Big[g^{\rho\mu}k_3^2 - k_3^\rho k_3^\mu - g^{\rho\mu}k_1^2 + k_1^\rho k_1^\mu\Big]$$

Assume $k_1^2 = 0$ (on-shell) and $\epsilon^*_\mu(k_1)k_1^\mu = 0$ (physical polarization). Then:

- the last two terms vanish
- the second term vanishes as well:

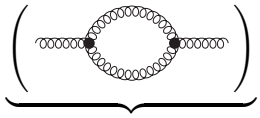
$$(-k_3^\rho)\bar v(p')\gamma_\rho u(p) = \bar v(p')[(\not p' + m) + (\not p - m)]u(p) = 0 \tag{2.77}$$

- the first term gives:

$$iM_3^{\mu\nu}\epsilon^*_\mu(k_1)k_{2\nu} = \epsilon^*_\mu(k_1) g^2 \bar v(p')\gamma^\mu u(p) f^{abc} t^c \tag{2.78}$$

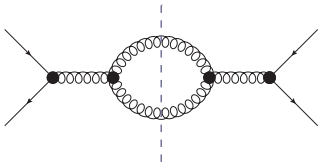
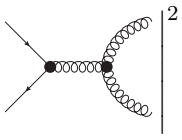
  and exactly cancels the contribution of the first two diagrams!

Happy end? — **No!** — a disaster in loop diagrams (true quantum effects)

$q_\nu \cdot$  $\neq 0$

↘ all polarizations in intermediate state (2.79)

Gluons with unphysical polarizations can be produced $\longrightarrow$ unitarity is broken:

2 Im  $\neq \int d\,(\text{phase space})$  ^{2} (2.80)

2.4.4 Faddeev-Popov ghosts

Solution: Faddeev, Popov (1967):
Modify QCD Lagrangian

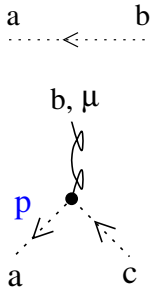
$$\mathcal{L}_{QCD} \longrightarrow \mathcal{L}_{QCD} + \bar c^a(x)\Big(-\partial^\mu D^{ab}_{\mu\nu}\Big)c^b(x)$$

$$D^{ab}_\mu = \partial_\mu + f^{abc}A^c_\mu \qquad \text{covariant derivative in adjoint representation} \tag{2.81}$$

ghost field $c^a(x)$:

- spin-0 field (scalar)
- $a = 1, 2, \ldots, 8$
  (adjoint representation, like gluon)
- Fermi-statistics (!?)
  $\{c(\vec{x}, t), \bar{c}(\vec{y}, t)\} = 0$

(new) Feynman rules:



$$= \frac{i\delta^{ab}}{p^2 + i\epsilon}$$

$$= -g f^{abc} p_\mu \tag{2.82}$$

- Derivation uses path integral formalism ⇒ QFT lectures in SS
- The role of ghosts is to subtract "wrong" gluon polarizations, hence "wrong" spin-statistics relation
- Ward identities are modified by terms including ghosts ⇒ Slavnov-Taylor identities
- We will see how this works in practical calculations (exercises)

3 Asymptotic freedom

3.1 Reminder: Renormalization and regularization in QED

This is only a reminder, see QED lectures for more.

- The idea of renormalization is to rewrite everything in terms of physical parameters instead of "bare" parameters that appear in the Lagrangian
- In QED the bare parameters are electron mass m_0 and electric charge e_0 and we want to write everything in terms of electron mass $m = 0.511\ldots$ MeV and the coupling $\alpha = e^2/(4\pi) = 1/137\ldots$

3.1.1 Exact electron propagator

$$\langle \Omega | T\{\psi_i(x)\bar{\psi}_j(y)\} | \Omega \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}\, S(p) \tag{3.1}$$

$S(p) =$ 

$$= \frac{i}{\not{p} - m_0 + \Sigma(\not{p})} \tag{3.2}$$

Electron self energy is defined as the sum of all 1PI diagrams with amputated legs:

$\Sigma(\not{p}) =$  (3.3)

We rewrite

$$\boxed{S(p) = \frac{iZ_2}{\not{p} - m + \Sigma^{(r)}(\not{p})} = Z_2 S^{(r)}(p)} \tag{3.4}$$

where

$$\begin{aligned} m - m_0 + \Sigma(m) = 0 \implies m = m(m_0, e_0) \qquad &\text{[pole position]} \\ \Sigma^{(r)}(\not{p}) = Z_2\Big[\Sigma(\not{p}) - \Sigma(m) - (\not{p} - m)\Sigma'(m)\Big] \\ Z_2 = \frac{1}{1 + \Sigma'(m)} \end{aligned} \tag{3.5}$$

- m is called a renormalized mass
- Z_2 is called a field renormalization constant
- $S^{(r)}(p)$ is called a renormalized propagator

Note that

$$\Sigma^{(r)}(\not{p}) \overset{\not{p}\to m}{=} \mathcal{O}\Big((\not{p} - m)^2\Big) \tag{3.6}$$

Therefore for slow electrons $\vec{p} \ll 0.5$ MeV/c

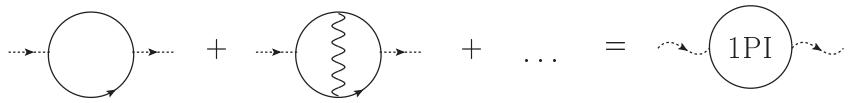
$$\begin{aligned} \langle\Omega|T\{\psi(x)\bar{\psi}(y)\}|\Omega\rangle\Big|_{x_0>y_0} &= \int \frac{dp_0 d^3\vec{p}}{(2\pi)^4} e^{-ip(x-y)} \frac{iZ_2}{\not{p} - m} \\ &= \sum_s \int \frac{d^3p}{(2\pi)^3 2E_p} \underbrace{\big[\sqrt{Z_2} u(p,s) e^{-iE_p x_0 + i\vec{p}\cdot\vec{x}}\big]}_{\Psi_{\vec{p}}(\vec{x},t)} \underbrace{\big[\sqrt{Z_2}\bar{u}(p,s) e^{iE_p y_0 - i\vec{p}\cdot\vec{y}}\big]}_{\Psi^*_{\vec{p}}(\vec{y},t)} \end{aligned} \tag{3.7}$$

Note:
- $E_p = \sqrt{\vec{p}^2 + m^2}$, i.e m is the physical mass in Einstein formula
- correct normalization of the free electron wave function ($\leftarrow$ role of Z_2) L8

3.1.2 Exact photon propagator

$$\begin{aligned} \langle\Omega|T\{A_\mu(x)\bar{A}_\nu(y)\}|\Omega\rangle &= \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} D_{\mu\nu}(k) \\ D_{\mu\nu}(k) &= \frac{g_{\mu\nu}}{k^2[1 - \Pi(k^2)]} \end{aligned} \tag{3.8}$$

where $\Pi(k^2)$ is called photon self energy:

$\Pi_{\mu\nu} =$ 

$$= (g_{\mu\nu}k^2 - k_\mu k_\nu)\Pi(k^2) \qquad \Longleftarrow k^\mu \Pi_{\mu\nu}(k) = 0 \tag{3.9}$$

We rewrite:

$$\boxed{D_{\mu\nu}(k) = g_{\mu\nu}\frac{Z_3}{k^2[1-\Pi^{(r)}(k^2)]} = Z_3 D^{(r)}_{\mu\nu}(k)} \tag{3.10}$$

where

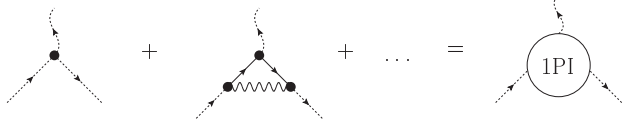
$$Z_3 = \frac{1}{1-\Pi(0)}\,, \qquad \Pi^{(r)}(k^2) = Z_3\Big[\Pi(k^2) - \Pi(0)\Big] \tag{3.11}$$

• photon mass is not renormalized because of local $U(1)$ symmetry

3.1.3 Exact interaction vertex

The interaction vertex is usually defined as the the sum of all 1PI diagrams with amputated legs

$-ie_0\Gamma_\mu(p_1,p_2) = -ie_0\Big[\gamma_\mu + \Lambda_\mu(p_1,p_2)\Big]$

$=$  (3.12)

It is possible to show [QED lectures] that

$$\Lambda_\mu(p_1,p_2)\Big|_{\substack{\not{p}_1\to m\\ \not{p}_2\to m}} = \gamma_\mu \Lambda(m,m) \tag{3.13}$$

We define

$$\boxed{-ie_0\Gamma_\mu(p_1,p_2) = -ie_0 Z_1^{-1}\Big[\gamma_\mu + \Lambda^{(r)}_\mu(p_1,p_2)\Big] = Z_1^{-1}\Gamma^{(r)}_\mu(p_1,p_2)} \tag{3.14}$$

where

$$Z_1 = \frac{1}{1+\Lambda(m,m)}$$

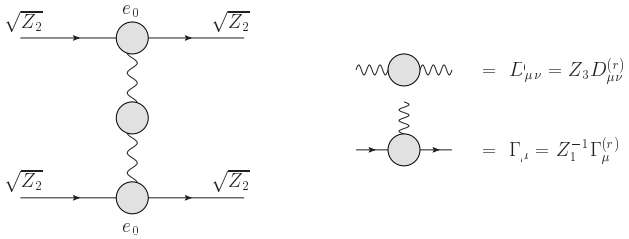
$$\Lambda^{(r)}_\mu = Z_1\Big[\Lambda_\mu(p_1,p_2) - \gamma_\mu\Lambda(m,m)\Big] \tag{3.15}$$

3.1.4 Effective charge

Consider electron-electron scattering at small angles

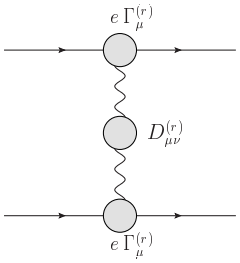
$$e(p_1) + e(p_2) \to e(p_1') + e(p_2')\,, \qquad q^2 = (p_1' - p_1)^2 \to 0 \quad (\ll m_e^2)$$

We expect to find a Coulomb law. In this limit

$iM =$ 

(3.16)

[For these diagrams $M \sim 1/q^2$, all others at most $M \sim \ln(1/q^2)$]
Collecting Z-factors can rewrite the result as the diagram made of renormalized propagators and the renormalized or effective charge:

$\equiv$ 

$$\boxed{e = e_0 Z_2 Z_1^{-1}\sqrt{Z_3}\,, \qquad \frac{e^2}{4\pi} = \frac{1}{137}} \tag{3.17}$$

e is the "correct" electric charge as known from Coulomb law

- Renormalization:

$$\boxed{m_0,\, e_0 \longrightarrow m = m(m_0, e_0), \quad e = e(m_0, e_0)} \tag{3.18}$$

- easy to generalize to the general case:
  all Feynman diagrams can be rewritten in terms of renormalized electron mass and charge
- in QED

$$Z_1 = Z_2 \implies e = e_0\sqrt{Z_3} \qquad [\Leftarrow \text{ Ward–Takahashi Identity}] \tag{3.19}$$

3.1.5 UV divergences and renormalizability

A quantum field theory is always defined with an UV cutoff: $\quad |k_\mu| < \Lambda$.
In particular

$$\text{QED} = \{\mathcal{L}_{QED}(m_0, e_0), \Lambda\} \tag{3.20}$$

- old argument (math):
  — integrals over loop momenta diverge for large k_μ (UV-divergences)

- new argument (phys):
  — we know nothing about interactions at very small distances / very large momenta;
  even if the integrals were finite, the contribution of very large k is most likely wrong

How such a theory make sense? Let

$$\begin{aligned} \text{Theory 1} &= \{\mathcal{L}_{QED}(m_0, e_0), \Lambda\} \\ \text{Theory 2} &= \{\mathcal{L}_{QED}(m_0', e_0'), \Lambda'\} \end{aligned} \tag{3.21}$$

and require that the parameters are adjusted such that

$$\begin{aligned} m(e_0, m_0, \Lambda) &= m(e_0', m_0', \Lambda') \\ e(e_0, m_0, \Lambda) &= e(e_0', m_0', \Lambda') \end{aligned} \tag{3.22}$$

This means that the bare mass and charge depend on the cutoff:

$$e_0 = e_0(\Lambda)\,, \qquad\qquad m_0 = m_0(\Lambda) \tag{3.23}$$

Accepted terminology "The theory is defined at the scale Λ"

A theory in which such tuning is possible is called "a renormalizable theory"

QED is renormalizable:

$$\begin{aligned} \{Z_1, Z_2, Z_3\} &= \{Z_1, Z_2, Z_3\}(e_0, m_0, \Lambda) \qquad [\Leftarrow \text{ depend on bare parameters and the cutoff}] \\ &\text{but} \\ \{\Sigma^{(r)}, D^{(r)}_{\mu\nu}, \Gamma^{(r)}_{\mu}\} &= \{\Sigma^{(r)}, D^{(r)}_{\mu\nu}, \Gamma^{(r)}_{\mu}\}(m, e) \qquad [\Leftarrow \text{ depend on physical parameters only}] \end{aligned} \tag{3.24}$$

Thus, the dependence on Λ disappears in the predictions for physical observables

- Regularization should not destroy gauge and Lorentz invariance
- If the dependence on Λ disappears, each mathematically consistent regularization possible.
  — Dimensional regularization:

$$\int d^4x \longrightarrow \int d^D x\,, \qquad\qquad D = 4 - \epsilon \tag{3.25}$$

3.2 Renormalization group (QED)

An explicit calculation gives (QED lectures)
Photon self energy, one loop:

$$\Pi(k) = 2\frac{\alpha_0}{\pi}\int_0^1 du\, u(1-u)\left[\ln\frac{m_0^2 - u(1-u)k^2 - i\epsilon}{\Lambda_{UV}^2} + 1\right] \tag{3.26}$$

For large momenta

$$m_0^2 \ll |k^2| \ll \Lambda_{UV}^2 \tag{3.27}$$

this expression simplifies to logarithmic accuracy to

$$\Pi(k) = -\frac{\alpha_0}{3\pi} \ln \frac{\Lambda^2}{-k^2 - i\epsilon} + \mathcal{O}(1)\,, \qquad \alpha_0 = \frac{e_0^2}{4\pi} \tag{3.28}$$

Let us clarify the meaning of renormalization procedure on this simple example.
Start with

$$D_{\mu\nu} = \frac{g_{\mu\nu}}{k^2 + i\epsilon} \frac{1}{1 - \Pi(k^2)} = \frac{g_{\mu\nu}}{k^2 + i\epsilon} \frac{1}{1 + \frac{\alpha_0}{3\pi} \ln \frac{\Lambda^2}{-k^2}} \tag{3.29}$$

and rewrite

$$\begin{aligned} \frac{1}{1 + \frac{\alpha_0}{3\pi} \ln \frac{\Lambda^2}{-k^2}} &= \frac{1}{1 + \frac{\alpha_0}{3\pi} \ln \frac{\Lambda^2}{m^2} - \frac{\alpha_0}{3\pi} \ln \frac{-k^2}{m^2}} \\ &= \frac{1}{1 + \frac{\alpha_0}{3\pi} \ln \frac{\Lambda^2}{m^2}} \times \frac{1}{1 - \frac{\frac{\alpha_0}{3\pi} \ln \frac{-k^2}{m^2}}{1 + \frac{\alpha_0}{3\pi} \ln \frac{\Lambda^2}{m^2}}} \\ &= \frac{Z_3}{1 - Z_3 \frac{\alpha_0}{3\pi} \ln \frac{-k^2}{m^2}} \; = \; \frac{Z_3}{1 - \frac{\alpha}{3\pi} \ln \frac{-k^2}{m^2}} \end{aligned} \tag{3.30}$$

where

$$Z_3 = Z_3(\Lambda, m) = \frac{1}{1 + \frac{\alpha_0}{3\pi} \ln \frac{\Lambda^2}{m^2}} \tag{3.31}$$

and I used that

$$e = \sqrt{Z_3} e_0 \Longrightarrow \alpha = Z_3 \alpha_0 \tag{3.32}$$

Thus we obtain

$$\begin{aligned} D_{\mu\nu} &= Z_3 \, D^{(r)}_{\mu\nu} \\ D^{(r)}_{\mu\nu} &= \frac{g_{\mu\nu}}{k^2 + i\epsilon} \frac{1}{1 - \frac{\alpha}{3\pi} \ln \frac{-k^2}{m^2}} \end{aligned} \tag{3.33}$$

- $D^{(r)}_{\mu\nu} = D^{(r)}_{\mu\nu}(e, m)$ as we want
- This is a good approximation so far as

$$\frac{\alpha}{3\pi} \ln \frac{-k^2}{m^2} \ll 1\,, \qquad \alpha = \frac{1}{137} \tag{3.34}$$

Question: L9
What to do if $\alpha << 1$ but $\alpha \ln \frac{-k^2}{m^2} \sim 1$?
— [In QED of academic interest, in QCD ($\alpha_{QCD} \sim 0.3$) important]

— One can show that two-loop diagrams produce $\alpha^2 \ln^2 \frac{-k^2}{m^2}$, three-loop $\alpha^3 \ln^3 \frac{-k^2}{m^2}$ etc

We can try to reorganize the perturbation theory:

- Fixed-order perturbation theory (usual):

  1-st order: α
  2-nd order: α^2, etc.

- Resummed perturbation theory:

$$\begin{aligned} &\text{LO (Leading Order)}: && \alpha \ln\frac{k^2}{m^2}\,,\ \alpha^2 \ln^2\frac{k^2}{m^2}\,,\ \ \dots\ \ ,(\alpha \ln\frac{k^2}{m^2})^n \\ &\text{NLO (next-to-LO)}: && \alpha^2 \ln\frac{k^2}{m^2}\,,\ \alpha^3 \ln^2\frac{k^2}{m^2}\,,\ \ \dots\ \ ,\alpha(\alpha \ln\frac{k^2}{m^2})^n \end{aligned} \tag{3.35}$$

but would need to calculate leading (subleading etc.) parts of Feynman diagrams to all orders.

A very powerful approach — **the renormalization group**

3.2.1 The running coupling, β-function

The idea:
Renormalization procedure is, basically, splitting the large logarithms in two parts:

$$\ln\frac{-k^2}{\Lambda^2} = \underbrace{\ln\frac{-k^2}{m^2}}_{\swarrow\ \Pi^{(r)},\Sigma^{(r)},\Lambda^{(r)}} + \underbrace{\ln\frac{m^2}{\Lambda^2}}_{\searrow\ Z-\text{factors}} \tag{3.36}$$

What we achieve by doing this:

1. Dependence on Λ disappears from physical observables; only enters $e = e(e_0, m_0, \Lambda)$
2. Renormalized e, m are the charge and mass for free electrons, as measured in low-energy expts.

   Renormalized $\Pi^{(r)}, \Sigma^{(r)}, \Lambda^{(r)}$ are zero for free particles
   $\Longrightarrow$ Renormalized propagators = free propagators for $k^2 = 0$, $p^2 = m^2$.

Note that property (1.) is crucial, (2.) is convenient

Let us change the prescription of how we renormalize:

$$\ln\frac{-k^2}{\Lambda^2} = \underbrace{\ln\frac{-k^2}{M^2}}_{\swarrow\ \Pi^{(r)},\Sigma^{(r)},\Lambda^{(r)}} + \underbrace{\ln\frac{M^2}{\Lambda^2}}_{\searrow\ Z-\text{factors}} \tag{3.37}$$

where M is an arbitrary mass parameter. For simplicity we will assume

$$m \ll M \ll \Lambda \tag{3.38}$$

In other words

$$\begin{aligned} &\Pi^{(r)}(k^2) := \Pi(k^2) - \Pi(k^2 = 0) \\ \Longrightarrow \\ &\Pi^{(r)}(k^2) := \Pi(k^2) - \Pi(-k^2 = M^2) \end{aligned} \tag{3.39}$$

and similar

$$\Lambda_\mu^{(r)}(p_1,p_2) := \Lambda_\mu(p_1,p_2) - \Lambda_\mu(p_1^2 = p_2^2 = (p_1-p_2)^2 = -M^2)$$
$$\Sigma^{(r)}(p) := \Sigma^{(r)}(p) - \Sigma(\not{p} = M) - (\not{p} - M)\Sigma'((\not{p} = M) \tag{3.40}$$

What happens in this situation?

- We must always specify the value of M explicitly, i.e.

$$\Pi^{(r)} = \Pi^{(r)}(k^2, M^2, e(M), m(M)) \text{ etc.}$$

- The first property remains valid
- The second property is lost; replaced by

Renormalized propagators (vertices) = free propagators (vertices) for $-k^2 = M^2$, $-p^2 = M^2$

$\hookleftarrow$ Accepted terminology: "the theory is renormalized on the scale M"

We can overtake all results with a simple substitution $m \to M$:

$$Z_3(\Lambda, M, \alpha_0) = \frac{1}{1 + \frac{\alpha_0}{3\pi}\ln\frac{\Lambda^2}{M^2}} \equiv Z_3(M)$$
$$\Pi^{(r)}(k) = -\frac{\alpha}{3\pi}\ln\frac{M^2}{-k^2} \qquad D_{\mu\nu}^{(r)}(k) = \frac{g_{\mu\nu}}{k^2}\,\frac{1}{1 - \frac{\alpha}{3\pi}\ln\frac{-k^2}{M^2}}$$
$$\alpha(M) = \frac{\alpha_0}{1 + \frac{\alpha_0}{3\pi}\ln\frac{\Lambda^2}{M^2}} \equiv Z_3(M)\,\alpha_0\,, \qquad \alpha_{\text{Coulomb}} \simeq \alpha(M = m) \tag{3.41}$$

- ! If we choose $M \sim |k|$ there are no large logs in the renormalized propagator
  — Problem solved? Not quite: we do not know the value of $\alpha(M)$ for large M
  — What happens when we change $M_1 \to M_2$?

Consider

$$\alpha(M_1) = \frac{\alpha_0}{1 + \frac{\alpha_0}{3\pi}\ln\frac{\Lambda^2}{M_1^2}} \qquad \alpha(M_2) = \frac{\alpha_0}{1 + \frac{\alpha_0}{3\pi}\ln\frac{\Lambda^2}{M_2^2}} \tag{3.42}$$

• both expressions are valid if $\ln(\Lambda^2/M_{1,2}^2) \ll 1/\alpha$

Idea: eliminate α_0:

$$\alpha(M_2) = \frac{\alpha_0}{1 + \frac{\alpha_0}{3\pi}\ln\frac{\Lambda^2}{M_1^2} + \frac{\alpha_0}{3\pi}\ln\frac{M_1^2}{M_2^2}} = \frac{\alpha_0}{1 + \frac{\alpha_0}{3\pi}\ln\frac{\Lambda^2}{M_1^2}} \times \frac{1}{1 + \frac{\frac{\alpha_0}{3\pi}\ln\frac{M_1^2}{M_2^2}}{1 + \frac{\alpha_0}{3\pi}\ln\frac{\Lambda^2}{M_1^2}}}$$
$$= \frac{\alpha(M_1)}{1 + \frac{\alpha(M_1)}{3\pi}\ln\frac{M_1^2}{M_2^2}} \tag{3.43}$$

• this relation is valid if $\ln(M_1^2/M_2^2) \ll 1/\alpha$; Λ can be arbitrary as it falls out

What to do if $M_1^2 \lll M_2^2$?
— Split the interval in smaller ones, $M_1^2 \le \mu_1^2 \le \mu_2^2 \le \ldots \le M_2^2$
— Apply the above relation for each smaller interval!

An intelligent way to do this: a differential equation

$$\frac{1}{\alpha(M)} = \frac{1}{\alpha_0}\left(1 + \frac{\alpha_0}{3\pi}\ln\frac{\Lambda^2}{M^2}\right)$$

$$\Longrightarrow \qquad M\frac{d}{dM}\frac{1}{\alpha(M)} = -\frac{1}{\alpha^2(M)}M\frac{d}{dM}\alpha(M) = -\frac{2}{3\pi} \tag{3.44}$$

or

$$M\frac{d}{dM}\alpha(M) = +\frac{2}{3\pi}\alpha^2(M) \tag{3.45}$$

A very important concept:

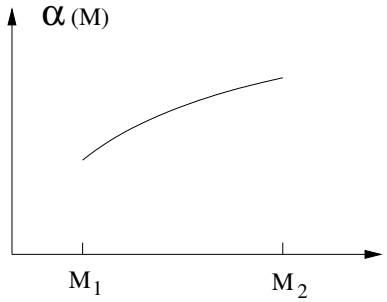
Beta-function (Gell-Mann–Low function)

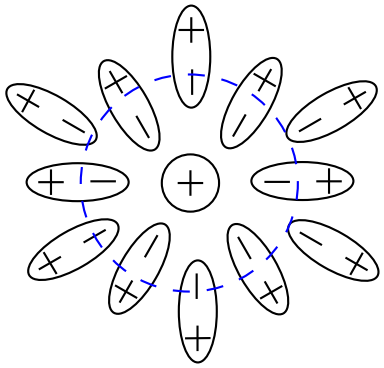
$$\boxed{M\frac{d}{dM}\alpha(M) = \beta(\alpha(M)) = \beta_0\alpha^2(M) + \beta_1\alpha^3(M) + \beta_2\alpha^4(M) + \ldots} \tag{3.46}$$

We have calculated

$$\boxed{\beta_0^{QED} = \frac{2}{3\pi} > 0} \tag{3.47}$$

Terminology: the running coupling
Interpretation: QED charge increases at small distances/large scales due to vacuum polarization





3.2.2 Electron propagator at large momenta; Callan-Symanzik and renormalization group equations

Consider electron propagator for $p \gg m$

$$S(p) = \frac{iZ_2}{\not p - m + \Sigma^{(r)}(\not p)}\,, \qquad \Sigma(\not p) = \not p\,\Sigma_1(p^2) + m\Sigma_2(p^2)\,, \qquad p^2 = \not p^2 \tag{3.48}$$

In this calculation we can neglect the electron mass $m \to 0$, thus

$$S(p) \simeq \frac{iZ_2}{\not{p}(1+\Sigma_1^{(r)}(p^2))} \;=\; ? \tag{3.49}$$

First order:

$$\Sigma_1^{\text{bare}}(p) = \quad = \frac{\alpha_0}{4\pi}\ln\frac{\Lambda^2}{-p^2} \qquad \text{[exercises]} \tag{3.50}$$

Renormalization:

$$\Sigma_1^{(r)}(p^2, M) = \Sigma_1(p^2) - \Sigma_1(-p^2 = M^2) = \frac{\alpha_0}{4\pi}\ln\frac{M^2}{-p^2} \simeq \frac{\alpha}{4\pi}\ln\frac{M^2}{-p^2}$$

$$Z_2(M) = \frac{1}{1+\dfrac{\alpha_0}{4\pi}\ln\dfrac{\Lambda^2}{M^2}} \tag{3.51}$$

• Both expressions are valid to $\mathcal{O}(\alpha^2)$

In general:

$$S^{\text{bare}}(p, \Lambda, \alpha_0) = Z_2(\Lambda, M, \alpha_0) S^{\text{ren}}(p, M, \alpha(M)) \tag{3.52}$$

Note that the l.h.s. does not depend on M. Thus

$$0 = M\frac{d}{dM}S^{\text{bare}} = \left(M\frac{d}{dM}Z_2\right) S^{\text{ren}} + Z_2\left[M\frac{\partial}{\partial M} + \left(M\frac{d}{dM}\alpha(M)\right)\frac{\partial}{\partial\alpha}\right] S^{\text{ren}} \tag{3.53}$$

We define

$$M\frac{d}{dM}\alpha(M) \;:=\; \beta(\alpha) \qquad\qquad \text{beta-function}$$

$$\boxed{\frac{1}{Z_2}M\frac{d}{dM}Z_2 \;:=\; 2\gamma(\alpha) \qquad\qquad \text{anomalous dimension}} \tag{3.54}$$

Callan-Symanzik equation:

$$\boxed{\left[M\frac{\partial}{\partial M} + \beta(\alpha)\frac{\partial}{\partial\alpha} + 2\gamma(\alpha)\right] S^{\text{ren}}(p, M, \alpha(M)) = 0} \tag{3.55}$$

• this is an example; the CS equation can written for other objects (later)
• why $2\gamma(\alpha)$: the number of external fermion legs (convenient)

In perturbation theory

$$\beta(\alpha) = \beta_0\alpha^2 + \beta_1\alpha^3 + \ldots \qquad\qquad \beta_0 = \frac{2}{3\pi}$$

$$\gamma(\alpha) = \gamma_0\alpha + \gamma_1\alpha^2 + \ldots \qquad\qquad \gamma_0 = \frac{1}{4\pi} \tag{3.56}$$

The self energy can depend on M through $\alpha(M)$, or dimensionless ratio p/M: L10

$$S^{\rm ren}(p,M,\alpha) = \frac{i}{\not{p}(1+\Sigma_1^{(r)}(p^2,M^2,\alpha))} = \frac{i}{\not{p}(1+\Sigma_1^{(r)}(p^2/M^2,\alpha))} \tag{3.57}$$

This implies that if we rescale $p_\mu \to tp_\mu$ and $M \to tM$ <u>for fixed α</u>:

$$S^{\rm ren}(tp,tM,\alpha) = t^{-1} S^{\rm ren}(p,M,\alpha) \tag{3.58}$$

Euler's Homogeneous Function Theorem:

$$\left[p_\mu \frac{\partial}{\partial p_\mu} + M\frac{\partial}{\partial M}\right] S^{\rm ren} = (-1)\, S^{\rm ren} \tag{3.59}$$

We can use this to convert partial derivatives in M into derivatives in p_μ! Obtain

Renormalization Group (RG) equation:

$$\boxed{\left[p_\mu \frac{\partial}{\partial p_\mu} - \beta(\alpha)\frac{\partial}{\partial \alpha} + 1 - 2\gamma(\alpha)\right] S^{\rm ren}(p,M,\alpha(M)) = 0} \tag{3.60}$$

- this is an example; the RG equation can written for other objects (later)
- this is a linear differential equation describing the momentum dependence

General solution:

$$S^{\rm ren}(p,M,\alpha(M)) = \frac{i}{\not{p}}\,\tilde{S}(\alpha(p))\,\exp\left\{2\int_{\alpha(M)}^{\alpha(p)} d\alpha'\,\frac{\gamma(\alpha')}{\beta(\alpha')}\right\}$$
$$\tilde{S}(\alpha(p)) := S^{\rm ren}(p=M,\alpha(M)) \qquad \text{boundary condition, only function of } \alpha(p) \tag{3.61}$$

where $\alpha(p) = \alpha(M^2 = -p^2)$ is the running coupling

$$p_\mu \frac{\partial}{\partial p_\mu}\alpha(p) = \beta(\alpha) \tag{3.62}$$

[insert in the equation and check that it is satisfied]

One-loop approximation:

$$\tilde{S}(\alpha(p)) = 1\,, \qquad \gamma(\alpha) = \gamma_0\alpha\,, \qquad \beta(\alpha) = \beta_0\alpha^2 \tag{3.63}$$

Then

$$\exp\{**\} = \exp\left\{2\int_{\alpha(M)}^{\alpha(p)} d\alpha'\,\frac{\gamma_0}{\beta_0\alpha'}\right\} = \exp\left\{2\frac{\gamma_0}{\beta_0}\ln\frac{\alpha(p)}{\alpha(M)}\right\}$$
$$= \left(\frac{\alpha(p)}{\alpha(M)}\right)^{2\gamma_0/\beta_0} = \left(\frac{\alpha(p)}{\alpha(M)}\right)^{3/4} \tag{3.64}$$

so that

$$\boxed{S^{\rm ren}(p, M, \alpha(M)) = \frac{i}{\not{p}} \left(\frac{\alpha(p)}{\alpha(M)} \right)^{3/4}} \tag{3.65}$$

If $|p|$ is not very much different from M, can use

$$\alpha(p) = \frac{\alpha(M)}{1 + \dfrac{\alpha(M)}{3\pi} \ln \dfrac{M^2}{-p^2}} \tag{3.66}$$

In this case

$$\left(\frac{\alpha(p)}{\alpha(M)} \right)^{2\gamma_0/\beta_0} = \left(1 + \frac{\alpha(M)}{3\pi} \ln \frac{M^2}{-p^2} \right)^{-2\gamma_0/\beta_0} = \left(1 + \frac{\beta_0}{2} \alpha(M) \ln \frac{M^2}{-p^2} \right)^{-2\gamma_0/\beta_0}$$

$$= 1 - \gamma_0 \alpha(M) \ln \frac{M^2}{-p^2} + \mathcal{O}(\alpha^2 \ln^2(\ldots)) = 1 - \frac{\alpha(M)}{4\pi} \ln \frac{M^2}{-p^2} \tag{3.67}$$

— in agreement with one-loop calculation

Our new result is also applicable for $p^2 \ggg M^2$:
The difference:

- The one-loop result valid if

$$\alpha \ll 1\,, \qquad \alpha \ln \frac{M^2}{p^2} \ll 1 \tag{3.68}$$

- The "RG-improved" result valid if

$$\alpha \ll 1\,, \qquad \alpha \ln \frac{M^2}{p^2} = \mathcal{O}(1) \tag{3.69}$$

- At the end can choose $M \to m$ if desired.

For a better (NLO) approximation have to calculate three new constants:

$$\beta(\alpha) = \beta_0 \alpha^2 + \beta_1 \alpha^3$$

$$\gamma(\alpha) = \gamma_0 \alpha + \gamma_1 \alpha^2$$

$$\tilde{S}(\alpha(p)) = 1 + s_1 \alpha(p) \tag{3.70}$$

— sum up all terms $\sim \alpha(\alpha \ln M^2/p^2)^k$, $k = 0, 1, \ldots$

Last but not least:
In dimensional regularization

$$Z_2 = 1 - \gamma_0 \alpha_0 \ln \frac{\Lambda^2}{M^2} \;\longrightarrow\; 1 - \frac{1}{4 - D} \gamma_0 \alpha_0 \tag{3.71}$$

Therefore, formally

$$\ln \frac{\Lambda^2}{M^2} \;\longrightarrow\; \frac{1}{4 - D} \;+\; \text{const} \tag{3.72}$$

[Details $\longrightarrow$ exercises]

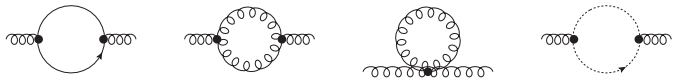
3.3 QCD Beta–function

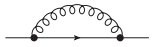
$$\alpha_s(\mu) \equiv \frac{g^2(\mu)}{4\pi} = Z_2^2(\mu) Z_1^{-2}(\mu) Z_3(\mu) \alpha_0 \,, \qquad \alpha_0 \equiv \alpha_s^{(0)}$$

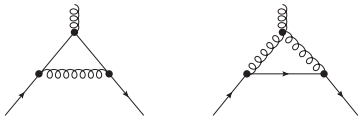
$$\mu \frac{d}{d\mu} \alpha_s(\mu) = \beta(\alpha_s) = \beta_0 \alpha_s^2 + \beta_1 \alpha_s^3 + \ldots \tag{3.73}$$

Difference to QED:

- Renormalization at $k^2 \to 0$, $\not{p} \to m$ makes no sense because of confinement
- $Z_1 \neq Z_2$, therefore have to calculate everything:

Z_3 : 

Z_2 : 

Z_1 : 

(3.74)

Let

$$Z_3(\mu) = 1 - \delta_3 \alpha_0 \ln \frac{\Lambda^2}{\mu^2}$$

$$Z_2(\mu) = 1 - \delta_2 \alpha_0 \ln \frac{\Lambda^2}{\mu^2}$$

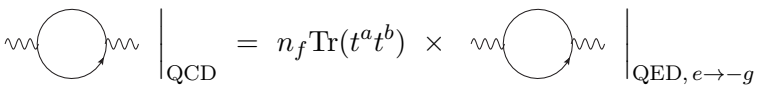
$$Z_1(\mu) = 1 - \delta_1 \alpha_0 \ln \frac{\Lambda^2}{\mu^2} \tag{3.75}$$

Then

$$\beta_0^{\mathrm{QCD}} = 2(\delta_3 + 2\delta_2 - 2\delta_1) \tag{3.76}$$

- Calculation using explicit UV cutoff is at best inconvenient
  $\Longrightarrow$ Dimensional regularization

The simplest case



(3.77)

where n_f is the number of existing quarks $u, d, s, \ldots$, $\mathrm{Tr}(t^a t^b) = 1/2 \delta^{ab}$

Compare:
1) Calculation with a cutoff:

$$\begin{aligned}\Pi^{\text{cutoff}}(k^2) &= 2\frac{\alpha_0}{\pi}\int_0^1 d\alpha\,\alpha(1-\alpha)\left[\ln\frac{m^2-\alpha(1-\alpha)k^2}{\Lambda^2}-1\right]\\ &= \frac{\alpha_0}{3\pi}\ln\frac{M^2}{\Lambda^2} \;+\; \text{finite terms}\\ &= \alpha_0\delta_3^{QED}\ln\frac{M^2}{\Lambda^2} \;+\; \text{finite terms}\end{aligned} \tag{3.78}$$

$\Longrightarrow$ δ_3 is a coefficient of $\alpha_0 \ln\frac{1}{\Lambda^2}$

2) Calculation in dim. reg.:

$$\begin{aligned}\Pi^{\text{dim.reg.}}(k^2) &= 2\frac{\alpha_0}{\pi}\int_0^1 d\alpha\,\alpha(1-\alpha)\left[-\frac{2}{4-D}+\gamma_E-\ln 4\pi+\ln\frac{m^2-\alpha(1-\alpha)k^2}{\mu^2}-1\right]\\ &= \frac{\alpha_0}{3\pi}\left(-\frac{2}{4-D}\right) \;+\; \text{finite terms}\end{aligned} \tag{3.79}$$

$\Longrightarrow$ δ_3 is a coefficient of $\alpha_0\left(-\frac{2}{4-D}\right)$

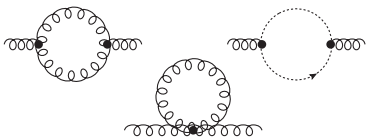
In both cases we need a divergent part only, with the correspondence

$$\boxed{\frac{2}{4-D} \iff \ln\Lambda_{UV}^2} \tag{3.80}$$

Explicit calculation (exercises) gives:



$$=(k^2 g^{\mu\nu}-k^\mu k^\nu)\delta^{ab}\left(-\frac{\alpha_s}{4\pi}\cdot\frac{2}{3}n_f\right)\Gamma(2-D/2)$$



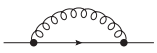
$$=(k^2 g^{\mu\nu}-k^\mu k^\nu)\delta^{ab}\left(+\frac{\alpha_s}{4\pi}\cdot\frac{5}{3}N_c\right)\Gamma(2-D/2) \tag{3.81}$$

$$\Gamma(2-D/2)=\frac{1}{2-D/2}\cdot(2-D/2)\Gamma(2-D/2)=\frac{1}{2-D/2}\Gamma(3-D/2)=\frac{2}{4-D} \;+\; \text{finite terms} \tag{3.82}$$

so that

$$\delta_3=\frac{1}{4\pi}\left(-\frac{5}{3}N_c+\frac{2}{3}n_f\right) \tag{3.83}$$

Next:



$$: \qquad \Sigma(p,m=0)=\frac{\alpha_0}{4\pi}\not{p}\,C_F\,\Gamma(2-D/2) \tag{3.84}$$

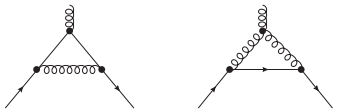
where for $SU(N_c)$

$$\sum_a t^a t^a := C_F \mathbb{I}\,, \qquad C_F = \frac{N_c^2-1}{2N_c} \to \frac{4}{3} \quad (N_c = 3) \tag{3.85}$$

$$S(p) = \frac{1}{\not{p}\left(1 + \frac{\alpha_0}{4\pi} C_F\, \Gamma(2 - D/2)\right)} \implies Z_2 = 1 - \frac{\alpha_0}{4\pi} C_F\, \Gamma(2-D/2) + \ldots$$

$$\implies \delta_2 = \frac{1}{4\pi} C_F \tag{3.86}$$

Finally (exercises)



$$\implies \delta_1 = \frac{1}{4\pi}(C_F + N_c) \tag{3.87}$$

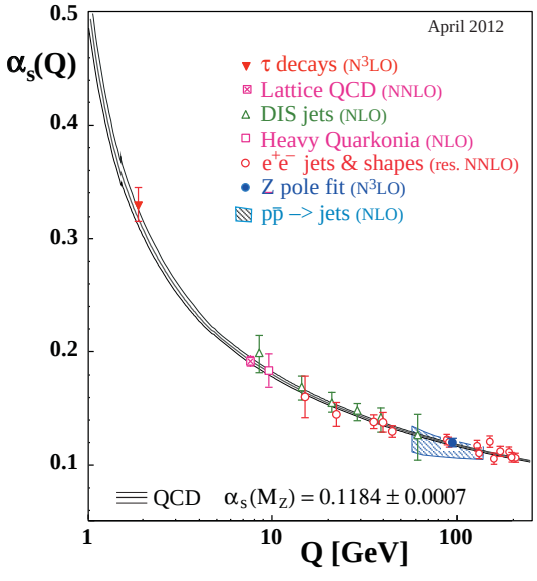
and we obtain (Gross, Wilczek, Politzer, 1973)

$$\beta_0^{\rm QCD} = 2(\delta_3 + 2\delta_2 - 2\delta_1) = \frac{2}{4\pi}\Big[\underbrace{-\frac{5}{3}N_c + \frac{2}{3}n_f}_{\delta_3} + \underbrace{2C_F}_{\delta_2} \underbrace{- 2C_F - 2N_c}_{\delta_1}\Big] \tag{3.88}$$

- Results for individual δ_k in Feynman gauge, the sum is gauge-invariant

$$\boxed{\beta_0^{\rm QCD} = \frac{1}{2\pi}\Big[\frac{2}{3}n_f - \frac{11}{3}N_c\Big] < 0 \qquad \beta_0^{\rm QED} = \frac{2}{3\pi} > 0} \tag{3.89}$$

Asymptotic freedom:



- Often a different definition is used:

$$\mu\frac{d}{d\mu}g(\mu)=\beta(g)\,,\qquad \frac{1}{g}\beta(g)=-b_0\frac{\alpha_s}{4\pi}-b_1\left(\frac{\alpha_s}{4\pi}\right)^2+\dots$$
$$b_0=\frac{11}{3}N_c-\frac{2}{3}n_f \tag{3.90}$$

- five terms in this expansion are known (five loops) [*Baikov, Chetyrkin, Kühn, 2017*]
- A simple parametrization

$$\mu\frac{d}{d\mu}\alpha_s(\mu)=\beta_0\alpha_s^2=-\frac{b_0}{2\pi}\alpha_s^2\implies\mu\frac{d}{d\mu}\frac{1}{\alpha_s(\mu)}=-\frac{b_0}{2\pi}\implies\alpha_s(\mu)=\frac{2\pi}{b_0\ln(\mu/\Lambda_{\rm QCD})} \tag{3.91}$$

with $\Lambda_{\rm QCD}\simeq 200-250$ MeV (not to be mixed with UV cutoff)

One can rewrite this relation as

$$\Lambda_{\rm QCD}=\mu\, e^{-2\pi/(b_0\alpha_s(\mu))}=\Lambda_{\rm UV}e^{-2\pi/(b_0\alpha_0)} \tag{3.92}$$

! The scale parameter arises because a field theory "remembers" about the UV cutoff
This phenomenon is called dimensional transmutation.
! Note that $e^{-1/(\beta_0\alpha_s(\mu))}$ has zero perturbative expansion

4 Unitarity and Feynman diagrams

4.1 Optical Theorem

A transition amplitude (probability amplitude) to observe a particular state in a scattering experiment can formally be thought of as a matrix element of the unitary operator in the Hilbert space

$$\langle q_1\dots q_n|S|k_1k_2\rangle\equiv\lim_{T\to\infty}\langle q_1\dots q_n|e^{-i\hat{H}_IT}|k_1k_2\rangle \tag{4.1}$$

Unitarity means

$$S^\dagger S=\mathbb{I} \tag{4.2}$$

i.e.

$$\sum_{q_f}\langle k_1'k_2'|S^\dagger|q_f\rangle\langle q_f|S|k_1k_2\rangle=\langle k_1'k_2'|S^\dagger S|k_1k_2\rangle=(2\pi)^6\delta^{(3)}(k_1-k_1')\delta^{(3)}(k_2-k_2') \tag{4.3}$$

We write

$$S:=\mathbb{I}+iT \qquad \text{T-matrix}$$
$$\langle A|iT|B\rangle:=(2\pi)^4\delta^{(4)}(p_A-p_B)\,iM(B\to A) \qquad \text{Amplitude} \tag{4.4}$$

Unitarity of the S-matrix implies

$$S^\dagger S=\mathbb{I}\implies(\mathbb{I}-iT^\dagger)(\mathbb{I}+iT)=\mathbb{I}$$
$$\implies\boxed{-i(T-T^\dagger)=T^\dagger T} \tag{4.5}$$

- a) Take matrix element $\langle p_1 p_2| \dots |k_1 k_2\rangle$
- b) Insert a complete set of states $T^\dagger T \to \sum_{q_f} T^\dagger |q_f\rangle\langle q_f| T$, i.e.

$$\langle p_1 p_2|T^\dagger T|k_1 k_2\rangle = \sum_n \left(\prod_{f=1}^n \int \frac{d^3 q_f}{(2\pi)^3} \frac{1}{2E_f} \right) \langle p_1 p_2|T^\dagger|q_1, \dots, q_n\rangle\langle q_1, \dots, q_n|T|k_1 k_2\rangle \quad (4.6)$$

One obtains:

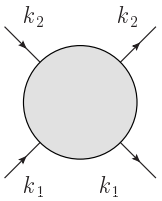
$$\begin{aligned} -i\Big[M(k_1 k_2 \to p_1 p_2) - M^*(p_1 p_2 \to k_1 k_2)\Big](2\pi)^4 \delta^{(4)}(k_1 + k_2 - p_1 - p_2) = \\ = \sum_n \left(\prod_{f=1}^n \int \frac{d^3 q_f}{(2\pi)^3} \frac{1}{2E_f} \right) M^*(p_1 p_2 \to q_f) M(k_1 k_2 \to q_f) \\ \times (2\pi)^4 \delta^{(4)}(k_1 + k_2 - \sum_f q_f)(2\pi)^4 \delta^{(4)}(k_1 + k_2 - p_1 - p_2) \end{aligned} \quad (4.7)$$

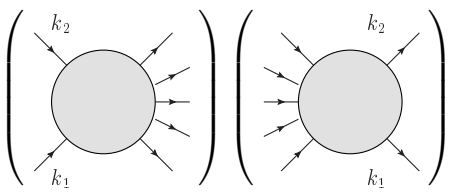
where the energy-conservation δ-functions can be cancelled on both sides.

An important special case is $(k_1, k_2) = (p_1, p_2)$ (forward scattering)

$$\boxed{2\,\mathrm{Im}\, M(k_1 k_2 \to k_1 k_2) = \sum_n \left(\prod_{f=1}^n \int \frac{d^3 q_f}{(2\pi)^3} \frac{1}{2E_f} \right) |M(k_1 k_2 \to q_f)|^2 (2\pi)^4 \delta^{(4)}(k_1 + k_2 - \sum_f q_f) \;=\; 4\sqrt{s}|p_{cm}|\sigma_{tot}} \quad (4.8)$$

— The optical theorem

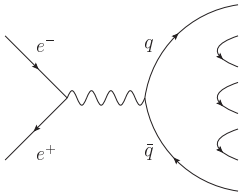
$$2\,\mathrm{Im}$$ 

$$= \sum_f \int d\Pi_f$$ 

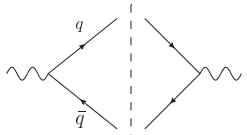
(4.9)

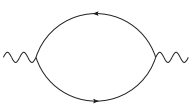
- this is an exact relation
- valid in each order of perturbation theory separately

Example: hadron production in electron-positron annihilation



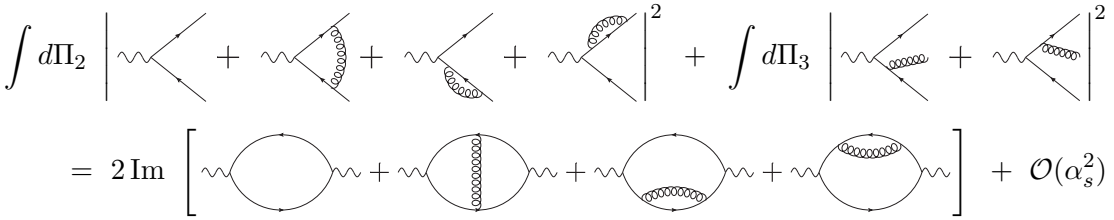
On the quark-gluon level, to leading order:

$$\int d\Pi_2$$ 

$$= 2\,\mathrm{Im}$$  (4.10)

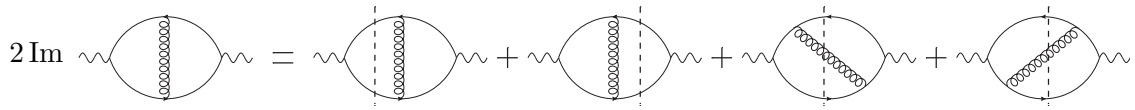
This explains why we always had $\ln(-k^2 - i\epsilon)$: Imaginary part can only occur for $k^2 > 0$ in which case real decay processes are possible

To the first order in α_s we can have quark-antiquark or quark-antiquark-gluon final states:



$$= 2\,\mathrm{Im}\left[\quad\right] + \mathcal{O}(\alpha_s^2) \qquad (4.11)$$

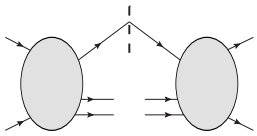
Very importantly, these rules are valid on a diagram-per-diagram level:



(4.12)

Cut diagrams

Consider phase space integration over one particular fermion line:



$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\sum_s M^*(p,\ldots)u(p,s)\bar{u}(p,s)M(p,\ldots)$$

$$= \int \frac{d^4p}{(2\pi)^3}\delta(p^2-m^2)\theta(p_0)\, M^*(p,\ldots)(\not{p}+m)M(p,\ldots) \qquad (4.13)$$

Note that

$$\frac{1}{p^2-m^2+i\epsilon} = \mathrm{PV}\frac{1}{p^2-m^2} - i\pi\delta(p^2-m^2) \qquad (4.14)$$

so that this can be written as

$$\ldots = \int \frac{d^4p}{(2\pi)^4}M^*(p,\ldots)\Big[-2\mathrm{Im}\,\frac{(\not{p}+m)}{p^2-m^2+i\epsilon}\Big]\theta(p_0)M(p,\ldots) \qquad (4.15)$$

Similarly for a gluon (photon) line:

$$\int \frac{d^3k}{(2\pi)^3}\frac{1}{2|k|}\sum_\lambda M^{\nu,*}(k,\ldots)\epsilon_\nu^{(\lambda)}(k)\epsilon_\mu^{*(\lambda)}(k)M^\mu(k,\ldots)$$

$$= \int \frac{d^4k}{(2\pi)^3}\delta(k^2)\theta(k_0)M^{\nu,*}(k,\ldots)(-g_{\mu\nu})M^\mu(k,\ldots)$$

$$= \int \frac{d^4k}{(2\pi)^4}M^{\nu*}(k,\ldots)\Big[-2\mathrm{Im}\,\frac{-g_{\mu\nu}}{k^2+i\epsilon}\Big]\theta(p_0)M^\mu(k,\ldots) \qquad (4.16)$$

— Thus, expressions for $\int d\Pi_n\, M^*\, M$ can be written similar to usual Feynman diagrams, adding new rules for "cut" propagators — the "cut diagrams"

— Thanks to the optical theorem, the sum over all cuts gives the total imaginary part for the particular Feynman diagram contribution to the forward scattering amplitude.

— This statement is actually more general and applies to arbitrary Feynman diagram. The imaginary part of a Feynman diagram can be calculated using the following algorithm:

- Cut through the diagram in all possible ways such that cut propagators can simultaneously be put on shell
- Use $k^2 + i\epsilon$ prescription for all propagators to the left of the cut and $k^2 - i\epsilon$ prescription to the right of the cut
- Use usual vertices to the left and complex conjugated to the right of the cut
- For all cut propagators replace

$$\frac{1}{p^2 - m^2 + i\epsilon} \implies -2\pi i\delta(p^2 - m^2) \tag{4.17}$$

  and add positive energy conditions $\theta(p_0)$ for all lines

- Sum the contributions of all cuts

These rules are often referred to as Cutkosky cutting rules (which they are not)

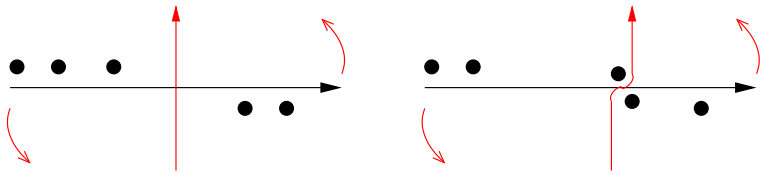
4.2 Singularities of Feynman diagrams

L12

It is easy to convince oneself that various "i" factors in propagators and vertices combine in such a way that the Feynman diagram is real unless the integrations become singular because the denominators vanish.

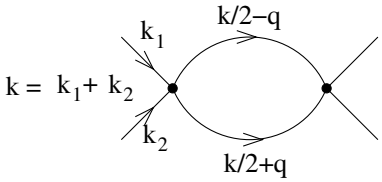
• Vanishing of one denominator is not enough because the integration contour can be moved away

• Imaginary parts arise when two poles come together and trap the integration countour so it cannot be moved



• This phenomenon is called "pinching"

Example: Scalar field theory with quartic interaction $\lambda\phi^4$



$$iM = \frac{\lambda^2}{2}\int \frac{d^4q}{(2\pi)^4}\frac{1}{[(k/2-q)^2 - m^2 + i\epsilon][(k/2+q)^2 - m^2 + i\epsilon]} \tag{4.18}$$

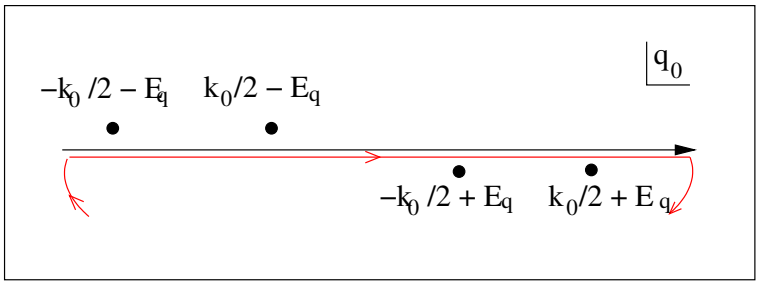
— $1/2$ is a symmetry factor for the diagram
— $\int d^4q/[(2\pi)^4 i]$ is real and positive after rotation to Euclidian space; thus M is naturally a real function

Let $k^2 > 0$, choose

$$k_\mu = (k_1 + k_2)_\mu = (k_0, 0, 0, 0) \tag{4.19}$$

and consider integration over q_0. The integrand has four poles located at

$$q_0 = \frac{1}{2}k_0 \pm (E_q - i\epsilon)\,, \qquad q_0 = -\frac{1}{2}k_0 \pm (E_q - i\epsilon)\,, \qquad E_q = \sqrt{\vec{q}^2 + m^2} \tag{4.20}$$



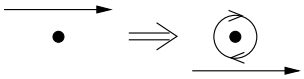
We choose to take the integral by closing the contour in the lower half-plane
— Thus have to sum the contributions of two poles at $q_0 = -\frac{1}{2}k_0 + E_q - i\epsilon$ and $q_0 = +\frac{1}{2}k_0 + E_q - i\epsilon$
— Only $q_0 = -\frac{1}{2}k_0 + E_q - i\epsilon$ can contribute to the imaginary part, neglect the second

Why:
Since $E_q > 0$, the only pair of poles that can pinch is

$$\frac{1}{2}k_0 - E_q + i\epsilon \;\leftrightarrow\; -\frac{1}{2}k_0 + E_q - i\epsilon\,, \qquad \Longrightarrow \text{ pinch at } E_q = 2k_0 \tag{4.21}$$

the rest always stay apart.

Picking the contribution of this pole corresponds to the following replacement in the integral:



$$\frac{1}{[(k/2+q)^2 - m^2 + i\epsilon]} \Longrightarrow -2\pi i \delta_+((k/2+q)^2 - m^2) \tag{4.22}$$

This gives:

$$\begin{aligned} iM &= -2\pi i \frac{\lambda^2}{2} \int \frac{d^3\vec{q}}{(2\pi)^4} \frac{1}{2E_q} \frac{1}{[(k^0 - E_q)^2 - E_q^2 + i\epsilon]} \\ &= -2\pi i \frac{\lambda^2}{2} \frac{4\pi}{(2\pi)^4} \int_m^\infty dE_q E_q |\vec{q}| \frac{1}{2E_q} \frac{1}{k_0[k_0 - 2E_q + i\epsilon]} \end{aligned} \tag{4.23}$$

The imaginary part appears because of the singularity at $k_0 = 2E_q$ which is exactly the pinching condition

$$\frac{1}{[k_0 - 2E_q + i\epsilon]} = \text{PV}\frac{1}{[k_0 - 2E_q]} - i\pi\delta(k_0 - 2E_q) \tag{4.24}$$

Taking ino account this delta-function contribution only, obtain

$$M \simeq \frac{1}{i}(-2\pi i)\frac{\lambda^2}{2}\frac{4\pi}{(2\pi)^4}\int_m^\infty dE_q\, E_q\sqrt{E_q^2-m^2}\,\frac{1}{2E_q}\frac{1}{k_0}(-i\pi)\delta(k_0-2E_q)$$

$$= \frac{i\lambda^2}{32\pi}\frac{1}{k_0}\sqrt{k_0^2-4m^2}\theta(k_0^2-4m^2) \tag{4.25}$$

or

$$\boxed{\mathrm{Im}\, M = \frac{\lambda^2}{32\pi}\sqrt{1-\frac{4m^2}{k^2}}\,\theta(k^2-4m^2)} \tag{4.26}$$

• note that the imaginary part must be positive (related to total cross section).
• This is consistent with $+i\epsilon$ prescription in propagators (and can be used to derive it)

Note also that picking up the pole is equivalent to replacing the original propagator by a delta-function

$$\frac{1}{[(k/2-q)^2-m^2+i\epsilon]} \Longrightarrow -2\pi i\delta_+((k/2-q)^2-m^2) \tag{4.27}$$

Going back to the original representation we can relabel the momenta in the loop as p_1 and p_2 and rewrite the momentum integration as

$$\int\frac{d^4q}{(2\pi)^4} = \int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p_1+p_2-k) \tag{4.28}$$

Then we can summarize our findings as

$$2i\,\mathrm{Im}M(k) = \frac{\lambda^2}{2}\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p_1+p_2-k)(-2\pi i)^2\delta_+(p_1^2-m^2)\delta_+(p_2^2-m^2)$$

$$= \frac{i}{2}\int\frac{d^3p_1}{(2\pi)^3}\frac{1}{2E_1}\int\frac{d^3p_2}{(2\pi)^3}\frac{1}{2E_2}|M(k)^2|(2\pi)^4\delta^{(4)}(p_1+p_2-k) \tag{4.29}$$

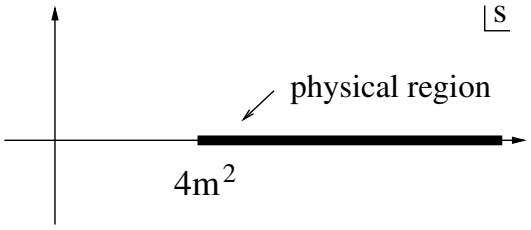
where to our accuracy on the r.h.s. $M(k)=\lambda$; factor $1/2$ takes into account identity of particles in the final state.

— the first line above gives a representation as a cut diagram
— the second line is an optical theorem

Last but not least:
Scattering amplitudes can be viewed as analytic functions of invariant energies. In our case

$$M(s=k^2=(k_1+k_2)^2) = M(s+i\epsilon) \tag{4.30}$$

has a cut in the complex s-plane:



Two-point functions do not have other singularities apart from unitarity cuts (can be proven).

Landau has given a general classification for singularity structure of arbitrary Feynman diagrams, hence arbitrary three-point, four-point etc. functions. [important topic, but too much for these lectures]

5 Electron-Positron annihilation

5.1 Total cross section: Leading-order analysis

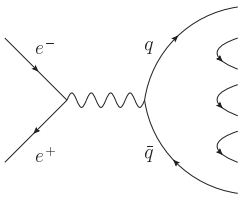
The best environment to study strong interactions: Electron-positron collisions at high energies

$$e^+(k_+) + e^-(k_-) \to \text{ hadrons} \tag{5.1}$$

pro: initial state exactly known, small backgrounds etc.
contra: electrons are more difficult to accelerate as protons because of radiation losses

The simplest quantity: L13

$$R(s) \;=\; \frac{\sigma_{tot}(e^+e^- \to \text{ hadrons})}{\sigma_{tot}(e^+e^- \to \mu+\mu^-)} \tag{5.2}$$



where

$$\sigma_{tot}(e^+e^- \to \mu+\mu^-) = \frac{4\pi\alpha^2}{3s}\,, \qquad \alpha = 1/137 \tag{5.3}$$

is used for normalization

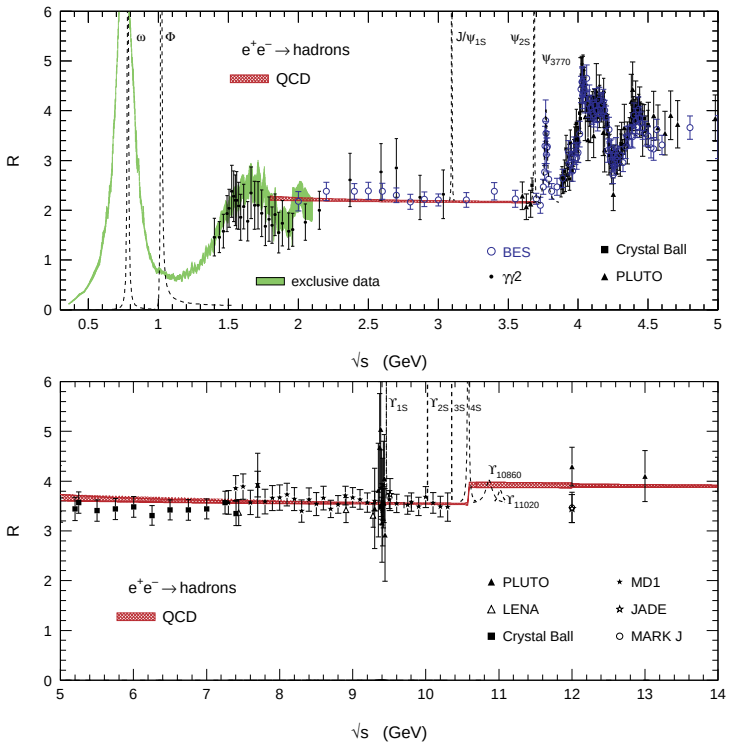
Heuristic discussion:

- The process involves two time scales (in CM system $q = k_+ + k_- = \{q_0, 0, 0, 0\}$, $s = q^2$):
  — quark-antiquark pair is produced at times $t \sim 1/q_0$ (uncertainty principle)
  — hadrons are produced when interaction becomes strong $T \sim 1/\Lambda_{\rm QCD} \sim 1$ fm
- If energy is large, $T \gg t$, these two (sub)processes cannot have any quantum interference; Probability to produce a given hadron state is a product of probabilities to produce a $q\bar{q}$ pair times the probability to produce a particular hadronic state
- For a total cross section it suffices to know that when q and $\bar{q}$ fly apart, *some* hadronic state will be produced with probability one
- Thus expect

  $$\sigma_{tot}(e^+e^- \to \text{ hadrons}) \simeq \sigma_{tot}(e^+e^- \to q\bar{q})$$

  or

  $$R(s) \;\simeq\; N_c\Big[\underbrace{e_u^2 + e_d^2 + e_s^2 + \ldots}_{\text{quarks with mass } < \sqrt{s}/2}\Big]\,, \qquad e_u = 2/3\,, \quad e_d = -1/3\,, \text{ etc.} \tag{5.4}$$



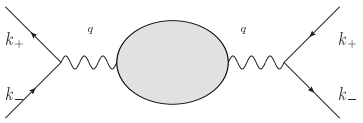
[Figure taken from: *Davier et al., Eur.Phys.J. C27:497-521,2003*]

- A strong argument for $N_c = 3$

We can make the two-time-scales argument more precise using optical theorem:

$$\sigma_{tot}(e^+e^- \to \text{ hadrons}) \;=\; \frac{1}{s}\,\mathrm{Im}\,M(e^+e^- \to e^+e^-) \tag{5.5}$$

[exact expression contains $1/(2\sqrt{s}p_{cm})$, reduces to $1/s$ for large energies]

$M \;=$  $= e^4 \bar{u}(k_-)\gamma^\mu v(k_+)\,\frac{1}{s}\,\Pi_{\mu\nu}(q)\,\frac{1}{s}\,\bar{v}(k_+)\gamma^\nu u(k_-)$ (5.6)

where

$$\Pi_{\mu\nu}(q) = i\int d^4x\, e^{iqx}\langle\Omega|T\{j_\mu(x)j_\nu(0)\}|\Omega\rangle = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2)\,, \qquad s = q^2 \tag{5.7}$$

and

$$j_\mu = \sum_q e_q \bar{\psi}_q \gamma_\mu \psi_q\,, \qquad e_u = \frac{2}{3}\,, e_d = -\frac{1}{3}\,, \ldots \qquad e = \sqrt{4\pi\alpha} \tag{5.8}$$

Note that both momenta *and* spins of the e^+ and e^- must coincide in initial and final state.

Neglecting the electron mass and averaging over spin directions obtain

$$\frac{1}{2}\cdot\frac{1}{2}\sum_{s,s'}\bar{u}(k_-,s)\gamma^\mu v(k_+,s')\,\bar{v}(k_+,s')\gamma^\nu u(k_-,s)\big[q_\mu q_\nu - s g_{\mu\nu}\big]\Pi(s)$$

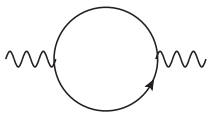
$$= \;-\frac{1}{4}s\Pi(s)\,\mathrm{Tr}\big[\not{k}_-\gamma^\mu\not{k}_+\gamma_\mu\big] = \frac{1}{2}s\Pi(s)\,4(k_-\cdot k_+) = s^2\Pi(s) \tag{5.9}$$

where I used $s = (k_- + k_+)^2 \simeq 2k_- \cdot k_+$.
Obtain therefore

$$\sigma_{tot}(e^+e^- \to \text{ hadrons}) \;=\; \frac{1}{s}\, s^2\, \mathrm{Im}\,\Pi(s) \frac{e^4}{s^2} = \frac{1}{s}(4\pi\alpha)^2 \mathrm{Im}\,\Pi(s) \tag{5.10}$$

and

$$\boxed{R(s) \;=\; 12\pi\, \mathrm{Im}\,\Pi(s)} \tag{5.11}$$

• Let us check this relation in leading order 

If quark masses can be neglected, calculations often become easier in coordinate space.
I choose the present example to illustrate this technique

Massless quark propagator

$$\langle 0|T\{\overbrace{\psi^i(x)\overline{\psi}^j}(0)\}|0\rangle = \delta^{ij} \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{i\not{p}}{p^2 + i\epsilon} \;=\; \frac{i}{2\pi^2} \frac{\not{x}}{(-x^2 + i\epsilon)^2} \tag{5.12}$$

Then, ignoring electric charges

$$\begin{aligned}
\Pi_{\mu\nu}(q) &= i \int d^4x\, e^{iqx} \langle 0|T\{\overbrace{\overline{\psi}(x)\gamma_\mu\psi(x)}\, \overbrace{\overline{\psi}(0)\gamma_\nu\psi(0)}\}|0\rangle \\
&= iN_c \left(\frac{i}{2\pi^2}\right)^2 \int d^4x\, e^{iqx}\, \mathrm{Tr}\left[\frac{\not{x}}{x^4}\gamma_\mu \frac{\not{x}}{x^4}\gamma_\nu\right] \\
&= \frac{-iN_c}{4\pi^4} \int d^4x \frac{e^{iqx}}{x^8} \cdot 4\big[2x_\mu x_\nu - g_{\mu\nu}x^2\big]
\end{aligned} \tag{5.13}$$

Fourier integral:

$$\int d^4x\, e^{ipx} \frac{x_\mu x_\nu}{(-x^2 + i\epsilon)^4} \;=\; \frac{i\pi^2}{48} \ln\left(\frac{\mu^2}{-p^2 - i\epsilon}\right) \big[2p_\mu p_\nu + p^2 g_{\mu\nu}\big] \tag{5.14}$$

Thus

$$\begin{aligned}
\Pi_{\mu\nu}(q) &= \frac{-iN_c}{4\pi^4} \frac{i\pi^2}{48} \ln\frac{\mu^2}{-q^2}\, 4\left[2(2q_\mu q_\nu + q^2 g_{\mu\nu}) - 6g_{\mu\nu}q^2\right] \\
&= \frac{N_c}{12\pi^2} \ln\frac{\mu^2}{-q^2}\big[q_\mu q_\nu - q^2 g_{\mu\nu}\big]
\end{aligned} \tag{5.15}$$

Finally

$$\mathrm{Im}\, \ln\frac{\mu^2}{-q^2 - i\epsilon} \;=\; \pi\,\theta(q^2) \qquad \Longrightarrow \qquad \mathrm{Im}\,\Pi(s) = \frac{N_c}{12\pi}\theta(s) \tag{5.16}$$

thus, up to quark electric charges

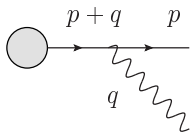
$$R(s) = N_c\Big[1 + \mathcal{O}(\alpha_s)\Big] \qquad \heartsuit \tag{5.17}$$

Corrections are due to radiation of "hard" gluons at small times $\sim 1/q_0$, so it should be possible to calculate them in perturbation theory

- the possibility to use perturbation theory at $q^2 \to +\infty$ is not obvious; we will develop a more rigorous approach later

5.2 Gluon Bremsstrahlung and jets

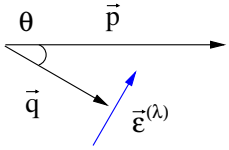
5.2.1 Soft and Collinear emission



$$\epsilon^{*(\lambda)}_\mu M^\mu \;\sim\; \bar{u}(p)\epsilon^{*(\lambda)}_\mu \gamma^\mu \frac{\not{p}+\not{q}+m}{(p+q)^2-m^2} \tag{5.18}$$

Assume that a gluon with small energy is emitted at small angle:

$$\begin{aligned} p_\mu &= \{p_0,0,0,p\} & p_0 = E \gg m \qquad & p_\mu p^\mu = m^2 \\ q_\mu &= \{q_0, q\sin\theta, 0, q\cos\theta\} & q_0 = \omega \ll E \qquad & q_\mu q^\mu = \lambda^2 \quad \text{"gluon mass"} \end{aligned} \tag{5.19}$$



$\theta << 1$

$$\cos\theta \simeq 1 - \frac{\theta^2}{2}$$

$$|p| \simeq E\Big(1 - \frac{m^2}{2E^2}\Big)$$

$$|q| \simeq \omega\Big(1 - \frac{\lambda^2}{2\omega^2}\Big)$$

To this accuracy

$$\begin{aligned} 2pq &= 2E\omega - 2|p||q|\cos\theta = E\omega\left(\theta^2 + \frac{m^2}{E^2} + \frac{\lambda^2}{\omega^2}\right) \\ \epsilon^*\cdot p &= E\sin\theta = E\,\theta \end{aligned} \tag{5.20}$$

and therefore L14

$$\epsilon^{*(\lambda)}_\mu M^\mu \sim \frac{\epsilon^*\cdot p}{2pq+\lambda^2} \;\sim\; \frac{1}{\omega}\,\frac{\theta}{\theta^2 + \frac{m^2}{E^2} + \frac{\lambda^2}{\omega^2} + \cancel{\frac{\lambda^2}{E\omega}}} \tag{5.21}$$

The corresponding cross section is

$$d\sigma \sim |\epsilon^* M|^2 \frac{d^3q}{2q_0} \;\sim\; |\epsilon^* M|^2 \omega d\omega\theta d\theta \;\sim\; \frac{d\omega}{\omega}\cdot\frac{\theta^2 d\theta^2}{\left[\theta^2 + \frac{m^2}{E^2} + \frac{\lambda^2}{\omega^2}\right]^2} \tag{5.22}$$

One can consider two cases:
1) $\frac{\lambda}{\omega} \ll \frac{m}{E}$:

$$\sigma \sim \ln\frac{E}{\lambda}\ln\frac{E}{m} \tag{5.23}$$

2) $\frac{\lambda}{\omega} \gg \frac{m}{E}$:

$$\sigma \sim \int_\lambda^E \frac{d\omega}{\omega} \ln \frac{\omega}{\lambda} \sim \ln^2 \frac{E}{\lambda} \tag{5.24}$$

The general pattern (common for QED and QCD)

$$\boxed{d\sigma \sim \frac{d\omega}{\omega} \frac{d\theta}{\theta}} \tag{5.25}$$

— soft and collinear emission

What is the meaning of the divergence at $\omega \to 0$ (called IR-divergence)?

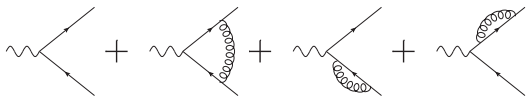
- Wrong question: — What is the total *number* of gluons emitted at a certain angle?
- Correct questions: — What is the total *energy* of gluons emitted at a certain angle?

  — What is the total *number* of gluons with energy $\omega > \omega_0$ (e.g. experimental resolution) emitted at a certain angle?

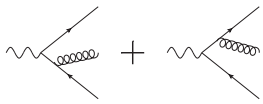
5.2.2 Total cross section to NLO

Continue our discussion of the e^+e^- annihilation

Our LO result can/will be modified by gluon emission at short distances $\sim 1/q_0$. To $\mathcal{O}(\alpha_s)$ accuracy there are two effects (cf. Section 4.1):
1) interaction between the outgoing quark and antiquark



2) gluon production



Explicit calculation gives (for $m_q = 0$), $Q = \sqrt{s}$:

$$\sigma_{e^+e^- \to \bar{q}q} = \sigma_0 + \sigma_0 \frac{4}{3} \frac{\alpha_s}{\pi} \left[-2 \ln^2 \frac{Q}{\lambda} + 3 \ln \frac{Q}{\lambda} - \frac{7}{4} + \frac{\pi^2}{6} \right]$$

$$\sigma_{e^+e^- \to \bar{q}qg} = \sigma_0 \frac{4}{3} \frac{\alpha_s}{\pi} \left[2 \ln^2 \frac{Q}{\lambda} - 3 \ln \frac{Q}{\lambda} + \frac{5}{2} - \frac{\pi^2}{6} \right] \tag{5.26}$$

Both expressions do not makes sense:
— what is λ?
— gluons do not exist as free particles

. . . but the total cross section is well defined:

$$\sigma_{tot} = \sigma_{e^+e^-\to\bar{q}q} + \sigma_{e^+e^-\to\bar{q}qg} = \sigma_0 \left[1 + \frac{\alpha_s}{\pi} + \mathcal{O}\left(\frac{\alpha_s}{\pi}\right)^2\right] \tag{5.27}$$

The cancellation of IR divergencies in the sum of contributions of real and virtual emission is referred to as the "Bloch-Nordsieck cancellation"

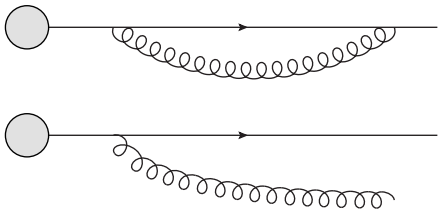
5.2.3 Sterman–Weinberg jets

— IR divergencies in QED signal that the question must be put in a more precise way
— IR divergencies in QCD signal that the quantity of interest recieves large contributions at long distances and cannot be calculated in perturbation theory at all; what to do?

1. Improve the theory. . .
2. Find a class of observables that are not sensitive to contributions of large distances and can be calculated within the theory that we have at present

Infrared-safe observables, that do not suffer from IR divergences, have a chance to be calculable.

Why did IR divergences cancel in the total cross section of e^+e^- annihilation?
— because for the total cross section it does not matter if the emitted gluon recombines with the quark at large distances or flies away:



"To be or not to be" . . .
(gluon)

Sterman and Weinberg formulated a simple criterium for such cancellations:

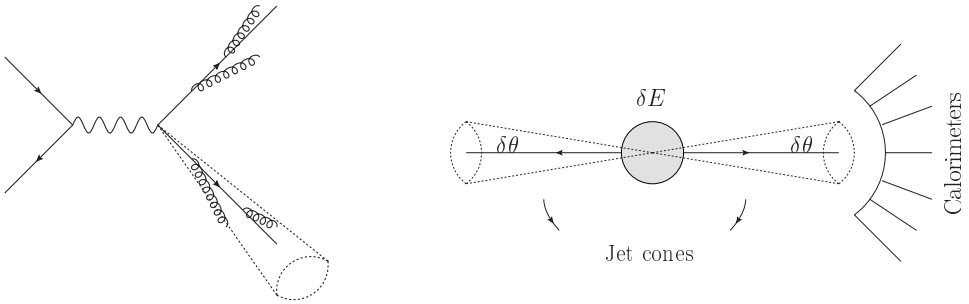
Large (potenitally divergent) contributions come from emission of soft gluons (small energy) or collinear gluons (small angles), therefore:

Observables that do not change if:
a) a "soft" gluon is emitted (with $\omega \to 0$) b) a "collinear" gluon is emitted (with $\theta \to 0$)
are IR safe and have a chance to be calculable

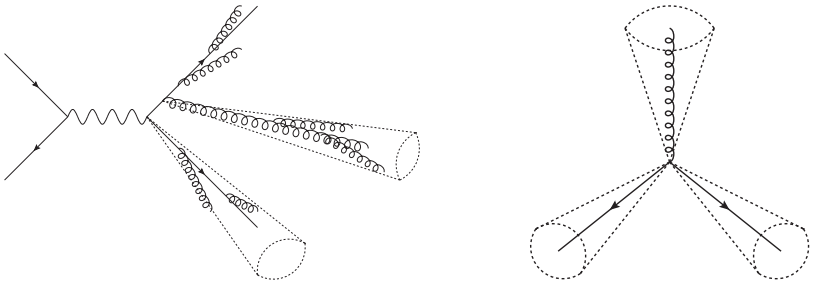
A jet: Collection of particles (spray, bundle,. . .) flying in more or less the same direction (inside a given solid angle $\delta\Omega$, called jet cone)

Jets are IR-stable because if a collinear particle is emitted it remains inside the jet cone, if a soft particle is emitted, the energy of the jet does not change.

A two-jet event:
Almost all energy in the c.m. frame $q_\mu = (Q, 0, 0, 0)$ is deposited in two narrow cones



A three-jet event:
Almost all energy in the c.m. frame $q_\mu = (Q, 0, 0, 0)$ is deposited in three narrow cones



Jet cross sections for given $\delta\theta$ and δE can be calculated in QCD:

$$\sigma_{e^+e^-\to 3\,\text{jets}} \sim \left| \quad + \quad \right|^2_{\substack{\theta>\theta_0 \\ \delta E<\epsilon Q}} \tag{5.28}$$

One obtains

$$\sigma_{e^+e^-\to 3\,\text{jets}} = \sigma_0 \frac{4}{3}\frac{\alpha_s}{\pi}\left[4\ln\frac{1}{\theta_0}\,\ln\frac{1}{\epsilon} + 3\ln\frac{1}{\theta_0} - \frac{7}{4} + \frac{\pi^2}{3}\right]$$

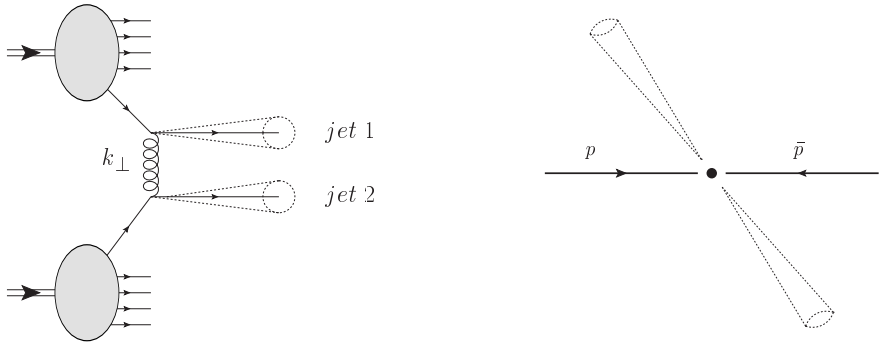
$$\sigma_{e^+e^-\to 2\,\text{jets}} = \sigma_{tot} - \sigma_{e^+e^-\to 3\,\text{jets}} \tag{5.29}$$



An example of real data collected from the OPAL detector on the Large Electron-Positron (LEP) collider at CERN, which ran between 1989 and 2000. Here a Z0 particle is produced in the collision between an electron and positron that then decays into a quark-antiquark pair. The quark pair is seen as a pair of hadron jets in the detector.

• Unphysical IR regulators (gluon mass) substituted by physical parameters of concrete experiment
• Observations of three-jet events (DESY, ca. 1983) gave direct evidence for existence of gluons

Similar:
Jet production in proton-antiproton collisions (Fermilab)



• Need to know quark distributions in protons — later

5.3 Unstable particles

L15

Most of the existing hadrons decay:

Weak decays	$n \to pe\bar{\nu}_e$	$\tau \sim 900s$	[life time]
	$\Lambda \to p\pi^-$	$\tau \sim 10^{-10}s$	
Strong decays	$\rho \to \pi\pi$	$\tau \sim 10^{-23}s$	(5.30)

The life time τ is defined through the decay rate [half-life $= \tau \ln 2$]

$$\tau = 1/\Gamma \tag{5.31}$$

where

$$\Gamma = \frac{\text{Number of decays per unit time}}{\text{Number of particles present}} \tag{5.32}$$

The decay rate can be calculated as (in the rest frame of the decaying particle A)

$$d\Gamma = \frac{1}{2m_A}|M(A \to \{p_f\})|^2 \left(\prod_f \frac{d^3p_f}{(2\pi)^3 2E_f}\right)(2\pi)^4\delta^{(4)}(P_A - \sum p_f) \tag{5.33}$$

For comparison:

$$d\sigma = \frac{1}{2E_A\,2E_B}\frac{1}{|v_A - v_B|}|M(A+B \to \{p_f\})|^2 \left(\prod_f \frac{d^3p_f}{(2\pi)^3 2E_f}\right)(2\pi)^4\delta^{(4)}(P_A + P_B - \sum p_f) \tag{5.34}$$

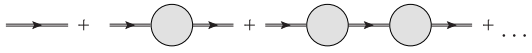
In the following I use a shorthand notation

$$d\Pi_f = \left(\prod_f \frac{d^3 p_f}{(2\pi)^2 2E_f} \right) (2\pi)^4 \delta^{(4)}(P_{\text{initial}} - P_{\text{final}}) \tag{5.35}$$

How can one describe unstable particles in quantum field theory?
— Consider a scalar particle as example

The propagator:

$$\frac{i}{p^2 - m_0^2 + \Sigma(p^2)} \;=\; \tag{5.36}$$



The pole (renormalized) mass:

$$m^2 - m_0^2 + \Sigma(m^2) = 0 \tag{5.37}$$

Renormalized propagator:

$$\begin{aligned} \frac{i}{p^2 - m_0^2 + \Sigma(p^2)} &= \frac{i}{p^2 - m^2 + \Sigma(p^2) - \Sigma(m^2)} \\ &= \frac{i}{(p^2 - m^2)[1 + \Sigma'(m^2)] + [\Sigma(p^2) - \Sigma(m^2) - (p^2 - m^2)\Sigma'(m^2)]} \\ &= \frac{iZ}{p^2 - m^2 - \Sigma^{(r)}(p^2)} \end{aligned} \tag{5.38}$$

Here we tacitly assumed that $\Sigma(m^2)$ is a real number. Is this always the case?

Optical theorem:

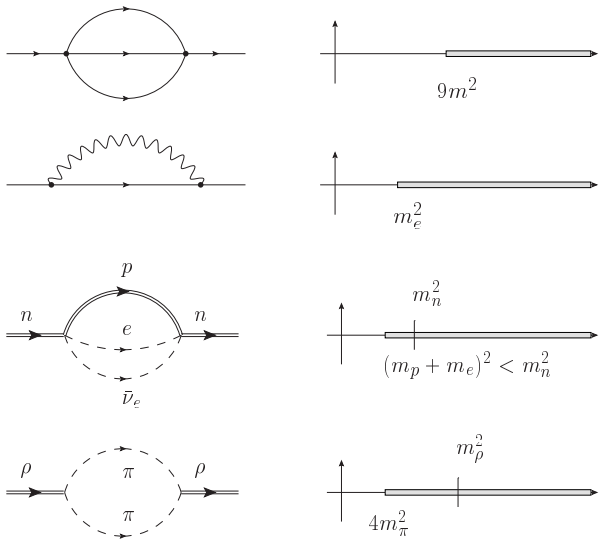
$$2\,\text{Im}\,\Sigma(p^2) \;=\; \sum_f d\Pi_f \, |M(P \to \{p_f\})|^2 \tag{5.39}$$

ϕ^4 – theory

QED/QCD

β – decay

ρ – meson



(5.40)

For unstable particles

$$\Sigma(m^2) \;=\; \mathrm{Re}\,\Sigma(m^2) \;+\; i\,\mathrm{Im}\,\Sigma(m^2) \tag{5.41}$$

One defines the pole mass as

$$m^2 - m_0^2 + \mathrm{Re}\,\Sigma(m^2) \;=\; 0 \tag{5.42}$$

In this case

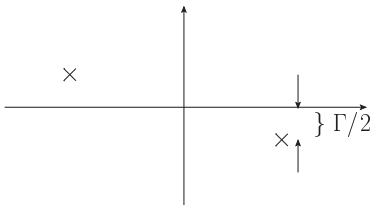
$$\begin{aligned}\frac{i}{p^2 - m_0^2 + \Sigma(p^2)} \;&=\; \frac{i}{p^2 - m^2 + \mathrm{Re}\,\Sigma(p^2) - \mathrm{Re}\,\Sigma(m^2) + i\,\mathrm{Im}\,\Sigma(p^2)} \\ &\overset{p^2\to m^2}{\simeq} \frac{iZ}{p^2 - m^2 + iZ\,\mathrm{Im}\,\Sigma(m^2)}\,, \qquad Z = 1/(1 + \mathrm{Re}\,\Sigma'(m))\end{aligned} \tag{5.43}$$

We define

$$Z\,\mathrm{Im}\,\Sigma(m^2) \;=\; m\Gamma \tag{5.44}$$

The renormalized propagator close to $p^2 = m^2$ is therefore

$$\frac{i}{p^2 - m^2 + im\Gamma}: \tag{5.45}$$



effectively

$$m \to m - i\Gamma/2$$

(mass acquires a *negative* imaginary part)

Note that our Γ is indeed the decay rate:

$$m\Gamma = Z\,\mathrm{Im}\,\Sigma(m^2) = \frac{1}{2}\sum_f \int d\Pi_f\,|M(P \to \{p_f\})|^2 \tag{5.46}$$

in agreement with the definition
[extra Z-factor corresponds to $\sqrt{Z}\cdot\sqrt{Z}$ for external legs (LSZ formula)]

- Why a negative imaginary part (in the mass) corresponds to a decay?

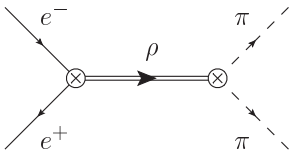
Wave function of a free particle is a plane wave

$$\phi_{\vec{p}}(\vec{x},t) \sim \frac{1}{\sqrt{2E}} e^{-ip_\mu x^\mu} \;\sim\; e^{-i(m - i\frac{1}{2}\Gamma)t} \tag{5.47}$$

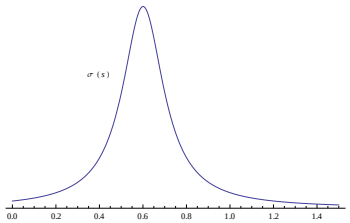
Total probability to find a particle is then

$$\int d^3\vec{x}\,\phi^*(\vec{x},t) i \overset{\leftrightarrow}{\frac{d}{dt}} \phi(\vec{x},t) \;\sim\; e^{-\Gamma t} \;=\; \text{decay} \tag{5.48}$$

Production cross section:



$$\sigma \sim \left| \frac{1}{s - m^2 + im\Gamma} \right|^2$$



(5.49)

This shape is called a Breit-Wigner resonance

This is a good approximation if $\Gamma \ll m$; in general case $\Gamma \to \Gamma(s)$

In the review by the Particle Data Group one finds for $J^{PC} = 1^{--}$:

$$\begin{aligned}
\rho(770): &\quad m_\rho = 775 \text{ MeV}, &\quad \Gamma = 149 \text{ MeV} &\quad \rho \to \pi\pi(\sim 100\%) \\
\omega(782): &\quad m_\omega = 783 \text{ MeV}, &\quad \Gamma = 8.49 \text{ MeV} &\quad \omega \to \pi\pi\pi(\sim 90\%) \\
\phi(1020): &\quad m_\phi = 1019 \text{ MeV}, &\quad \Gamma = 4.26 \text{ MeV} &\quad \phi \to K\,K(\sim 85\%)
\end{aligned} \tag{5.50}$$

ω does not decay in two pions because it has negative G-parity $I^G = 0^-$ compared to $I^G = 1^+$ for ρ-meson.

6 Operator Product Expansion

Motivation:
Find a generalization of the Taylor series for operator products of the type

$$\mathrm{T}\{\phi(0)\phi(x)\} \stackrel{?}{=} \sum_n \frac{1}{n!} x_{\mu_1} \dots x_{\mu_n} \phi(0) \partial_{\mu_1} \dots \partial_{\mu_n} \phi(0) \tag{6.1}$$

or, more generally

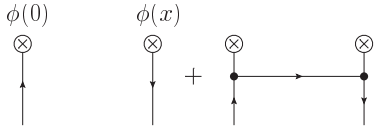
$$\mathrm{T}\{j(0)j(x)\} \stackrel{?}{=} \sum_n C^{(n)}(x) \mathcal{O}_n(0) \tag{6.2}$$

where $\mathcal{O}(0)$ are local operators built of fields and derivatives and $C(x)$ are some functions, i.e. for any states

$$\langle A|\mathrm{T}\{j(0)j(x)\}|B\rangle \stackrel{?}{=} \sum_n C^{(n)}(x) \langle A|\mathcal{O}_n(0)|B\rangle \tag{6.3}$$

This is useful if the sum can be truncated after a few terms:
— x-dependence is "universal" for all matrix elements
— can use RG methods (later)

A naive Taylor expansion does not work because the point $x = 0$ is singular:



$$\langle p|\mathrm{T}\{\phi(0)\phi(x)\}|p\rangle \sim 1 + a\lambda^2 \ln x^2 p^2 + \dots, \qquad a \in \mathbb{R} \tag{6.4}$$

The trick is to introduce a factorization scale

$$\ln x^2 p^2 \;=\; \ln x^2\mu_F^2 + \ln\frac{p^2}{\mu_F^2}\,, \qquad\qquad 1/x^2 \gg \mu_F^2 \gg p^2 \tag{6.5}$$

[Note obvious similarity with *renormalization scale*] and rewrite

$$1 + a\lambda^2 \ln x^2 p^2 + \ldots = \underbrace{\left(1 + a\lambda^2 \ln x^2\mu_F^2\right)}_{\text{coef. function}} \underbrace{\left(1 + a\lambda^2 \ln p^2/\mu_F^2\right)}_{\text{operator matrix element}} + \ldots \tag{6.6}$$

Thus, we envisage an expansion of the type

$$\mathrm{T}\{j(0)j(x)\} = \sum_n C^{(n)}(x,\mu_F)[\mathcal{O}_n]^{(\mu_F)}(0) \tag{6.7}$$

where the coefficient functions $C^{(n)}(x,\mu_F)$ contain all x-dependence and thus all singularities at $x \to 0$, and the operators' matrix elements have to be calculated with an UV cutoff μ_F (or other regulator)

• The result must be independent on $\mu_F \Longrightarrow$ RG machinery: Callan-Symanzik equation etc.

6.1 Example: scalar field theory in 6 dimensions

L16

It turns out that the structure of loop integrals in scalar theory in D=6 dimensions is similar to QED/QCD integrals in physical D=4 dimensions. We consider this theory as a toy model:

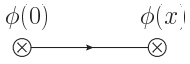
$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3\,, \qquad x_\mu = \{x_0 = ct, \vec{x}\}\,, \qquad \vec{x} = \{x_1,\ldots,x_5\} \tag{6.8}$$

One-particle states

$$|p\rangle = a^\dagger(p)|0\rangle\,, \qquad \langle p'|p\rangle = 2E_p(2\pi)^5\delta^{(5)}(\vec{p}-\vec{p'}) \tag{6.9}$$

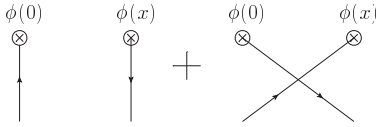
Feynman rules are as usual in scalar theory.

In leading order we have two terms:



$$\langle p'|\,\overbrace{\phi(0)\phi(x)}\,|p\rangle \;=\; \int \frac{d^Dk}{(2\pi)^D}e^{ipx}\frac{i}{k^2+i\epsilon} \times \langle p'||p\rangle \tag{6.10}$$

— this is a disconnected contribution, neglect it



$$\langle p'|:\phi(0)\phi(x):|p\rangle \;=\; e^{-ipx} + e^{ip'x} \tag{6.11}$$

↖ This follows from $\phi(x)|p\rangle = \phi(x)a^\dagger(p)|0\rangle = e^{-ipx}$
The expansion

$$e^{-ipx} \;=\; 1 - ipx + \frac{1}{2}(-ipx)^2 + \ldots \tag{6.12}$$

exactly corresponds to the expansion of the operator

$$\phi(0)\phi(x) \;=\; \phi(0)\phi(0) + x_\mu \phi(0)\partial^\mu\phi(0) + \frac{1}{2}x_\mu x_\nu \phi(0)\partial^\mu\partial^\nu\phi(0) + \dots \tag{6.13}$$

because $\hat{P}_\mu = i\partial_\mu$ is the momentum operator, $\hat{P}_\mu|p\rangle = p_\mu|p\rangle$.
Thus $\partial_\nu\phi(0)|p\rangle = \partial/\partial x^\nu \phi(x)|p\rangle|_{x->0} = \partial/\partial x^\nu e^{-ipx} = (-ip_\nu)e^{-ipx}$.

For simplicity assume $p' = p$; in this case the matrix element of the operator with one derivative vanishes

Local operators can be classified by dimension and spin. Let:

$$\begin{aligned}
\mathcal{O}_1 &= \phi(0)\phi(0)\,, \\
\mathcal{O}_2^{\mu\nu} &= \phi(0)\partial^\mu\partial^\nu\phi(0) - \frac{1}{6}g^{\mu\nu}\phi(0)\partial^2\phi(0) \\
\mathcal{O}_3 &= \phi(0)\partial^2\phi(0)
\end{aligned} \tag{6.14}$$

In six dimensions, $[\phi] = 2$, therefore $\mathcal{O}_1$ has dimension 4 and $\mathcal{O}_{2,3}$ have dimension 6; operator $\mathcal{O}_2$ has spin two, the other two are scalars.

Thus, to leading order we have

$$\langle p|\mathrm{T}\{\phi(0)\phi(x)\}|p\rangle^{LO}_{\text{connected}} = C_1^{LO}\langle p|\mathcal{O}_1|p\rangle^{LO} + C^{LO}_{2;\mu\nu}\langle p|\mathcal{O}_2^{\mu\nu}|p\rangle^{LO} + C_3^{LO}\langle p|\mathcal{O}_3|p\rangle^{LO} + \dots \tag{6.15}$$

with

$$\begin{aligned}
C_1^{LO} &= 1\,, & \langle p|\mathcal{O}_1|p\rangle^{LO} &= 2\,, \\
C_2^{LO} &= \frac{1}{2}x_\mu x_\nu\,, & \langle p|\mathcal{O}_2^{\mu\nu}|p\rangle^{LO} &= -2\Big[p^\mu p^\nu - \frac{1}{6}g^{\mu\nu}m^2\Big] \\
C_3^{LO} &= -\frac{1}{12}x^2\,, & \langle p|\mathcal{O}_3|p\rangle^{LO} &= -2m^2\,,
\end{aligned} \tag{6.16}$$

This is nothing but the Taylor expansion. Note that operators of higher dimension are multiplied by increasing powers of x. Thus if $|x_\mu| \to 0$ only a few operators are important.

Perturbation theory in $g\phi^3$ theory is conveniently organized in terms of the coupling

$$a = \frac{g^2}{(4\pi)^3} \tag{6.17}$$

We write

$$\begin{aligned}
\langle p|\mathrm{T}\{\phi(0)\phi(x)\}|p\rangle &= \langle p|\mathrm{T}\{\phi(0)\phi(x)\}|p\rangle^{LO} + a\langle p|\mathrm{T}\{\phi(0)\phi(x)\}|p\rangle^{NLO} + \dots \\
\langle p|[\mathcal{O}_1]^{(\mu_F)}|p\rangle &= \langle p|[\mathcal{O}_1]^{(\mu_F)}|p\rangle^{LO} + a\langle p|[\mathcal{O}_1]^{(\mu_F)}|p\rangle^{NLO} + \dots
\end{aligned} \tag{6.18}$$

etc.
The conjectured OPE has the form

$$\begin{aligned}\langle p|\mathrm{T}\{\phi(0)\phi(x)\}|p\rangle &= \langle p|\mathrm{T}\{\phi(0)\phi(x)\}|p\rangle^{LO} + a\langle p|\mathrm{T}\{\phi(0)\phi(x)\}|p\rangle^{NLO} \\ &:= \Big[1 + aC_1^{NLO}(x,\mu_F)\Big]\Big[\langle p|[\mathcal{O}_1]^{(\mu_F)}|p\rangle^{LO} + a\langle p|[\mathcal{O}_1]^{(\mu_F)}|p\rangle^{NLO}\Big] \\ &+ \Big[\frac{1}{2}x_\alpha x_\beta + aC_{2;\alpha\beta}^{NLO}(x,\mu_F)\Big]\Big[\langle p|[\mathcal{O}_2^{\alpha\beta}]^{(\mu_F)}|p\rangle^{LO} + a\langle p|[\mathcal{O}_2^{\alpha\beta}]^{(\mu_F)}|p\rangle^{NLO}\Big] \\ &+ \Big[-\frac{1}{12}x^2 + aC_3^{NLO}(x,\mu_F)\Big]\Big[\langle p|[\mathcal{O}_3]^{(\mu_F)}|p\rangle^{LO} + a\langle p|[\mathcal{O}_3]^{(\mu_F)}|p\rangle^{NLO}\Big] + \dots\end{aligned} \tag{6.19}$$

Comparing terms $\mathcal{O}(a)$ on the l.h.s and r.h.s.

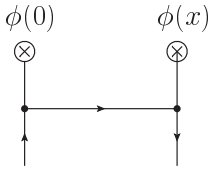
$$\begin{aligned}\langle p|\mathrm{T}\{\phi(0)\phi(x)\}|p\rangle^{NLO} = {} & C_1^{NLO}(x,\mu_F)\times 2 && + 1\times\langle p|[\mathcal{O}_1]^{(\mu_F)}|p\rangle^{NLO} \\ & + C_{2;\alpha\beta}^{NLO}(x,\mu_F)\times\Big(-2p^\alpha p^\beta + \frac{1}{3}g^{\alpha\beta}m^2\Big) && + \frac{1}{2}x_\alpha x_\beta\times\langle p|[\mathcal{O}_2^{\alpha\beta}]^{(\mu_F)}|p\rangle^{NLO} + \dots \\ & + C_3^{NLO}(x,\mu_F)\times(-2m^2) && + \Big(-\frac{1}{12}x^2\Big)\times\langle p|[\mathcal{O}_3]^{(\mu_F)}|p\rangle^{NLO} + \dots\end{aligned} \tag{6.20}$$

Thus in order to calculate the coefficient functions $C_{1,2}^{NLO}$ I need to do *two* calculations:

- Calculate the amplitude $\langle p|\mathrm{T}\{\phi(0)\phi(x)\}|p\rangle$ to one-loop accuracy
- Calculate the (renormalized) operator matrix elements $\langle p|[\mathcal{O}_n]^{(\mu_F)}|p\rangle$ to one-loop accuracy
- Identify different structures and subtract

Let us follow this procedure for the simplest example.

Step 1: In first order in g^2 we have the following diagram:



$+ \quad crossed$ (6.21)

Explicit calculation gives $\bar{u} = 1 - u$

$$\langle p|\mathrm{T}\{\phi(0)\phi(x)\}|p\rangle^{NLO} = 2\int_0^1 du\,\bar{u}\Big[e^{-uipx} + e^{uipx}\Big]K_0(\sqrt{m^2(1-u\bar{u})(-x^2+i\epsilon)}) \tag{6.22}$$

Expanding this result for $|x|\to 0$ obtain

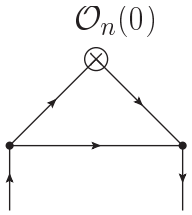
$$\begin{aligned}\langle p|\mathrm{T}\{\phi(0)\phi(x)\}|p\rangle^{NLO} = {} & \Big[-\ln(-x^2m^2) + d_1\Big] - \frac{1}{6}(px)^2\Big[-\ln(-x^2m^2) + d_2\Big] \\ & + \frac{5}{24}(-x^2m^2)\Big[-\ln(-x^2m^2) + d_3\Big] + \dots\end{aligned} \tag{6.23}$$

where e.g.

$$d_1 = -\frac{\pi}{\sqrt{3}} + 2 - 2\gamma_E + 2\ln 2$$
$$d_2 = -\sqrt{3}\pi + \frac{17}{6} - 2\gamma_E + 2\ln 2 \tag{6.24}$$

etc.

Step 2: In first order in g^2 we have the following diagram:



(6.25)

By explicit calculation in dimensional regularization $g \to g\mu_F^{3-D/2}$

$$\begin{aligned} & \langle p|\mathcal{O}_1|p\rangle^{NLO} \;=\; \frac{1}{3-D/2} - \gamma_E + \ln 4\pi + \ln\frac{\mu_F^2}{m^2} - \frac{\pi}{\sqrt{3}} + 2 \\ \Longrightarrow \\ & \langle p|[\mathcal{O}_1]^{(\mu_F)}|p\rangle^{NLO} \;=\; \ln\frac{\mu_F^2}{m^2} - \frac{\pi}{\sqrt{3}} + 2 \end{aligned} \tag{6.26}$$

and therefore

$$C_1^{NLO}(x,\mu_F) \;=\; -\frac{1}{2}\ln(-x^2\mu_F^2) - \gamma_E + \ln 2 \tag{6.27}$$

Similar

$$\langle p|\mathcal{O}_2^{\mu\nu}|p\rangle^{NLO} = -\frac{1}{3}\left(p^\mu p^\nu - \frac{1}{6}g^{\mu\nu}m^2\right)\left[\frac{1}{3-D/2} - \gamma_E + \ln 4\pi + \ln\frac{\mu_F^2}{m^2} - \sqrt{3}\pi + \frac{17}{6}\right] \tag{6.28}$$

leading to

$$C_{2;\mu\nu}^{NLO}(x,\mu_F) \;=\; \frac{1}{16}x_\mu x_\nu\Big[-\frac{1}{2}\ln(-x^2\mu_F^2) - \gamma_E + \ln 2\Big] \tag{6.29}$$

Collecting everything we obtain

$$\begin{aligned} \langle p|\mathrm{T}\{\phi(0)\phi(x)\}|p\rangle =& \Big\{1 + a\Big[-\frac{1}{2}\ln(-4x^2\mu_F^2) - \gamma_E\Big] + \dots\Big\}\,\langle p|[\mathcal{O}_1]^{(\mu_F)}|p\rangle \\ & + \frac{1}{2}x_\alpha x_\beta\Big\{1 + \frac{a}{4}\Big[-\frac{1}{2}\ln(-4x^2\mu_F^2) - \gamma_E\Big] + \dots\Big\}\,\langle p|[\mathcal{O}_2^{\mu\nu}]^{(\mu_F)}|p\rangle + \dots \\ \equiv\; & C_1(x,\mu_F)\,\langle p|[\mathcal{O}_1]^{(\mu_F)}|p\rangle + C_{2,\alpha\beta}(x,\mu_F)\,\langle p|[\mathcal{O}_2^{\mu\nu}]^{(\mu_F)}|p\rangle + \dots \end{aligned} \tag{6.30}$$

The calculation can be continued iteratively to each order of perturbation theory

The main statement is that the coefficient functions are universal and do not depend on the particular matrix element $\langle p|\ldots|p\rangle$ that I have taken for this calculation. Thus we have a relation between operators

$$\begin{aligned}\mathrm{T}\{\phi(0)\phi(x)\} =&\Big\{1+a\Big[-\frac{1}{2}\ln(-4x^2\mu_F^2)-\gamma_E\Big]+\ldots\Big\}\,[\mathcal{O}_1]^{(\mu_F)}\\ &+\frac{1}{2}x_\alpha x_\beta\Big\{1+\frac{a}{4}\Big[-\frac{1}{2}\ln(-4x^2\mu_F^2)-\gamma_E\Big]+\ldots\Big\}\,[\mathcal{O}_2^{\alpha\beta}]^{(\mu_F)}+\ldots\\ \equiv& C_1(x,\mu_F)\,[\mathcal{O}_1]^{(\mu_F)}+C_{2,\alpha\beta}(x,\mu_F)\,[\mathcal{O}_2^{\alpha\beta}]^{(\mu_F)}+\ldots\end{aligned}\tag{6.31}$$

which means that it is valid for arbitrary matrix elements.

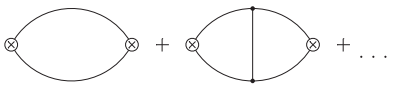
What is the use of this expansion? L17

- Separation of scales (Factorization):
  — By construction, contributions of large momenta $|k|>\mu_F$ are included in coefficient functions (CFs), contributions of small momenta $|k|<\mu_F$ are absorbed in matrix elements (MEs)
  — In QCD, this means that CFs can be calculated in PT
- Universality: The same CFs for arbitrary MEs. — Can calculate CFs using simplest possible ME (quark state) and use for complicated MEs (nucleon state)
- Universality: The same MEs appear in the OPE of different correlation functions (physical amplitudes)

  Example:

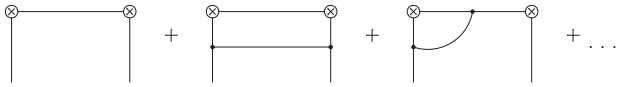
$$\mathrm{T}\{\phi^2(0)\phi^2(x)\} =\tilde{C}_0(x)\,\mathbb{I}+\tilde{C}_1(x,\mu_F)\,[\mathcal{O}_1]^{(\mu_F)}+\tilde{C}_{2,\mu\nu}(x,\mu_F)\,[\mathcal{O}_2^{\mu\nu}]^{(\mu_F)}+\ldots\tag{6.32}$$

  The unity operator: contributes for vacuum ME $\langle 0|\mathbb{I}|0\rangle=1$:



(6.33)

  The rest:



(6.34)

  Expansion of the propagator

$$\begin{aligned}\int\frac{d^Dk}{(2\pi)^D}e^{ikx}\frac{i}{k^2+m^2}=&\int\frac{d^Dk}{(2\pi)^D}e^{ikx}\left(\frac{i}{k^2}-\frac{im^2}{k^4}+\ldots\right)\\ =&\frac{1}{4\pi^3x^4}-\frac{m^2}{16\pi^3x^2}+\mathcal{O}(\ln x^2)\qquad\text{[everything for D=6!]}\end{aligned}\tag{6.35}$$

Then

$$\tilde{C}_0(x) = \frac{1}{8\pi^6 x^8}\left[1 + c_0 a(\mu^2 = 1/x^2) + \dots\right]$$
$$\tilde{C}_1(x) = \frac{1}{2\pi^3 x^4}\left[1 + c_1 a \ln(-x^2\mu_F^2) + \dots\right] \tag{6.36}$$

etc.

In QCD applications, CFs depend on the process and can be calculated; operator MEs are typically unknown, but can be introduced as free parameters and their values determined from a dedicated experiment. They can be used afterwards to describe other experiments.

6.2 Renormalization of composite operators and renormalization group

CFs of composite operators have the generic structure

$$C(x, \mu_f) \sim 1 + \gamma g^2 \ln(-x^2\mu_F^2) + \dots \tag{6.37}$$

If $|x| \to 0$, the logarithms eventually become large. If $g^2 \ll 1$ but $g^2 \ln(-x^2\mu_F^2) \sim \mathcal{O}(1)$ one has to account the whole series of contributions $\sim [g^2 \ln(-x^2\mu_F^2)]^k$ but can still neglect terms with $\sim g^2[g^2 \ln(-x^2\mu_F^2)]^k$.

This is the same situation as we had with the polarization operator or electron propagator in QED, so we can treat this problem using the same methods.

For a generic operator $\mathcal{O}$ we can write

$$\begin{array}{ccccc} \mathcal{O}(\Lambda, \alpha_0) & = & Z_O(\Lambda, \mu, \alpha_0)[\mathcal{O}]^{(\mu)} & & \\ \uparrow & & \swarrow & \searrow & \\ \text{UV-cutoff} & & \text{ren.const} & \text{ren.operator} & \end{array} \tag{6.38}$$

Renormalized operator is defined through the renormalization condition.
Typical requirement: MEs of the renormalized operator coincide with the MEs of the same operator in the free theory at a certain scale.

Since the l.h.s. of this equation does not depend on μ, we obtain

$$\mu\frac{d}{d\mu}\mathcal{O}(\Lambda) = 0 \implies \mu\frac{d}{d\mu}\left(Z_O\right)[\mathcal{O}]^{(\mu)} + Z_O\left[\mu\frac{\partial}{\partial\mu} + \left(\mu\frac{d}{d\mu}\alpha\right)\frac{\partial}{\partial\alpha}\right][\mathcal{O}]^{(\mu)} = 0 \tag{6.39}$$

Define

$$\mu\frac{d}{d\mu}\alpha(\mu) = \beta(\alpha) = \beta_0\alpha^2 + \dots$$
$$\frac{1}{Z_O}\,\mu\frac{d}{d\mu}Z_O = \gamma^O(\alpha) = \gamma_0^O\alpha + \dots \tag{6.40}$$

$\implies$ Callan-Symanzik equation

$$\boxed{\left\{\mu\frac{\partial}{\partial\mu} + \beta(\alpha)\frac{\partial}{\partial\alpha} + \gamma^O(\alpha)\right\}[\mathcal{O}]^{(\mu)} = 0} \tag{6.41}$$

The solution of this equation is, as we have seen

$$[\mathcal{O}]^{(\mu_1)} = \left(\frac{\alpha(\mu_1^2)}{\alpha(\mu_2^2)}\right)^{-\gamma_0^O/\beta_0} [\mathcal{O}]^{(\mu_2)} \tag{6.42}$$

The operator product expansion can be formulated as the following

Theorem (K. Wilson)

Let $[\mathcal{O}_1]^{(\mu)}$ and $[\mathcal{O}_2]^{(\mu)}$ be renormalized operators. Then

$$[\mathcal{O}_1]^{(\mu)}(x)[\mathcal{O}_2]^{(\mu)}(0) \;=\; \sum_n C_{12}^n(x,\mu)[\mathcal{O}_n]^{(\mu)}(0) \tag{6.43}$$

where the sum goes over the complete set of renormalized operators $[\mathcal{O}_n]^{(\mu)}$ with suitable quantum numbers and $C_{12}^n(x,\mu)$ are (complex valued) functions.

Each of the operators in the OPE satisfies its own Callan-Symanzik equation:

$$\begin{aligned}\left\{\mu\frac{\partial}{\partial\mu} + \beta(\alpha)\frac{\partial}{\partial\alpha} + \gamma_{1,2}\right\}[\mathcal{O}]_{1,2}^{(\mu)} = 0 \\ \left\{\mu\frac{\partial}{\partial\mu} + \beta(\alpha)\frac{\partial}{\partial\alpha} + \gamma_{n}\right\}[\mathcal{O}]_{n}^{(\mu)} = 0\end{aligned} \tag{6.44}$$

Since the μ-dependence of the two sides of the OPE must agree, the CFs have to satisfy a similar equation:

$$\left\{\mu\frac{\partial}{\partial\mu} + \beta(\alpha)\frac{\partial}{\partial\alpha} + \gamma_1 + \gamma_2 - \gamma_n\right\}C_{12}^n(x,\mu) = 0 \tag{6.45}$$

Let d_1, d_2 and d_n be the dimensions of the operators $[\mathcal{O}_{1,2,n}]^{(\mu)}$, respectively. Since dimensions in all terms in the OPE must be the same, the CFs have to be of the form

$$C_{12}^n(x,\mu) = \left(\frac{1}{|x|}\right)^{d_1+d_2-d_n} \tilde{C}(x\cdot\mu) \tag{6.46}$$

where $\tilde{C}$ is a dimensionless function that depends on the product $x\mu$ only.

The general solution (cf. electron propagator)

$$\begin{aligned}C_{12}^n(x,\mu) &= \left(\frac{1}{|x|}\right)^{d_1+d_2-d_n} \widetilde{C}_{12}^n(\alpha(1/|x|))\exp\left\{\int_{\alpha(\mu)}^{\alpha(1/|x|)} d\alpha' \frac{1}{\beta(\alpha')}\Big[\gamma_n(\alpha') - \gamma_1(\alpha') - \gamma_2(\alpha')\Big]\right\} \\ &\simeq \left(\frac{1}{|x|}\right)^{d_1+d_2-d_n} \widetilde{C}_{12}^n \left(\frac{\alpha(1/|x|)}{\alpha(\mu)}\right)^{-(\gamma_0^n-\gamma_0^1-\gamma_0^2)/\beta_0}\end{aligned} \tag{6.47}$$

Special case: It can happen that $\beta(\alpha) = 0$ or at least $\beta(\alpha^*) = 0$ for a particularly chosen α^* (critical point, e.g. point of phase transition in condensed matter). In this case the solution is

$$C_{12}^n(x) = \widetilde{C}_{12}^n \left(\frac{1}{|x|}\right)^{D_1+D_2-D_n}, \qquad \begin{aligned}D_1 &= d_1 + \gamma^1(\alpha^*) \\ D_2 &= d_2 + \gamma^2(\alpha^*) \\ D_n &= d_n + \gamma^n(\alpha^*)\end{aligned} \tag{6.48}$$

This explains why γ^n are called anomalous dimensions

7 Electron-Positron annihilation II

7.1 OPE analysis and QCD sum rules

We have found that

$$R(s) \;=\; \frac{\sigma_{tot}(e^+e^- \to \text{ hadrons})}{\sigma_{tot}(e^+e^- \to \mu+\mu^-)} \;=\; 12\pi\,\mathrm{Im}\,\Pi(s) \tag{7.1}$$

where

$$\Pi_{\mu\nu}(q) = i\int d^4x\, e^{iqx}\langle\Omega|T\{j_\mu(x)j_\nu(0)\}|\Omega\rangle = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2)\,, \qquad s=q^2 \tag{7.2}$$

Our program is now:

1. Find a way to relate $\Pi(s)$ at large s to $T\{j_\mu(x)j_\nu(0)\}$ at small $|x|$
2. Study $T\{j_\mu(x)j_\nu(0)\}$ using OPE

7.1.1 Disperion relations

Causality $\Longrightarrow$ $\Pi(q^2)$ is an analytic function of q^2 with a cut at positive real $q^2 = s > 4m_\pi^2$.

It is easy to see that the expansion of $T\{j_\mu(x)j_\nu(0)\}$ in powers of $|x|$ produces a series in $1/q$ after the Fourier transform, but convergence properties of this series can depend on the direction in the complex q^2 plane.

There are reasons to suspect that convergence becomes bad (no uniform convergence) for q^2 approaching the cut at real positive values.

Example 1: (extreme)
Assume $\Pi(q^2)$ has a contribution

$$\Pi(q^2) = \ldots + e^{q^2} \tag{7.3}$$

For $q^2 < 0$ it is exponentially suppressed, will not be seen in OPE in any finite order, but it explodes at $q^2 > 0$

Example 2: (more realistic)
Assume $\Pi(q^2)$ has a contribution

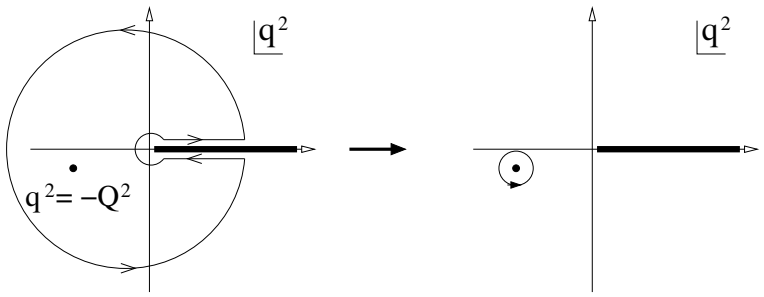
$$\begin{aligned}
&\Pi(q^2) = \ldots + K_0(\sqrt{-q^2 - i\epsilon}) \\
&q^2 \to -\infty: \qquad K_0(\sqrt{-q^2}) \simeq \text{const.} \times \left(\frac{1}{-q^2}\right)^{-1/4} e^{-\sqrt{-q^2}} \\
&q^2 \to +\infty: \qquad K_0(\sqrt{-q^2}) \simeq \text{const.} \times \left(\frac{1}{q^2}\right)^{-1/4} e^{i\sqrt{q^2} - i\pi/4}
\end{aligned} \tag{7.4}$$

For $q^2 < 0$ it is exponentially suppressed, will not be seen in OPE in any finite order, but produces an oscillating correction $\sim 1/\sqrt{q^2}$ at $q^2 > 0$

Thus, it is believed that the OPE has to work at $q^2 \to -\infty$ (called Euclidian region) but there

might be subtleties for $q^2 \to +\infty$ which so far nobody was able to quantify.

The trick is to connect these two regions using dispersion relations L18



(7.5)

If the integral over the large circle can be neglected

$$\Pi(q^2) = \frac{1}{2\pi i}\oint ds \frac{\Pi(s)}{s-q^2} = \frac{1}{2\pi i}\int_0^\infty \frac{ds}{s-q^2}\underbrace{\Big[\Pi(s+i\epsilon)-\Pi(s-i\epsilon)\Big]}_{2i\,\mathrm{Im}\,\Pi(s)} \tag{7.6}$$

so that

$$\Pi(q^2) = \frac{1}{\pi}\int_0^\infty \frac{ds}{s-q^2}\,\mathrm{Im}\,\Pi(s) = \frac{1}{12\pi^2}\int_0^\infty \frac{ds}{s-q^2}\,R(s) \tag{7.7}$$

— a dispersion relation.

Unfortunately it does not work that way, because for large q^2 we derived $\Pi(q^2) \sim \ln(q^2)$. A simple estimate shows that in this case the large circle contribution does not vanish.

A possible way out: do the same for the derivative:

$$\frac{d}{dq^2}\Pi(q^2) = \frac{1}{12\pi^2}\int_0^\infty \frac{ds}{(s-q^2)^2}\,R(s) \tag{7.8}$$

or write a dispersion relation with a subtraction:

$$\Pi(q^2) = \Pi(0) + q^2\frac{1}{12\pi^2}\int_0^\infty \frac{ds}{s-q^2}\frac{1}{s}\,R(s) \tag{7.9}$$

How to use this:

- Choose large $Q^2 = -q^2$, much larger than hadron masses
- Calculate the r.h.s. inserting exp. data and integrating numerically
- Calculate the l.h.s. in QCD using PT and/or OPE
- Compare and draw conclusions

7.1.2 Operator product expansion

We want to write

$$\langle\Omega|T\{j_\mu(x)j_\nu(0)\}|\Omega\rangle = \sum_n C^n_{\mu\nu}\langle\Omega|O_n|\Omega\rangle \tag{7.10}$$

Which operators can contribute? $\Longleftarrow$ Gauge and Lorentz invariance

Only gauge-invariant scalar operators can have nonzero vacuum expectation value (VEV):

1) Unity Operator	$\mathbb{I}$	$d_{\mathbb{I}} = 0\,, \quad d_j = 3$	$\Rightarrow$	$C_{\mathbb{I}} \sim \frac{1}{x^6}$
2) Quark Condensate	$\langle\Omega\|\bar\psi\psi\|\Omega\rangle$	$d_{\bar\psi\psi} = 3\,, \quad d_j = 3$	$\Rightarrow$	$C_{\bar\psi\psi} \sim \frac{1}{x^3}$
3) Gluon Condensate	$\langle\Omega\|G^A_{\mu\nu}G^{\mu\nu;A}\|\Omega\rangle$	$d_{GG} = 4\,, \quad d_j = 3$	$\Rightarrow$	$C_{GG} \sim \frac{1}{x^2}$
4) Mixed Condensate	$\langle\Omega\|\bar\psi g\sigma_{\mu\nu}G^{\mu\nu}\psi\|\Omega\rangle$	$d_{\bar\psi G\psi} = 5$	$\Rightarrow$	$C_{\bar\psi G\psi} \sim \frac{1}{x^1}$
5)	$\langle\Omega\|(\bar\psi\Gamma\psi)^2\|\Omega\rangle$	$d_{(\bar\psi\psi)^2} = 6$	$\Rightarrow$	$C_{(\bar\psi\psi)^2} \sim \ln x$
etc.				

(7.11)

After Fourier trafo

$$\int d^4x\, e^{iqx}\, C_{\mathbb{I}}(x) \;\sim q^2 \qquad\qquad = (q_\mu q_\nu - q^2 g_{\mu\nu})\cdot\mathcal{O}(q^0) \tag{7.12}$$

so that the corresponding contribution to $\Pi(q^2)$ is $\mathcal{O}(q^0)$ [In reality it is $\ln q^2/\mu^2$]

Similar

$$\int d^4x\, e^{iqx}\, C_{GG}(x) \sim \frac{1}{q^2}\,, \qquad \text{etc.} \tag{7.13}$$

Therefore OPE takes the form

$$\Pi(q^2) = \underbrace{\widetilde{C}_{\mathbb{I}}}_{\searrow\ \text{pert. theory}} + \frac{m_\psi}{q^4}\widetilde{C}_{\bar\psi\psi}\langle\bar\psi\psi\rangle + \frac{1}{q^4}\widetilde{C}_{GG}\langle G^2\rangle + \dots \tag{7.14}$$

where $\widetilde{C}_i$ are dimensionless functions (may contain $\ln q^2/\mu^2$)

Explicit calculation (Shifman, Vainstein, Zakharov 1979)

$$\begin{aligned}\Pi(q^2) = {} & \frac{1}{4\pi^2}\ln\frac{\mu^2}{-q^2}\Big[1 + \frac{\alpha_s(-q^2)}{\pi} + \dots\Big] + \frac{m_u + m_d}{6q^4}\langle\bar\psi\psi\rangle \\ & + \frac{1}{12q^4}\Big\langle\frac{\alpha_s}{\pi}G^2\Big\rangle + \frac{224}{81q^6}\alpha_s\cdot\pi\langle\bar\psi\psi\rangle^2 + \dots\end{aligned} \tag{7.15}$$

- in NLO, all coefficient functions receive corrections in $\alpha_s(|q^2|)$
- Perturbation theory is correct to $\mathcal{O}(1/q^4)$ accuracy: There exist no gauge-invariant operators

with dimension two.

- Leading corrections beyond perturbation theory involve two parameters

$$\langle\bar{\psi}\psi\rangle \simeq -(250\,\mathrm{MeV})^3$$
$$\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle \simeq 0.012 - 0.020\,\mathrm{GeV}^4 \tag{7.16}$$

- Interpretation: (background field)

Separate formally all field operators in "fast" and "slow" components:

$$\phi(x) = \left(\int\limits_{|k|>\mu}\frac{d^3k}{(2\pi)^3 2E_k} + \int\limits_{|k|<\mu}\frac{d^3k}{(2\pi)^3 2E_k}\right)\left[\hat{a}^\dagger(k)e^{ikx} + \hat{a}(k)e^{-ikx}\right] = \phi_{\mathrm{fast}}(x) + \phi_{\mathrm{slow}}(x) \tag{7.17}$$

so for quarks and gluons

$$\psi(x) = \psi_{\mathrm{fast}}(x) + \psi_{\mathrm{slow}}(x)\,, \qquad A^\mu(x) = A^\mu_{\mathrm{fast}}(x) + A^\mu_{\mathrm{slow}}(x) \tag{7.18}$$

Fast and slow components can be viewed as independent fields in the Lagrangian. One can, for example, consider "slow" fields as given external fields (think of a magnet in a lab) and develop a Feynman diagram technique for calculation of fast field propagation in the background of given slow fields (think of an electron propagating and emitting photons inside a magnet)

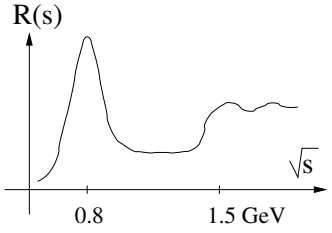
By this reason, fast and slow fields are often referred to as "quantum" and "classical"

Integrations over the fast ("quantum") fields in QCD can be done in perturbation theory; they correspond to the coefficient functions in the OPE (by construction, see scalar examples) and the result is expressed in terms of slow ("classical") fields in the vacuum, which cannot be described perturbatively.

Vacuum condensates are parametrizations of average properties of nonperturbative vacuum fields — a variant of the mean field aproach adapted for QFT.

7.1.3 QCD sum rules

Experiment:



where the peak corresponds to the ρ-meson. Let us calculate this contribution.

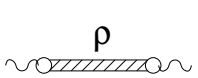
$$\Pi_{\mu\nu}(q) = \int d^4x\, e^{iqx}\langle\Omega|\,\mathrm{T}\,\{j_\mu(x)j_\nu(0)\}|\Omega\rangle$$
$$= \int d^4x\, e^{iqx}\Big[\theta(x_0)\langle\Omega|j_\mu(x)|\rho\rangle\langle\rho|j_\nu(0)\}|\Omega\rangle + \theta(-x_0)\langle\Omega|j_\nu(0)|\rho\rangle\langle\rho|j_\mu(x)\}|\Omega\rangle\Big] \tag{7.19}$$

Define L19

$$\langle\Omega|j_\mu(0)|\rho^{(\lambda)}(p)\rangle \;=\; m_\rho^2\frac{\sqrt{2}}{g_\rho}\epsilon_\mu^{(\lambda)}\,, \qquad \lambda=1,2,3\,, \qquad \epsilon_\mu^{(\lambda)}p^\mu=0 \tag{7.20}$$

Note that a ρ-meson can have three polarizations, two transverse ones and one longitudinal

The contribution of interest corresponds to a simple Feynman diagram



$$= \sum_\lambda m_\rho^2\frac{\sqrt{2}}{g_\rho}\epsilon_\mu^{(\lambda)}\cdot\frac{1}{m_\rho^2-q^2}\epsilon_\nu^{(\lambda)*}m_\rho^2\frac{\sqrt{2}}{g_\rho} \tag{7.21}$$

The sum over polarizations can be done using

$$\sum_\lambda \epsilon_\mu^{(\lambda)}\,\epsilon_\nu^{(\lambda)*} = -g_{\mu\nu}+\frac{q_\mu q_\nu}{m_\rho^2} \tag{7.22}$$

Obtain

$$\begin{aligned}\ldots \;&=\; m_\rho^4\frac{2}{g_\rho^2}\frac{1}{m_\rho^2-q^2}\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{m_\rho^2}\right)=\frac{2m_\rho^2}{g_\rho^2}\frac{1}{m_\rho^2-q^2}\left(-g_{\mu\nu}m_\rho^2+q_\mu q_\nu\right)\\ &=\frac{2m_\rho^2}{g_\rho^2}\frac{1}{m_\rho^2-q^2}\left(-g_{\mu\nu}q^2+q_\mu q_\nu\right)\;+\;\text{term}\sim g_{\mu\nu}\,\text{without the pole}\end{aligned} \tag{7.23}$$

? Is this extra term in contradiction with gauge invariance
— no, it will be cancelled by contributions of other states

Thus

$$\boxed{\Pi(q^2)\Big|_{\rho-\text{meson}} \;=\; \frac{2m_\rho^2}{g_\rho^2}\frac{1}{m_\rho^2-q^2-i\epsilon}} \tag{7.24}$$

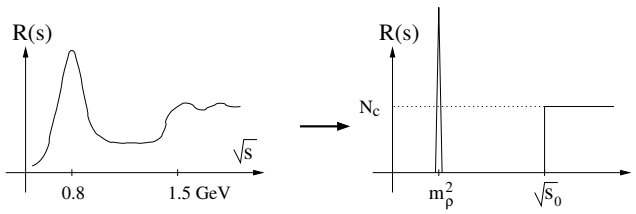
and therefore

$$R_\rho(s) \;=\; 12\pi\,\mathrm{Im}\,\Pi(s) \;=\; 12\pi^2\frac{2m_\rho^2}{g_\rho^2}\delta(s-m_\rho^2) \tag{7.25}$$

In this derivation we assumed that ρ is a stable state (for simplicity);
taking into account decays $\rho\to\pi\pi$ will modify the δ-function to a Breit-Wigner resonance.

Idea: (Shifman, Vainstein, Zakharov)

Consider a simplified model of the spectrum



with free parameters m_ρ^2, g_ρ and s_0

Then on the one hand

$$\Pi^{\rm model}(q^2) = \frac{1}{12\pi^2}\int^{\mu^2}\frac{ds}{s-q^2}R^{\rm model}(s) = \frac{2m_\rho^2}{g_\rho^2}\frac{1}{m_\rho^2-q^2} + \frac{1}{12\pi^2}\int_{s_0}^{\mu^2}\frac{ds}{s-q^2}\cdot N_c \tag{7.26}$$

and on the other hand

$$\Pi^{\rm QCD}(q^2) = \frac{N_c}{12\pi^2}\ln\frac{\mu^2}{-q^2} + \frac{1}{12q^4}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle + \frac{224}{81q^6}\alpha_s\cdot\pi\langle\bar\psi\psi\rangle^2 + \ldots \tag{7.27}$$

The sum rule:

$$\boxed{\Pi^{\rm model}(q^2)\simeq\Pi^{\rm QCD}(q^2) \qquad \text{for moderately large negative } q^2} \tag{7.28}$$

Why moderatively large:
For small q^2 need many terms in the OPE, for large q^2 loose sensitivity to resonance

First trick: The pert. term in the OPE can be written as ($\mu^2 \gg q^2$)

$$\frac{N_c}{12\pi^2}\ln\frac{\mu^2}{-q^2} \;=\; \frac{N_c}{12\pi^2}\int_0^{\mu^2}\frac{ds}{s-q^2} = \int_0^{s_0} + \int_{s_0}^{\mu^2} \tag{7.29}$$

Subtracting the $\int_{s_0}^{\mu^2}$ part from the both sides obtain

$$\frac{2m_\rho^2}{g_\rho^2}\frac{1}{m_\rho^2-q^2} := \frac{N_c}{12\pi^2}\int_0^{s_0}\frac{ds}{s-q^2} + \frac{1}{12q^4}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle + \frac{224}{81q^6}\alpha_s\cdot\pi\langle\bar\psi\psi\rangle^2 \tag{7.30}$$

This must work for $q^2\sim-(1-2)$ GeV^2, $\Longrightarrow$ three parameters m_ρ^2, g_ρ and s_0

Second trick: Borel transformation $q^2\to M^2$ = Borel parameter

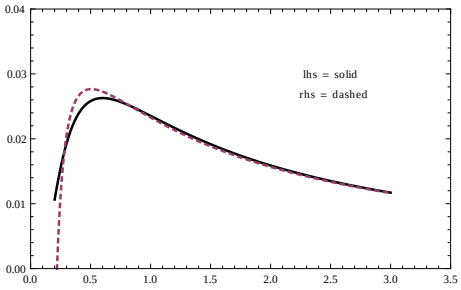
$$\begin{aligned} B\left[\frac{1}{m^2-q^2}\right] &\;\Rightarrow\; \frac{1}{M^2}e^{-m^2/M^2} \\ B\left[\frac{1}{(-q^2)^n}\right] &\;\Rightarrow\; \frac{1}{(n-1)!}\frac{1}{M^{2n}} \end{aligned} \tag{7.31}$$

Why:
— Contributions of higher mass states are suppressed exponentially e^{-s/M^2} instead of $1/(s-q^2)$
— Higher-order $1/q^{2n}$ terms in the OPE are suppressed by $1/(n-1)!$ factors

$\Longrightarrow$ SVZ sum rule

$$\frac{2m_\rho^2}{g_\rho^2}e^{-m_\rho^2/M^2}\frac{1}{M^2} \;:=\; \frac{N_c}{12\pi^2M^2}\int_0^{s_0}ds\,e^{-s/M^2} + \frac{1}{12M^4}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle - \frac{112}{81M^6}\alpha_s\cdot\pi\langle\bar\psi\psi\rangle^2 \tag{7.32}$$

A numerical analysis:



$$m_\rho^2 \simeq 0.5 - 0.6 \text{ GeV}^2$$
$$g_\rho^2 \simeq 28 \pm 2 \qquad (7.33)$$
$$s_0 \simeq 1.5 \pm 0.5 \text{ GeV}^2$$

in fair agreement with experiment.

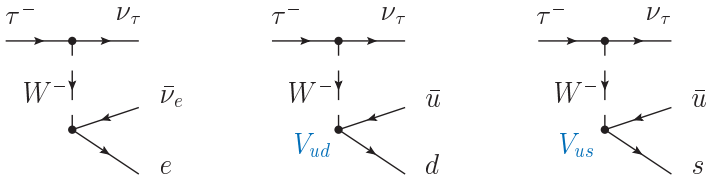
- A large subfield of research inside QCD
- Original SVZ paper is one of the most cited in HEP, has 5607 citations as of 30.11.2022



7.2 τ–decay and duality

Heavy τ-lepton: (e, μ, τ) $m_\tau = 1.777$ GeV

The Standard Model:



$$m_W = 80.4\,\text{GeV} \gg m_\tau\,, \qquad \text{hence for low energies} \qquad \frac{1}{m_W^2 - q^2} \simeq \frac{1}{m_W^2}$$

$\Longrightarrow$ Effective four-fermion interaction (Fermi)

$$\tau \to e\nu_\tau\bar{\nu}_e: \qquad \mathcal{L}_{eff} = -\frac{G_F}{\sqrt{2}}\left[\bar{e}\gamma^\mu(1-\gamma_5)\nu_e\right]\left[\bar{\nu}_\tau\gamma_\mu(1-\gamma_5)\tau\right]$$
$$\tau \to \text{ hadrons}: \qquad \mathcal{L}_{eff} = -V_{ud}\frac{G_F}{\sqrt{2}}\left[\bar{d}\gamma^\mu(1-\gamma_5)u\right]\left[\bar{\nu}_\tau\gamma_\mu(1-\gamma_5)\tau\right] + \ldots \qquad (7.34)$$

Explicit calculation yields (exercises):

$$\Gamma(\tau \to e\nu_\tau\bar{\nu}_e) = \frac{G_F^2 m_\tau^5}{192\pi^3} \qquad (7.35)$$

We expect, qualitatively

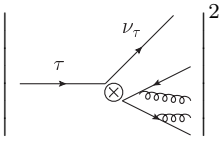
$$R_\tau = \frac{\Gamma(\tau \to \text{ hadrons})}{\Gamma(\tau \to e\nu_\tau\bar{\nu}_e)} \simeq \left(|V_{ud}|^2 + |V_{us}|^2\right) \cdot N_c \qquad (7.36)$$

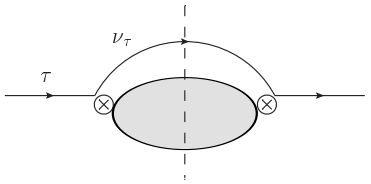
This quantity can be measured very precisely, let us try to calculate it in QCD.

For simplicity set $V_{ud} \to 1$, $V_{us} \to 0$ and neglect quark masses $m_{u,d} \to 0$

Using unitarity (optical theorem)

$$\Gamma(\tau \to \text{ hadrons}) \quad \simeq \quad \simeq \quad 2\,\text{Im} \tag{7.37}$$





where the shaded blob is given by

$$\begin{aligned} \Pi^\tau_{\mu\nu} &= i \int d^4x\, e^{iqx} \langle \Omega | \,\mathrm{T}\, \{ \bar{d}(x) \gamma_\mu (1-\gamma_5) u(x)\, \bar{u}(0) \gamma_\nu (1-\gamma_5) d(0) \} | \Omega \rangle \\ &= (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi^\tau(q^2) \end{aligned} \tag{7.38}$$

[transversality holds for $m_u = m_d = 0$, add an extra $\sim g_{\mu\nu}$ function otherwise]

Explicit calculation yields (exercises):

$$R_\tau \;=\; 12\pi \int\limits_0^{m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + \frac{2s}{m_\tau^2}\right) \,\text{Im}\, \Pi^\tau(s) \tag{7.39}$$

which can be compared with e^+e^- annihilation cross section

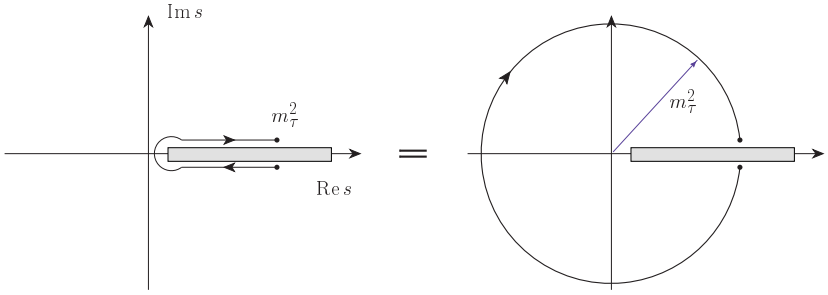
$$R_{e^+e^-}(s) = 12\pi\, \text{Im}\, \Pi_{e^+e^-}(s) \tag{7.40}$$

The extra s-integration corresponds to neutrino energy in the final state L20

<u>The trick:</u>
write

$$2i\, \text{Im}\, \Pi^\tau(s) = \Pi^\tau(s + i\epsilon) - \Pi^\tau(s - i\epsilon) \tag{7.41}$$

and transform the integration contour to a circle of radius m_τ^2 in the complex plane:



Thus

$$R_\tau \;=\; \frac{12\pi}{2i} \oint\limits_C \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + \frac{2s}{m_\tau^2}\right) \,\Pi^\tau(s) \tag{7.42}$$

In this form

- Integration contour is far away from $s = 0$, $|s| = m_\tau^2 \gg \Lambda^2_{\rm QCD}$; thus perturbation theory (+ OPE corrections) can be used, apart from possibly a small region close to the real axis
- Luck: The contribution from the dangerous region $s \to m_\tau^2$ suppressed by the factor $\left(1 - s/m_\tau^2\right)^2$
- Luck: The LO contributioon ofthe gluon condensate $\sim 1/s^2 = 1/q^4$ vanishes upon integration (accident)

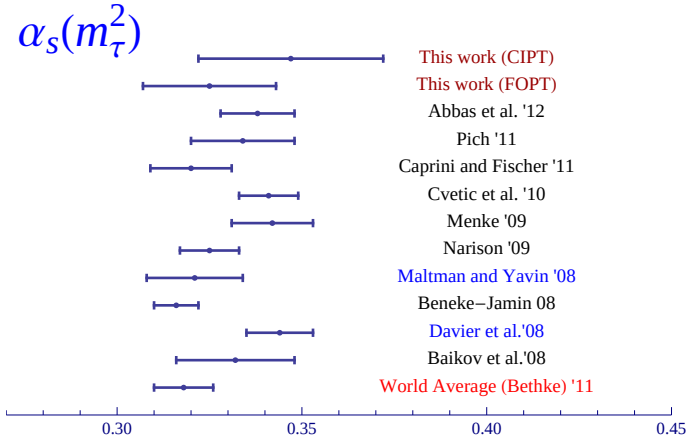
$\Longrightarrow$ a very accurate prediction possible

OPE:

$$\Pi(s) = \frac{1}{4\pi^2} \ln \frac{\mu^2}{-s} \Big[1 + a_1\alpha_s(s) + a_2\alpha_s^2(s) + a_3\alpha_s^3(s)\Big] + \Big[1 + b_1\alpha_s(s)\Big] \frac{1}{12s} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle + \Big[c_0 + c_1\alpha_s(s)\Big] \pi\alpha_s \langle\bar\psi\psi\rangle^2 \tag{7.43}$$

Experiment:

$$R_\tau = 3.4771 \pm 0.0084 \qquad \text{HFAG,} \quad \texttt{http://www.slac.stanford.edu/xorg/hfag/} \tag{7.44}$$



Recent α_s determinations from τ-decays using different hypothesises for the resummation of high orders in perturbation theory. Table taken from: D. Boito *et al.* [arXiv:1212.0091]

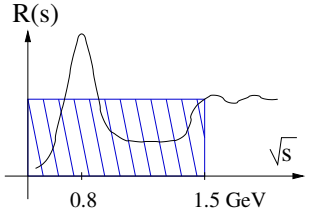
The trick that we used here is rather general.

Let $\Pi(s)$ be a certain two-point correlation function. Then $R(s) \sim \operatorname{Im}\Pi(s)$ for small $s \sim \Lambda^2_{QCD}$ cannot be calculated in perturbation theory, but the integral

$$\int_0^{s_0} ds\, w(s) \operatorname{Im}\Pi(s) \tag{7.45}$$

where $w(s)$ is a smooth function is often calulable (or at least can be estimated).

E.g. for the total cross section of e^+e^- annihilation



$\Leftarrow$ the area is the same

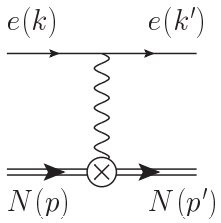
Accepted terminology: A resonance is "dual" to the $\bar qq$ pair in the "interval of duality" s_0

8 Deep–Inelastic Lepton–Nucleon Scattering

8.1 Elastic Lepton–Nucleon Scattering

Elastic scattering of electrons from protons (neutrons):

$$e(k) + N(p) \to e(k') + N(p')$$



Differential cross section for unpolarized particles (Rosenbluth)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha_{\rm em}^2}{4E^2 \sin^4\frac{\theta}{2}} \frac{1}{1 + (2E/m)\sin^2\frac{\theta}{2}} \left[\left(F_1^2 - \frac{q^2}{4m^2} F_2^2 \right) \cos^2\frac{\theta}{2} - \frac{q^2}{2m^2} \Big(F_1 + F_2 \Big)^2 \sin^2\frac{\theta}{2} \right] \quad (8.1)$$

where

E : electron energy in the lab frame
m : nucleon mass
θ : scattering angle in the lab frame
$q = k - k' = p' - p$: momentum transfer; $\qquad Q^2 \equiv -q^2 = 4E^2 sin^2\frac{\theta}{2}$

and (definition):

$$\langle N(p')|j_\mu^{\rm em}(0)|N(p)\rangle \;:=\; \bar{u}(p') \left[\gamma_\mu F_1(Q^2) + \frac{i}{2m} \sigma_{\mu\nu} q^\nu F_2(Q^2) \right] u(p) \quad (8.2)$$

The functions $F_1(Q^2)$ and $F_2(Q^2)$ are called Dirac and Pauli form factors, respectively

$$\begin{array}{lll} F_1^p(0) = 1\,, & F_1^n(0) = 0\,, & \text{electric charge} \\ F_2^p(0) = 2.792847\,\mu_N\,, & F_2^n(0) = -1.913043\,\mu_N & \text{magnetic moment} \quad \boxed{\mu_N = e\hbar/2m_N} \end{array} \quad (8.3)$$

In a non-relativistic theory form factor is a Fourier transform of the charge/current distribution:

$$F(\vec{q}^{\,2}) \;=\; \int d^3x\, e^{i\vec{q}\vec{x}}\, \rho(\vec{x}) \quad (8.4)$$

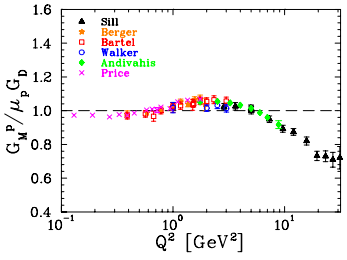
Hence a deviation from $F(Q^2) = $ const signals that a particle has internal structure
For $Q^2 \leq 1$ GeV^2 (Hofstadter)

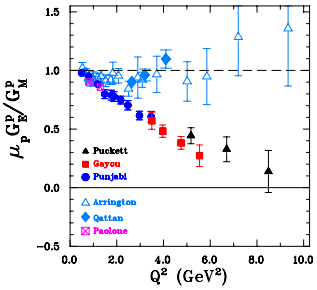
$$F_{1,2} \simeq \frac{1}{[1 + Q^2/Q_0^2]^2} \quad \text{with} \quad Q_0^2 \simeq 0.71\,\text{GeV}^2 \quad (8.5)$$

$\Longleftarrow$ corresponds to the proton radius ca. $R_p \simeq 0.85$ fm

Studies of form factors at large $Q^2 \gg 1$ GeV^2 is an active research topic (theory complicated)

Recent results from Jefferson Laboratory: [Arrington *et al.*, arXiv:1102.2463]

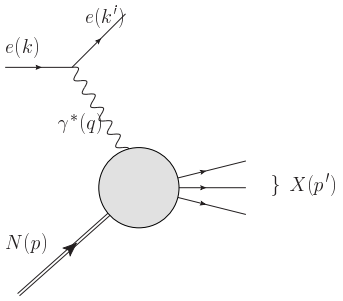




8.2 Deep-Inelastic Scattering and Bjorken scaling

In 1968 J. Bjorken proposed a different type of experiment: Sum over all hadronic final states

$$e(k) + N(p) \to e(k') + X(p')$$



$\Longleftarrow$ the Deep-Inelastic Lepton-Hadron Scattering (DIS)

- "Sum over all states" in practice means that hadrons are not identified (measured); the only detected particle is the scattered electron in the final state — one measures its scattering angle θ and energy E'

L21

Kinematic variables

- m: the nucleon mass
- $E = \frac{pk}{m}$: beam electron energy (in L.S. $p_\mu = \{m, 0, 0, 0\}$)
- $E' = \frac{pk'}{m}$: scattered electron energy (in L.S.)
- θ: electron scattering angle (in L.S.)

$$q \;=\; k - k' \;=\; p' - p \qquad\qquad Q^2 = -q^2 = 4EE' \sin^2 \frac{\theta}{2}$$

- $\nu = \frac{pq}{m}$: energy transfer to the hadronic system (in L.S.)
- $y = \nu/E$: energy transfer/maximum energy transfer (in L.S.) $\boxed{0 < y < 1}$
- $W^2 = (p+q)^2$: invariant mass squared of the hadronic system

- $x_B = \frac{Q^2}{2pq} = \frac{Q^2}{2m\nu}$: Bjorken's scaling variable $\boxed{0 < x < 1}$

  Kinematic bound:

$$W^2 = m^2 + 2m\nu - Q^2 \geq m^2 \qquad \Longrightarrow \qquad x_B \leq 1$$

  Elastic scattering corresponds to $x_B = 1$ in which case $2m\nu = Q^2$

The Bjorken limit

$$\boxed{W \to \infty\,, \qquad Q^2 \to \infty \qquad \text{such that} \qquad x_B = \frac{Q^2}{2pq} = \text{const.}}$$

The transition amplitude for a given final state is

$$iM(eN \to eX) \;=\; (-ie)\bar{u}(k',\lambda')\gamma^\mu u(k,\lambda)\frac{-i}{q^2}(ie)\langle X(p')|j_\mu(0)|N(p)\rangle \tag{8.6}$$

where

$$j_\mu(x) = e_u\bar{u}(x)\gamma_\mu u(x) + e_d\bar{d}(x)\gamma_\mu d(x) + \ldots \tag{8.7}$$

The cross section summed over hadronic states is then

$$d\sigma \;=\; \frac{1}{2s}\frac{d^3k'}{(2\pi)^3 2|k'|}\frac{1}{2}\sum_{\lambda,\lambda'}\sum_X \int d\Pi_X |iM(eN \to eX(p')|^2(2\pi)^4\delta^4(p+q-p') \tag{8.8}$$

↖ As usual, we sum over spins of final state electrons and average over spins of initial electrons. This corresponds to the simplest experimental setup where spin is not measured

Thus

$$\frac{d\sigma}{d^3k'} \;=\; \frac{1}{2s}\frac{1}{(2\pi)^3 2|k'|}\frac{1}{2}\sum_{\lambda,\lambda'}\bar{u}(k,\lambda)\gamma^\mu u(k',\lambda')\bar{u}(k',\lambda')\gamma^\nu u(k,\lambda)\frac{e^4}{Q^4}(2\pi)W_{\mu\nu}(p,q) \tag{8.9}$$

where

$$(2\pi)W_{\mu\nu}(p,q) = \sum_X \int d\Pi_X \langle N(p)|j_\mu(0)|X(p')\rangle\langle X(p')|j_\nu(0)|N(p)\rangle(2\pi)^4\delta^4(p+q-p') \tag{8.10}$$

Using the optical theorem

$$W_{\mu\nu}(p,q) = \frac{1}{\pi}\,\mathrm{Im}\,T_{\mu\nu}(p,q) \tag{8.11}$$

where

$$T_{\mu\nu} = i\int d^4x\, e^{iqx}\langle N(p)|\,\mathrm{T}\,\{j_\mu(x)j_\nu(0)|\}|N(p)\rangle \tag{8.12}$$

is called *the forward Compton amplitude*:

$$iM(\gamma N \to \gamma N) = e^2\epsilon^*_\mu(q)\epsilon_\nu(q)T^{\mu\nu}\,, \qquad q^2 = 0 \tag{8.13}$$

We will use this representation later

Conservation of the electromagnetic current (Ward identity) implies

$$q^\mu W_{\mu\nu}(p,q) = q^\nu W_{\mu\nu}(p,q) = 0 \tag{8.14}$$

Therefore $W_{\mu\nu}(p,q)$ must have the form

$$\begin{aligned} W_{\mu\nu}(p,q) = &\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1(pq,Q^2) + \left(p_\mu - q_\mu \frac{pq}{q^2}\right)\left(p_\nu - q_\nu \frac{pq}{q^2}\right) W_2(pq,Q^2) \\ &+ i\epsilon_{\mu\nu\alpha\beta} q^\alpha p^\beta W_3(pq,Q^2) \end{aligned} \tag{8.15}$$

Parity conservation in strong and electromagnetic interactions $\implies$ $W_3(\nu,Q^2) = 0$

The scalar functions $W_{1,2}$ depend on the invariants $W(pq,Q^2)$ or equivalently $W(x_B,Q^2)$
They are called *structure functions*

Further we have

$$\frac{1}{2}\sum_{\lambda,\lambda'}[\bar{u}(k,\lambda)\gamma^\mu u(k',\lambda')][\bar{u}(k',\lambda')\gamma^\nu u(k,\lambda)] = \frac{1}{2}\,\mathrm{Tr}\,[\not{k}\gamma^\mu \not{k}'\gamma^\nu] = 2[k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu}(kk')] \tag{8.16}$$

and

$$d^3k' \;=\; |k'|^2 d|k'|\, d\phi\, d\cos\theta \;=\; (E')^2 dE' d\Omega' \tag{8.17}$$

so that after some algebra

$$\frac{d\sigma}{dE' d\cos\theta} = \frac{8\pi\alpha_{\mathrm{em}}^2 E'^2}{Q^4}\left\{\frac{1}{m}W_1(\nu,Q^2)\,\sin^2\frac{\theta}{2} + \frac{m}{2}W_2(\nu,Q^2)\,\cos^2\frac{\theta}{2}\right\} \tag{8.18}$$

Thus $W_1(\nu,Q^2)$ and $W_2(\nu,Q^2)$ can be measured experimentally from the E',θ of scattered electrons

A more convenient representation

$$(E',\cos\theta) \;\to\; (x_B,y) \tag{8.19}$$

Since

$$s = (p+k)^2 \simeq 2mE \qquad y = \frac{2pq}{2pk} = \frac{E-E'}{E} \qquad x_B = \frac{4EE'\sin^2\frac{\theta}{2}}{2m(E-E')} \tag{8.20}$$

obtain

$$\sin^2\frac{\theta}{2} = \frac{m^2 x_B}{s}\frac{y}{1-y} \qquad \frac{\partial(x_B,y)}{\partial(E',\cos\theta)} = \frac{2E'}{2m(E-E')} = \frac{2E'}{ys} \tag{8.21}$$

and neglecting terms $\mathcal{O}(m/E)$

$$\boxed{\frac{d\sigma}{dxdy} = \frac{\pi\alpha_{\mathrm{em}}^2 s}{Q^4}\Big[2x_B y^2 W_1(\nu,Q^2) + y(1-y)sW_2(\nu,Q^2)\Big]} \tag{8.22}$$

The prediction by Bjorken was that in the synchronous high-energy and high-Q^2 limit

$$W_1(\nu, Q^2) = F_1(x_B) + \mathcal{O}(1/Q^2)$$
$$\frac{1}{4} y s W_2(\nu, Q^2) = F_2(x_B) + \mathcal{O}(1/Q^2) \tag{8.23}$$

— the Bjorken scaling

confusing notation: Structure functions F_1 and F_2 are not Dirac F_1 and Pauli F_2 form factors !!

As we will see, this behaviour indicates the existence of free pointlike charged particles in hadrons

8.3 The Parton Model

The physical picture of DIS becomes more transparent in a special reference frame, the *Breit frame*:

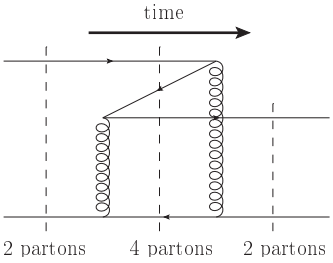
We choose

$$q_\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -Q \end{pmatrix} \qquad p_\mu = \begin{pmatrix} \sqrt{p_z^2 + m^2} \\ 0 \\ 0 \\ p_z \end{pmatrix} \tag{8.24}$$

- Since $q_0 = 0$, electron has the same energy before and after the collision (in this frame)
- Since $pq = Qp_z$ and $x_B = Q^2/(2pq)$ it follows that $p_z = Q/(2x_B) \to \infty$ in the Bjorken limit

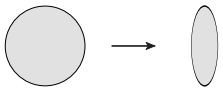
Assume that nucleon is a bound states of pointlike (small-size) constitients — *partons* (e.g. quarks and gluons).
— In nonrelativistic QM such a state would be described by a wave function $\Psi(x_1, \ldots, x_n)$.
— In a relativistic theory this description is lost, in general, because the number of partons is not conserved

Z-graphs:



However, in a fast moving hadron all longitudinal distances are contracted and all processes are slowed down by the Lorentz factor $\gamma = 1/\sqrt{1 - v^2/c^2}$

- For an observer in the Breit frame the proton "looks" like a pancake



- An interaction between partons in the proton rest frame requires time $\tau_0 \sim R/c$; it becomes $t \simeq \tau_0 \frac{p_z}{m} \to \infty$ in Breit frame — partons do not have time to "talk" to each other

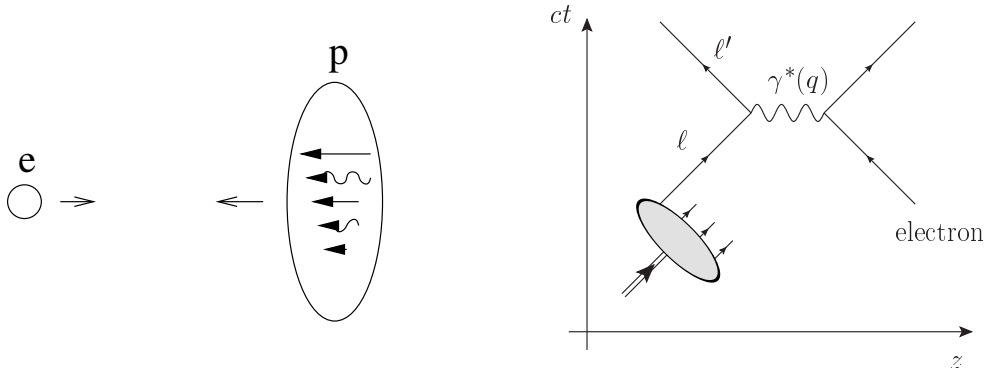
Thus: L22

- A fast-moving hadron can be viewed as a collection of partons which fly in the same direction

- Since the partons have no time to interact, each of them can be considered as free and carrying a fixed momentum fraction of the proton
- A "hard" external probe (virtual photon) can only interact with one parton (unless parton density is very large)
- Scattering processes on different partons do not interfere quantum-mechanically; the cross section is given by a sum of cross sections on individual partons

The overall picture that arises in this way has become known as *the QCD parton model*

In our case (DIS)



parton (quark) momentum in the proton:

$$\ell_\mu = (\xi p_z, 0, 0, \xi p_z) \tag{8.25}$$

parton momentum after the collision:

$$\ell'_\mu = \ell_\mu + q = (\xi p_z, 0, 0, \xi p_z - Q) \tag{8.26}$$

parton remains to be on-shell:

$$0 = \ell'^2 = (\xi p_z)^2 - (\xi p_z - Q)^2 = 2\xi p_z Q - Q^2 \qquad \Longrightarrow \qquad \xi = \frac{Q}{2p_z} \tag{8.27}$$

since $p_z = Q/(2x_B)$ (see above) we obtain

$$\xi = \frac{Q}{2p_z} = \frac{Q}{2}\frac{2}{Q}x_B \qquad \Longrightarrow \qquad \boxed{\xi = x_B} \tag{8.28}$$

— parton momentum fraction is equal to the Bjorken variable

The parton content of the nucleon is described by *parton distributions*. Let

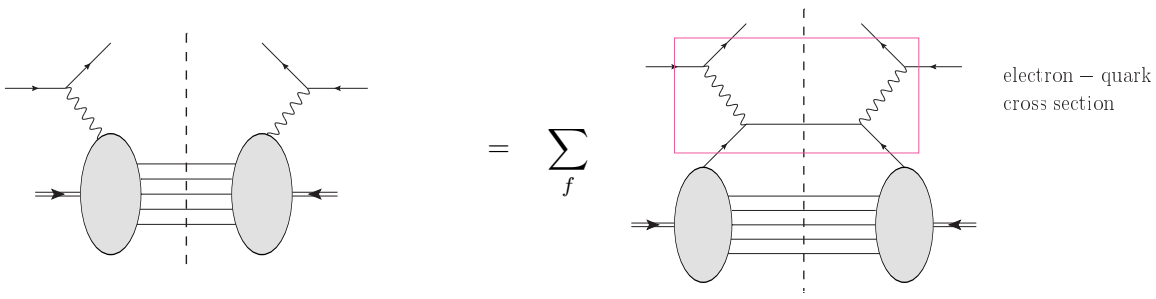
$$F_f(\xi)d\xi \qquad\qquad f = q, \bar{q}, g \tag{8.29}$$

be the number of quark, antiquark and gluon partons with momentum fractions in the interval between ξ and $\xi + d\xi$

The deep-inelastic cross section from a proton (in the parton model) is given by the incoherent sum of the DIS cross sections from the partons:

$$\boxed{\sigma_{e(k)N(p)\to e(k')X} = \sum_f \int_0^1 d\xi\, F_f(\xi)\, \sigma_{e(k)f(\xi p)\to e(k')f(\xi p+q)}} \tag{8.30}$$

This formula can be illustrated by the following picture:



Quantum-mechanical intepretation:

Transition amplitide between two states is proportional to the overlap between the wave functions:

$$\begin{aligned} F_{|1\rangle\to|2\rangle}(q) &= \int d^3x\, e^{-iqx}\Psi_1(x)\Psi_2^*(x) = \int d^3x\, e^{-iqx}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3}e^{ipx-ip'x}\Psi_1(p)\Psi_2^*(p') \\ &= \int\frac{d^3p}{(2\pi)^3}\Psi_1(p)\Psi_2^*(p+q) \qquad \sim\ 1/|q|^k \quad \text{or} \quad \sim\ e^{-R|q|} \quad \text{for large q} \end{aligned} \tag{8.31}$$

If we sum over the final states this suppression disappears:

$$\begin{aligned} \sum_{|2\rangle}|F_{|1\rangle\to|2\rangle}(q)|^2 &= \sum_{|2\rangle}\int\frac{d^3p}{(2\pi)^3}\Psi_1(p)\Psi_2^*(p+q)\int\frac{d^3p'}{(2\pi)^3}\Psi_1^*(p')\Psi_2(p'+q) \\ &= \int\frac{d^3p}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3}\Psi_1(p)\Psi_1^*(p')\,(2\pi)^3\delta^3(p-p') = 1 \end{aligned} \tag{8.32}$$

where I used the completeness condition

$$\sum_{|2\rangle}\Psi_2^*(p+q)\Psi_2(p'+q) = (2\pi)^3\delta^3(p-p') \tag{8.33}$$

Thus the cross section becomes much larger and, crucially, we can calculate it without any knowledge of the "true" eigenstates of the Hamiltonian:

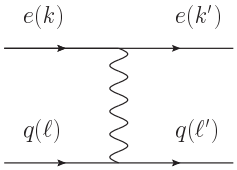
$$\sum_{|2\rangle:\text{eigenstates of H}}|F_{|1\rangle\to|2\rangle}(q)|^2 = \sum_{|2\rangle:\text{plane waves}}|F_{|1\rangle\to|2\rangle}(q)|^2 \tag{8.34}$$

Example: photodisintegration cross section for deuteron

In our context: we can neglect interaction of hit quark with the remnant partons in the final state

Electron-quark scattering

— a close analogue to electron-muon scattering (QED):



$$\frac{d\sigma}{d\Omega_{\rm CM}} \;=\; \frac{1}{64\pi^2\hat{s}}|M|^2 \;=\; \frac{\alpha^2 e_q^2}{2\hat{s}}\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}$$

where

$$\ell_\mu \simeq \xi p_\mu \qquad \text{quark four-momentum}$$
$$\ell^2 = \ell'^2 = 0\,, \qquad k^2 = k'^2 = 0$$

Mandelstam variables for the electron-quark scattering:

$$\begin{aligned}
\hat{s} &= (k+\ell)^2 = (k+\xi p)^2 = 2\xi(kp) + \cancel{p^2} \simeq \xi(k+p)^2 = \xi s \\
\hat{t} &= (k'-k)^2 = -Q^2 \\
\hat{u} &= (k-\ell')^2 = -\hat{s}-\hat{t}\,, \qquad \Longleftarrow \quad \hat{s}+\hat{t}+\hat{u}=0
\end{aligned} \tag{8.35}$$

In the physical region for the electron-quark scattering

$$\hat{s} \geq |\hat{t}| \qquad \Longrightarrow \qquad \xi s \geq Q^2 \tag{8.36}$$

In the CM system:

$$d\Omega_{\rm CM} = d\phi\, d\cos\theta_{\rm CM} \qquad\qquad \hat{t} = -\frac{1}{2}\hat{s}(1-\cos\theta_{\rm CM}) \tag{8.37}$$

Therefore

$$\frac{d\sigma}{dt} = \underbrace{(2\pi)}_{\int d\phi}\cdot\frac{2}{\hat{s}}\cdot\frac{\alpha^2 e_q^2}{2\hat{s}}\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2} \;=\; \frac{2\pi\alpha^2 e_q^2}{Q^4}\frac{\hat{s}^2+\hat{u}^2}{\hat{s}^2} \tag{8.38}$$

The last factor can be rewritten as $(\hat{s}^2+\hat{u}^2)/\hat{s}^2 = 1+\left[(-\hat{s}-\hat{t})/\hat{s}\right]^2 = 1+\left(1-Q^2/(\xi s)\right)^2$
It follows that

$$\begin{aligned}
\frac{d\sigma^{\rm DIS}}{dQ^2} &= \sum_f \int_0^1 d\xi\, F_f(\xi) e_f^2 \frac{2\pi\alpha^2}{Q^4}\left[1+\frac{\hat{u}^2}{\hat{s}^2}\right]\theta(\xi s - Q^2) \\
&:= \int_0^1 dx_B\, \frac{d\sigma^{\rm DIS}}{dx_B dQ^2}
\end{aligned} \tag{8.39}$$

Since $\xi = x_B$ this means that

$$\frac{d\sigma^{\rm DIS}}{dx_B dQ^2} = \sum_f e_f^2 F_f(x_B)\frac{2\pi\alpha^2}{Q^4}\left[1+\frac{\hat{u}^2}{\hat{s}^2}\right]\theta(x_B s - Q^2) \tag{8.40}$$

It remains to express everything in terms of x_B, y variables:

$$\begin{aligned} y :=& \frac{2pq}{2pk} = \frac{2p(k-k')}{2pk} \cdot \frac{\xi}{\xi} = \frac{2\ell(k-k')}{2\ell k} = \frac{\hat{s}+\hat{u}}{\hat{s}} \\ & \Longrightarrow \quad \frac{\hat{u}}{\hat{s}} = -(1-y) \\ Q^2 =& \frac{Q^2}{2pq} \cdot \frac{2pq}{2pk} \cdot 2pk = x_B \cdot y \cdot s \\ & \Longrightarrow \quad dx_B dQ^2 = \frac{dQ^2}{dy} dx_B dy = x_B s \, dx_B dy \\ & \Longrightarrow \quad \frac{d\sigma^{\rm DIS}}{dx_B dy} = x_B s \, \frac{d\sigma^{\rm DIS}}{dx_B dQ^2} \end{aligned} \tag{8.41}$$

We obtain

$$\frac{d\sigma^{\rm DIS}}{dx_B dy} = \left(\sum_f e_f^2 x_B F_f(x_B) \right) \frac{2\pi\alpha^2 s}{Q^4} \Big[1 + (1-y)^2 \Big] \tag{8.42}$$

On the other hand, we derived before

$$\frac{d\sigma^{\rm DIS}}{dx_B dy} = \frac{\pi\alpha_{\rm em}^2 s}{Q^4} \Big[2x_B y^2 W_1(\nu, Q^2) + y(1-y) s W_2(\nu, Q^2) \Big] \tag{8.43}$$

so that for $F_1 \equiv W_1$, $F_2 \equiv (ys/4) W_2$

$$F_2(x_B) = x_B F_1(x_B) = \sum_f e_f^2 x_B F_f(x_B) \tag{8.44}$$

- The structure functions F_1 and F_2 do not depend on y (or Q^2) (Bjorken scaling)
- Let

$$\begin{aligned} u(x_B): &\qquad u\text{-quark momentum fraction distribution} \\ d(x_B): &\qquad d\text{-quark momentum fraction distribution} \\ &\qquad\qquad \ldots \end{aligned} \tag{8.45}$$

  The structure function F_1 is given by the sum of quark distributions weighted with electric charges squared:

$$F_1(x_B) = \frac{4}{9} u(x_B) + \frac{1}{9} d(x_B) + \ldots \tag{8.46}$$

- The structure function F_2 is expressed in terms of F_1 through the Callan-Gross relation

$$F_2(x_B) = x_B F_1(x_B) \tag{8.47}$$

Let

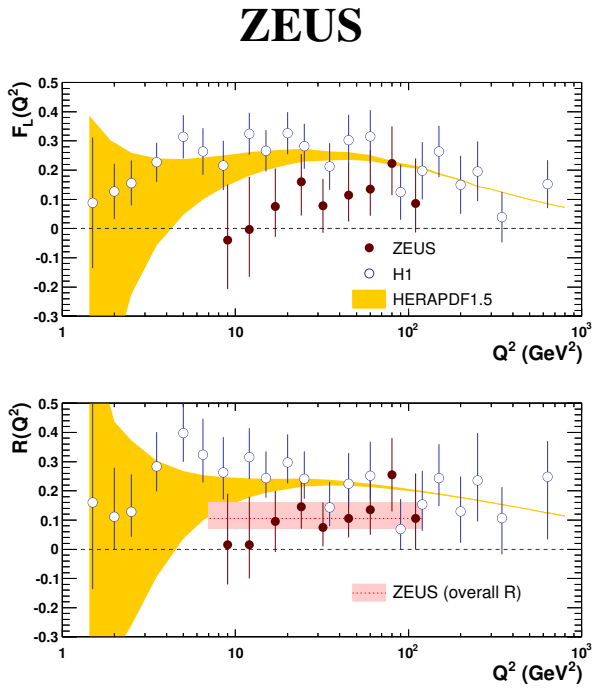
$$R(x_B) = \frac{F_2(x_B) - x_B F_1(x_B)}{F_2(x_B)} \tag{8.48}$$

One can show (exercise) that in the parton model

$$R(x) \;=\; \begin{cases} 0, & \text{if all charged partons have spin } 1/2 \\ 1, & \text{if all charged partons have spin zero} \end{cases} \tag{8.49}$$

$\Longrightarrow$ Experimental confirmation that partons have spin 1/2 (quarks)

Typical values: Shown: new ZEUS F_L (a) and R (b) measurements (solid points) in comparison with H1 measurements (open points) and NNLO HERAPDF



[Figure taken from: *Zhiqing Zhang, for the H1, ZEUS Collaborations, arXiv:1412.6328*]

The power of the parton model is that it is applicable for many reactions. Example:

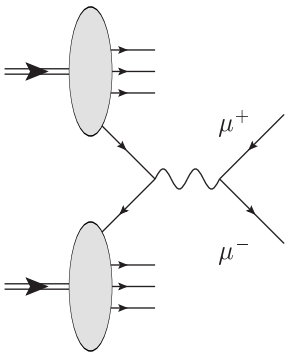
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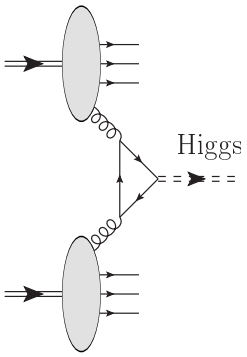
The Drell-Yan Process

$$N(p_1) + N(p_2) \;\to\; \mu^+(k_1) + \mu^-(k_2) + X$$

or

$$N(p_1) + N(p_2) \;\to\; \text{Higgs}(M) + X \tag{8.50}$$





Bjorken limit:

$$s = (p_1 + p_2)^2 \to \infty\,,$$
$$M^2 = (k_1 + k_2)^2 \to \infty\,,$$
$$M^2/s = \text{ const}$$

— Two "pancakes" approach each other at larger velocity and collide

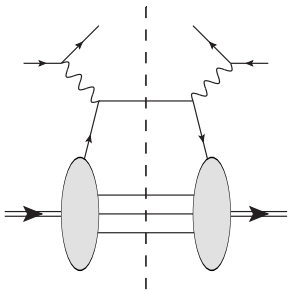
In the parton model

$$\frac{d\sigma}{dM^2}(NN \to \mu^+\mu^- + X) = \sum_q \int_0^1 d\xi_1\, F_q^N(\xi_1) \int_0^1 d\xi_2\, F_{\bar{q}}^N(\xi_2) \frac{d\sigma}{dM^2}(q\bar{q} \to \mu^+\mu^- + X)$$
$$= \frac{8\pi\alpha^2}{9M^4} \sum_q e_q^2 \int_0^1 d\xi_1\, F_q^N(\xi_1) \int_0^1 d\xi_2\, F_{\bar{q}}^N(\xi_2) \delta\left(\xi_1\xi_2 - \frac{M^2}{s}\right) \quad (8.51)$$

Most importantly, $F_{q,\bar{q}}^N(\xi)$ are *the same* functions as in DIS, so that e.g. Higgs cross section can be predicted

9 Factorization and Parton Distributions

Aim of this section is to provide a QCD derivation of the parton model and go beyond it.
Let us summarize what we know about DIS



$$T_{\mu\nu} = i \int d^4x\, e^{iqx} \langle N(p)|\, \mathrm{T}\,\{j_\mu(x) j_\nu(0)|\}|N(p)\rangle$$
$$W_{\mu\nu}(p,q) = \frac{1}{\pi}\, \mathrm{Im}\, T_{\mu\nu}(p,q)$$

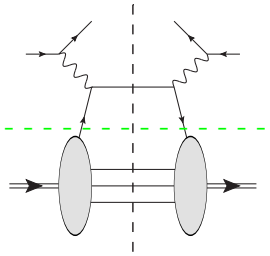
where

$$(2\pi) W_{\mu\nu}(p,q) = \sum_X \int d\Pi_X \langle N(p)|j_\mu(0)|X(p')\rangle\langle X(p')|j_\nu(0)|N(p)\rangle (2\pi)^4 \delta^4(p+q-p') \quad (9.1)$$

can be expressed in terms of the structure functions (observable quantities)

$$W_{\mu\nu}(p,q) = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) F_1(x_B, Q^2) + \left(p_\mu - q_\mu \frac{pq}{q^2}\right)\left(p_\nu - q_\nu \frac{pq}{q^2}\right) \frac{2}{pq} F_2(x_B, Q^2) \quad (9.2)$$

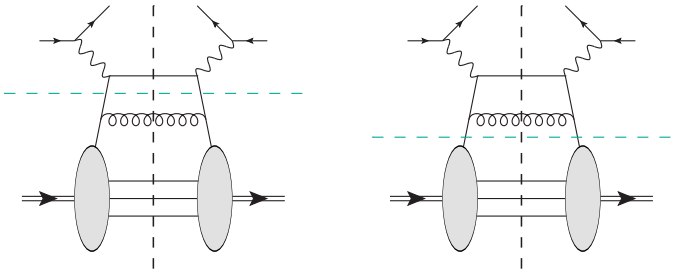
Parton model suggests that we can separate (factorize) the DIS cross section in two subprocesses:



↗ electron-quark cross section

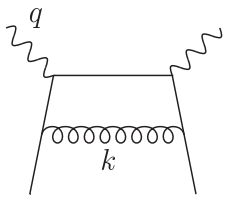
↘ quark momentum fraction distribution

QCD quarks interact by exchanging gluons; we will have more complicated Feynman diagrams like



Where to draw the line?
— The first possibility: gluon is just an "other" parton in the nucleon, part of "X"
— The second possiblity: gluon exchange is a correction to the electron-quark cross section

The guiding principle is whether the gluon is emitted at short distances or long time ago, as part of the preparation of the parton wave function. It is easy to see, however, that the integral over gluon momenta is logarithmic



$$\sim \alpha_s \int_m^Q \frac{d^4k}{k^4} \quad \sim \quad \alpha_s \ln Q^2/m^2$$

Logarithmic integrals are specific in that there is no dominant integration region — the answer comes from contributions of all momenta, both large ($\sim Q$) and small ($\sim m$).

This situation is similar to what we have seen in the operator product expansion.
Thus we need to make an explicit separation by introducing an intermediate *factorization scale*, schematically

$$\ln \frac{Q^2}{m^2} \;=\; \ln \frac{Q^2}{\mu_F^2} + \ln \frac{\mu_F^2}{m^2}\,, \qquad\qquad Q^2 \gg \mu_F^2 \gg m^2 \;(\Lambda^2_{\rm QCD}) \tag{9.3}$$

and rewrite, in the sum with the LO diagram

$$1 + a\alpha_s \ln \frac{Q^2}{m^2} + \ldots = \underbrace{\left(1 + a\alpha_s \ln \frac{Q^2}{\mu_F^2}\right)}_{e^-q \text{ cross section}} \underbrace{\left(1 + a\alpha_s \ln \frac{\mu_F^2}{m^2}\right)}_{\text{parton distribution}} + \ldots \tag{9.4}$$

In this way, the space-time picture of the parton model is recovered at the cost that the parton cross sections and parton distributions become *scale dependent*:

$$d\sigma_{eN} = \sum_f \int_0^1 d\xi\, F_f(\xi,\mu_F)\, d\sigma_{ef}(\mu_f) \tag{9.5}$$

or

$$F_{1,2}(x_B,Q^2) \;=\; \sum_f e_f^2 \int d\xi\, c_{1,2}(x_B,\xi,\frac{Q^2}{\mu_f^2}) F_f(\xi,\mu_f) \tag{9.6}$$

The scale-dependence of parton distributions can be studied using renormalization group.
After this is done, we can choose $\mu_f^2 = Q^2$ so that the coefficient functions will be calculable as a series in $\alpha_s(Q^2)$ without large logarithms.
To the leading order we will obtain e.g.

$$F_1(x_B,Q^2) = \frac{4}{9}u(x_B,Q^2) + \frac{1}{9}d(x_B,Q^2) + \ldots \tag{9.7}$$

Thus we will develop a systematic approach:

- We predict that Bjorken scaling is violated by (small) logarithmic effects
- We can calculate the Q^2 dependence of the structure functions to arbitrary accuracy in perturbation theory (in principle)
- We can generalize this method to other processes, e.g. Drell-Yan or Higgs production

9.1 Leading-order calculation

As I discussed already in connection with vacuum condensates, the scale separation can be introduced at the level of quantum fields in the Lagrangian:

Separate formally all field operators in "fast" and "slow" components:

$$\phi(x) = \left(\int\limits_{|k|>\mu} \frac{d^3k}{(2\pi)^3 2E_k} + \int\limits_{|k|<\mu} \frac{d^3k}{(2\pi)^3 2E_k} \right) \left[\hat{a}^\dagger(k) e^{ikx} + \hat{a}(k) e^{-ikx} \right] \;=\; \phi_{\rm fast}(x) + \phi_{\rm slow}(x) \tag{9.8}$$

so for quarks and gluons

$$\psi(x) = \psi_{\rm fast}(x) + \psi_{\rm slow}(x)\,, \qquad\qquad A^\mu(x) = A^\mu_{\rm fast}(x) + A^\mu_{\rm slow}(x) \tag{9.9}$$

Fast and slow fields are often referred to as "quantum" and "classical" and can be viewed as independent fields in the Lagrangian:

$$\psi_q(x) \equiv \psi_{\rm fast}(x)\,, \qquad\qquad \psi_c(x) \equiv \psi_{\rm slow}(x) \tag{9.10}$$

For gauge fields one requires

$$\begin{aligned} A^\mu_q(x) &\longrightarrow V(x) A^\mu_q(x) V^\dagger(x) \\ A^\mu_c(x) &\longrightarrow V(x) \left(A^\mu_c(x) + \frac{i}{g}\partial_\mu \right) V^\dagger(x) \end{aligned} \tag{9.11}$$

under gauge transformations. This has two advantages:

- Renormalization is simplified because A_q^μ transforms homogeneously ($\to gA_q^\mu$ is not renormalized)
- If the integrals over all "quantum" fields are taken (loops), the result must be gauge-invariant under the transformations of "classical" fields. Hence one can use different gauge conditions for "quantum" and "classical" fields.

Propagators of quantum fields are formally defined with an IR cutoff, e.g. in Feynman gauge

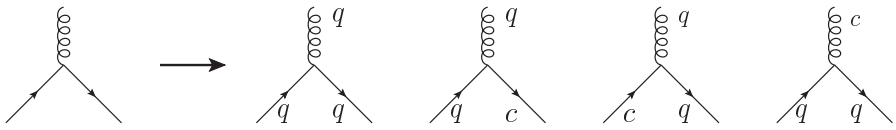
$$\langle 0|\,\mathrm{T}\,\{A_q^\mu(x)A_q^\nu(0)\}|0\rangle = \int\limits_{|k|>\mu} \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{ig^{\mu\nu}}{k^2+i\epsilon} \tag{9.12}$$

but the cutoff dependence only matters if the integrals are IR divergent (and in this case can be regularized dimensionally)

Classical and quantum fields are orthogonal:

$$\langle 0|\,\mathrm{T}\,\{A_q^\mu(x)A_c^\nu(0)\}|0\rangle = 0 \tag{9.13}$$

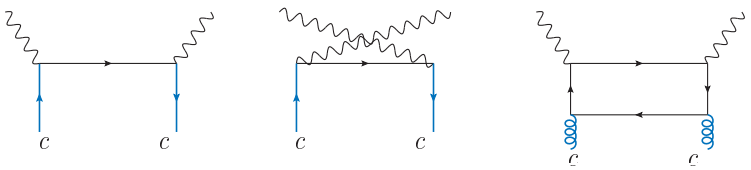
and all interaction vertices must contain at least two quantum fields



Let us do the leading-order calculation using this logic (we expect to reproduce the parton model).

L24

$$\begin{aligned}
T_{\mu\nu} &= i\int d^4x\, e^{iqx}\langle N|\,\mathrm{T}\,\{\overline{\psi}(x)\gamma_\mu\psi(x)\overline{\psi}(0)\gamma_\nu\psi(0))|\}|N\rangle \\
&= i\int d^4x\, e^{iqx}\langle N|\,\mathrm{T}\,\{\overline{\psi}_c(x)\gamma_\mu\overbrace{\psi_q(x)\overline{\psi}_q}(0)\gamma_\nu\psi_c(0))|\}|N\rangle \\
&\quad + i\int d^4x\, e^{iqx}\langle N|\,\mathrm{T}\,\{\overbrace{\overline{\psi}_q(x)\gamma_\mu\psi_c(x)\overline{\psi}_c}(0)\gamma_\nu\overbrace{\psi_q(0))|\}|N\rangle \\
&\quad + i\int d^4x\, e^{iqx}\langle N|\,\mathrm{T}\,\{\overbrace{\overline{\psi}_q(x)\gamma_\mu\overbrace{\psi_q(x)\overline{\psi}_q}(0)\gamma_\nu\psi_q(0))|\}|N\rangle + \ldots
\end{aligned} \tag{9.14}$$



(massless) quark propagator in coordinate space:

$$\langle 0|\,\mathrm{T}\,\{\overbrace{\psi_q(x)\overline{\psi}_q}(0)\}|0\rangle = \int\limits_{|k|>\mu} \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{i\not{p}}{p^2+i\epsilon} \simeq \frac{i}{2\pi^2}\frac{\not{x}}{[-x^2+i\epsilon]^2} \tag{9.15}$$

Thus to LO

$$T_{\mu\nu} = i\frac{i}{2\pi^2}\int d^4x\,\frac{e^{iqx}}{x^4}\langle N|\Big[\bar\psi_c(x)\gamma_\mu \not x\gamma_\nu\psi_c(0) - \bar\psi_c(0)\gamma_\nu\not x\gamma_\mu\psi_c(x)\Big]|N\rangle \tag{9.16}$$

In what follows we will omit the subscript "classical"; just remember that all matrix elements include small frequencies, less than μ.

- First, use

$$\gamma_\mu\not x\gamma_\nu = x_\mu\gamma_\nu + x_\nu\gamma_\mu - g_{\mu\nu}\not x + i\epsilon_{\mu\rho\nu\sigma}x^\rho\gamma_5\gamma^\sigma \tag{9.17}$$

The matrix element of $\gamma_5\gamma^\sigma$ must be a pseudovector

$$\langle N|\bar\psi(x)\gamma_5\gamma^\sigma\psi(0)|N\rangle \;\sim\; S^\rho \qquad \text{[nucleon spin vector]}$$

For unpolarized nucleons

$$\frac{1}{2}\sum_s\langle N(p,s)|\ldots|N(p,s)\rangle$$

such matrix elements vanish

- Second, the relevant matrix elements have the following general structure:

$$\langle N|\bar\psi(x)\gamma^\sigma\psi(0)|N\rangle \;\sim\; p^\sigma f_1(px,x^2) + x^\sigma f_2(px,x^2) \tag{9.18}$$

If $Q^2\to\infty$ (Bjorken limit) the second structure and also terms in the expansion of $f_1(px,x^2)$ in powers of x^2 produce corrections $\sim 1/Q^2$ and can be neglected

Indeed, one can easily derive

$$\begin{aligned}\int d^4x\,\frac{e^{iqx}}{x^4}x_\alpha &= 2\pi^2\frac{q_\alpha}{q^2}\\ \int d^4x\,\frac{e^{iqx}}{x^2}x_\alpha &= 8\pi^2\frac{q_\alpha}{q^4} \qquad \longleftarrow \quad \text{note extra } 1/q^2\end{aligned} \tag{9.19}$$

If $x^2=0$ the MEs are functions of one variable (px) and can be written in the form of a Fourier integral

$$\begin{aligned}\langle N|\bar\psi(x)\gamma^\sigma\psi(0)|N\rangle\Big|_{x^2=0} &= 2p^\sigma\int_{-1}^1 du\,e^{iupx}F(u)\\ \langle N|\bar\psi(0)\gamma^\sigma\psi(x)|N\rangle\Big|_{x^2=0} &= \langle N|\bar\psi(-x)\gamma^\sigma\psi(0)|N\rangle\Big|_{x^2=0} \;=\; 2p^\sigma\int_{-1}^1 du\,e^{-iupx}F(u)\end{aligned} \tag{9.20}$$

[why $\int_{-1}^1 du$: the function $F(u)$ vanishs outside of $u\in(-1,1)$; will motivate later; formal proof complicated]

We obtain:

$$T_{\mu\nu} = -\frac{1}{2\pi^2} 2 \int \frac{d^4x}{x^4} \int_{-1}^{1} du\, F(u) \Big\{ e^{iupx+iqx} \Big[x_\mu p_\nu + x_\nu p_\mu - g_{\mu\nu} px \Big] - e^{-iupx+iqx} \Big[x_\mu p_\nu + x_\nu p_\mu - g_{\mu\nu} px \Big] \Big\}$$

$$= -2 \int_{-1}^{1} du\, F(u) \Big\{ \frac{1}{(q+up)^2 + i\epsilon} \Big[(q+up)_\mu p_\nu + (q+up)_\nu p_\mu - g_{\mu\nu} p \cdot (q+up) \Big]$$

$$- \frac{1}{(q-up)^2 + i\epsilon} \Big[(q-up)_\mu p_\nu + (q-up)_\nu p_\mu - g_{\mu\nu} p \cdot (q-up) \Big] \Big\} \tag{9.21}$$

Let $p^2 = m_N^2 \to 0$, then

$$\begin{aligned} (q+up)^2 &= q^2 + 2uqp = -Q^2 + 2uqp = 2qp(u - x_B) \\ (q-up)^2 &= 2qp(-u - x_B) \end{aligned} \tag{9.22}$$

so that

$$T_{\mu\nu} = -2 \frac{1}{2qp} \int_{-1}^{1} du\, F(u) \Big\{ \frac{1}{u - x_B + i\epsilon} \Big[(q+up)_\mu p_\nu + (q+up)_\nu p_\mu - g_{\mu\nu} pq \Big]$$

$$- \frac{1}{-u - x_B + i\epsilon} \Big[(q-up)_\mu p_\nu + (q-up)_\nu p_\mu - g_{\mu\nu} pq \Big] \Big\} \tag{9.23}$$

Next

$$\mathrm{Im} \frac{1}{u - x_B + i\epsilon} = -\pi \delta(u - x_B)\,, \qquad \mathrm{Im} \frac{1}{-u - x_B + i\epsilon} = -\pi \delta(u + x_B) \tag{9.24}$$

so that

$$\frac{1}{\pi} \mathrm{Im}\, T_{\mu\nu} = \frac{1}{qp} \Big\{ F(x_B) \Big[(q + x_B p)_\mu p_\nu + (q + x_B p)_\nu p_\mu - g_{\mu\nu} qp \Big]$$

$$- F(-x_B) \Big[(q + x_B p)_\mu p_\nu + (q + x_B p)_\nu p_\mu - g_{\mu\nu} qp \Big] \Big\} \tag{9.25}$$

Finally, we can rewrite

$$(q + x_B p)_\mu p_\nu + (q + x_B p)_\nu p_\mu - g_{\mu\nu} qp = pq \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) + 2x_B \left(p_\mu + q_\mu \frac{1}{2x_B} \right) \left(p_\nu + q_\nu \frac{1}{2x_B} \right) \tag{9.26}$$

and thus

$$\boxed{\frac{1}{\pi} \mathrm{Im}\, T_{\mu\nu} = \Big[F(x_B) - F(-x_B) \Big] \Big\{ \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) + \frac{2x_B}{pq} \left(p_\mu + q_\mu \frac{1}{2x_B} \right) \left(p_\nu + q_\nu \frac{1}{2x_B} \right) \Big\}} \tag{9.27}$$

or

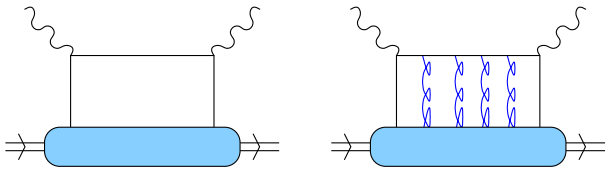
$$F_1(x_B, Q^2) = F(x_B) - F(-x_B)$$
$$F_2(x_B, Q^2) = x_B F_1(x_B, Q^2) \tag{9.28}$$

thus reproducing the parton model, with the identification

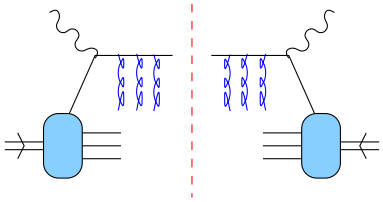
$$\begin{aligned} F(x_B) &= F_q(x_B): && \text{quark distribution} \\ -F(-x_B) &= F_{\bar{q}}(x_B): && \text{antiquark distribution} \end{aligned} \tag{9.29}$$

Thus:

- We are on the right track, but:
- Our definition of parton distributions is not gauge invariant. Result gauge dependent?
- We used free quark propagator in this calculation. In reality the quark propagates inside the nucleon and could interact with "clasical" fields theerein. Can we neglect this interaction or not?

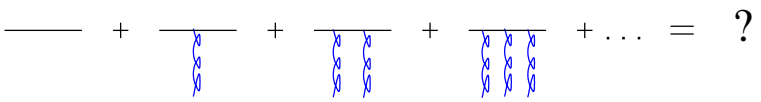


Note that this is a reformulation of the question whether we can neglect the final state interaction



9.2 Light-cone expansion of the quark propagator in the background field

Quark propagator in the background field



Light-cone expansion:

$$S(x) = \frac{1}{x^4} S_1(x, A_\mu) + \frac{1}{x^2} S_2(x, A_\mu) + \ln x^2 S_3(x, A_\mu) + \dots \tag{9.30}$$

Only the most singular term $\sim 1/x^4$ is relevant in the present context

$$
\begin{aligned}
\text{———} \quad &= \overbrace{\psi(x)\overline{\psi}}(0) \;=\; \frac{i}{2\pi^2}\frac{\not x}{x^4} \\
\text{———}\!\!\sim\!\!\sim \quad &= \overbrace{\psi(x) ig \int d^4z\, \overline{\psi}}(z) A_\mu(z)\gamma^\mu \overbrace{\psi(z)\overline{\psi}}(0) \\
&= ig\left(\frac{i}{2\pi^2}\right)^2 \int d^4z\, \frac{\not x - \not z}{(x-z)^4} A_\mu(z)\gamma^\mu \frac{\not z}{z^4} \qquad\qquad \boxed{\bar u = 1-u} \\
&= -\frac{ig}{4\pi^4}\int_0^1 du\, u(1-u) \int d^4z\, A_\mu(z)(\not x - \not z)\gamma^\mu \not z \frac{\Gamma(4)}{[(z-ux)^2 + u\bar u x^2]^4} \\
&= -\frac{ig}{4\pi^4}\int_0^1 du\, u\bar u \int d^4z\, A_\mu(z+ux)(\bar u \not x - \not z)\gamma^\mu(\not z + u\not x)\frac{\Gamma(4)}{[z^2 + u\bar u x^2]^4}
\end{aligned}
\tag{9.31}
$$

Since there is no dependence on $z\cdot x$ in the denominator, the expansion of the field L25

$$
A_\mu(z+ux) \;=\; A_\mu(ux) + z^\nu \partial_\nu A_\mu(ux) + \ldots \tag{9.32}
$$

will produce subleading terms that are less singular at $x^2 \to 0$.

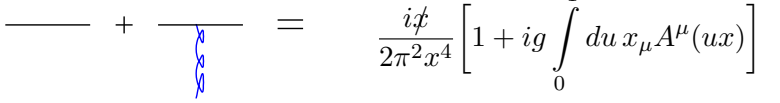
We need

$$
\begin{aligned}
\int d^4z \frac{\Gamma(4)}{[z^2+A]^4} &= -i\pi^2 \frac{1}{A^2} \\
\int d^4z \frac{\Gamma(4)}{[z^2+A]^4} z_\alpha z_\beta &= -i\pi^2 \left(\frac{1}{2} g_{\alpha\beta}\right)\frac{1}{A}
\end{aligned}
\tag{9.33}
$$

After some algebra one obtains

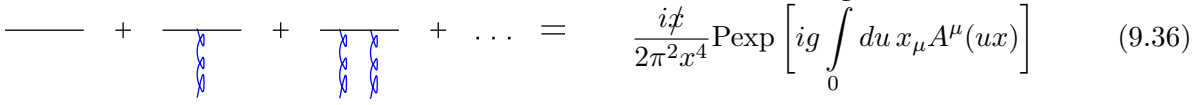
$$
\ldots = -\frac{g}{4\pi^2}\int_0^1 du \left\{ \frac{2x_\mu}{x^4} A_\mu(ux)\not x - \frac{1}{2x^2}\left[\partial^\alpha A^\beta(ux)\Big(u\gamma_\alpha\gamma_\beta \not x - \bar u \not x \gamma_\beta\gamma_\alpha\Big) + x_\xi u\bar u \not x \partial^2 A^\xi(ux)\right]\right\} + \mathcal{O}(\ln x^2) \tag{9.34}
$$

Here we only need the first term:



$$
\frac{i\not x}{2\pi^2 x^4}\left[1 + ig\int_0^1 du\, x_\mu A^\mu(ux)\right] \tag{9.35}
$$

The leading contributions are easy to calculate to all orders in the field. One obtains



$$
\frac{i\not x}{2\pi^2 x^4}\mathrm{Pexp}\left[ig\int_0^1 du\, x_\mu A^\mu(ux)\right] \tag{9.36}
$$

where

$$
\mathrm{Pexp}\Big[**\Big] = 1 + ig\int_0^1 du\, x_\mu A^\mu(ux) + (ig)^2 \int_0^1 du \int_0^u dv\, x_\mu A^\mu(ux) x_\nu A^\nu(vx) + \ldots \tag{9.37}
$$

The fields cannot be interchanged because

$$A_\mu \equiv A_\mu^a t^a = A_\mu^a \frac{\lambda^a}{2} \qquad 3\times 3 \,\text{matrix in color space} \tag{9.38}$$

[full analogy with time-ordered exponent in QM time-dependent perturbation theory]

A convenient shorthand notation:

$$[x,y] = \mathrm{Pexp}\left[ig\int\limits_0^1 du\,(x-y)_\mu A^\mu(ux+\bar u y)\right] \qquad \bar u = 1-u \tag{9.39}$$

Thus we obtain the quark propagator in the background gluon field (Gross, Treiman '71)

$$\boxed{S(x) = \overbrace{\psi_q(x)\bar\psi}_q(0) = \frac{i\not x}{2\pi^2 x^4}[x,0]_c + \mathcal{O}\left(\frac{1}{x^2}\right)} \tag{9.40}$$

9.3 Parton model revisited

Now go back to the calculation of the DIS cross section. The first contribution becomes

$$\begin{aligned} T_{\mu\nu} &= i\int d^4x\, e^{iqx}\langle N|\,\mathrm{T}\,\{\bar\psi_c(x)\gamma_\mu \overbrace{\psi_q(x)\bar\psi}_q(0)\gamma_\nu\psi_c(0))|\}|N\rangle + \ldots \\ &= i\frac{i}{2\pi^2}\int d^4x\,\frac{e^{iqx}}{x^4}\langle N|\bar\psi_c(x)\gamma_\mu \not x\gamma_\nu[x,0]_c\psi_c(0)|N\rangle \end{aligned} \tag{9.41}$$

The calculation remains the same, but the operators become decorated with a gauge-link factor (called Wilson line) connecting the quark fields. Thus a better definition of the parton distribution is

$$\boxed{\langle N(p)|\bar\psi(x)\gamma^\sigma[x,0]\psi(0)|N(p)\rangle\Big|_{x^2=0} = 2p^\sigma\int_{-1}^1 du\, e^{iupx}F(u)} \tag{9.42}$$

Let us check that the operator on the l.h.s. is gauge invariant. For simplicity I consider abelian gauge trafos (like in QED):

$$\bar\psi(x)\gamma^\sigma e^{ig\int_0^1 du\,x_\mu A^\mu(ux)}\psi(0) \longrightarrow \bar\psi(x)\gamma^\sigma e^{-i\alpha(x)}e^{ig\int_0^1 du\,x_\mu(A^\mu(ux)+\frac{1}{g}\partial^\mu\alpha(ux))}e^{i\alpha(0)}\psi(0) \tag{9.43}$$

Thus the extra term under the exponent is

$$-i\alpha(x) + i\alpha(0) + i\int\limits_0^1 du\, x^\mu\partial_\mu\alpha(ux) \tag{9.44}$$

Euler's Homogeneous Function Theorem implies

$$x^\mu\frac{\partial}{\partial x^\mu}\alpha(ux) = u\frac{d}{du}\alpha(ux) \tag{9.45}$$

Therefore

$$\int_0^1 du\, x^\mu \partial_\mu \alpha(ux) = \int_0^1 du\, x^\mu \frac{\partial}{\partial(ux^\mu)} \alpha(ux) = \int_0^1 du\, \frac{1}{u} u \frac{d}{du} \alpha(ux) = \alpha(x) - \alpha(0) \tag{9.46}$$

and all α-dependent terms cancel as they should

Another possibility to see gauge invariance: Let $x \to 0$ (all components); then

$$\begin{aligned} \bar\psi(x) &= \bar\psi(0)[1 + x^\mu \overleftarrow{\partial}_\mu + \ldots] \\ [x,0] &= 1 + ig \int_0^1 du\, x_\mu A^\mu(ux) + \ldots = 1 + igx_\mu A^\mu(0) + \ldots \end{aligned} \tag{9.47}$$

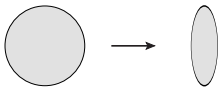
Thus to first order

$$\bar\psi(x)\Gamma[x,0]\psi(0) = \bar\psi(0)\Gamma\psi(0) + x^\mu \bar\psi(0)\Gamma[\overleftarrow{\partial}_\mu + igA_\mu]\psi(0) = \bar\psi(0)\Gamma\psi(0) + x^\mu \bar\psi(0)\Gamma \overleftarrow{D}_\mu \psi(0) \tag{9.48}$$

— the expansion of nonlocal operators with Wilson lines goes over covariant derivatives. This is true to all orders:

$$\bar\psi(x)\Gamma[x,0]\psi(0) \;=\; \sum_n \frac{1}{n!} x_{\mu_1} \ldots x_{\mu_n} \bar\psi(0)\Gamma \overleftarrow{D}_{\mu_1} \ldots \overleftarrow{D}_{\mu_n} \psi(0) \tag{9.49}$$

The parton model is largerly based on intuition that fast moving hadrons are Lorentz-contracted, similar to what we know about rigid bodies. But is it really obvious?



Example I: Assume that a proton can be modelled as a point-like charge which is a source of a scalar potential (Yukawa with mass zero)

$$\phi(\vec{x}) = \frac{q}{|\vec{x}|} \qquad \longleftarrow \quad \text{for proton at rest} \tag{9.50}$$

The field is certainly spherically symmetric in this frame.
Making a Lorentz transformation (boost along the x_3-direction) we obtain

$$\phi'(x') = \frac{q}{[x_\perp^2 + \gamma^2(vt' - x_3')^2]^{1/2}} \qquad x_\perp^2 = x_1^2 + x_2^2 \tag{9.51}$$

where $\gamma = 1/\sqrt{1-v^2}$. As $\gamma \to \infty$ the field vanishes as $1/\gamma$ except for a narrow strip (pancake) around $x_3' \simeq vt'$, as expected.

Example II: A Coulomb potential in Eigenkoordinatensystem (QED)

$$A_\mu(\vec{x}) = \frac{q\delta_{\mu 0}}{|\vec{x}|} \qquad \longleftarrow \quad \text{for proton at rest} \tag{9.52}$$

Then

$$A'_0(x') = \frac{q\gamma}{[x_\perp^2 + \gamma^2(vt' - x'_3)^2]^{1/2}}$$
$$A'_3(x') = \frac{-qv\gamma}{[x_\perp^2 + \gamma^2(vt' - x'_3)^2]^{1/2}}$$
$$A'_\perp(x') = 0 \tag{9.53}$$

In this case for $\gamma \to \infty$ both $A'_0(x')$ and $A'_3(x') \sim const(?!)$ Where is my pancake???

The resolution of this paradox is that

$$\text{for} \quad \gamma \to \infty \qquad\qquad A'_\mu(x') \sim q\partial_\mu \ln(vt' - x'_3)$$

— can be removed by a gauge transformation. Thus, although the four-potential is *not* Lorentz-contracted, the electric and magnetic fields are contracted to a pancake, e.g.

$$E'_3(x') = \frac{-q\gamma(vt' - x'_3)}{[x_\perp^2 + \gamma^2(vt' - x'_3)^2]^{3/2}} \tag{9.54}$$

The parton model is recovered in a certain gauge where the Wilson line can be neglected

$$x^\mu A_\mu(ux) = 0 \tag{9.55}$$

where x^μ is the 4-vector in the direction of the outgoing (struck) quark.

$$\begin{aligned} &\text{light-cone gauge} && n^\mu A_\mu(x) = 0\,, && n^2 = 0 \\ &\text{Fock-Schwinger gauge} && x^\mu A_\mu(x) = 0\,, && A_\mu(0) = 0 \end{aligned} \tag{9.56}$$

9.4 The DGLAP Evolution equation

9.4.1 Preliminary remarks

Let us look at our definition of the parton distribution more closely. We have

$$\langle N(p)|\bar\psi_c(x)\gamma^\sigma[x,0]\psi_c(0)|N(p)\rangle\Big|_{x^2=0} \;=\; 2p^\sigma \int_{-1}^{1} du\, e^{iupx} F(u,\mu) \tag{9.57}$$

where I restored the subscript "classical" and added the argument μ to $F(u,\mu)$ to remind that only low-frequency part of the fields is included. The role of the cutoff is that it allows the limit $x^2 \to 0$ to be taken.

Consider the similar matrix element with "full" fields and finite x^2: L26

$$\langle N(p)|\bar\psi(x)\gamma^\sigma[x,0]\psi(0)|N(p)\rangle\Big|_{x^2p^2\ll 1} \;=\; 2p^\sigma \int_{-1}^{1} du\, e^{iupx} \tilde F(u,x^2) \tag{9.58}$$

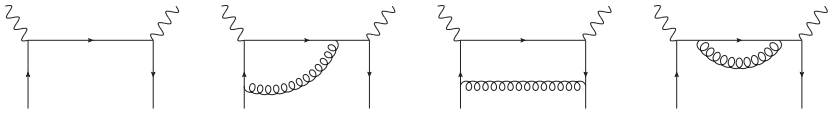
The only UV divergences in this ME are due to field and coupling renormalization: adding $\sqrt{Z_2}$ for each quark field and expressing the result in terms of the renormalized coupling the result is finite. However, the limit $x^2 \to 0$ cannot be taken. Schematically

$$\langle N(p)|\bar\psi(x)\gamma^\sigma[x,0]\psi(0)|N(p)\rangle\Big|_{x^2p^2\ll 1} \sim 1 + c\alpha_s \ln\frac{1}{x^2p^2} + \dots \tag{9.59}$$

The role of the cutoff in the fields is that it regularizes additional UV divergences that appear in the $x^2 \to 0$ limit:

$$\langle N(p)|\bar{\psi}_c(x)\gamma^\sigma[x,0]\psi_c(0)|N(p)\rangle\Big|_{x^2=0} \sim 1 + c\alpha_s \ln\frac{\mu^2}{p^2} + \dots \tag{9.60}$$

The dependence on μ will be compensated by the similar dependence of the Feynman diagrams for $T_{\mu\nu}$ in higher orders:



These diagrams (for quark legs on-shell) would be IR divergent if we forget that only frequencies above μ are included in "quantum" fields. The sum is schematically

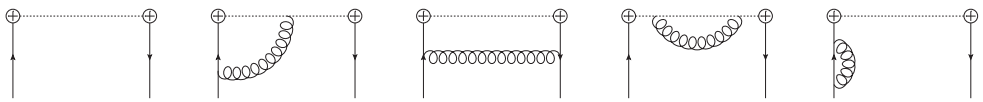
$$1 + c\alpha_s \ln\frac{1}{\mu^2 x^2} \qquad \overset{\text{Fourier}}{\longrightarrow} \quad 1 + c\alpha_s \ln\frac{Q^2}{\mu^2} \tag{9.61}$$

In order to avoid large logarithms in the coefficient functions we should set the scale $\mu \sim Q$. Therefore our leading-order result to the structure functions actually involves parton distributions at the scale $\sim Q$

$$F_1(x_B, Q^2) = F(x_B, \mu = Q) - F(-x_B, \mu = Q) = \frac{4}{9}u(x,Q) + \frac{4}{9}\bar{u}(x,Q) + \frac{1}{9}d(x,Q) + \frac{1}{9}\bar{d}(x,Q) + \dots \tag{9.62}$$

and we expect that Bjorken scaling will be violated by logarithmic corrections corresponding to the scale-dependence of parton distributions.

To calculate this dependence, split "slow" fields that appear in the operator matrix elements in
— "moderately slow" , with frequencies $\mu_0 < |k| < \mu$
— "very slow", with frequencies $|k| < \mu_0$
We can treat "moderately slow" fields as "quantum", that is involved the loops and "very slow" as "classical" external fields



Integrating over the "moderately slow" fields we obtain operator matrix elements of "very slow" operators corresponding to parton distributions at the scale μ_0:

$$[\text{operator at the scale}\,\mu] \sim (1 + c\alpha_s \ln\frac{\mu}{\mu_0})\,[\text{operator at the scale}\,\mu_0] \tag{9.63}$$

and obtain the finite-difference equation

$$[\text{operator at the scale}\,\mu] - [\text{operator at the scale}\,\mu_0] \sim c\alpha_s \ln\frac{\mu}{\mu_0}\,[\text{operator at the scale}\,\mu_0] \tag{9.64}$$

which we will rewrite as a differential renormalization group equation which is our goal.

A technicality:
as we know, a calculation with momentum cutoffs is very awkward. One scale can be introduced via the dimensional regularization but for the second scale we need another regulator.

A convenient choice: use
— finite x^2 as a substitute for $\mu^2 \sim 1/|x^2|$ (larger scale);
— dimensional regularization for μ_0^2 (smaller scale, thus an IR divergence) and, simultaneously, the remaining UV divergences (coupling and field renormalization)

[One can prove that both IR and UV divergences can be regularized dimensionally]

9.4.2 One-loop calculation

To avoid an open vector index consider the operator multiplied by extra x^σ so that

$$\langle N(p)|\bar\psi(x)\not{x}[x,0]\psi(0)|N(p)\rangle \;=\; 2(px)\int_{-1}^{1} du\, e^{iupx}F(u) + \mathcal{O}(x^2) \tag{9.65}$$

Let us calculate the diagram corresponding to gluon emission from the Pexp.
We write

$$A^\mu(z) = A_q^\mu(z) + A_c^\mu(z) \tag{9.66}$$

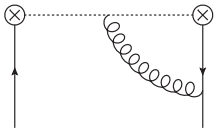
and consider the first term in the expansion of Pexp[...] in the "quantum" gluon field

$$[x,0] = [x,0]_c + ig\int_0^1 du\,[x,ux]_c x_\mu A_q^\mu(ux)[ux,0]_c \tag{9.67}$$

We choose

$$\begin{aligned} A_q^\mu &: && \text{Feynman gauge} \\ A_c^\mu &: && \text{Fock-Schwinger gauge} \qquad [x,0]_c = [x,ux]_c = [ux,0]_c = 1 \end{aligned} \tag{9.68}$$

Then



$$\longleftrightarrow \quad \bar\psi_c(x) ig\int_0^1 du\; x_\mu \overbrace{A_q^\mu(ux)\not{x}\psi_q(0)\Big[ig\int d^dz\,\bar\psi_q(z)\not{A}_q(z)\psi_c(z)\Big]} \tag{9.69}$$

Use

$$\begin{aligned} \overbrace{\psi_a(z)\bar\psi_b}(0) &= \delta_{ab}\frac{i}{2\pi^{d/2}}\not{z}\frac{\Gamma(d/2)}{[-z^2+i\epsilon]^{d/2}} \\ \overbrace{A_\mu^A(ux)A_\nu^B}(z) &= -\delta_{AB}g_{\mu\nu}\frac{1}{4\pi^{d/2}}\frac{\Gamma(d/2-1)}{[-(ux-z)^2+i\epsilon]^{d/2-1}} \end{aligned} \tag{9.70}$$

leading to

$$\ldots \;=\; (ig)^2 \int\limits_0^1 du \int d^dz \bar\psi_c(x) \not{x} \frac{i}{2\pi^{d/2}} \frac{-1}{4\pi^{d/2}} t^A (-\not{z}) \frac{\Gamma(d/2)}{[-z^2]^{d/2}} \frac{\Gamma(d/2-1)}{[-(ux-z)^2]^{d/2-1}} \not{x} t^A \psi_c(z) \qquad (9.71)$$

Next, combine the propagators

$$\frac{\Gamma(d/2)}{[-z^2]^{d/2}} \frac{\Gamma(d/2-1)}{[-(ux-z)^2]^{d/2-1}} = \int\limits_0^1 dv\, v^{d/2-2} \bar v^{d/2-1} \frac{\Gamma(d-1)}{[-(z-uvx)^2 - x^2u^2\bar v v]^{d-1}} \qquad (9.72)$$

and shift the integration variable $z \to z + uvx$ to obtain

$$\ldots \;=\; \frac{-ig^2}{8\pi^d} C_F \int\limits_0^1 du \int\limits_0^1 dv \int d^dz \frac{v^{d/2-2} \bar v^{d/2-1} \Gamma(d-1)}{[-z^2 - x^2u^2\bar v v]^{d-1}} \bar\psi_c(x) \not{x} (\not{z} + uv\not{x}) \not{x} \psi_c(z+uvx) \qquad (9.73)$$

We are looking for the contributions of the type

$$\ln \frac{1}{x^2\mu^2} \qquad \longleftarrow \qquad \frac{\Gamma(d/2-2)}{[x^2\mu^2]^{d/2-2}}\,, \qquad g^2 \to g^2\mu^{4-d} \qquad (9.74)$$

Note:

- do not need terms with positive powers of x^2 $\quad\Rightarrow$ can expand the field

$$\psi_c(z+uvx) = \psi(uvx) + z_\alpha \partial^\alpha \psi(uvx) + \ldots$$

- argument of the Gamma-function: $\Gamma(d/2-2)$ stands for IR divergence, $\Gamma(2-d/2)$ stands for UV divergence.

Why:

$$\int \frac{d^dk}{k^4} = ? \qquad \leftarrow \begin{cases} \text{choose d} < 4 \text{ to suppress the large-k region (UV)} \\ \text{choose d} > 4 \text{ to suppress the small-k region (IR)} \end{cases} \qquad (9.75)$$

One can regularize IR or UV divergence "by hand" so that only the other one remains:

$$\begin{aligned} \int \frac{d^dk}{(k^2+a^2)^2} &= i\pi^{d/2} \frac{\Gamma(2-d/2)}{[-a^2]^{2-d/2}} \qquad \text{finite for } d<4 \\ \int \frac{d^dk}{k^4} e^{2ika} &= i\pi^{d/2} \frac{\Gamma(d/2-2)}{[-a^2]^{d/2-2}} \qquad \text{finite for } d>4 \end{aligned} \qquad (9.76)$$

In general situation the identification — UV or IR — is not simple, one needs to consider each case separately.

Using generic integrals

$$\begin{aligned} \int dz \frac{\Gamma(\alpha)}{[-z^2-a^2]^\alpha} &= -i\pi^{d/2} \frac{\Gamma(\alpha-d/2)}{[-a^2]^{\alpha-d/2}} \\ \int dz \frac{\Gamma(\alpha)}{[-z^2-a^2]^\alpha} z_\mu z_\nu &= -i\pi^{d/2} \left(-\frac{g_{\mu\nu}}{2}\right) \frac{\Gamma(\alpha-d/2-1)}{[-a^2]^{\alpha-d/2-1}} \end{aligned} \qquad (9.77)$$

obtain L27

$$\ldots = \frac{-ig^2}{8\pi^d} C_F(-i\pi^{d/2}) \int_0^1 du \int_0^1 dv\, v^{d/2-2}\bar{v}^{d/2-1}\Big\{x^2uv\bar{\psi}_c(x)\not{x}\psi_c(uvx)\frac{\Gamma(d/2-1)}{[-x^2u^2v\bar{v}]^{d/2-1}}$$

$$+\,\bar{\psi}_c(x)\not{x}\gamma_\mu\not{x}\partial_\nu\psi_c(uvx)\left(-\frac{g_{\mu\nu}}{2}\right)\frac{\Gamma(d/2-2)}{[-x^2u^2v\bar{v}]^{d/2-2}}\Big\}$$

$$= \frac{g^2C_F}{8\pi^{d/2}}\int_0^1 du\int_0^1 dv\bar{\psi}_c(x)\Big\{\frac{\Gamma(d/2-1)}{[-x^2]^{d/2-2}}u^{3-d}\not{x} + \frac{1}{2}\frac{\Gamma(d/2-2)}{[-x^2u^2]^{d/2-2}}\not{x}\gamma_\mu\not{x}\bar{v}\partial^\mu\Big\}\psi_c(uvx) \quad (9.78)$$

The first term:

$$\int_0^1 \frac{du}{u^{d-3}}: \qquad \text{finite for } d<4 \qquad \Rightarrow \qquad \text{UV divergence at } u\to 0 \quad (9.79)$$

$\Longrightarrow$ contributes to coupling/field renormalization, irrelevant for us

In the second term can replace

$$\frac{1}{2}\not{x}\gamma_\mu\not{x}\partial^\mu \to \not{x}(x\partial) + \mathcal{O}(x^2)$$

so it becomes

$$\frac{g^2C_F}{8\pi^{d/2}}\int_0^1 du\int_0^1 dv\,\bar{v}\frac{\Gamma(d/2-2)}{[-x^2u^2]^{d/2-2}}\bar{\psi}_c(x)\not{x}(x\partial)\psi_c(uvx) \quad (9.80)$$

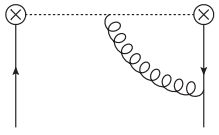
The familiar trick:

$$(x\partial)\psi_c(uvx) = x^\mu\frac{\partial}{\partial(uvx^\mu)}\psi_c(uvx) = \frac{1}{uv}u\frac{d}{du}\psi_c(uvx) = \frac{1}{v}\frac{d}{du}\psi_c(uvx) \quad (9.81)$$

The dependence on u in $[-x^2u^2]^{d/2-2}$ can be neglected to logarithmic accuracy ($\ln 1/(x^2\mu^2)+$ const) leaving us with

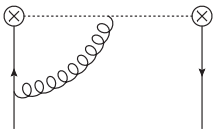
$$\int_0^1 du\,\frac{d}{du}\psi_c(uvx) = \psi_c(vx) - \psi_c(0) \quad (9.82)$$

Thus the result is

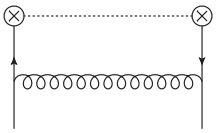


$$\longleftrightarrow \quad \frac{g^2C_F}{8\pi^{d/2}}\frac{\Gamma(d/2-2)}{[-x^2\mu_{IR}^2]^{d/2-2}}\int_0^1 dv\,\frac{\bar{v}}{v}\Big[\bar{\psi}_c(x)\not{x}\psi_c(vx) - \bar{\psi}_c(x)\not{x}\psi_c(0)\Big] \quad (9.83)$$

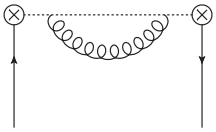
where I restored the IR scale μ_{IR} dependence coming from $g^2 \to g^2\mu^{4-d}$.
Other contributions:



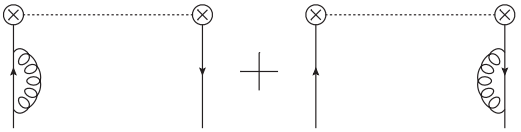
$$\longleftrightarrow \quad \frac{g^2 C_F}{8\pi^{d/2}} \frac{\Gamma(d/2-2)}{[-x^2\mu_{IR}^2]^{d/2-2}} \int\limits_0^1 dv\, \frac{v}{\bar v}\Big[\bar\psi_c(vx) \not x \psi_c(0) - \bar\psi_c(x)\not x\psi_c(0)\Big]$$



$$\longleftrightarrow \quad \frac{g^2 C_F}{8\pi^{d/2}} \frac{\Gamma(d/2-2)}{[-x^2\mu_{IR}^2]^{d/2-2}} \int\limits_0^1 du \int\limits_0^u dv\, \bar\psi_c(ux) \not x \psi_c(vx)$$



$$\longleftrightarrow \quad \text{only UV divergences, coupling renormalization in Pexp}$$



$$\longleftrightarrow \quad \sqrt{Z_2}\cdot\sqrt{Z_2} \quad \text{field renormalization } |x^2| \to \mu_{IR}^2 \tag{9.84}$$

The final result:

$$\bar\psi(x)\not x\psi(0) = \bar\psi_c(x)\not x\psi_c(0) + \frac{g^2 C_F}{8\pi^{d/2}} \frac{\Gamma(d/2-2)}{[-x^2\mu_{IR}^2]^{d/2-2}} \int\limits_0^1 du \int\limits_0^u dv\, K(u,v)\, \bar\psi_c(ux)\not x\psi_c(vx) \tag{9.85}$$

where

$$K(u,v) = \delta(\bar u)\left[\frac{\bar v}{v}\right]_+ + \delta(v)\left[\frac{u}{\bar u}\right]_+ + 1 - \frac{1}{2}\delta(\bar u)\delta(v) \tag{9.86}$$

The "plus-distribution":

$$\begin{aligned}
\int_0^1 dv\, \left[\frac{\bar v}{v}\right]_+ f(v) &:= \int_0^1 dv\, \frac{\bar v}{v}[f(v) - f(0)] \\
\int_0^1 du\, \left[\frac{u}{\bar u}\right]_+ f(u) &:= \int_0^1 du\, \frac{u}{\bar u}[f(u) - f(1)]
\end{aligned} \tag{9.87}$$

It remains a little bit of cosmetics.

- We are only interested in matrix elements between the states with the same momenta. In this case

$$\begin{aligned}
\langle N(p)|\bar\psi(ux)\not x\psi(vx)|N(p)\rangle &= \langle N(p)|e^{+i\hat P vx}\bar\psi((u-v)x)\not x\psi(0)e^{-i\hat P vx}|N(p)\rangle \\
&= \langle N(p)|\bar\psi((u-v)x)\not x\psi(0)|N(p)\rangle
\end{aligned} \tag{9.88}$$

where $\hat{P}$ is the momentum operator: $\hat{P}_\mu|N(p)\rangle = p_\mu|N(p)\rangle$
Thus e.g.

$$\int_0^1 du \int_0^u dv\, \bar{\psi}((u-v)x)\not{x}\psi(0) = \int_0^1 du \int_0^u dt\, \bar{\psi}(tx)\not{x}\psi(0) = \int_0^1 dt\,(1-t)\bar{\psi}(tx)\not{x}\psi(0) \quad (9.89)$$

- Expanding at $d = 4-\epsilon$

$$\frac{g^2}{8\pi^{d/2}} C_F \left(-\frac{2}{\epsilon}\right)\left[1 - \frac{2}{\epsilon}\ln\frac{1}{-x^2\mu_{IR}^2}\right] \quad \longrightarrow \quad \frac{\alpha_s}{2\pi} C_F \ln\frac{1}{-x^2\mu_{IR}^2} \quad + \text{const.} \quad (9.90)$$

Remember that the goal of the calculation was to change the definition of the "classical" field from the cutoff $\mu_1^2 \equiv 1/-x^2 \sim Q^2$ to a lower cutoff $\mu_2^2 \equiv \mu_{IR}^2 < \mu_1^2$. Thus we get, for $x^2 = 0$

$$[\bar{\psi}(x)\not{x}\psi(0)]_{\mu_1^2} = [\bar{\psi}(x)\not{x}\psi(0)]_{\mu_2^2} + \frac{\alpha_s C_F}{2\pi}\ln\frac{\mu_1^2}{\mu_2^2}\int_0^1 du\, K(u)\,[\bar{\psi}(ux)\not{x}\psi(0)]_{\mu_2^2} \quad (9.91)$$

with

$$K(u) = 2\left[\frac{u}{\bar{u}}\right]_+ + \bar{u} - \frac{1}{2}\delta(\bar{u}) \quad (9.92)$$

This relation is valid if

$$\alpha_s(\mu_2)\ln\frac{\mu_1^2}{\mu_2^2} \ll 1$$

- The last step, we have to take the matrix element $\langle N|\ldots|N\rangle$. Consider quark contribution:

$$\langle N(p)|[\bar{\psi}(x)\not{x}[x,0]\psi(0)]_{\mu^2}|N(p)\rangle \;=\; 2(px)\int_0^1 d\xi\, e^{i\xi px} F(\xi,\mu^2) \quad (9.93)$$

We obtain

$$\cancel{2(px)}\int_0^1 d\xi\, e^{i\xi px}\Big[F(\xi,\mu_1^2) - F(\xi,\mu_2^2)\Big] = \frac{\alpha_s C_F}{2\pi}\ln\frac{\mu_1^2}{\mu_2^2}\int_0^1 du\, K(u)\,\cancel{2(px)}\int_0^1 d\xi\, e^{iu\xi px} F(\xi,\mu_2^2) \quad (9.94)$$

Next, insert

$$1 = \int_0^1 dv\,\delta(v-u\xi)$$

in the integral on the r.h.s.:

$$\int_0^1 du\, K(u)\int_0^1 d\xi\, e^{iu\xi px}F(\xi)\cdot\int_0^1 dv\,\delta(v-u\xi) = \int_0^1 dv\, e^{ivpx}\int_0^1 du\, K(u)\underbrace{\int_0^1 d\xi\, F(\xi)\,\delta(v-u\xi)}_{\frac{1}{u}F\left(\frac{v}{u}\right)\theta(u-v)}$$

$$= \int_0^1 dv\, e^{ivpx}\int_v^1 \frac{du}{u}\, K(u)\, F\Big(\frac{v}{u}\Big) \quad (9.95)$$

and finally

$$F_q(v,\mu_1^2) - F_q(v,\mu_2^2) \;=\; \frac{\alpha_s}{2\pi} C_F \ln\frac{\mu_1^2}{\mu_2^2} \int_v^1 \frac{du}{u}\, K(u)\, F_q\Big(\frac{v}{u},\mu_2^2\Big) \tag{9.96}$$

or

$$\boxed{\mu^2 \frac{d}{d\mu^2} F_q(v,\mu^2) \;=\; \frac{\alpha_s(\mu)}{2\pi} C_F \int_v^1 \frac{du}{u}\, K(u)\, F_q\Big(\frac{v}{u},\mu^2\Big)} \tag{9.97}$$

— the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation

Accepted notation:

$$P_{q\to q}(u) \equiv C_F K(u) \;=\; C_F \left[\frac{1+u^2}{[1-u]_+} + \frac{3}{2}\delta(1-u)\right] \tag{9.98}$$

- We could consider antiquarks instead of quarks, replacing $\int_0^1 d\xi e^{i\xi px} F(\xi) \to \int_{-1}^0 d\xi e^{i\xi px} F(\xi)$. To this order in perturbation theory there will be no difference:

$$P_{\bar q\to \bar q}(u) \;=\; P_{q\to q}(u) \tag{9.99}$$

— quarks and antiquarks have autonomous (and the same) scale dependence to leading order.

9.5 Solution of the DGLAP equation

The DGLAP equation can be solved by numerical integration, but a better approach is to consider *moments*:

$$M_q^N(Q^2) \;=\; \int_0^1 dx_B\, x_B^{N-1} F_q(x_B,Q^2) \tag{9.100}$$

Taking moments

$$\begin{aligned}
\mu^2 \frac{d}{d\mu^2} \int_0^1 dv\, v^{N-1} F_q(v,\mu^2) &= \frac{\alpha_s(\mu)}{2\pi} \int_0^1 dv\, v^{N-1} \int_v^1 \frac{du}{u}\, P_{qq}(u)\, F_q\Big(\frac{v}{u},\mu^2\Big) \\
&= \frac{\alpha_s(\mu)}{2\pi} \int_0^1 du\, P_{qq}(u)\, u^{N-1} \int_0^u \frac{dv}{u}\, \Big(\frac{v}{u}\Big)^{N-1} F_q\Big(\frac{v}{u},\mu^2\Big) \\
&= \frac{\alpha_s(\mu)}{2\pi} \int_0^1 du\, P_{qq}(u)\, u^{N-1} \int_0^1 dt\, t^{N-1} F_q\Big(t,\mu^2\Big)\,; \qquad \boxed{t = \frac{v}{u}}
\end{aligned} \tag{9.101}$$

Thus we obtain

$$\mu^2 \frac{d}{d\mu^2} M_q^N(\mu^2) \;=\; -\frac{\alpha_s(\mu)}{2\pi} \gamma_{qq}^N\, M_q^N(\mu^2) \tag{9.102}$$

where

$$\gamma_{qq}^N = -\int_0^1 du\, P_{qq}(u)\, u^{N-1} = -C_F \int_0^1 du\, u^{N-1} \left\{ 2\left[\frac{u}{\bar{u}}\right]_+ + \bar{u} - \frac{1}{2}\delta(\bar{u}) \right\}$$

$$= -C_F \left\{ 2\int_0^1 du\, \frac{u}{\bar{u}}(u^{N-1}-1) + \left(\frac{1}{N} - \frac{1}{N+1}\right) - \frac{1}{2} \right\} \tag{9.103}$$

The remaining integral:

$$\int_0^1 du\, \frac{1}{1-u}(u^N - u) = \int_0^1 du\, \frac{1}{1-u}(u^N - 1 + 1 - u) = -\int_0^1 du\, \frac{1-u^N}{1-u} + 1$$

$$= -\int_0^1 du\, (1 + u + \ldots + u^{N-1}) + 1 = -\sum_{j=1}^N \frac{1}{j} + 1 = -\sum_{j=2}^N \frac{1}{j} \tag{9.104}$$

We obtain

$$\boxed{\gamma_{qq}^N = \frac{4}{3}\left\{ 2\sum_{j=2}^N \frac{1}{j} - \frac{1}{N(N+1)} + \frac{1}{2} \right\} \qquad \text{Gross, Wilczek '73}} \tag{9.105}$$

The solution of Eq. (9.102) is then

$$M_q^N(Q^2) = \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu_0^2)}\right)^{2\gamma_{qq}^N/b} M_q^N(\mu_0^2)\,; \qquad b = \frac{11}{3}N_c - \frac{2}{3}n_f \tag{9.106}$$

Check:

$$Q^2\frac{d}{dQ^2}M_q^N(Q^2) = \frac{1}{[\alpha_s(\mu_0^2)]^{2\gamma_{qq}^N/b}} M_q^N(\mu_0^2) \frac{2\gamma_{qq}^N}{b} [\alpha_s(Q^2)]^{2\gamma_{qq}^N/b-1} Q^2 \frac{d}{dQ^2}\alpha_s(Q^2) \tag{9.107}$$

The β-function:

$$\mu\frac{d}{d\mu}\alpha_s(\mu) \;=\; \beta(\alpha_s) \;=\; -\frac{1}{2\pi}b\alpha_s^2 + \mathcal{O}(\alpha_s^3)$$

$$Q^2\frac{d}{dQ^2} \;=\; \frac{1}{2}Q\frac{d}{dQ} \qquad \longleftarrow \quad \text{extra factor } 1/2 \tag{9.108}$$

Thus we obtain

$$Q^2\frac{d}{dQ^2}M_q^N(Q^2) = -\frac{1}{2\pi}\gamma_{qq}^N\alpha_s(Q^2) \cdot \underbrace{\left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu_0^2)}\right)^{2\gamma_{qq}^N/b} M_q^N(\mu_0^2)}_{M_q^N(Q^2)} \qquad \boxed{\text{OK}} \tag{9.109}$$

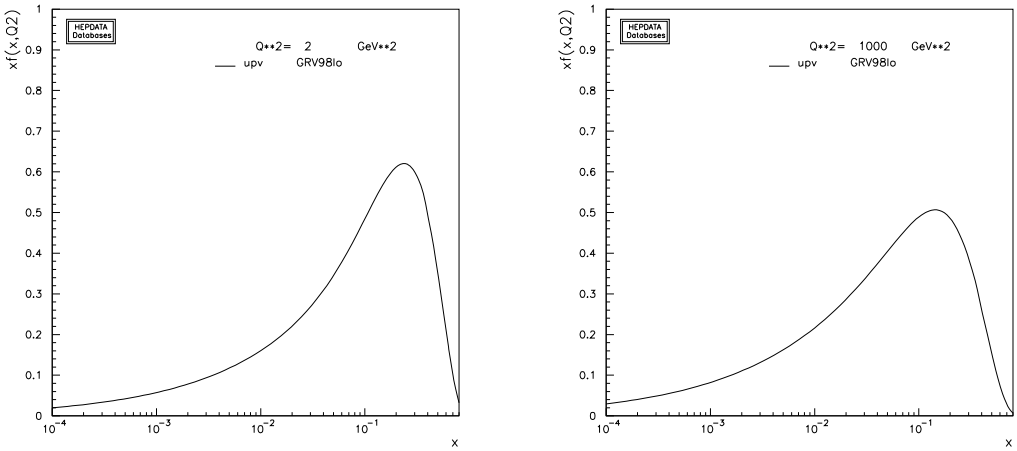
Since $\alpha_s(Q^2)$ decreases with Q^2, the moments $M_q^N(Q^2)$ decrease as well: the quark looses momentum because of gluon radiation

The parton distributions can be reconstructed from the moments using *inverse Mellin transform*

$$F_q(x) \;=\; \frac{1}{2\pi i}\int\limits_{1/2-i\infty}^{1/2+i\infty} dN\, x^{-N} M_q^N \tag{9.110}$$

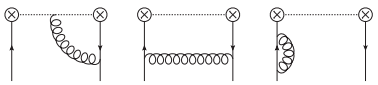
The example below shows the so-called valence u-quark distribution

$$x u_v(x,Q^2) \;=\; x\big[u(x,Q^2)-\bar{u}(x,Q^2)\big] \tag{9.111}$$

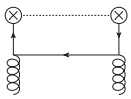


This is, however, not the end of the story...

$$[\psi(x)\not{x}\psi(0)]_{\mu_1^2} \;=\; [\psi(x)\not{x}\psi(0)]_{\mu_2^2}$$



$$+\;\frac{\alpha_s}{2\pi}C_F\ln\frac{\mu_1^2}{\mu_2^2}\int\limits_0^1 du\, K_{qq}(u)\,[\bar{\psi}(ux)\not{x}\psi(0)]_{\mu_2^2}$$



$$+\;\frac{\alpha_s}{2\pi}C_F\ln\frac{\mu_1^2}{\mu_2^2}\int\limits_0^1 du\, K_{qg}(u)\, x^\nu x^\alpha [F_{\mu\nu}(ux)F^\mu{}_\alpha(0)]_{\mu_2^2} \tag{9.112}$$

— our calculation is not complete

9.6 Gluon parton distribution

The definition:

$$x^\mu x^\nu \langle N(p)|[G^A_{\mu\xi}(x)[x,0]_{AB}G^B_{\nu\xi}(0)]_{\mu^2}|N(p)\rangle \;\overset{x^2=0}{=}\; 2(px)^2\int\limits_{-1}^{1} du\, e^{iupx} u F_g(u,\mu^2) \tag{9.113}$$

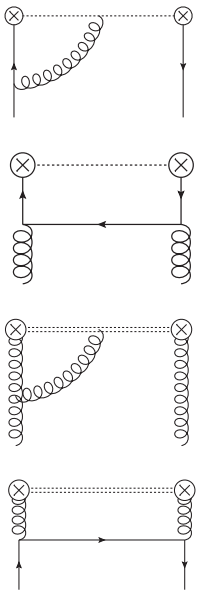
- There exist no "antigluons":

$$F_g(u) = F_g(-u) \tag{9.114}$$

- The gauge link is in adjoint representation:

$$
\begin{aligned}
[x,0] &= \mathrm{Pexp}\left\{ig\int_0^1 du\, x^\mu A_\mu(ux)\right\} \\
A_\mu &= A_\mu^A \big(T^A\big)_{BC}\,, \qquad \big(T^A\big)_{BC} = -if_{ABC} \qquad \longleftarrow \quad 8\times 8\,\text{matrix}
\end{aligned}
\tag{9.115}
$$

Explicit calculation:



$$
\begin{aligned}
&+\ldots \quad \longrightarrow \quad P_{qq} \\
&\quad\;\; \longrightarrow \quad P_{qg} \\
&+\ldots \quad \longrightarrow \quad P_{gg} \\
&\quad\;\; \longrightarrow \quad P_{gq}
\end{aligned}
\tag{9.116}
$$

Obtain the system of coupled equations:

$$
\begin{aligned}
Q^2\frac{d}{dQ^2}F_g(x,Q^2) &= \frac{\alpha_s(Q)}{2\pi}\int_x^1 \frac{dy}{y}\left\{P_{gq}(y)\sum_q\Big[F_q(\frac{x}{y},Q)+F_{\bar q}(\frac{x}{y},Q)\Big]+P_{gg}(y)F_g(\frac{x}{y},Q)\right\} \\
Q^2\frac{d}{dQ^2}F_q(x,Q^2) &= \frac{\alpha_s(Q)}{2\pi}\int_x^1 \frac{dy}{y}\left\{P_{qq}(y)F_q(\frac{x}{y},Q)+P_{qg}(y)F_g(\frac{x}{y},Q)\right\} \\
Q^2\frac{d}{dQ^2}F_{\bar q}(x,Q^2) &= \frac{\alpha_s(Q)}{2\pi}\int_x^1 \frac{dy}{y}\left\{P_{qq}(y)F_{\bar q}(\frac{x}{y},Q)+P_{qg}(y)F_g(\frac{x}{y},Q)\right\}
\end{aligned}
\tag{9.117}
$$

where

$$
\begin{aligned}
P_{qq}(y) &= C_F\left[\frac{1+y^2}{[1-y]_+}+\frac{3}{2}\delta(1-y)\right] \\
P_{gq}(y) &= C_F\left[\frac{1+(1-y)^2}{y}\right] \\
P_{qg}(y) &= \frac{1}{2}\Big[y^2+(1-y)^2\Big] \\
P_{gg}(y) &= 2N_c\left[\frac{1-y}{y}+\frac{y}{(1-y)_+}+y(1-y)+\frac{1}{12}\left(\frac{11}{3}N_c-\frac{2}{3}n_f\right)\delta(1-y)\right]
\end{aligned}
\tag{9.118}
$$

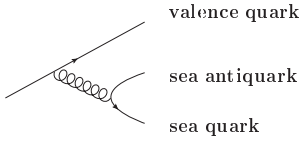
• Gluons only couple to $F_q(x) + F_{\bar{q}}(x)$ (sum of quarks and antiquarks). Thus we obtain

— a separate equation for $F_q(x) - F_{\bar{q}}(x)$ which is the same as we had before, and
— a coupled system of two equations for $F_q(x) + F_{\bar{q}}(x)$ and $F_g(x)$

For example for the moments

$$\begin{aligned} Q^2\frac{d}{dQ^2}\left(M_q^N - M_{\bar{q}}^N\right) &= -\frac{\alpha_s(Q)}{2\pi}\gamma_{qq}^N\left(M_q^N - M_{\bar{q}}^N\right) \\ Q^2\frac{d}{dQ^2}\begin{pmatrix} M_q^N + M_{\bar{q}}^N \\ 2M_g^N \end{pmatrix} &= -\frac{\alpha_s(Q)}{2\pi}\begin{pmatrix} \gamma_{qq}^N & \gamma_{qg}^N \\ \gamma_{gq}^N & \gamma_{gg}^N \end{pmatrix}\begin{pmatrix} M_q^N + M_{\bar{q}}^N \\ 2M_g^N \end{pmatrix} \end{aligned} \tag{9.119}$$

• A proton can be viewed as a collection of three "valence quarks" that carry the quantum numbers, extra quark-antiquark pairs that are called "sea quarks(antiquarks)", and gluons



$$q(x) = q_v(x) + q_s(x)\,, \quad \bar{q}(x) = \bar{q}_s(x) \tag{9.120}$$

Then

$$F_{q_v}(x) = F_q(x) - F_{\bar{q}}(x) \tag{9.121}$$

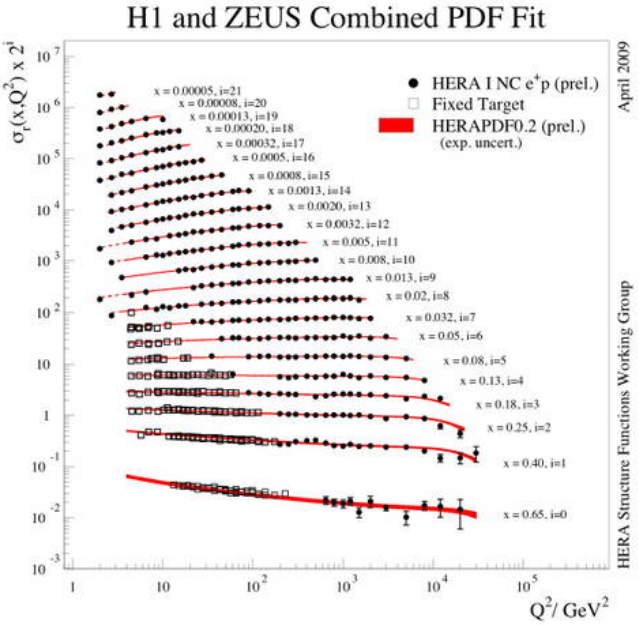
can be interpreted as the valence quark distribution.

(The number of valence quarks)/3 = baryon charge, so we expect that for any Q^2

$$\int_0^1 dx\,[F_q(x,Q^2) - F_{\bar{q}}(x,Q^2)] = 3 \tag{9.122}$$

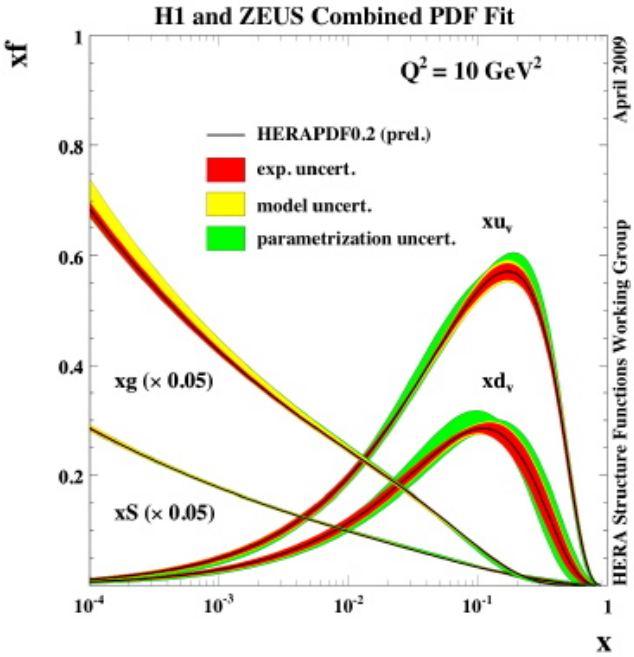
This is consistent with the evolution equation since

$$Q^2\frac{d}{dQ^2}\int_0^1 dx\,[F_q(x,Q^2) - F_{\bar{q}}(x,Q^2)] = -\frac{\alpha_s(Q)}{2\pi}\underbrace{\gamma_{qq}^{N=1}}_{\uparrow = 0!}\int_0^1 dx\,[F_q(x,Q^2) - F_{\bar{q}}(x,Q^2)] \tag{9.123}$$



Measured reaction rates (reduced cross sections) for neutral current processes, as a function of the scaling variable Q2 for different values of x, the proton momentum fraction carried by the partons. These results represent the combined analysis of H1 and ZEUS. Scaling violations, i.e. a Q2 - dependence of the rates, are clearly observed.

Presented by V. Radescu at Int. Conference DIS2009, April 2009, Madrid, Spain



Parton distribution functions of the proton as obtained by the HERA Structure Functions Working Group, at a scale of $Q^2 = 10$ GeV^2. The PDFs (solid lines) are shown separately for the gluon (xg), the sea quarks (xS), (both scaled down for visibility by a factor of 20), and the valence quarks up xuv and down xdv. The uncertainties are indicated by the coloured bands.

Presented by V. Radescu at Int. Conference DIS2009, April 2009, Madrid, Spain

Why is the gluon distribution so large for $x \to 0$?

$$P_{gg}(y) \simeq 2N_c \frac{1}{y} \tag{9.124}$$

The DGLAP equation is not in danger since it involves a cutoff $y > x$, but this behavior implies that

$$F_g(x) \overset{x\to 0}{\geq} \frac{1}{x} \tag{9.125}$$

so that the number of gluons in the proton is infinite:

$$\int_0^1 dx\, F_g(x) = \infty \tag{9.126}$$

Let us see why this happens. For $Q_1^2 \sim Q_0^2$ and small x

$$F_g(x,Q_1^2) - F_g(x,Q_0^2) \simeq \frac{3\alpha_s}{\pi} \ln\frac{Q_1^2}{Q_0^2} \int_x^1 \frac{dy}{y}\frac{1}{y} F_g(\frac{x}{y}, Q_0^2) \tag{9.127}$$

Try an ansatz

$$F_g(x,Q_0^2) = \text{ const} \tag{9.128}$$

Then

$$F_g(x,Q_1^2) = \text{ const} \left(1 + \frac{3\alpha_s}{\pi}\frac{1}{x}\ln\frac{Q_1^2}{Q_0^2}\right) \tag{9.129}$$

$\longleftarrow$ the gluon distribution cannot be "flat" at $x \to 0$, our ansatz was bad.

Second try:

$$F_g(x,Q_0^2) = \text{ const} \cdot \frac{1}{x}, \qquad \text{or} \qquad xF_g(x,Q_0^2) = \text{ const} \tag{9.130}$$

In this case obtain

$$xF_g(x,Q_1^2) = \text{ const} \left(1 + \frac{3\alpha_s}{\pi}\ln\frac{1}{x}\ln\frac{Q_1^2}{Q_0^2}\right) = xF_g(x,Q_0^2)\left(1 + \frac{3\alpha_s}{\pi}\ln\frac{1}{x}\ln\frac{Q_1^2}{Q_0^2}\right) \tag{9.131}$$

One can show that this structure is general: In the limit $x \to 0$ and $Q^2 \to \infty$ each power of α_s is accompanied by two logarithms:

$$1 + c_1\alpha_s \ln\frac{1}{x}\ln\frac{Q_1^2}{Q_0^2} + c_2\left(\alpha_s \ln\frac{1}{x}\ln\frac{Q_1^2}{Q_0^2}\right)^2 + \ldots \tag{9.132}$$

It is possible to sum this series to all orders (the so-called double-log approximation)

$$xF_g(x,Q^2)\Big|_{\substack{x\to 0\\ Q^2\to\infty}} \sim \exp\sqrt{\frac{48}{b}\ln\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\ln\frac{1}{x}} \tag{9.133}$$

What to do if $x \to 0$ but $Q^2/Q_0^2 \sim 1$?

In higher orders

$$\frac{\alpha_s}{2\pi}P_{gg}(x)\Big|_{x\to 0} = \frac{3\alpha_s}{\pi x}\left[1 + 9\zeta(3)\left(\frac{\alpha_s}{\pi}\ln\frac{1}{x}\right)^3 + \frac{81}{20}\zeta(5)\left(\frac{\alpha_s}{\pi}\ln\frac{1}{x}\right)^5 + \mathcal{O}\left[\left(\frac{\alpha_s}{\pi}\ln\frac{1}{x}\right)^6\right]\right] \tag{9.134}$$

$\hookleftarrow$ this series is known to all orders $\longleftarrow$ BFKL equation

Resummation

$$\sum_k (\alpha_s \ln 1/x)^k \qquad\qquad xF_g(x,Q^2) \sim x^{-12\ln 2\frac{\alpha_s}{\pi}} \tag{9.135}$$

Too strong! What stops this rise? — Active field of research

10 Effective field theories

Remember the background field method:
Separate formally all field operators in "fast" and "slow" components:

$$\phi(x) = \left(\int\limits_{|k|>\mu} \frac{d^3k}{(2\pi)^3 2E_k} + \int\limits_{|k|<\mu} \frac{d^3k}{(2\pi)^3 2E_k} \right) \left[\hat{a}^\dagger(k) e^{ikx} + \hat{a}(k) e^{-ikx} \right] \; = \; \phi_{\rm fast}(x) + \phi_{\rm slow}(x) \quad (10.1)$$

so for quarks and gluons

$$\psi(x) = \psi_{\rm fast}(x) + \psi_{\rm slow}(x)\,, \qquad\qquad A^\mu(x) = A^\mu_{\rm fast}(x) + A^\mu_{\rm slow}(x) \quad (10.2)$$

The QCD Lagrangian

$$\mathcal{L}_{\rm QCD}(\phi_{\rm fast} + \phi_{\rm slow}) = \mathcal{L}_{\rm QCD}(\phi_{\rm fast}) + \mathcal{L}_{\rm QCD}(\phi_{\rm slow}) + \delta\mathcal{L}_{\rm I}(\phi_{\rm fast}, \phi_{\rm slow}) \quad (10.3)$$

where $\delta\mathcal{L}_{\rm I}(\phi_{\rm fast}, \phi_{\rm slow})$ describes interaction between fast and slow modes. This construction ia useful if one asks the question: what happens if we change the UV cutoff in our theory $\Lambda_{\rm UV} \to M \ll \Lambda_{\rm UV}$ such that we "integrate out" the fast fields. This terminology actually refers to the path-integral formulation of a field theory (next semester). Schematically, all correlation functions in the theory can be generated from the so-called generating functional

$$Z(J) \equiv \int \mathcal{D}\phi \exp\left[i \int d^4x \left[\mathcal{L}(\phi(x) + J(x)\phi_{\rm slow}(x) \right] \right] \quad (10.4)$$

where $J(x)$ are some external sources that we assume only couple to slow fields. The statement that fast and slow fields can be treated as independent means that

$$\mathcal{D}\phi = \mathcal{D}\phi_{\rm slow}\mathcal{D}\phi_{\rm fast} \equiv \mathcal{D}\phi_{\rm s}\mathcal{D}\phi_{\rm f}$$

and we can formally integrate (take an average) over all fast fields leading to

$$\begin{aligned} Z(J) &= \int \mathcal{D}\phi_{\rm s}\mathcal{D}\phi_{\rm f} \exp\left[i \int d^4x \left[\mathcal{L}(\phi_{\rm s}) + \mathcal{L}(\phi_{\rm f}) + \delta\mathcal{L}(\phi_{\rm s}, \phi_{\rm f}) + J(x)\phi_{\rm s}(x) \right] \right] \\ &= \int \mathcal{D}\phi_{\rm s} \exp\left[i \int d^4x \left[\mathcal{L}_{\rm eff}(\phi_{\rm s}) + J(x)\phi_{\rm s}(x) \right] \right] \end{aligned} \quad (10.5)$$

where $\mathcal{L}_{\rm eff}(\phi_{\rm s})$ is called effective Lagrangian. In our previous discussion we tacitly assumed that $\mathcal{L}_{\rm eff}(\phi_{\rm s})$ for QCD is just the usual QCD Langranqian. This is not entirely correct: there are in fact corrections suppressed as a power of μ but these corrections are small if all external momenta are $|p| \ll \mu$ and was sufficient for our purposes.

Can we say smth about $\mathcal{L}_{\rm eff}(\phi_{\rm s})$ without a calculation? Yes, if we have a reason to claim that our effective theory (with smaller cutoff) has to have some symmetries. In particular in QCD case if we require that the effective theory is gauge- and Lorentz invariant, the only terms that we might add to the usual Lagrangian are scalar higher-dimensional gauge-invariant operators, say $(\bar{q}q)^2$ or $F_{\mu\nu}F^{\nu\alpha}F^\mu_\alpha$ that have to be accompanied by $1/\mu^2$ by dimension counting reasons.

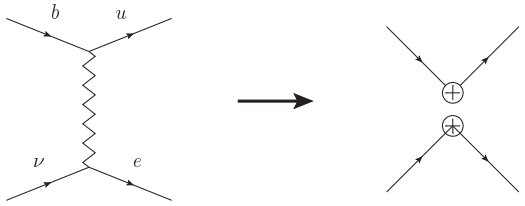
In what follows I will sketch some examples that are somewhat more general and the effective theories are different of the original ones in essential way. The effective field theory (EFT) construction has become the main technical tool in making advanced calculations in QFT and has also changed our way of thinking of what is a "renormalizable" theory.

10.1 Effective weak interactions

Weak interactions between quarks and leptons are mediated by very heavy particles so that their propagators at small momenta can be expanded

$$\frac{1}{m_W^2 - p^2} \simeq \frac{1}{m_W^2}\Big[1 + \frac{p^2}{M_W^2} + \dots\Big]$$

and the leading terms give rize to local four-fermion interactions:



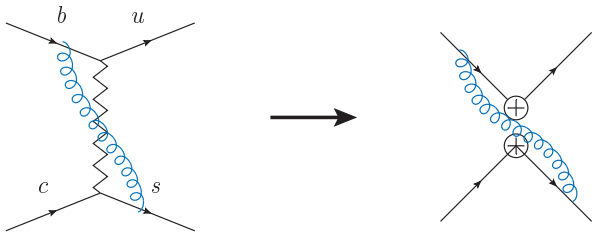
$$\mathcal{L}_{weak}^{eff} = -\frac{g^2}{8M_W^2}\left[J_\mu^- J^{+\mu} + \frac{1}{M_W^2} J_\mu^- (\partial^\mu\partial^\nu - g_{\mu\nu}\partial^2) J^{+\nu} + \dots\right] \tag{10.6}$$

where

$$J_\mu^+ = V_{ub}\bar{u}\gamma_\mu(1-\gamma_5)b + \bar{\nu}\gamma_\mu(1-\gamma_5)e + \dots,, \qquad J_\mu^- = \left(J_\mu^+\right)^\dagger \tag{10.7}$$

$V_{ub} = 0.0035\pm?$ is the so-called CKM mixing angle and $g^2/8M_W^2 \equiv G_F/\sqrt{2}$ defines the Fermi constant $G_F = 1.16639 \cdot 10^{-5}\,\mathrm{GeV}^{-2}$. The corrections in $1/M_W^2$ are almost always too small to be relevant but what happens if we switch on strong interactions (QCD)?

Consider a somewhat more interesting case $b \to u\bar{c}s$ that corresponds to e.g. the decay $\bar{B}^0 \to \pi^+ D_S^-$ and add one gluon exchange, e.g.,



At tree level a minor difference

$$\mathcal{L}_{weak}^{tree} = -\frac{4G_F}{\sqrt{2}} V_{cs}^* V_{ub} (\bar{s}_L^a \gamma_\mu c_L^a)(\bar{u}_L^b \gamma^\mu b_L^b)\,, \qquad q_L = (1-\gamma_5)q\,, \qquad a,b \leftarrow \text{color} \tag{10.8}$$

but with a gluon it easy to see that the left diagram is finite whereas the right one is UV divergent. Why so? Because

$$\int d^4p \frac{1}{M_W^2 - p^2} f(p) \neq \frac{1}{M_W^2}\int d^4p \left(1 + \frac{p^2}{M_W^2} + \dots\right) f(p) \tag{10.9}$$

The difference arises because this expansion does not hold for large loop momentum $p \sim M_W$.

The physical reason is that M_W^2 acts as an UV regulator in the (box) diagram in the full theory so that we have a familiar situation

$$\underbrace{1+\alpha_s \ln \frac{M_W^2}{-k^2}}_{\text{full theory}} = \underbrace{\left(1+\alpha_s \ln \frac{M_W^2}{\mu^2}\right)}_{C(\mu)} \cdot \underbrace{\left(1+\alpha_s \ln \frac{\mu^2}{-p^2}\right)}_{\text{effective theory}} \tag{10.10}$$

where k is some external momentum.

The QCD coupling is small at large momenta and therefore $C(\mu)$ can be calculated in perturbation theory. One obtains

$$\mathcal{L}_{weak}^{eff} = -\frac{4G_F}{\sqrt{2}} V_{cs}^* V_{ub} \left[C_1(\mu)[(\bar{s}_L \gamma_\xi c_L)(\bar{u}_L \gamma^\xi b_L)]_\mu + C_2(\mu)[(\bar{s}_L \gamma_\mu b_L)(\bar{u}_L \gamma^\mu c_L)]_\mu \right] \tag{10.11}$$

with

$$\begin{aligned} C_1(\mu) &= 1 + \frac{3}{N_c}\frac{\alpha_s(\mu)}{4\pi}\left(\ln\frac{M_W^2}{\mu^2} - \frac{11}{6}\right) + \mathcal{O}(\alpha_s^2) \\ C_2(\mu) &= 0 - 3\frac{\alpha_s(\mu)}{4\pi}\left(\ln\frac{M_W^2}{\mu^2} - \frac{11}{6}\right) + \mathcal{O}(\alpha_s^2) \end{aligned} \tag{10.12}$$

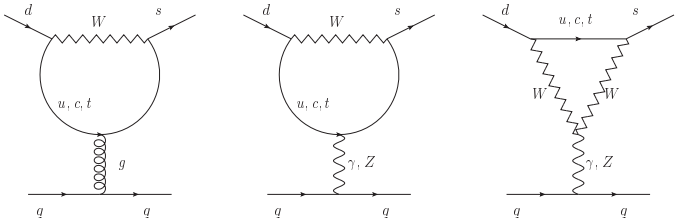
— Note that at $\mathcal{O}(\alpha_s)$ a new operator arises that was not present at tree level.

Can there be more operators? Yes! And the most interesting are the ones that lead to physical processes that are forbidden at tree level - Flavor Changing Neutral Currents (FCNC).

Main principle: every operator that is not forbidden by symmetries will appear sooner or later (in some order of PT). In addition to the above, we might have additional dimension-6 operators

$$\begin{aligned} &(\bar{s}_L \gamma_\mu b_L)(\bar{q}_R \gamma^\mu q_R) && (\bar{s}_L \gamma_\mu q_R)(\bar{q}_R \gamma^\mu b_L) \\ &g_s m_b \bar{s}_L \sigma_{\mu\nu} G^{\mu\nu} b_R && e m_b \bar{s}_L \sigma_{\mu\nu} F^{\mu\nu} b_R \quad \leftarrow \text{photon} \end{aligned} \tag{10.13}$$

and also more operators involving leptons. These are typical diagrams that give rize to such contributions:



— "penguin" operators

In addition we will have a large amount of dimension 7,8,9... operators but they are suppressed by powers of $1/M_W$ and usually irrelevant **unless** the corresponding process is forbidden at dimension-6 level. This can be used to probe whether the Standard Model is the correct theory at small distances.

10.2 Heavy Quark Effective Theory

In the effective weak Lagrangian we were able to integrate out contributions of heavy W, Z bosons because we tacitly assumed that they are not present in the initial and final state. If heavy particles are present the above construction has to be modified. This is practically relevant for processes involving B-mesons where mass of the b quark $m_B \sim 4.8\,\text{GeV} \gg \Lambda_{\text{QCD}}$ and one can try an expansion in $\Lambda_{\text{QCD}}/m_b \sim 1/20$ to make quantitative predictions.

Consider an ingoing heavy quark with the four momentum $p_\mu = mv_\mu$, v_μ is a four-velocity vector $v^2 = 1$, that starts interacting with gluons with $|k| << m$ such that its momentum changes but not very strongly:

$$p_\mu = mv_\mu + \hat{p}_\mu\,, \qquad |\hat{p}_\mu| \ll m$$

The free propagator is then approximately

$$\frac{m + m\not{v} + \not{\hat{p}}}{m^2 - (mv + \hat{p})^2 - i\epsilon} \simeq \left(\frac{1+\not{v}}{2}\right) \frac{1}{-p\cdot v + i\epsilon} \tag{10.14}$$

Note that

$$P_+ = \frac{1+\not{v}}{2}\,, \qquad P_- = \frac{1-\not{v}}{2}\,, \tag{10.15}$$

are projection operators:

$$P_+^2 = P^+\,, \qquad P_-^2 = P_-\,, \qquad P_+P_- = 0 \tag{10.16}$$

and for the simplest choice $v = (1,0,0,0)$ are simply the projections onto the two upper/lower components of the Dirac bispinor (as can be expected).

The interaction is simplified as well because the vertex is always surrounded by two propagators

$$\left(\frac{1+\not{v}}{2}\right)\gamma_\mu\left(\frac{1+\not{v}}{2}\right) = v_\mu\left(\frac{1+\not{v}}{2}\right) \tag{10.17}$$

so that effectively $\gamma_\nu \to v_\mu$.

How to construct the corresponding effective theory framework? Let Q be the heavy quark field, then QCD action can be written as

$$S_{\text{QCD}} = \int d^4x \left\{\bar{Q}(i\not{D} - m_Q)Q + \underbrace{\mathcal{L}^{light}_{QCD}}_{\text{gluons+light quarks}}\right\} \tag{10.18}$$

Define

$$Q_+(x) = \frac{1}{2}(1+\not{v})Q(x)\,, \qquad Q_-(x) = \frac{1}{2}(1-\not{v})Q(x) \tag{10.19}$$

and also split the covariant derivative in contributions along and orthogonal to v:

$$\not{D} = \not{v}(v\cdot D) + \not{D}^\perp, \qquad \not{D}^\perp = \gamma^\mu\left(g_{\mu\nu} - v_\mu v_\nu\right)D^\nu \equiv \gamma^\mu g^\perp_{\mu\nu} D^\nu \tag{10.20}$$

Useful:

$$\{\not{D}^\perp, \not{v}\} = 0\,, \qquad \{\not{D}^\perp, P_+\} = \not{D}^\perp\,. \tag{10.21}$$

The heavy-quark part of the action then becomes

$$S^{\text{heavy}} = \int d^4x \left\{ \bar{Q}_+[i(vD) - m_Q]Q_+ - \bar{Q}_-[i(vD) + m_Q]Q_- + \bar{Q}_+[i\not{D}^\perp]Q_- + \bar{Q}_-[i\not{D}^\perp]Q_+ \right\} \tag{10.22}$$

Field redefiniton — remove rapid oscillations (this is just a reparametrization)

$$Q_+(x) = e^{-im_Q(vx)}h_V\,, \qquad\qquad Q_-(x) = e^{-im_Q(vx)}H_V\,, \tag{10.23}$$

Then

$$S^{\text{heavy}} = \int d^4x \left\{ \bar{h}_v i(vD)h_v - \bar{H}_v[i(vD) + 2m_Q]H_v + \bar{h}_v[i\not{D}^\perp]H_v + \bar{H}_v[i\not{D}^\perp]h_+ \right\} \tag{10.24}$$

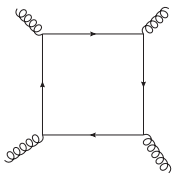
Suppose external sources only create h_v or $\bar{h}_v$, that is the heavy quarks involved in the process have large P_+ component. Then the H_v fields can be integrated out:

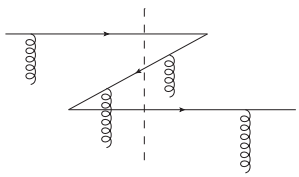
$$\begin{aligned} S^{\text{eff}}_{\text{heavy}} &= \int d^4x \left\{ \bar{h}_v i(vD)h_v - \bar{h}_v[i\not{D}^\perp] \left(\frac{1}{i(vD) + 2m_Q - i\epsilon} \right) [i\not{D}^\perp]h_v \right\} \\ &= \int d^4x \left\{ \bar{h}_v i(vD)h_v + \frac{1}{2m_Q}K_v - \frac{1}{2m_Q}\Sigma_v + \mathcal{O}(1/m_Q^2) \right\} \end{aligned} \tag{10.25}$$

where

$$\begin{aligned} K_v &= \bar{h}_v[iD^\mu g^\perp_{\mu\nu} iD^\nu]h_v\,, && \Leftarrow \text{NR kinetic energy} \\ \Sigma_v &= \frac{1}{2}\bar{h}_v \sigma_{\mu\nu} g G^{\mu\nu} h_v\,, && \Leftarrow \text{Pauli term} \end{aligned} \tag{10.26}$$

Difference to the above: here we integrate out Z-graphs:



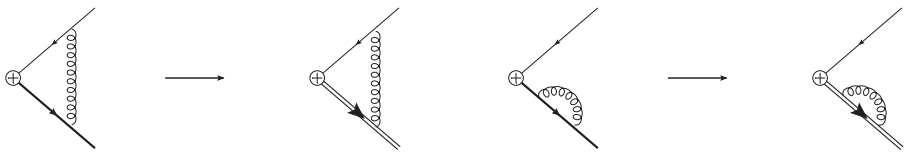


Feynman rules:

$$\text{Propagator:} \qquad \overbrace{h_v(x)\bar{h}_v}(0) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{i}{vk + i\epsilon} = \int_0^\infty ds\delta^{(4)}(x - sv) = \theta(vx)\delta^{(3)}_\perp(x)$$

$$\text{Vertex:} \qquad igt^a\gamma_\mu \mapsto igt^a v_\mu \tag{10.27}$$

Matching to the full theory, heavy-light currents:



$$\bar{q}\gamma_\mu Q = \left(1 - C_F\frac{\alpha_s(m_Q)}{\pi}\right)\left[\bar{q}\gamma_\mu h_v\right]^{\mu=m_Q} + \frac{1}{2}C_F\frac{\alpha_s}{\pi}\left[\bar{q}v_\mu h_v\right]^{\mu=m_Q} + \frac{1}{2m_b}\left[\bar{q}\gamma_\mu i\vec{\not{D}}h_v\right]^{\mu=m_Q}$$
$$\bar{q}\gamma_\mu\gamma_5 Q = \left(1 - C_F\frac{\alpha_s(m_Q)}{\pi}\right)\left[\bar{q}\gamma_\mu\gamma_5 h_v\right]^{\mu=m_Q} - \frac{1}{2}C_F\frac{\alpha_s}{\pi}\left[\bar{q}v_\mu\gamma_5 h_v\right]^{\mu=m_Q} + \frac{1}{2m_b}\left[\bar{q}\gamma_\mu\gamma_5 i\vec{\not{D}}h_v\right]^{\mu=m_Q} \tag{10.28}$$

Anomalous dimensions:

$$\left[\bar{q}\Gamma h_v\right]^{m_Q} = \left(\frac{\alpha_s(m_Q)}{\alpha_s(\mu)}\right)^{\gamma_0/2\beta_0}\left[\bar{q}\Gamma h_v\right]^{m_Q}, \qquad \gamma_0 = -4, \qquad b = \frac{11}{3}N_c - \frac{2}{3}n_f \tag{10.29}$$

How can one use this? — e.g. to compare decay constants for different heavy mesons

$$\langle 0|\bar{q}\gamma_\mu\gamma_5 c|D(p)\rangle = ip_\mu\frac{f_D}{\sqrt{m_D}} \qquad \text{vs.} \qquad \langle 0|\bar{q}\gamma_\mu\gamma_5 b|B(p)\rangle = ip_\mu\frac{f_B}{\sqrt{m_B}} \tag{10.30}$$

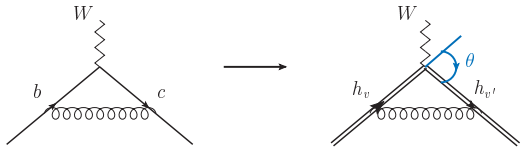
can be measured (difficult) in $D^- \to \mu^-\bar{\nu}_\mu$ and $B^- \to \tau^-\bar{\nu}_\tau$. Neglecting terms $1/m_b, 1/m_c$ we can predict

$$\frac{f_B}{f_D} = \sqrt{\frac{m_D}{m_B}}\left(\frac{\alpha_s(m_c)}{\alpha_s(m_b)}\right)^{6/25}\left(1 + 0.894\frac{\alpha_s(m_c) - \alpha_s(m_b)}{\pi}\right) \simeq 0.69 \tag{10.31}$$

Unfortunately c-quark is not sufficiently heavy so that this does not work very well.

Heavy-to-heavy transitions

E.g. $B \to De\nu$ or $B \to D^*e\nu$: $b \to ce\nu$ on quark level; Involves V_{cb}



Note that each heavy quark with a different velocity gives rise to a new field in the effective Lagrangian so that for such processes

$$S^{\text{eff}}_{\text{heavy}} = \int d^4x\left\{\bar{h}_v i(vD)h_v + \bar{h}_{v'}i(v'D)h_{v'} + \ldots\right\} \tag{10.32}$$

Soft gluons cannot change quark velocity so that h_v cannot turn to $h_{v'}$ unless there is some external force that enforces this transition (weak decay!). One finds

$$\left[\bar{h}_v\Gamma h_{v'}\right]^{\mu_1} = \left(\frac{\alpha_s(\mu_1)}{\alpha_s(\mu_2)}\right)^{\gamma_0^{hh}/2\beta_0}\left[\bar{h}_v\Gamma h_{v'}\right]^{\mu_2} \tag{10.33}$$

Cusp anomalous dimension (Polyakov, 1980)

$$\Gamma_{\text{cusp}}(\theta,\alpha_s) = \frac{\alpha_s}{\pi}C_F\Big(\theta\coth\theta - 1\Big) + \ldots = \frac{\alpha_s}{4\pi}\gamma_0^{hh} + \mathcal{O}(\alpha_s^2), \qquad \boxed{\cosh\theta = v\cdot v'} \tag{10.34}$$

From $B \to D\ell\nu_\ell$ and $B \to D^*\ell\nu_\ell$ one obtains

$$|V_{cb}| = 0.0405 \pm 0.0015 \qquad \text{vs.} \quad |V_{cb}| = 0.0422 \pm 0.0008 \quad \text{inclusive decays} \tag{10.35}$$

10.3 The strategy of regions

The strategy of regions is a technique which allows one to carry out asymptotic expansions of loop integrals in dimensional regularization around various limits. The expansion is obtained by splitting the integration in different regions and appropriately expanding the integrand in each case. We will later formulate an effective theory, called SCET, where the different regions will be represented by different effective theory fields. The expanded integrals obtained by means of the strategy of regions technique are in one-to-one correspondence to the Feynman diagrams of effective field theories regularized in dimensional regularization. My presentation follows a book by T.Becher, A.Broggio, A.Ferroglia: *Introduction to Soft-Collinear Effective Theory*. For a higher-level discussion see B. Jantzen, arXiv:1111.2589.

10.3.1 A simple example

In order to illustrate the main idea of the strategy of regions we start by considering a simple integral, which we will expand using different methods, first using a cutoff to separate two different regions and then with dimensional regularization:

$$I = \int_0^\infty dk \frac{k}{(k^2+m^2)(k^2+M^2)} = \frac{1}{M^2-m^2} \ln \frac{M}{m} \,. \tag{10.36}$$

We assume $m^2 \ll M^2$ and will discuss an expansion in the small parameter m/M. Obviously

$$I = \frac{1}{M^2} \left(1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \cdots \right) \ln \frac{M}{m} \,. \tag{10.37}$$

Our goal is to reproduce this result by expanding the integrand in Eq. (10.36) before carrying out the integration. This is helpful in cases where the full result is not available.

A naive expansion of the integrand

$$\frac{k}{(k^2+m^2)(k^2+M^2)} = \frac{k}{k^2(k^2+M^2)} \left(1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} + \cdots \right) \tag{10.38}$$

does not work as it gives rise to IR divergent integrals. This was to be expected: If one could simply Taylor expand the integrand in m/M and integrate term by term, the result would necessarily be an analytic function of m because the integrals would simply give the Taylor coefficients of the expansion in m. But the result for I is not analytic in m/M, it contains a logarithm.

The problem with the naive expansion is obviously that it is valid only for $k \gg m^2$ while the integration domain in Eq. (10.36) includes a region in which $k^2 \sim m^2$. As a warm up, we can avoid this problem introducing a scale $m \ll \Lambda \ll M$ to separate the two regions:

$$I = \underbrace{\int_0^\Lambda dk \frac{k}{(k^2+m^2)(k^2+M^2)}}_{I_{(I)}} + \underbrace{\int_\Lambda^\infty dk \frac{k}{(k^2+m^2)(k^2+M^2)}}_{I_{(II)}} \,. \tag{10.39}$$

In the first region we use that $k, m \ll M$ so that

$$I_{(I)} = \int_0^\Lambda dk \frac{k}{(k^2+m^2)(k^2+M^2)} = \int_0^\Lambda dk \frac{k}{(k^2+m^2)M^2} \left(1 - \frac{k^2}{M^2} + \frac{k^4}{M^4} + \cdots \right) \,. \tag{10.40}$$

In the second region we use $m \ll k, M$ so that

$$I_{(II)} = \int_\Lambda^\infty dk \frac{k}{(k^2+m^2)(k^2+M^2)} = \int_\Lambda^\infty dk \frac{k}{k^2(k^2+M^2)} \left(1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} + \cdots\right) . \tag{10.41}$$

Taking into account the first two terms in $I_{(I)}$ and the leading term only in $I_{(II)}$

$$I_{(I)} \simeq \frac{M^2+m^2}{2M^4} \ln\left(1+\frac{\Lambda^2}{m^2}\right) - \frac{\Lambda^2}{2M^4} = -\frac{1}{M^2}\ln\left(\frac{m}{\Lambda}\right) - \frac{\Lambda^2}{2M^4} + \mathcal{O}\left(\frac{\Lambda^4}{M^6}, \frac{m^2}{M^4}\log\left(\frac{\Lambda}{m}\right)\right) ,$$

$$I_{(II)} \simeq \frac{1}{2M^2} \ln\left(1+\frac{M^2}{\Lambda^2}\right) = -\frac{1}{M^2}\ln\left(\frac{\Lambda}{M}\right) + \frac{\Lambda^2}{2M^4} + \mathcal{O}\left(\frac{\Lambda^4}{M^6}\log\left(\frac{M}{\Lambda}\right)\right) . \tag{10.42}$$

and summing up

$$I = I_{(I)} + I_{(II)} = -\frac{1}{M^2}\ln\left(\frac{m}{M}\right) + \mathcal{O}\left(\frac{m^2}{M^4}\log\left(\frac{M}{m}\right)\right) , \tag{10.43}$$

which is the expected result. All terms depending on Λ cancel as they must.

Thus the trick works, but it is well known that the use of hard cutoffs is impractical in calculations of Feynman diagrams (apart from the simplest cases). We want to find out whether a separation of different integration regions can be achieved using dimensional regularization. To this end, consider

$$I = \int_0^\infty dk\, k^{-\varepsilon} \frac{k}{(k^2+m^2)(k^2+M^2)} , \tag{10.44}$$

where we will eventually send $\varepsilon \to 0$ at the end of the calculation.

Using a low-energy expansion of the integrand, $k, m \ll M$, as in (10.40)

$$I_{(I)} = \int_0^\infty dk\, k^{-\varepsilon} \frac{k}{(k^2+m^2)M^2} \left(1 - \frac{k^2}{M^2} + \frac{k^4}{M^4} + \cdots\right) . \tag{10.45}$$

Here, for each term separately, we can choose the dimensional regulator $\varepsilon > 0$ such that the integral will converge both for $k \to 0$ (IR finite) and for $k \to \infty$ (UV finite).

Similarly, by performing a large-energy expansion $k \gg \Lambda$ (cf. (10.41)) one obtains

$$I_{(II)} = \int_0^\infty dk\, k^{-\varepsilon} \frac{k}{k^2(k^2+M^2)} \left(1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} + \cdots\right) . \tag{10.46}$$

and we can choose $\varepsilon < 0$, to avoid IR divergences in the region where $k \to 0$. Taking into account the first terms only one finds

$$I_{(I)} = \frac{m^{-\varepsilon}}{2M^2}\Gamma\left(1-\frac{\varepsilon}{2}\right)\Gamma\left(\frac{\varepsilon}{2}\right) = \frac{1}{M^2}\left(\frac{1}{\varepsilon} - \ln m + \mathcal{O}(\varepsilon)\right) . \tag{10.47}$$

$$I_{(II)} = -\frac{M^{-\varepsilon}}{2M^2}\Gamma\left(1-\frac{\varepsilon}{2}\right)\Gamma\left(\frac{\varepsilon}{2}\right) = \frac{1}{M^2}\left(-\frac{1}{\varepsilon} + \ln M + \mathcal{O}(\varepsilon)\right) . \tag{10.48}$$

The poles in ε cancel in the sum, and the final result is again as expected!

This looks like magic, because at first sight there are at least two suspect issues:

• First: can we choose $\varepsilon > 0$ in the low-energy region and $\varepsilon < 0$ in the high-energy region and then combine the two?
— This is legitimate, because dimensionally regularized expressions are defined for *arbitrary* ε: we only choose $\varepsilon > 0$ to be able to evaluate $I_{(I)}$ as a standard integral, but by analytic continuation the resulting function on the right-hand side is uniquely defined for any complex-valued ε and can be combined with $I_{(II)}$.
• Second: The integration domain in both Eq. (10.45) and Eq. (10.46) is not restricted to a low/high energy region. Since we integrate the high-energy part over the low-energy region (and vice versa), one could worry that this leads to a double counting?
— To see that this does not happen, observe that the low-energy integral $I_{(I)} \sim m^{-\varepsilon}$, while the high-energy integral $I_{(II)} \sim M^{-\varepsilon}$. This statement remains true also for the subleading terms. Keeping the complete dependence on m and M the result for our integral (for finite ε) is

$$I = \frac{1}{2}\Gamma\left(1-\frac{\varepsilon}{2}\right)\Gamma\left(\frac{\varepsilon}{2}\right)\,\frac{m^{-\varepsilon}-M^{-\varepsilon}}{M^2-m^2}\,. \tag{10.49}$$

and the low-energy/high-energy parts just pick up the pieces $\sim m^{-\varepsilon}$ and $\sim M^{-\varepsilon}$, respectively. Even though we integrate twice over the full integration domain, there is no double counting, since the two pieces scale differently: the low-energy integrals can never produce a term $M^{-\varepsilon}$ since they depend analytically on the large scale, and vice-versa.

— Let us see what happens if we insist in restricting the integration domain of the low- and high-energy region integrals when using dimensional regularization. The integral in the low-energy region would become in this case

$$\begin{aligned} I^{\Lambda}_{(I)} &= \int_0^{\Lambda} dk\, k^{-\varepsilon}\frac{k}{(k^2+m^2)M^2}\left(1-\frac{k^2}{M^2}+\frac{k^4}{M^4}+\dots\right) \\ &= \left[\int_0^{\infty} dk - \int_{\Lambda}^{\infty} dk\right] k^{-\varepsilon}\frac{k}{(k^2+m^2)M^2}\left(1-\frac{k^2}{M^2}+\frac{k^4}{M^4}+\dots\right) \\ &= I_{(I)} - R_{(I)}\,. \end{aligned} \tag{10.50}$$

To calculate the remainder $R_{(I)}$ we can use that $k \geq \Lambda \gg m^2$ to expand in the small m limit:

$$\begin{aligned} R_{(I)} &= \int_{\Lambda}^{\infty} dk\, k^{-\varepsilon}\frac{k}{(k^2+m^2)M^2}\left(1-\frac{k^2}{M^2}+\dots\right) \\ &= \int_{\Lambda}^{\infty} dk\, k^{-\varepsilon}\frac{k}{k^2M^2}\left(1-\frac{m^2}{k^2}-\frac{k^2}{M^2}+\dots\right)\,. \end{aligned} \tag{10.51}$$

Note that in this way we performed two expansions already: First, expanding the integrand in the limit $M \to \infty$ to separate part $I_{(I)}$, and, second, expanding the result for $m \to 0$.

Similarly, consider the high-energy integral $I_{(II)}$ in Eq. (10.46) with a lower cutoff Λ on the integration. As above, it can be written as an integral without a cutoff and a remainder

$$\begin{aligned} R_{(II)} &= \int_0^{\Lambda} dk\, k^{-\varepsilon}\frac{k}{k^2(k^2+M^2)}\left(1-\frac{m^2}{k^2}+\dots\right) \\ &= \int_0^{\Lambda} dk\, k^{-\varepsilon}\frac{k}{k^2M^2}\left(1-\frac{m^2}{k^2}-\frac{k^2}{M^2}+\dots\right)\,. \end{aligned} \tag{10.52}$$

In this remainder, we have again expanded the integrand in both the limit of small m and also in the limit of large M, but in the opposite order as in $R_{(I)}$. However, the two expansions commute so that the integrands of $R_{(I)}$ and $R_{(II)}$ are identical. Adding up the two pieces, we find that

$$R = R_{(I)} + R_{(II)} = \int_0^\infty dk\, k^{-\varepsilon} \frac{k}{k^2 M^2} \left(1 - \frac{m^2}{k^2} - \frac{k^2}{M^2} + \dots\right) . \tag{10.53}$$

The dependence on Λ disappears, as expected, and the remaining integrals are of the type

$$\int_0^\infty \frac{dk}{k} \frac{1}{k^{n+\varepsilon}} =? \qquad n = 0, \pm 2, \dots \tag{10.54}$$

Integrals of this type are called *scaleless integrals* as they do not involve dimensionful parameters.

Take $n = 0$ as example (the leading term). As written, the integral is ill-defined since a nonzero $\varepsilon > 0$ does not help to eliminate the divergence for $k \to 0$ and $k \to \infty$ simultaneously. To define this integral properly one needs to introduce an additional regulator — e.g. restore the scale separation

$$\int_0^\infty \frac{dk}{k}\, k^{-\epsilon} = \int_0^\Lambda \frac{dk}{k}\, k^{-\epsilon} + \int_\Lambda^\infty \frac{dk}{k}\, k^{-\epsilon} \tag{10.55}$$

such that the first and the second pieces can be regularized choosing $\varepsilon = \varepsilon_{IR} < 0$ or $\varepsilon = \varepsilon_{UV} > 0$, respectively:

$$\int_0^\Lambda \frac{dk}{k}\, k^{-\epsilon_{IR}}\Big|_{\varepsilon_{IR}<0} = -\frac{1}{\varepsilon_{IR}} \Lambda^{-\varepsilon_{IR}} , \text{ and } \qquad \int_\Lambda^\infty \frac{dk}{k}\, k^{-\epsilon_{UV}}\Big|_{\varepsilon_{UV}>0} = +\frac{1}{\varepsilon_{UV}} \Lambda^{-\varepsilon_{UV}} \tag{10.56}$$

The main point is that by analytic continuation the resulting functions on the r.h.s. are uniquely defined for any complex-valued ε so that we can take $\varepsilon_{IR} = \varepsilon_{UV}$ at the end, such that the whole integral vanishes. This observation is crucial for the applications of dimensional regularization:

all scaleless integrals can be put to zero

To summarize, the remainder (10.53) vanishes because it is given by a series of scaleless integrals, so that there is no double counting.

10.3.2 Kinematics: light-cone coordinates

For particles with large energy moving along predominantly z (or $-z$) direction, the usual decomposition of momenta/coordinates as $p_\mu = (p_0, p_x, p_y, p_z)$ is not convenient and one usually goes over to

$$(p_0, p_x, p_y, p_z) \mapsto (\underbrace{p_0 + p_z}_{p_+}, \underbrace{p_0 - p_z}_{p_-}, \underbrace{p_x, p_y}_{\vec{p}_\perp}) \tag{10.57}$$

This can be formalized introducing light-like vectors n and $\bar{n}$ defined as

$$n_\mu = (1, 0, 0, 1) , \quad \text{and} \quad \bar{n}_\mu = (1, 0, 0, -1) , \tag{10.58}$$

so that

$$n^2 = n \cdot n = 0 , \qquad \bar{n}^2 = \bar{n} \cdot \bar{n} = 0 , \qquad n \cdot \bar{n} = 2 . \tag{10.59}$$

For each four-vector p^μ we can define the *four-vectors* p_+^μ and p_-^μ by using the light-like vectors n and $\bar{n}$:

$$p_+^\mu \equiv (n\cdot p)\,\frac{\bar{n}^\mu}{2}\,, \qquad p_-^\mu \equiv (\bar{n}\cdot p)\,\frac{n^\mu}{2}\,, \tag{10.60}$$

so that the vector p^μ can be written in terms of these vectors or as a list of components in the light-cone basis defined by n and $\bar{n}$:

$$p^\mu = p_+^\mu + p_-^\mu + p_\perp^\mu \equiv (n\cdot p\,, \bar{n}\cdot p\,, \vec{p}_T)\ . \tag{10.61}$$

The four-vector $p_\perp^\mu$ is related to *two-dimensional* vector $\vec{p}_T$ by $p_\perp^2 = -|\vec{p}_T|^2 \equiv -p_T^2$.

The scalar product of two vectors p^μ and q^μ can then be written as

$$p\cdot q = p_+\cdot q_- + p_-\cdot q_+ + p_\perp\cdot q_\perp = \frac{1}{2}\,(n\cdot p)\,(\bar{n}\cdot q) + \frac{1}{2}\,(n\cdot q)\,(\bar{n}\cdot p) - \vec{p}_T\cdot\vec{q}_T\,, \tag{10.62}$$

Consequently, the square of p is

$$p^2 = 2p_+\cdot p_- + p_\perp^2 = (n\cdot p)\,(\bar{n}\cdot p) - p_T^2\,. \tag{10.63}$$

Some confusion can arise (leading to lost factors two) as the subscripts $\pm$ are often used to indicate the light-cone *components* rather than the four-vectors:

$$p_+ \equiv n\cdot p = p_0 + p_z \qquad\qquad p_- \equiv \bar{n}\cdot p = p_0 - p_z. \tag{10.64}$$

Then, e.g., $2(p_+\cdot p_-) = p_+p_-$ and $p^2 = p_+p_- - p_T^2$.

Using light-cone *components*, the integration over momentum space can be written as

$$\begin{aligned}\int d^dk &= \frac{1}{2}\int_{-\infty}^{+\infty} dk_+ \int_{-\infty}^{+\infty} dk_- \int d^{d-2}k_\perp\,,\\ \int d^dk\,\delta(k^2)\theta(k^0) &= \frac{1}{2}\int_0^{\infty} dk_+ \int_0^{\infty} dk_- \int d^{d-2}k_\perp\,\delta(k_+k_- - k_T^2)\,.\end{aligned} \tag{10.65}$$

The second integral arises in "cut" diagrams. A common notation is $\delta(k^2)\theta(k^0) \equiv \delta_+(k^2)$.

10.3.3 The Sudakov form factor

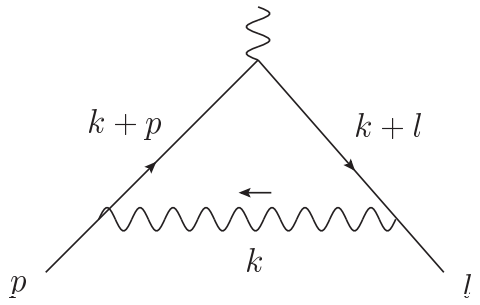
The general strategy to obtain the expansion of a given Feynman integral in a given kinematic limit is the following:

i) Identify all regions of the integrand which lead to singularities in the limit under consideration,

ii) Expand the integrand in each region and integrate each expansion over the full phase space.

iii) Add the result of the integrations over the different regions to obtain the expansion of the original full integral.

In this section we will discuss the so-called off-shell Sudakov form factor. This example is of practical importance, and also allows us to discuss possible subtleties. To one-loop accuracy, it is given by the Feynman diagram:



and requires evaluation of the following integral (we will ignore the numerator)

$$I = i\pi^{-d/2}\mu^{4-d}\int d^dk \frac{1}{(k^2+i0)\,[(k+l)^2+i0]\,[(k+p)^2+i0]}\,, \tag{10.66}$$

where $d = 4-2\varepsilon$. The μ^{4-d} factor is inserted to make the dimension of I independent of the value of d. We introduce the following notation:

$$L^2 \equiv -l^2 - i0\,, \qquad P^2 \equiv -p^2 - i0\,, \qquad Q^2 \equiv -(l-p)^2 - i0\,. \tag{10.67}$$

The goal is to calculate the integral in Eq. (10.66) in the limit in which $L^2 \sim P^2 \ll Q^2$ that is, in the case in which the external legs carrying momenta l and p have large energies but small invariant masses.

It is convenient to choose the frame in which $q^\mu = (0,\vec{q})$ and the two particles have large momenta in opposite z-directions, so that, e.g., ℓ^μ has a large "plus" component and p^μ a large "minus" component, respectively:

$$p^\mu \approx Q\frac{n^\mu}{2}\,, \qquad \ell^\mu \approx Q\frac{\bar{n}^\mu}{2}\,, \tag{10.68}$$

so that

$$p^\mu = (\underbrace{(n\cdot p)}_{\text{small}}, \underbrace{(\bar{n}\cdot p)}_{\text{large}}, \vec{p}_\perp)\,, \qquad \ell^\mu = (\underbrace{(n\cdot \ell)}_{\text{large}}, \underbrace{(\bar{n}\cdot \ell)}_{\text{small}}, \vec{\ell}_\perp)\,. \tag{10.69}$$

To proceed, introduce a power counting parameter λ that vanishes in the limit $L^2 \sim P^2 \ll Q^2$

$$\lambda^2 \sim \frac{P^2}{Q^2} \sim \frac{L^2}{Q^2}\,. \tag{10.70}$$

The (plus, minus, perp)-components of p and l will then typically scale as follows

$$p^\mu \sim \left(\lambda^2, 1, \lambda\right) Q\,, \qquad \text{and} \qquad l^\mu \sim \left(1, \lambda^2, \lambda\right) Q\,. \tag{10.71}$$

For the transverse components there is some freedom: changing the direction of n^μ slightly, we could choose e.g. $p^\mu \sim \left(\lambda^2, 1, \lambda^{1+a}\right) Q$ with $a > 0$ which is also consistent with $p^2 \sim \lambda^2 Q^2$.

Our task is to calculate the integral (10.66) by separating it in contributions of several regions with different scaling behavior of the momentum $k^\mu \sim (\lambda^a, \lambda^b, \lambda^c)Q$. In simple cases, it turns out

that only the regions with $k^\mu \sim (\lambda^a, \lambda^b, \lambda^c)Q$, with $a + b = 2c$ are important. Specifically, upon expanding (10.66), one finds that only the following four regions give non-vanishing contributions:

hard region (denoted by h)	$k^\mu \sim (1, 1, 1)\, Q$
collinear to p (denoted by c)	$k^\mu \sim (\lambda^2, 1, \lambda)\, Q$
collinear to ℓ (denoted by $\bar{c}$)	$k^\mu \sim (1, \lambda^2, \lambda)\, Q$
soft region (denoted by s)	$k^\mu \sim (\lambda^2, \lambda^2, \lambda^2)\, Q$

All other regions, with the scaling $k^\mu \sim (\lambda^a, \lambda^b, \lambda^c)\, Q$ and with a, b, c not matching one of the four cases listed above, give rise to scaleless integrals only, and therefore they do not contribute. In particular, **do not contribute** to the Sudakov form factor:

semi-hard region (denoted by sh)	$k^\mu \sim (\lambda, \lambda, \lambda)\, Q$
Glauber region (denoted by G)	$k^\mu \sim (\lambda^2, \lambda^2, \lambda)\, Q$

- For example, consider a potential contribution of the semihard region. In this case

$$(k+l)^2 = \underbrace{k^2}_{\mathcal{O}(\lambda^2)} + 2(\underbrace{k_+ \cdot l_-}_{\mathcal{O}(\lambda^3)} + \underbrace{k_- \cdot l_+}_{\mathcal{O}(\lambda)} + \underbrace{k_\perp \cdot l_\perp}_{\mathcal{O}(\lambda^2)}) + \underbrace{l^2}_{\mathcal{O}(\lambda^2)} = 2k_- \cdot l_+ + \mathcal{O}(\lambda^2)\,, \tag{10.72}$$

and analogously

$$(k+p)^2 = 2k_+ \cdot p_- + \mathcal{O}(\lambda^2)\,, \tag{10.73}$$

so that we need to evaluate

$$I_{sh} = i\pi^{-d/2}\mu^{4-d} \int d^d k \frac{1}{(k^2 + i0)\,(2k_- \cdot l_+ + i0)\,(2k_+ \cdot p_- + i0)}\,. \tag{10.74}$$

This integrals vanishes: $I_{sh} = 0$; the explicit calculation $\Rightarrow$ next section. Exercise: show that also the Glauber region $k^\mu \sim (\lambda^2, \lambda^2, \lambda)Q$ gives a vanishing contribution to the form factor integral.

- Hard region, $k^\mu \sim (1, 1, 1)\, Q$

In this case $k^2 \sim \lambda^0 Q^2$; for the other two propagators one finds

$$(k+l)^2 = \underbrace{k^2}_{\mathcal{O}(1)} + 2(\underbrace{k_+ \cdot l_-}_{\mathcal{O}(\lambda^2)} + \underbrace{k_- \cdot l_+}_{\mathcal{O}(1)} + \underbrace{k_\perp \cdot l_\perp}_{\mathcal{O}(\lambda)}) + \underbrace{l^2}_{\mathcal{O}(\lambda^2)} = k^2 + 2k_- \cdot l_+ + \mathcal{O}(\lambda)\,, \tag{10.75}$$

and, similarly

$$(k+p)^2 = k^2 + 2k_+ \cdot p_- + \mathcal{O}(\lambda)\,. \tag{10.76}$$

so that

$$I_h = i\pi^{-d/2}\mu^{4-d} \int d^d k \frac{1}{(k^2 + i0)\,(k^2 + 2k_- \cdot l_+ + i0)\,(k^2 + 2k_+ \cdot p_- + i0)}\,. \tag{10.77}$$

It coincides with the form factor integral with on-shell external legs $p^2 = l^2 = 0$. [Why: in this case can choose $n, \bar{n}$ vectors such that $p^\mu = (Q/2)n^\mu$, $\ell^\mu = (Q/2)\bar{n}^\mu$; the normalization is fixed by $-Q^2 = (\ell - p)^2 = -2\ell \cdot p$].

The integral evaluates to

$$\begin{aligned} I_h &= \frac{\Gamma(1+\varepsilon)}{2l_+ \cdot p_-} \frac{\Gamma^2(-\varepsilon)}{\Gamma(1-2\varepsilon)} \left(\frac{\mu^2}{2l_+ \cdot p_-}\right)^\varepsilon \\ &= \frac{\Gamma(1+\varepsilon)}{Q^2} \left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \ln \frac{\mu^2}{Q^2} + \frac{1}{2} \ln^2 \frac{\mu^2}{Q^2} - \frac{\pi^2}{6}\right) + \mathcal{O}(\varepsilon) \,. \end{aligned} \tag{10.78}$$

The poles in ε are of infrared origin. This calculation is rather straightforward.

- collinear region to p, $k^\mu \sim (\lambda^2, 1, \lambda)Q$.

In this region $k^2 \sim \lambda^2 Q^2$, while

$$(k+l)^2 = 2k_- \cdot l_+ + \mathcal{O}(\lambda^2) \,, \qquad (k+p)^2 = \mathcal{O}(\lambda^2) \,. \tag{10.79}$$

The **collinear region** integral is obtained by keeping only the leading term in each propagator

$$\begin{aligned} I_c &= i\pi^{-d/2} \mu^{4-d} \int d^d k \frac{1}{(k^2 + i0)\,(2k_- \cdot l_+ + i0)\,[(k+p)^2 + i0]} \\ &= -\frac{\Gamma(1+\varepsilon)}{2l_+ \cdot p_-} \frac{\Gamma^2(-\varepsilon)}{\Gamma(1-2\varepsilon)} \left(\frac{\mu^2}{P^2}\right)^\varepsilon \\ &= \frac{\Gamma(1+\varepsilon)}{Q^2} \left(-\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon} \ln \frac{\mu^2}{P^2} - \frac{1}{2} \ln^2 \frac{\mu^2}{P^2} + \frac{\pi^2}{6}\right) + \mathcal{O}(\varepsilon) \,. \end{aligned} \tag{10.80}$$

The calculation $\Rightarrow$ next section. Note that the integral scales as $P^{-2\varepsilon}$.

- collinear region to ℓ, $k^\mu \sim (1, \lambda^2, \lambda)Q$.

The calculation and the result are the same up to the replacement $P^2 \to L^2$ in the final result.

- soft region $k^\mu \sim (\lambda^2, \lambda^2, \lambda^2)Q$.

Here

$$k^2 = \mathcal{O}(\lambda^4) \,, \quad (k+l)^2 = 2k_- \cdot l_+ + l^2 + \mathcal{O}(\lambda^3) \,, \text{ and } \quad (k+p)^2 = 2k_+ \cdot p_- + p^2 + \mathcal{O}(\lambda^3) \,, \tag{10.81}$$

and therefore the integral in the **soft region** is

$$\begin{aligned} I_s &= i\pi^{-d/2} \mu^{4-d} \int d^d k \frac{1}{(k^2 + i0)\,(2k_- \cdot l_+ + l^2 + i0)\,(2k_+ \cdot p_- + p^2 + i0)} \\ &= -\frac{\Gamma(1+\varepsilon)}{2l_+ \cdot p_-} \Gamma(\varepsilon) \Gamma(-\varepsilon) \left(\frac{2l_+ \cdot p_- \mu^2}{L^2 P^2}\right)^\varepsilon \\ &= \frac{\Gamma(1+\varepsilon)}{Q^2} \left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \ln \frac{\mu^2 Q^2}{L^2 P^2} + \frac{1}{2} \ln^2 \frac{\mu^2 Q^2}{L^2 P^2} + \frac{\pi^2}{6}\right) + \mathcal{O}(\varepsilon) \,. \end{aligned} \tag{10.82}$$

The poles in the last line of Eq. (10.82) are of ultraviolet origin. Note that the result depends on the "new" soft scale $\Lambda^2_{\rm soft} \sim P^2L^2/Q^2$, much smaller than virtualties of the external momenta!

- The sum

We find

$$
\begin{aligned}
I_h &= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\ln\frac{\mu^2}{Q^2}+\frac{1}{2}\ln^2\frac{\mu^2}{Q^2}-\frac{\pi^2}{6}+\mathcal{O}(\lambda)\right)\\
I_c &= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(-\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}\ln\frac{\mu^2}{P^2}-\frac{1}{2}\ln^2\frac{\mu^2}{P^2}+\frac{\pi^2}{6}+\mathcal{O}(\lambda)\right)\\
I_{\bar c} &= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(-\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}\ln\frac{\mu^2}{L^2}-\frac{1}{2}\ln^2\frac{\mu^2}{L^2}+\frac{\pi^2}{6}+\mathcal{O}(\lambda)\right)\\
I_s &= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\ln\frac{\mu^2Q^2}{L^2P^2}+\frac{1}{2}\ln^2\frac{\mu^2Q^2}{L^2P^2}+\frac{\pi^2}{6}+\mathcal{O}(\lambda)\right)
\end{aligned}
\tag{10.83}
$$

so that summing up

$$
I \equiv I_h+I_c+I_{\bar c}+I_s = \frac{1}{Q^2}\left(\ln\frac{Q^2}{L^2}\ln\frac{Q^2}{P^2}+\frac{\pi^2}{3}+\mathcal{O}(\lambda)\right). \tag{10.84}
$$

- The final result does not depend on the dimensional regulator ε.
- It coincides with the one that would be obtained by evaluating directly the integral in Eq. (10.66) and then expanding the result in the $\lambda \to 0$ limit.
- IR divergences found in the hard region cancel out against the UV divergences found in the sum of the soft and collinear contributions. This feature is general and requires a nontrivial interplay of the logarithms found in the various integrals:

$$
\underbrace{-\frac{1}{\varepsilon}\ln\frac{\mu^2}{P^2}-\frac{1}{\varepsilon}\ln\frac{\mu^2}{L^2}+\frac{1}{\varepsilon}\ln\frac{\mu^2Q^2}{L^2P^2}}_{IR} = \underbrace{-\frac{1}{\varepsilon}\ln\frac{\mu^2}{Q^2}}_{UV}. \tag{10.85}
$$

- Each contribution (h, c, $\bar c$, s) depends on only one scale (will be important).
- Overlap between collinear and soft regions? — No, leads to scaleless integrals.
- Life can be more complicated: For some observables the simple separation of the integral in hard, soft, and collinear regions breaks down because the different momentum regions are not well defined if one does not introduce additional regulators on top of dimensional regularization.

10.3.4 Sample calculations

- **The collinear integral I_c**

$$I_c = i\pi^{-d/2}\mu^{4-d}\int d^dk \frac{1}{(k^2+i0)\,(2k_-\cdot l_+ + i0)\,[(k+p)^2+i0]} \tag{10.86}$$

This integral involves a propagator which is linear in the loop momentum k. To combine linear and quadratic propagators, it is convenient to use the representation

$$\frac{1}{ab} = \int_0^\infty dy\, \frac{1}{(a+by)^2}\,, \tag{10.87}$$

in cases where a is a standard propagator, and b linear in the loop momentum. Note that the y integral runs up to infinity. By performing a variable change from y to x, with $y = x/(1-x)$, one recovers the standard Feynman parameterization. This can be generalized to the case of n-propagators as follows

$$\frac{1}{a_1a_2a_3\cdots a_n} = \int_0^\infty dy_1\int_0^\infty dy_2\cdots\int_0^\infty dy_{n-1}\frac{(n-1)!}{(a_1+a_2y_1+a_3y_2+\cdots a_ny_{n-1})^n}\,. \tag{10.88}$$

For higher power of propagators, one uses

$$\frac{1}{a^nb^m} = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)}\int_0^\infty dy\,\frac{y^{m-1}}{(a+yb)^{n+m}}\,. \tag{10.89}$$

For our case we use $n=3$

$$\frac{1}{abc} = \int_0^\infty dx_1\int_0^\infty dx_2\,\frac{2}{(a+bx_1+cx_2)^3}\,, \tag{10.90}$$

where $a=k^2$, $c = 2l_+k_- \simeq 2l_+\cdot k$, and $b=(k+p)^2$. In this way one obtains

$$I_c = i\pi^{-\frac{d}{2}}\mu^{4-d}\int_0^\infty dx_1\int_0^\infty dx_2\int d^dk\,\frac{2}{\left[(1+x_1)\,(k^2+2k\cdot V - M^2)\right]^3}\,, \tag{10.91}$$

with

$$V^\mu = \frac{x_1p^\mu + x_2l_+^\mu}{(1+x_1)}\,,\qquad M^2 = \frac{x_1P^2}{(1+x_1)}\,,\qquad P^2\equiv -p^2\,. \tag{10.92}$$

The loop momentum integral can now be taken using

$$\int d^dk\frac{\Gamma(a)}{(-k^2-2k\cdot V+M^2-i0)^a} = i\pi^{\frac{d}{2}}\frac{\Gamma\left(a-\frac{d}{2}\right)}{(M^2+V^2)^{a-\frac{d}{2}}} \tag{10.93}$$

Then

$$\begin{aligned} I_c &= \mu^{2\varepsilon}\Gamma(1+\varepsilon)\int_0^\infty dx_1\frac{1}{(1+x_1)^3}\int_0^\infty dx_2\left(\frac{P^2x_1+2l_+\cdot p_-x_1x_2}{(1+x_1)^2}\right)^{-1-\varepsilon} \\ &= \mu^{2\varepsilon}\Gamma(1+\varepsilon)\int_0^\infty dx_1\frac{x_1^{-1-\varepsilon}}{(1+x_1)^{1-2\varepsilon}}\int_0^\infty dx_2\left(P^2+2l_+\cdot p_-x_2\right)^{-1-\varepsilon}\,. \end{aligned} \tag{10.94}$$

The integrals over x_1 and x_2 decouple and one finds

$$\begin{aligned} I_c &= \mu^{2\varepsilon}\Gamma\left(1+\varepsilon\right)P^{-2-2\varepsilon}\frac{\Gamma(1-\varepsilon)\Gamma(-\varepsilon)}{\Gamma(1-2\varepsilon)}\int_0^\infty dx_2\left(1+rx_2\right)^{-1-\varepsilon}\,, \qquad \boxed{r=2l_+\cdot p_-/P^2} \\ &= \left(\frac{\mu^2}{P^2}\right)^\varepsilon \frac{\Gamma\left(1+\varepsilon\right)}{2l_+\cdot p_-}\frac{\Gamma(1-\varepsilon)\Gamma(-\varepsilon)}{\Gamma(1-2\varepsilon)}\int_0^\infty dx_2'\left(1+x_2'\right)^{-1-\varepsilon} \qquad \boxed{x_2\mapsto x_2'/r} \\ &= \left(\frac{\mu^2}{P^2}\right)^\varepsilon \frac{\Gamma\left(1+\varepsilon\right)}{2l_+\cdot p_-}\frac{\Gamma(1-\varepsilon)\Gamma(-\varepsilon)}{\varepsilon\Gamma(1-2\varepsilon)} \;=\; -\frac{\Gamma\left(1+\varepsilon\right)}{2l_+\cdot p_-}\frac{\Gamma^2(-\varepsilon)}{\Gamma(1-2\varepsilon)}\left(\frac{\mu^2}{P^2}\right)^\varepsilon\,, \end{aligned} \tag{10.95}$$

which is the result advertized in (10.86).

- **The soft and semi-hard integrals I_s, I_{sh}**

$$\begin{aligned} I_s &= i\pi^{-d/2}\mu^{4-d}\int d^dk\frac{1}{\left(k^2+i0\right)\left(2k_-\cdot l_+ + l^2+i0\right)\left(2k_+\cdot p_- + p^2+i0\right)} \\ I_{sh} &= i\pi^{-d/2}\mu^{4-d}\int d^dk\frac{1}{\left(k^2+i0\right)\left(2k_-\cdot l_+ +i0\right)\left(2k_+\cdot p_- +i0\right)}\,. \end{aligned} \tag{10.96}$$

Note that I_{sh} is given by the same integral as I_s but for $p^2=\ell^2=0$. We will find that this integral vanishes.

Start using

$$\frac{1}{abc} = \int_0^\infty dx_1\int_0^\infty dx_2\,\frac{2}{(a+bx_1+cx_2)^3} \tag{10.97}$$

with $a=k^2$, $b=2p_-\cdot k+p^2$, and $c=2l_+\cdot k+l^2$. The denominator becomes

$$a+bx_1+cx_2 = k^2+2k\cdot\left(p_-x_1+l_+x_2\right)+p^2x_1+l^2x_2\,. \tag{10.98}$$

and doing the k-integration we obtain

$$I_s = \mu^{2\epsilon}\Gamma\left(1+\epsilon\right)\int_0^\infty dx_1\int_0^\infty dx_2\left(x_1P^2+x_2L^2+2l_+\cdot p_-x_1x_2\right)^{-1-\epsilon}\,, \tag{10.99}$$

Changes of variables:

- $x_1\to x_1'/P^2, x_2\to x_2'/L^2$:

$$I_s = \mu^{2\varepsilon}\frac{\Gamma\left(1+\varepsilon\right)}{P^2L^2}\int_0^\infty dx_1\int_0^\infty dx_2\left(x_1+x_2+ax_1x_2\right)^{-1-\varepsilon}\,, \qquad \boxed{a=\frac{2l_+\cdot p_-}{P^2L^2}}$$

Next

- $x_2\to x_1x_2'$:

$$I_s = \mu^{2\varepsilon}\frac{\Gamma\left(1+\varepsilon\right)}{P^2L^2}\int_0^\infty dx_1\int_0^\infty dx_2x_1^{-\varepsilon}\left(1+x_2+ax_1x_2\right)^{-1-\varepsilon}$$

Next

- $x_2\to x_2'/(1+ax_1)$:

$$I_s = \mu^{2\varepsilon}\frac{\Gamma\left(1+\varepsilon\right)}{P^2L^2}\int_0^\infty dx_1\frac{x_1^{-\varepsilon}}{1+ax_1}\int_0^\infty dx_2\left(1+x_2\right)^{-1-\varepsilon}\,. \tag{10.100}$$

The two integrals are now factorized and we get

$$I_s = \mu^{2\varepsilon}\frac{\Gamma(1+\varepsilon)}{P^2L^2}a^{-1+\varepsilon}\underbrace{\int_0^\infty dx_1'\frac{{x_1'}^{-\varepsilon}}{1+x_1'}}_{=\Gamma(1-\varepsilon)\Gamma(\varepsilon)}\underbrace{\int_0^\infty dx_2\,(1+x_2)^{-1-\varepsilon}}_{=\frac{1}{\varepsilon}} \qquad \boxed{x_1 \to x_1'/a}$$

$$= \frac{\Gamma(1+\varepsilon)}{P^2L^2}\left(\frac{P^2L^2}{2l_+\cdot p_-}\right)\left(\frac{2l_+\cdot p_-\mu^2}{P^2L^2}\right)^{\varepsilon}\frac{(-\varepsilon)\Gamma(\varepsilon)\Gamma(-\varepsilon)}{\varepsilon}\,,$$

$$= -\frac{\Gamma(1+\varepsilon)}{2l_+\cdot p_-}\Gamma(\varepsilon)\Gamma(-\varepsilon)\left(\frac{2l_+\cdot p_-\mu^2}{P^2L^2}\right)^{\varepsilon}\,, \tag{10.101}$$

which is the result advertized in (10.82).

If we set $p^2 = l^2 = 0$ at the beginning of the calculation, the integral vanishes, even if $p\cdot l \neq 0$. This is seen immediately by setting $p^2 = l^2 = 0$ in Eq. (10.99). By doing this one obtains

$$I_s\left(p^2=0, l^2=0\right) = \mu^{2\varepsilon}\Gamma(1+\varepsilon)\int_0^\infty dx_1\int_0^\infty dx_2\,(2l_+\cdot p_-)^{-1-\varepsilon}\,(x_1x_2)^{-1-\varepsilon}\,, \tag{10.102}$$

where the two integrals in x_1 and x_2 factorize and both vanish (scaleless integrals, see example above). Thus $I_{sh} = 0$, as advertized.

An alternative way of proving that

$$\forall p_1, p_2: \qquad \int d^dk\frac{1}{k^2(k\cdot p_1)(k\cdot p_2)} = 0 \tag{10.103}$$

involves integration by parts identities. This is a very important technique so I want to illustrate its application on this example.
The starting observation is that in dim. reg.

$$\int d^dk\,\frac{\partial}{\partial k^\mu}\frac{v^\mu}{k^2(k\cdot p_1)(k\cdot p_2)} = 0\,, \tag{10.104}$$

for any v^μ. There are no boundary terms at infinite momentum because one can always choose the dimension d in such a way that the integrand goes to zero sufficiently fast for $k^\mu \to \infty$. By choosing $v^\mu = k^\mu$ and applying the derivative to the integrand one obtains

$$\begin{aligned}\frac{\partial}{\partial k^\mu}\frac{k^\mu}{k^2(k\cdot p_1)(k\cdot p_2)} &= d\frac{1}{k^2(k\cdot p_1)(k\cdot p_2)} - k_\mu\cdot\frac{2k^\mu}{k^4(k\cdot p_1)^2(k\cdot p_2)} \\ &\quad - k_\mu\cdot\frac{p_1^\mu}{k^2(k\cdot p_1)^2(k\cdot p_2)} - k_\mu\cdot\frac{p_2^\mu}{k^2(k\cdot p_1)(k\cdot p_2)^2} \\ &= (d-4)\frac{1}{(k^2)(k\cdot p_1)(k\cdot p_2)}\,.\end{aligned} \tag{10.105}$$

Thus

$$(d-4)\int d^dk\,\frac{1}{(k^2)(k\cdot p_1)(k\cdot p_2)} = 0 \tag{10.106}$$

Since in dimensional regularization one works in $d \neq 4$ (and then one takes the limit $\varepsilon \to 0$) this relation implies Eq. (10.103)!

A somewhat more complicated example, consider

$$I(\alpha,\beta) = \int d^d k \frac{1}{(2k\cdot l + l^2)^\alpha (2k\cdot p + p^2)^\beta} = ? \tag{10.107}$$

for $\alpha, \beta \geq 1$. By considering the integration by parts identities

$$\int d^d k \frac{\partial}{\partial k^\mu} \frac{v^\mu}{(2k\cdot l + l^2)^\alpha (2k\cdot p + p^2)^\beta} = 0\,, \tag{10.108}$$

with $v = \ell$ and $v = p$ one finds

$$\begin{aligned} &\alpha l^2 I(\alpha+1,\beta) + \beta\,(l\cdot p)\, I(\alpha,\beta+1) = 0\,,\\ &\alpha\,(l\cdot p)\, I(\alpha+1,\beta) + \beta p^2 I(\alpha,\beta+1) = 0\,. \end{aligned} \tag{10.109}$$

This implies

$$I(\alpha+1,\beta) = I(\alpha,\beta+1) = 0 \quad \text{unless} \quad (l\cdot p)^2 - l^2 p^2 = 0\,. \tag{10.110}$$

A one more relation comes from setting $v = k$. One finds

$$(d-\alpha-\beta) I(\alpha,\beta) + \alpha l^2 I(\alpha+1,\beta) + \beta p^2 I(\alpha,\beta+1) = 0\,, \tag{10.111}$$

which, taken together with Eq. (10.110), implies (unless $d = \alpha+\beta$)

$$I(\alpha,\beta) = 0\,. \tag{10.112}$$

The IBP identities (K. Chetyrkin, F.Tkachev, 1981) prove to be very powerful in higher loops. This is currently the leading technique (starting point of) the calculation of multiloop diagrams.

10.4 Scalar SCET

10.4.1 The Lagrangian

We now construct an effective field theory whose Feynman rules directly yield the above hard, collinear, and soft integrals for the Sudakov form factor. We will do it here for scalar ϕ^3 theory:

$$\mathcal{L}\,(\phi) = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{g}{3!}\phi^3(x)\,, \tag{10.113}$$

In order to derive the SCET effective Lagrangian needed for the calculation of the Sudakov form factor in this theory, we need to split the scalar field in the sum of a hard field, field collinear to the momentum p, a field collinear to the momentum l, and a soft field:

$$\phi(x) \to \phi_h(x) + \phi_c(x) + \phi_{\bar c}(x) + \phi_s(x)\,. \tag{10.114}$$

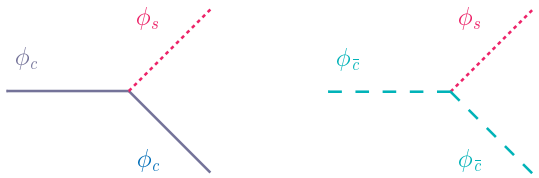
Contributions of the "hard" field will be integrated out and absorbed in the coefficient functions of the operators built of soft and collinear fields.

At three level we can simply ignore terms with ϕ_h and the effective Lagrangian can be written as the sum of four terms:

$$\mathcal{L}(\phi) = \underbrace{\mathcal{L}(\phi_c)}_{\equiv\mathcal{L}_c} + \underbrace{\mathcal{L}(\phi_{\bar{c}})}_{\equiv\mathcal{L}_{\bar{c}}} + \underbrace{\mathcal{L}(\phi_s)}_{\equiv\mathcal{L}_s} + \mathcal{L}_{c+s}(\phi_c, \phi_{\bar{c}}, \phi_s)\,. \tag{10.115}$$

The first three terms are copies of the original Lagrangian, and the last one describes the interactions between soft and collinear fields:

$$\mathcal{L}_{c+s}(\phi_c, \phi_{\bar{c}}, \phi_s) = -\frac{g}{2}\phi_c{}^2\phi_s - \frac{g}{2}\phi_{\bar{c}}{}^2\phi_s\,, \tag{10.116}$$



All others vertices are forbidden by momentum conservation.

Next, one needs to expand the interaction terms to keep the proper components in the $\lambda \to 0$ limit. This is called a derivative (or multipole) expansion. Consider, e.g.,

$$\int d^dx\, \phi_c^2(x)\phi_s(x) = \int d^dx \int \frac{d^dp_1}{(2\pi)^d} \int \frac{d^dp_2}{(2\pi)^d} \int \frac{d^dp_s}{(2\pi)^d} \tilde{\phi}_c(p_1)\tilde{\phi}_c(p_2)\tilde{\phi}_s(p_s) e^{-i(p_1+p_2+p_s)\cdot x}\,, \tag{10.117}$$

We assume $p_{1,2}^\mu \sim (\lambda^2, 1, \lambda)\, Q$ and $p_s^\mu \sim (\lambda^2, \lambda^2, \lambda^2)\, Q$ so that also

$$p_1^\mu + p_2^\mu + p_s^\mu \sim \left(\lambda^2, 1, \lambda\right) Q\,. \tag{10.118}$$

Consequently the components of x must scale as

$$x^\mu \sim \left(1, \frac{1}{\lambda^2}, \frac{1}{\lambda}\right) \frac{1}{Q} \tag{10.119}$$

and

$$p_s \cdot x = \underbrace{(p_s)_+ \cdot x_-}_{\mathcal{O}(1)} + \underbrace{(p_s)_- \cdot x_+}_{\mathcal{O}(\lambda^2)} + \underbrace{(p_s)_\perp \cdot x_\perp}_{\mathcal{O}(\lambda)}\,. \tag{10.120}$$

This means that the soft field $\phi_s(x)$ can be Taylor-expanded around the $x_-^\mu = (x\cdot\bar{n})n^\mu/2$

$$\phi_s(x) = \phi_s(x_-) + \underbrace{x_\perp \cdot \partial_\perp}_{\mathcal{O}(\lambda)} \phi_s(x_-) + \underbrace{x_+ \cdot \partial_-}_{\mathcal{O}(\lambda^2)} \phi_s(x_-) + \frac{1}{2}\Big(\underbrace{x_\perp^\mu x_\perp^\nu \partial_\mu\partial_\nu}_{\mathcal{O}(\lambda^2)} \phi_s(x_-)\Big) + \ldots \tag{10.121}$$

where each derivative counts as $\sim \lambda^2$ (as soft momentum). Thus we can write the interaction term as

$$\int d^dx\, \phi_c^2(x)\phi_s(x) = \int d^dx\, \phi_c^2(x)\phi_s(x_-)\left(1 + \mathcal{O}(\lambda)\right). \tag{10.122}$$

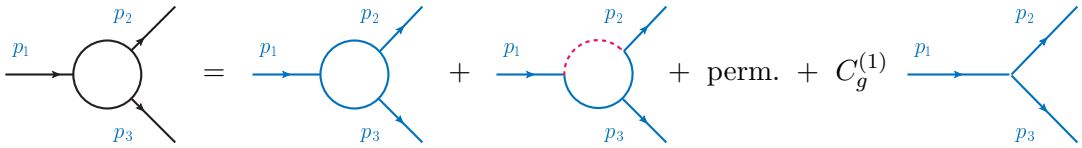
Thus the full leading-power tree-level scalar SCET Lagrangian has the following form

$$\mathcal{L}_{\rm SCET} = \frac{1}{2}\partial_\mu\phi_c(x)\partial^\mu\phi_c(x) - \frac{g}{3!}\phi_c^3(x) + \frac{1}{2}\partial_\mu\phi_{\bar c}(x)\partial^\mu\phi_{\bar c}(x) - \frac{g}{3!}\phi_{\bar c}^3(x) + \frac{1}{2}\partial_\mu\phi_s(x)\partial^\mu\phi_s(x) - \frac{g}{3!}\phi_s^3(x) - \frac{g}{2}\phi_c^2(x)\phi_s(x_-) - \frac{g}{2}\phi_{\bar c}^2(x)\phi_s(x_+)\,. \tag{10.123}$$

In general, one should expect that taking into account loop corrections the effective Lagrangian will be modified, e.g. for the interaction term

$$-\frac{g}{3!}\phi_c^3(x) \to -\frac{g_{\rm eff}}{3!}\phi_c^3(x) \equiv -\frac{g}{3!}\left(1 + g^2 C_g^{(1)} + g^4 C_g^{(2)} + \ldots\right)\phi_c^3(x)\,. \tag{10.124}$$

In order to fix the coefficient $C^{(1)}$ one requires that the corrections of order g^2 to the interaction of three scalar fields are the same in the full theory and in the effective theory. In the full theory these corrections coincide with the one loop corrections to the ϕ^3 vertex, while in the effective theory one finds contributions originating from one loop graphs and contributions proportional to $C^{(1)}$:



where (important!) all three momenta are collinear: $p_{1,2,3} \sim (\lambda^2, 1, \lambda)$. This is an equation on $C_g^{(1)}$.

The first diagram on the r.h.s. is identical to the original loop integral on the l.h.s. (same Feynman rules). A non-zero $C_g^{(1)}$ would therefore only be needed to compensate a contribution with a soft particle in the loop. Fortunately, these additional one-loop diagrams all vanish. Starting in position space we would get [†]

$$\begin{aligned}
&\iiint d^dx_1 d^dx_2 d^dx_3\, e^{-ix_1\cdot p_1 + ix_2\cdot p_2 + ix_3\cdot p_3} \\
&\times \underbrace{\int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2} e^{-i(x_2-x_1)\cdot\bar n k_+}}_{\phi_s(x_{1-})\phi_s(x_{2-})} \int \frac{d^dk_1}{(2\pi)^d}\frac{i}{k_1^2} e^{-i(x_3-x_2)\cdot k_1} \int \frac{d^dk_2}{(2\pi)^d}\frac{i}{k_2^2} e^{-i(x_1-x_3)\cdot k_2} \\
&= (2\pi)^d i^3 \delta^d(p_1 - p_2 - p_3) \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2(p_2 - \bar n k_+)^2(p_1 - \bar n k_+)^2}
\end{aligned} \tag{10.125}$$

Thus these diagrams involve

$$\int d^dk \frac{1}{(k^2+i0)(-2k_+\cdot(p_1)_- + p_1^2 + i0)(-2k_+\cdot(p_2)_- + p_2^2 + i0)} \tag{10.126}$$

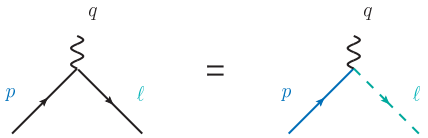
which is similar to the soft integral in (10.96) but with $\ell \mapsto p_2$. Following the calculation in the previous section one can see that this replacement produces a scaleless integral. Thus $C_g^{(1)} = 0$!

[†] thanks to Jakob Schoenleber. Note that $\phi_s(x) \mapsto \phi_s(x_\pm)$ in the Lagrangian effectively eliminates the terms $\sim k^2$ in the collinear propagators.

A general argument why no corrections are necessary for purely collinear diagrams: one can choose all external momenta p_i^μ in a single direction $p_i^\mu \propto p^\mu$ and require $p^2 = 0$. In this case all loop integrals are scaleless and vanish, so that necessarily $C^{(1)} = 0$. This argument also applies for higher loops; thus the SCET Lagrangian receives no corrections to all orders in perturbation theory. [The same kind of reasoning can be applied also to soft-collinear interaction terms.]

10.4.2 Matching for composite operators

Sudakov form factor corresponds to a process when a fast particle is abruptly changing its flight direction because of a hard interaction with some external source. At tree level, obviously,



or, on the operator level

$$J(x) \equiv \frac{1}{2}\phi^2(x) = \phi_c(x)\phi_{\bar c}(x) \tag{10.127}$$

but including loop corrections the situation becomes much more interesting. One can envisage an expression of the type

$$J = \underbrace{C_2\phi_c\phi_{\bar c}}_{=J_2} + \underbrace{\frac{C_3}{2!}\left(\phi_c^2\phi_{\bar c} + \phi_c\phi_{\bar c}^2\right)}_{=J_3} + \dots\,, \tag{10.128}$$

but, most importantly, in addition to operators with multiple fields, one should also consider operators involving derivatives. A derivative of the collinear field in a given direction scales as the corresponding component of the momentum, therefore

$$n\cdot\partial\phi_c(x) \sim \lambda^2\phi_c(x)\,, \quad \partial_\perp^\mu\phi_c(x) \sim \lambda\phi_c(x)\,, \quad \bar n\cdot\partial\phi_c(x) \sim \lambda^0\phi_c(x)\,, \tag{10.129}$$

and similarly

$$\bar n\cdot\partial\phi_{\bar c}(x) \sim \lambda^2\phi_{\bar c}(x)\,, \quad \partial_\perp^\mu\phi_{\bar c}(x) \sim \lambda\phi_{\bar c}(x)\,, \quad n\cdot\partial\phi_{\bar c}(x) \sim \lambda^0\phi_{\bar c}(x)\,. \tag{10.130}$$

The derivatives $\partial_-\phi_c \equiv (\bar n\cdot\partial)\phi_c$ and $\partial_+\phi_{\bar c} \equiv (n\cdot\partial)\phi_{\bar c}$ are not power suppressed, because the collinear fields carry large momentum in these directions. Thus <u>all</u> operators of the type $(\partial_-^{k_1}\phi_c)(\partial_+^{k_2}\phi_{\bar c})$ contribute at leading power in λ and have to be taken into account.

A convenient way to include the operators with derivatives in given directions is to go over to a non-local representation

$$\begin{aligned} J_2(x) &= \iint_{-\infty}^{+\infty} ds dt\, C_2(s,t,\mu)\,\phi_c\left(x+s\bar n\right)\phi_{\bar c}\left(x+tn\right), \\ J_3(x) &= \iiint_{-\infty}^{+\infty} ds dt_1 dt_2\, C_3(s,t_1,t_2,\mu)\phi_c(x+s\bar n)\phi_{\bar c}(x+t_1 n)\phi_{\bar c}(x+t_2 n) + (c\leftrightarrow\bar c)\,, \end{aligned} \tag{10.131}$$

etc. The SCET operators are thus non-local along light-cone directions corresponding to large energies. The coefficient functions are usually called matching coefficients in this context:

$$C_2(s,t,\mu) = \delta(s)\delta(t) + g^2 C_2^{(1)}(s,t,\mu) + g^4 C_2^{(2)}(s,t,\mu) + \dots$$
$$C_3(s,t_1,t_2,\mu) = g C_3^{(0)}(s,t_1,t_2,\mu) + g^3 C_3^{(1)}(s,t_1,t_2,\mu) + \dots \tag{10.132}$$

In practice it can be more convenient to consider Fourier transformed coefficients, e.g.

$$\tilde{C}_2(p_- l_+,\mu) = \iint ds dt\, e^{-is\bar{n}\cdot p} e^{itn\cdot l} C_2(s,t,\mu)\,, \qquad p_- \equiv (\bar{n}\cdot p), \quad l_+ \equiv (n\cdot l) \tag{10.133}$$

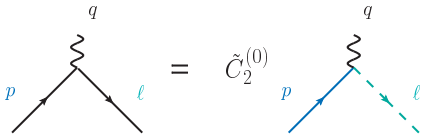
Note that $\tilde{C}_2$ will only depend on the product $p_- l_+ = Q^2$ but not p_- and l_+ separately. One way to understand this is to note that our formalism is invariant under a rescaling $n^\mu \to \alpha n^\mu$ and simultaneously $\bar{n}^\mu \to 1/\alpha n^\mu$ which is seen already in the momentum decomposition $p^\mu = \frac{1}{2}(n\cdot p)\bar{n}^\mu + \frac{1}{2}(\bar{n}\cdot p) n^\mu + p_\perp^\mu$.

In order to calculate the coefficients C_2, C_3 we have to require that matrix elements of J and $J_2 + J_3 + \dots$ coincide on a suitable set of states

$$\langle l|J(x)|p\rangle \stackrel{!}{=} \langle l|J_2(x) + J_3(x) + \dots|p\rangle$$
$$\langle l_1 l_2|J(x)|p\rangle \stackrel{!}{=} \langle l_1 l_2|J_2(x) + J_3(x) + \dots|p\rangle$$
$$\dots \tag{10.134}$$

In order to make the calculation simple we will take $p^2 = l^2 = 0$ (free particles states).

- At order $\mathcal{O}(g^0)$

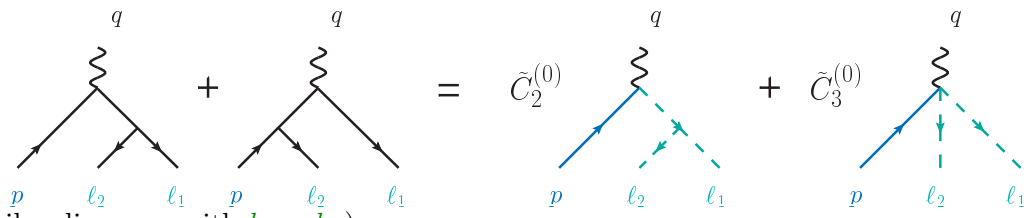


$$\text{l.h.s.} = \frac{1}{2}\langle l|\phi^2(x)|p\rangle = e^{ilx - ipx} \qquad \boxed{\phi(x)|p\rangle = e^{-ipx}}$$

$$\text{r.h.s.} = \iint ds dt\, C_2^{(0)}(s,t,\mu)\langle l|\phi_c(x+s\bar{n})\phi_{\bar{c}}(x+tn)|p\rangle = \iint ds dt\, C_2^{(0)}(s,t,\mu) e^{-ip(x+s\bar{n})} e^{il(x+tn)}$$
$$= e^{ilx-ipx}\iint ds dt\, C_2^{(0)}(s,t,\mu) e^{-isp_-} e^{itl_+} = e^{ilx-ipx}\tilde{C}_2^{(0)}(p_- l_+,\mu)\,, \tag{10.135}$$

so that $\tilde{C}_2^{(0)} = 1$.

- At order $\mathcal{O}(g^1)$



(and similar diagrams with $l_1 \leftrightarrow l_2$.)

The first diagram on the l.h.s. involves only one propagator with virtuality $(l_1+l_2)^2$. Since $\tilde{C}_2^{(0)}=1$, this diagram and the first diagram on the r.h.s. give identical contributions and drop out of the equation. Thus $\tilde{C}_3^{(0)}$ is determined by the second diagram on the l.h.s.:

$$\tilde{C}_3^{(0)}\left(n\cdot l_1, n\cdot l_2, \bar{n}\cdot p, \mu\right)=\frac{1}{(p-l_2)^2+i0}=\frac{1}{-\left(n\cdot l_2\right)\left(\bar{n}\cdot p\right)+i0}\,. \tag{10.136}$$

The corresponding expression in position space is

$$C_3(s,t_1,t_2,\mu)=\theta(-s)\delta(t_1)\theta(t_2)+\mathcal{O}(g^3)\,, \tag{10.137}$$

which can be verified by carrying out the Fourier transform

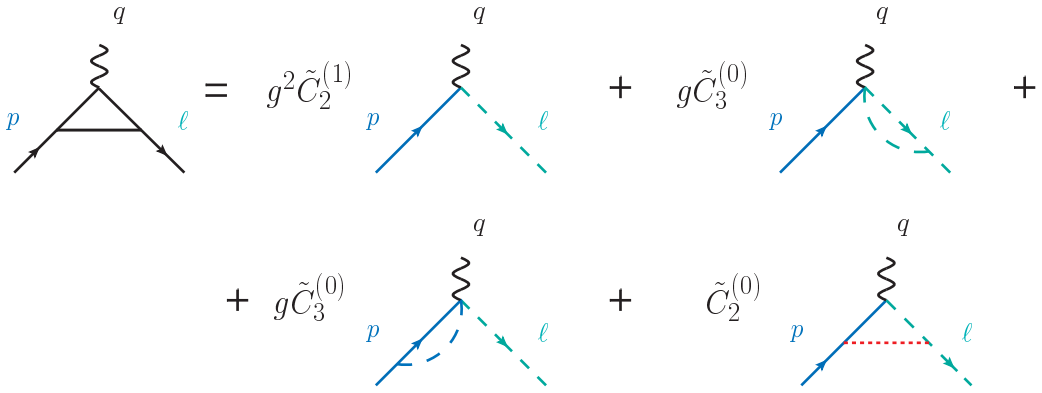
$$\begin{aligned}\tilde{C}_3^{(0)} &= \iiint_{-\infty}^{+\infty} ds dt_1 dt_2\, e^{-is\bar{n}\cdot p} e^{it_1 n\cdot l_1} e^{it_2 n\cdot l_2}\theta(-s)\theta(t_2)\delta(t_1)\\ &= \int_{-\infty}^{0} ds \int_0^{\infty} dt_2\, e^{-is\bar{n}\cdot p} e^{it_2 n\cdot l_2} = \frac{1}{-(\bar{n}\cdot p)(n\cdot l_2)}\,.\end{aligned} \tag{10.138}$$

Thus we get

$$\begin{aligned}J_3(x) &= \iiint_{-\infty}^{+\infty} ds dt_1 dt_2\, C_3(s,t_1,t_2,\mu)\phi_c(x+s\bar{n})\phi_{\bar{c}}(x+t_1 n)\phi_{\bar{c}}(x+t_2 n)+(c\leftrightarrow\bar{c})\,,\\ &= g\int_{-\infty}^{0} ds\int_0^{\infty} dt_2\,\phi_c(x+s\bar{n})\phi_{\bar{c}}(x)\phi_{\bar{c}}(x+t_2 n)+(c\leftrightarrow\bar{c})+\mathcal{O}(g^3)\end{aligned} \tag{10.139}$$

- At order $\mathcal{O}(g^2)$

The matching condition is, diagrammatically



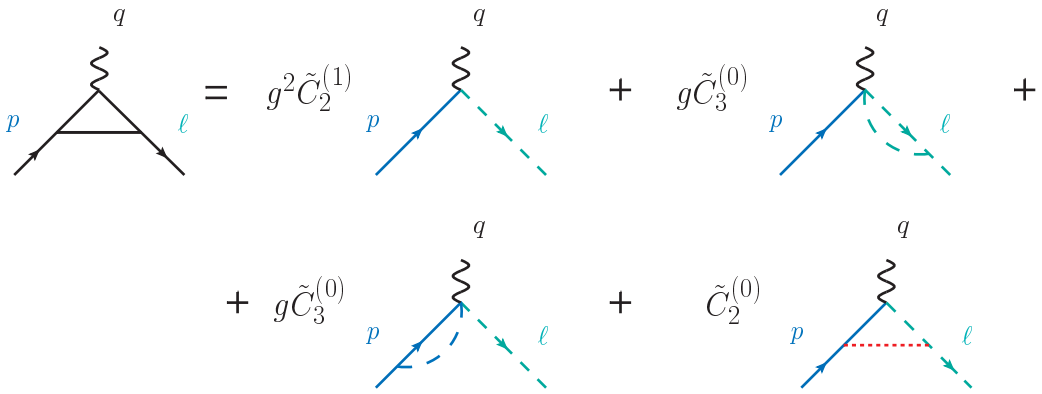
This looks complicated but for $p^2=l^2=0$ all diagrams on the r.h.s. except for the first one vanish. E.g. the last diagram corresponds to the soft region integral (in the region decomposition) with on-shell momenta and vanishes. At the same time Since $p^2=l^2=0$ the diagram on the l.h.s. for $p^2=l^2=0$ becomes identical to the contribution of the hard region in (10.77). Thus we get

$$\tilde{C}_2^{(1)}\propto I_h=\frac{1}{\mu^2}\Gamma(3-\tfrac{d}{2})\frac{\Gamma^2(\frac{d}{2}-2)}{\Gamma(d-3)}\left(\frac{\mu^2}{l_+p_-}\right)^{3-\frac{d}{2}}\,,\qquad l_+p_-=Q^2 \tag{10.140}$$

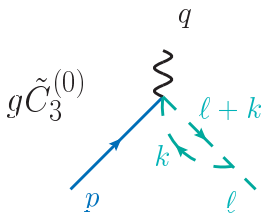
cf. (10.78).

10.4.3 Sudakov form factor in scalar SCET

Let us return to the Sudakov form factor: we want to employ SCET machinery to calculate the triangle diagram that we did before using the strategy of regions. One obtains at order $\mathcal{O}(g^2)$



The diagrams are the same as above in the matching calculation, but now we take p^2 and l^2 small but nonzero. It is straightforward to check that the four diagrams on the r.h.s. exactly correspond to the hard-region integral, the two collinear region integrals, and the soft-region integral which were found with the strategy of regions. For example, consider



$$i\int d^dx\, e^{iqx}\langle l|J(x)|p\rangle \ni i\int d^dx\, e^{iqx}\langle l|\left\{i\int d^dy\left(-\frac{g}{6}\right))\phi_{\bar{c}}^3(y)\right\}\left\{g\int_{-\infty}^0 ds\int_0^\infty dt\,\phi_c(x+s\bar{n})\phi_{\bar{c}}(x)\phi_{\bar{c}}(x+tn)\right\}|p\rangle \tag{10.141}$$

Use $\phi_c(x+s\bar{n})|p\rangle = e^{-ip(x+s\bar{n})}$, $\langle l|\phi_{\bar{c}}(y) = e^{ily}$ and usual expressions for the propagators:

$$T\{\overbrace{\phi(x)}\phi(y)\} = \int\frac{d^dk}{(2\pi)^d}e^{-ik(x-y)}\frac{i}{k^2+i0} \tag{10.142}$$

Get

$$\ldots = -g^2\int d^dx\, e^{iqx}\int d^dy\int_{-\infty}^0 ds\int_0^\infty dt\, e^{-ip(x+s\bar{n})}e^{ily}\int\frac{d^dk_1}{(2\pi)^d}\frac{e^{-ik_1(y-x)}}{k_1^2+i0}\int\frac{d^dk_2}{(2\pi)^d}\frac{e^{-ik_2(x+tn-y)}}{k_2^2+i0} \tag{10.143}$$

Here use

$$\int_{-\infty}^0 ds\int_0^\infty dt\, e^{-is(p\cdot\bar{n})}e^{-it(k_2\cdot n)} = \frac{1}{(p\cdot\bar{n})(k_2\cdot n)} \equiv \frac{1}{2(p_-\cdot(k_2)_+)} \tag{10.144}$$

The integrals over x and y produce δ-functions: $(2\pi)^d\delta^d(-p+k_1-k_2+q)$ and $(2\pi)^d\delta^d(l-k_1+k_2)$, respectively, so that one gets, renaming $k_2\to k$

$$\ldots = -g^2(2\pi)^d\delta^4(q-p+l)\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+i0)[(k+l)^2+i0]}\underbrace{\frac{1}{2(p_-\cdot k_+)+i0}}_{C_3^{(0)}} \tag{10.145}$$

This is exactly the integral for the (anti)collinear region $I_{\bar c}$, cf. (10.86).

For order-by-order calculations, the direct application of the strategy of regions is more efficient. However, SCET allows one to study all-order properties of scattering amplitudes, such as factorization theorems (an example follows), and also apply renormalization group equations to resum to all orders large logarithms of the ratio p^2/Q^2 (maybe later).

10.4.4 Factorization theorem for the Sudakov form factor in $d=6$

So far the space-time dimension was arbitrary. In this section we take $d=6$ in which case the coupling is dimensionless (which is what have in QCD, e.g.). Setting $\hbar=1$ the action of the theory has to be dimensionless:

$$S=\int d^dx\left[\frac{1}{2}\partial_\mu\phi(x)\,\partial^\mu\phi(x)-g\phi^3(x)\right], \tag{10.146}$$

for the mass dimensions of the field and the coupling, it follows that

$$\begin{aligned}
&[\phi]=\frac{d-2}{2}; \qquad [\phi]=1 \quad \text{in} \quad d=4, \qquad [\phi]=2 \quad \text{in} \quad d=6,\\
&[g]=\frac{6-d}{2}; \qquad [g]=1 \quad \text{in} \quad d=4, \qquad [g]=0 \quad \text{in} \quad d=6.
\end{aligned} \tag{10.147}$$

At this stage, we want to study how the various fields in the effective theory scale in terms of powers of λ in $d=6$. This will allow us to assign a power of λ to any operator in the effective theory to determine which parts of the effective Lagrangian contribute at a given order; this process is called *power counting*. To this end, consider the propagators

$$\begin{aligned}
&\langle 0|\,T\{\phi_c(x)\phi_c(0)\}\,|0\rangle \sim \int \underbrace{d^6p}_{\lambda^6}\,\underbrace{e^{-ip\cdot x}}_{\lambda^0}\,\underbrace{\frac{i}{p^2}}_{\lambda^{-2}} \sim \lambda^4, \qquad p^\mu\sim\{\lambda^2,1,\underbrace{\lambda,\lambda,\lambda,\lambda}_{d-2\text{"perp" directions}}\}Q\\
&\langle 0|\,T\{\phi_s(x)\phi_s(0)\}\,|0\rangle \sim \int \underbrace{d^6p}_{\lambda^{12}}\,\underbrace{e^{-ip\cdot x}}_{\lambda^0}\,\underbrace{\frac{i}{p^2}}_{\lambda^{-4}} \sim \lambda^8, \qquad p^\mu\sim\{\lambda^2,\lambda^2,\underbrace{\lambda^2,\lambda^2,\lambda^2,\lambda^2}_{d-2\text{"perp" directions}}\}Q
\end{aligned} \tag{10.148}$$

where from conclude that the collinear and soft fields scale differently

$$\phi_c\sim\lambda^2, \qquad \phi_s\sim\lambda^4. \tag{10.149}$$

For different terms which appear in the SCET Lagrangian, keeping in mind that the scaling of the integration measure is given by the components of x, the conjugate variable to p, one finds

$$\begin{aligned}
&\int d^6x\, \tfrac{1}{2}\partial_\mu\phi_c(x)\,\partial^\mu\phi_c(x) &&\sim \tfrac{1}{\lambda^6}\left(\lambda\lambda^2\right)^2 &&= \lambda^0\,,\\
&\int d^6x\, \tfrac{1}{2}\partial_\mu\phi_s(x)\,\partial^\mu\phi_s(x) &&\sim \tfrac{1}{\lambda^{12}}\left(\lambda^2\lambda^4\right)^2 &&= \lambda^0\,,\\
&\int d^6x\, g\,\phi_c^3(x) &&\sim \tfrac{1}{\lambda^6}\left(\lambda^2\right)^3 &&= \lambda^0\,,\\
&\int d^6x\, g\,\phi_s^3(x) &&\sim \tfrac{1}{\lambda^{12}}\left(\lambda^4\right)^3 &&= \lambda^0\,,\\
&\int d^6x\, g\,\phi_c^2(x)\,\phi_s(x) &&\sim \tfrac{1}{\lambda^6}\left(\lambda^2\right)^2\lambda^4 &&= \lambda^2 \quad\Longrightarrow\quad \text{Suppressed}\,.
\end{aligned} \tag{10.150}$$

In turn, for the contributions to the matching for the current $J(x)$

$$\begin{aligned}
&\int d^6x\,\phi_c(x)\,\phi_{\bar c}(x) &&\sim \frac{1}{\lambda^4}\lambda^2\lambda^2 = \lambda^0,\\
&\int d^6x\,\phi_c^2(x)\,\phi_{\bar c}(x) &&\sim \frac{1}{\lambda^4}\left(\lambda^2\right)^2\lambda^2 = \lambda^2 \quad\Longrightarrow\quad \text{Suppressed}\,,\\
&\int d^6x\,\phi_c(x)\,\phi_{\bar c}(x)\,\phi_s(x) &&\sim \frac{1}{\lambda^4}\left(\lambda^2\right)^2\lambda^4 = \lambda^4 \quad\Longrightarrow\quad \text{Suppressed}\,.
\end{aligned} \tag{10.151}$$

[Observe that the integration measure d^6x scales in this case as $1/\lambda^4$ because the plus and minus components of x^μ are both $\sim\lambda^0$, as they are conjugate to a momentum which is a sum of l-collinear and p-collinear momenta. Therefore $d^6x\sim(p_\perp)^{-4}\sim\lambda^{-4}$.]
We conclude that to leading power of λ only the two-particle contributions (still nonlocal!) have to be retained.

In summary, in six dimensions the Lagrangian simplifies to

$$\int d^6x\,\mathcal{L}_{\rm SCET} = \int d^6x\,\left[\mathcal{L}_c+\mathcal{L}_{\bar c}+\mathcal{L}_s\right]+\mathcal{O}(\lambda^2)\,, \tag{10.152}$$

while for the current one finds

$$\int d^6x\, e^{iqx}\,J(x)\to\int d^6x\, e^{iqx}\iint ds\,dt\,C_2(s,t,\mu)\,\phi_c\left(x+s\bar n\right)\phi_{\bar c}\left(x+tn\right)+\mathcal{O}\left(\lambda^2\right). \tag{10.153}$$

- **Factorization theorem**

Consider the three-point Green function

$$\begin{aligned}
G(p,l,\mu) &= \int d^6x_1\, e^{-ip\cdot x_1}\int d^6x_2\, e^{il\cdot x_2}\langle 0|T\left\{\phi_c(x_1)J(0)\phi_{\bar c}(x_2)\right\}|0\rangle\\
&= \int d^6x_1\, e^{-ip\cdot x_1}\int d^6x_2\, e^{il\cdot x_2}\iint ds\,dt\,C_2(s,t,\mu)\langle 0|T\left\{\phi_c(x_1)\phi_c(s\bar n)\phi_{\bar c}(tn)\phi_{\bar c}(x_2)\right\}|0\rangle\,,\\
&= \int d^6x_1\, e^{-ip\cdot x_1}\int d^6x_2\, e^{il\cdot x_2}\iint ds\,dt\,C_2(s,t,\mu)\langle 0|T\left\{\phi_c(x_1)\phi_c(s\bar n)\right\}|0\rangle\langle 0|T\left\{\phi_{\bar c}(tn)\phi_{\bar c}(x_2)\right\}|0\rangle\,,
\end{aligned} \tag{10.154}$$

Since the soft-collinear interactions are power-suppressed, the fields ϕ_c and $\phi_{\bar{c}}$ do not interact with each other. Up to power suppressed terms, we now deal with two separate theories and the matrix element in the first line reduces to the correlator of the ϕ_c fields times a correlator of the $\phi_{\bar{c}}$ fields.

Translation invariance implies that

$$\begin{aligned}\langle 0|T\left\{\phi_c(x_1)\phi_c(s\bar{n})\right\}|0\rangle &= \langle 0|T\left\{\phi_c(x_1-s\bar{n})\phi_c(0)\right\}|0\rangle\,,\\ \langle 0|T\left\{\phi_{\bar{c}}(tn)\phi_{\bar{c}}(x_2)\right\}|0\rangle &= \langle 0|T\left\{\phi_{\bar{c}}(0)\phi_{\bar{c}}(x_2-tn)\right\}|0\rangle\end{aligned} \tag{10.155}$$

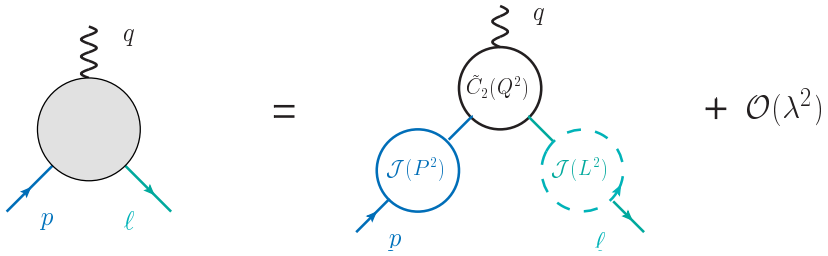
Changing variables $x_1 \to x_1 + s\bar{n}$ and $x_2 \to x_2 + tn$, one obtains

$$\begin{aligned}G(p,l,\mu) &= \iint ds\,dt\,C_2(s,t,\mu)e^{-isp\cdot\bar{n}+itl\cdot n}\mathcal{J}\left(p^2,\mu\right)\mathcal{J}\left(l^2,\mu\right)\\ &= \widetilde{C}_2(\underbrace{(p\cdot\bar{n})(l\cdot n)}_{Q^2},\mu)\mathcal{J}\left(p^2,\mu\right)\mathcal{J}\left(l^2,\mu\right)\end{aligned} \tag{10.156}$$

with

$$\begin{aligned}\mathcal{J}\left(p^2,\mu\right) &\equiv \int d^6x_1\,e^{-ip\cdot x_1}\langle 0|T\left\{\phi_c(x_1)\phi_c(0)\right\}|0\rangle\,,\\ \mathcal{J}\left(l^2,\mu\right) &\equiv \int d^6x_2\,e^{il\cdot x_2}\langle 0|T\left\{\phi_{\bar{c}}(0)\phi_{\bar{c}}(x_2)\right\}|0\rangle\,.\end{aligned} \tag{10.157}$$

Diagrammatically:



We have factorized the Green function G into a product of a *hard function* $\tilde{C}_2$ and two *jet functions* $\mathcal{J}$. In other words, we have managed to factor a function of three variables into a product of three functions of a single variable. All entries can be calculated within the full theory since the collinear Lagrangian is identical to the complete ϕ^3 Lagrangian, and the hard function coincides with the form factor at $p^2 - l^2 = 0$.

To one-loop accuracy our result is consistent with the calculation of the triangle diagram above, which is given by the above Green function with amputated legs, e.g. removing the jet functions. To see that, notice that for $d = 6$ [‡]

$$I_h \sim \Gamma(3-\tfrac{d}{2})\frac{\Gamma^2(\frac{d}{2}-2)}{\Gamma(d-3)}\left(\frac{\mu^2}{Q^2}\right)^{3-\frac{d}{2}} = \mathcal{O}(\lambda^0)\,. \tag{10.158}$$

and it is easy to check that

$$I_c \sim \frac{P^2}{Q^2} = \mathcal{O}(\lambda^2)\,, \qquad I_{\bar{c}} \sim \frac{L^2}{Q^2} = \mathcal{O}(\lambda^2)\,, \qquad I_s \sim \frac{P^2L^2}{Q^4} = \mathcal{O}(\lambda^4)\,, \tag{10.159}$$

[‡]We have to replace in (10.78) $\varepsilon \mapsto 2 - d/2$ in order that it is valid for all dimensions (and adjust the power of μ^2)

so that collinear and soft regions all produce power corrections and can be dropped. The hard contribution, in turn, develops an UV divergence and to logarithmic accuracy is $\sim \ln Q^2/\mu^2$. [§] In other words, the triangle diagram in $d = 6$ remains finite at $P^2 \to 0$ and $L^2 \to 0$ and coincides with the hard region contribution alone (up to power corrections). The only logarithms that one encounters in higher orders will be of the form

$$\left(\frac{g^2}{(4\pi)^3}\right)^n \ln^n\left(\frac{\mu_{UV}^2}{Q^2}\right).$$

This suppression of collinear and soft contributions in the Sudakov form factor is a pecularity of the $d = 6$ theory, it will be more complicated for QCD.

10.5 SCET in QCD

In the QCD case three complications arise:

i) Different components of the quark field $\psi(x)$ and of the gluon field $A_\mu(x)$ scale differently with the expansion parameter λ;

ii) The theory is invariant under gauge transformations, but it is necessary to make sure that they respect the scaling of the fields;

iii) Non-local operators involve Wilson lines to ensure gauge invariance.

10.5.1 Power counting and effective Lagrangian

- The starting point is similar. We split

$$\begin{aligned}
\psi(x) &\to \psi_h(x) + \psi_c(x) + \psi_{\bar c}(x) + \psi_s(x)\,,\\
\bar\psi(x) &\to \bar\psi_h(x) + \bar\psi_c(x) + \bar\psi_{\bar c}(x) + \bar\psi_s(x)\,,\\
A^\mu(x) &\to A_h^\mu(x) + A_c^\mu(x) + A_{\bar c}^\mu(x) + A_s^\mu(x)\,.
\end{aligned} \tag{10.160}$$

Contributions of the "hard" fields will be integrated out and absorbed in the coefficient functions of the operators built of soft and collinear fields.

At tree level we can ignore the hard fields and simply substitute these expressions in the QCD Lagrangian. At loop level, matching corrections can arise, but will only affect contributions containing both c and $\bar c$ fields (as in scalar case).

- The first difference is that fermion and gauge fields have several components, which scale differently with the expansion parameter λ. To read off the scaling of the fields one can consider the propagators. Start with gluon:

$$\langle 0|\,T\Big\{A_\mu^a(x)A_\nu^b(0)\Big\}\,|0\rangle = \delta^{ab}\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i0}e^{-ip\cdot x}\left[-g_{\mu\nu}+(1-\xi)\frac{p_\mu p_\nu}{p^2}\right]\delta^{ab}\,. \tag{10.161}$$

[§]Note that $\Gamma(3-\frac{d}{2})\Gamma(\frac{d}{2}-2) = \Gamma(3-\frac{d}{2})(\frac{d}{2}-3)\Gamma(\frac{d}{2}-3) = -\Gamma(4-\frac{d}{2})\Gamma(\frac{d}{2}-3)$ so that from the structure of Γ-functions alone, one cannot tell whether the divergence at $d \to 6$ is IR or UV, one needs a separate analysis.

The position argument x^μ of the fields is conjugate to the momentum p^μ in the Fourier exponent $p\cdot x \sim \mathcal{O}(\lambda^0)$. The part of the propagator involving the gauge parameter scales like $d^4p/(p^2)^2\, p_\mu p_\nu \sim p_\mu p_\nu$. In a generic covariant gauge, the gluon field thus scales exactly like the momentum $A_\mu \sim p_\mu$:

$$\begin{aligned}
(n\cdot A_s\,,\, \bar n\cdot A_s\,,\, A_{s\perp}^\mu) &\sim (\lambda^2\,\lambda^2\,\lambda^2)\,,\\
(n\cdot A_c\,,\, \bar n\cdot A_c\,,\, A_{c\perp}{}^\mu) &\sim (\lambda^2,\,1\,,\lambda)\,,\\
(n\cdot A_{\bar c}\,,\, \bar n\cdot A_{\bar c}\,,\, A_{\bar c\perp}{}^\mu) &\sim (1\,,\lambda^2\,,\lambda)\,.
\end{aligned} \tag{10.162}$$

From the scaling we immediately see that in terms involving both s and c gluons, the soft gluons are power suppressed, except for the $n\cdot A_s$ component, and in terms with s and $\bar c$ gluons only the $\bar n\cdot A_s$ will be relevant.

- Next, consider the fermion propagators. For a soft fermion

$$\langle 0|\,T\{\psi_s(x)\bar\psi_s(0)\}\,|0\rangle = \int\frac{d^4p}{(2\pi)^4}\frac{i\not p}{p^2+i0}e^{-ip\cdot x} \sim (\lambda^2)^4\frac{\lambda^2}{\lambda^4} = \lambda^6\,, \tag{10.163}$$

so that the soft fermions scale as $\psi_s(x)\sim\lambda^3$. The collinear case is more complicated because different momenta components scale differently, and numerator of the propagator must be decomposed into

$$\not p = \underbrace{n\cdot p}_{\mathcal{O}(\lambda^2)}\frac{\not{\bar n}}{2} + \underbrace{\bar n\cdot p}_{\mathcal{O}(\lambda^0)}\frac{\not n}{2} + \underbrace{\not p^\perp}_{\mathcal{O}(\lambda^1)}\,. \tag{10.164}$$

This implies that different parts of the fermion spinor scale differently. Write

$$\psi_c = P_+\,\psi_c + P_-\,\psi_c \equiv \xi_c + \eta_c \tag{10.165}$$

using the projection operators

$$P_+ = \frac{\not n\not{\bar n}}{4}\,,\qquad P_- = \frac{\not{\bar n}\not n}{4}\,,\qquad \text{such that}\quad P_+ + P_- = 1\,,\qquad P_+P_- = 0\,,\qquad P_\pm^2 = P_\pm \tag{10.166}$$

Then

$$\begin{aligned}
\langle 0|T\{\xi_c(x)\bar\xi_c(0)\}|0\rangle &= \int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot x}\frac{\not n\not{\bar n}}{4}\frac{i\not p}{p^2+i0}\frac{\not{\bar n}\not n}{4} = \int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot x}\frac{i\bar n\cdot p\frac{\not n}{2}}{p^2+i0} \sim \lambda^4\frac{1}{\lambda^2} = \lambda^2\,,\\
\langle 0|T\{\eta_c(x)\bar\eta_c(0)\}|0\rangle &= \int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot x}\frac{\not n\not{\bar n}}{4}\frac{i\not p}{p^2+i0}\frac{\not{\bar n}\not n}{4} = \int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot x}\frac{in\cdot p\frac{\not{\bar n}}{2}}{p^2+i0} \sim \lambda^4\frac{1}{\lambda^2}\lambda^2 = \lambda^4\,,
\end{aligned} \tag{10.167}$$

where from

$$\xi_c\sim\lambda\,,\qquad \eta_c\sim\lambda^2\,,\qquad \psi_s(x)\sim\lambda^3 \tag{10.168}$$

Observe that soft fermions are suppressed compared to the collinear ones.

- Now that we know the scaling of all fields, we can plug the decomposition (10.160) into the QCD action and read off the tree-level effective field theory Lagrangian,

$$S = S_s + S_c + S_{\bar{c}} + S_{c+s} + S_{\bar{c}+s} + \dots \tag{10.169}$$

The purely soft part S_s
is just the usual QCD action, except that the fields are replaced with their soft parts:

$$S_s = \int d^4x \left\{ \bar{\psi}_s i \not{D}_s \psi_s - \frac{1}{4} (F_s^a)_{\mu\nu} (F_s^a)^{\mu\nu} \right\}, \tag{10.170}$$

with

$$D_s^\mu = \partial^\mu - ig A_s^\mu \,, \qquad [D_s^\mu, D_s^\nu] = -ig (F_s^a t^a)_{\mu\nu} \,. \tag{10.171}$$

Here $\int d^4x \sim \lambda^{-8}$ and it is easy to check that all terms are of order $\mathcal{O}(\lambda^0)$, no further simplification is possible.

The purely collinear part S_c
is again a simple copy of the QCD action, but we decompose the fermion field $\psi_c = \xi_c + \eta_c$ and get

$$\begin{aligned} S_c &= \int d^4x \, (\bar{\xi}_c + \bar{\eta}_c) \left[\frac{\not{n}}{2} i\bar{n}\cdot D_c + \frac{\not{\bar{n}}}{2} in\cdot D_c + i\not{D}_{c\perp} \right] (\xi_c + \eta_c) - \int d^4x \, \frac{1}{4} (F_c^a)_{\mu\nu} (F_c^a)^{\mu\nu} \\ &= \int d^4x \left[\bar{\xi}_c \frac{\not{\bar{n}}}{2} in\cdot D_c \, \xi_c + \bar{\xi}_c \, i\not{D}_{c\perp} \, \eta_c + \bar{\eta}_c \, i\not{D}_{c\perp} \, \xi_c + \bar{\eta}_c \frac{\not{n}}{2} i\bar{n}\cdot D_c \, \eta_c \right] - \int d^4x \, \frac{1}{4} (F_c^a)_{\mu\nu} (F_c^a)^{\mu\nu} \,. \end{aligned} \tag{10.172}$$

where of course

$$D_c^\mu = \partial^\mu - ig A_c^\mu \,, \qquad [D_c^\mu, D_c^\nu] = -ig (F_c^a t^a)_{\mu\nu} \,. \tag{10.173}$$

Here $\int d^4x \sim \lambda^{-4}$ and, again, all terms are of order $\mathcal{O}(\lambda^0)$. Thus all terms have to be kept, but this form of the Lagrangian is inconvenient since it involves both the large component $\xi_c \sim \lambda$ and the power suppressed field $\eta_c \sim \lambda^2$ and the two mix into each other.

This complication can be avoided by integrating out the small components of the fermion field, similar to how one integrates out the H_v field in HQET. To do so, we can rewrite the fermion part of the Lagrangian as[¶]

$$\bar{\xi}_c \frac{\not{\bar{n}}}{2} \left[in\cdot D_c + i\not{D}_{c\perp} \frac{1}{i\bar{n}\cdot D_c} i\not{D}_{c\perp} \right] \xi_c + \left(\bar{\eta}_c - \bar{\xi}_c \, i\not{D}_{c\perp} \frac{1}{i\bar{n}\cdot D_c} \frac{\not{\bar{n}}}{2} \right) \frac{\not{n}}{2} i\bar{n}\cdot D_c \left(\eta_c + \frac{\not{\bar{n}}}{2} \frac{1}{i\bar{n}\cdot D_c} i\not{D}_{c\perp} \, \xi_c \right)$$

and shift the field

$$\eta_c \mapsto \eta_c - \frac{\not{\bar{n}}}{2} \frac{1}{i\bar{n}\cdot D_c} i\not{D}_{c\perp} \, \xi_c \tag{10.174}$$

[¶] care has to be taken to define the inverse operator $\frac{1}{i\bar{n}\cdot D_c}$, e.g. as $\frac{1}{i\bar{n}\cdot D_c + i0}$

After the shift, the fermion part of the Lagrangian takes the form

$$\bar{\xi}_c \frac{\bar{n}\!\!\!/}{2}\left[in\cdot D_c + i\not{D}_{c\perp}\frac{1}{i\bar{n}\cdot D_c} i\not{D}_{c\perp}\right]\xi_c + \bar{\eta}_c \frac{n\!\!\!/}{2} i\bar{n}\cdot D_c\,\eta_c \tag{10.175}$$

and the mixing between ξ_c and η_c is eliminated.

Assuming there are no η_c external lines, the remaining $\bar{\eta}_c \frac{n\!\!\!/}{2} i\bar{n}\cdot D_c\,\eta_c$ only produces closed loops (in gluon background), which all vanish because singularities of the η_c propagator appear to be always on the same side of the integration contour. Thus finally

$$S_p = \int d^4x \left\{\bar{\xi}_c \frac{\bar{n}\!\!\!/}{2}\left[in\cdot D_c + i\not{D}_{c\perp}\frac{1}{i\bar{n}\cdot D_c} i\not{D}_{c\perp}\right]\xi_c - \frac{1}{4}(F_c^a)_{\mu\nu}(F_c^a)^{\mu\nu}\right\} \tag{10.176}$$

The soft-collinear interaction term S_{c+s}

The soft-collinear interactions in ϕ^3 theory were obtained by replacing

$$-\frac{g}{3!}\int d^4x\,\phi^3(x) \longrightarrow -\frac{g}{2!}\int d^4x\,\phi_c^2(x)\phi_s(x_-)\,. \tag{10.177}$$

In QCD it is more tricky. Remind that

$$\begin{aligned}(n\cdot A_c, \bar{n}\cdot A_c, A_{c\perp}) &\sim (\lambda^2, 1, \lambda)\,,\\ (n\cdot A_s, \bar{n}\cdot A_s, A_{s\perp}) &\sim (\lambda^2, \lambda^2, \lambda^2)\,,\\ \xi_c &\sim \lambda\,, \qquad \psi_s \sim \lambda^3\,.\end{aligned} \tag{10.178}$$

Thus

- ψ_s is power suppressed compared to the collinear fermion fields, so that no soft quarks are present in leading-power interactions with collinear fields.
- Since $\bar{n}\cdot A_s$ and $A_{s\perp}^\mu$ are power-suppressed compared to their collinear counterparts, only the component $n\cdot A_s$ has to be added.

Therefore to leading power one only needs to replace

$$A^\mu(x) \longrightarrow (n\cdot A_c(x) + n\cdot A_s(x_-))\frac{\bar{n}^\mu}{2} + \bar{n}\cdot A_c(x)\frac{n^\mu}{2} + A_{c\perp}^\mu(x) \tag{10.179}$$

in the quark and gluon collinear Lagrangians discussed above. Note $A_s(x) \to A_s(x_-)$: E.g. for soft gluons interacting with collinear quarks

$$\begin{aligned}S_{c+s} &\ni \int d^4x\,\bar{\xi}_c(x)\frac{\bar{n}\!\!\!/}{2} n\cdot A_s(x)\xi_c(x)\\ &= \int d^4x\,\bar{\xi}_c(x)\frac{\bar{n}\!\!\!/}{2}\Big[1 + \underbrace{x_\perp\cdot\partial_\perp}_{\mathcal{O}(\lambda)} + \underbrace{x_+\cdot\partial_-}_{\mathcal{O}(\lambda^2)} + \ldots\Big] n\cdot A_s(x)\Big|_{x=x_-}\xi_c(x)\\ &= \int d^4x\,\bar{\xi}_c(x)\frac{\bar{n}\!\!\!/}{2} n\cdot A_s(x_-)\xi_c(x) + \mathcal{O}(\lambda)\,.\end{aligned} \tag{10.180}$$

This is similar to scalar theory.

To summarize, the SCET Lagrangian for QCD can be written in a compact form as follows

$$\mathcal{L}_{\rm SCET} = \bar\psi_s i\not{D}_s \psi_s + \bar\xi \frac{\not{\bar n}}{2}\left[in\cdot\mathbf{D} + i\not{D}_{c\perp}\frac{1}{i\bar n\cdot D_c} i\not{D}_{c\perp}\right]\xi - \frac{1}{4}\left(F^{s,a}_{\mu\nu}\right)^2 - \frac{1}{4}\left(\mathbf{F}^{c,a}_{\mu\nu}\right)^2 . \quad (10.181)$$

The covariant derivatives are defined as

$$iD^s_\mu = i\partial_\mu + gA^s_\mu , \qquad iD^c_\mu = i\partial_\mu + gA^c_\mu , \qquad i\mathbf{D}_\mu = iD^c_\mu + n\cdot A_s(x_-)\frac{\bar n_\mu}{2}$$
$$in\cdot\mathbf{D} = in\cdot D^c_\mu + g\, n\cdot A_s(x_-) = in\cdot\partial + g\, n\cdot A_c(x) + g\, n\cdot A_s(x_-) . \quad (10.182)$$

The field strengths are

$$igF^s_{\mu\nu} = \left[iD^s_\mu, iD^s_\nu\right] , \qquad igF^c_{\mu\nu} = \left[i\mathbf{D}_\mu, i\mathbf{D}_\nu\right] . \quad (10.183)$$

Note that the $\left(\mathbf{F}^a_{\mu\nu}\right)^2$ term includes interaction of soft and collinear gluon fields.

As written above, the SCET Lagrangian includes only one collinear sector, but in practical applications one needs two or more. As in the scalar ϕ^3 theory, we will in the following consider two collinear momenta $p \sim (\lambda^2, 1, \lambda)$ and $l \sim (1, \lambda^2, \lambda)$. The l-collinear sector in the Lagrangian can be obtained by replacing $n^\mu \leftrightarrow \bar n^\mu$ (which implies $x_+ \leftrightarrow x_-$) in the p-collinear sector.

10.5.2 Gauge transformations

QCD is a gauge-invariant theory. However, in the same way in which we expanded the Lagrangian, it is necessary to expand the gauge trafos: we want to avoid the situation when, e.g. a gauge trafo converts a soft field into a collinear field. Thus it makes sense to split all possible trafos into two classes: the soft gauge transformation

$$V_s(x) = \exp\left[i\alpha^a_s(x)t^a\right] , \quad (10.184)$$

and the collinear gauge trafo

$$V_c(x) = \exp\left[i\alpha^a_c(x)t^a\right] , \quad (10.185)$$

where the difference is that $\alpha_s(x)$ is a slowly-varying function in all directions, i.e. $\partial_\mu\alpha_s(x) \sim (\lambda^2, \lambda^2, \lambda^2)\alpha_s(x)$, whereas $\alpha_c(x)$ has a collinear scaling, i.e. $\partial_\mu\alpha_c(x) \sim (\lambda^2, 1, \lambda)\alpha_c(x)$. We have to *define* the action of these trafos on our fields.

- **Soft gauge trafos**

We choose

$$\psi_s(x) \to V_s(x)\psi_s(x) ,$$
$$A^\mu_s(x) \to V_s(x)A^\mu_s(x)V^\dagger_s(x) + \frac{i}{g}V_s(x)\partial^\mu V^\dagger_s(x) , \quad (10.186)$$

which is the standard trafo for the full fields. This is natural because we want out theory to remain gauge-invariant also if collinear fields are absent.

The collinear fields transform under soft trafos as follows

$$\begin{aligned}\xi(x) &\to V_s(x_-)\xi(x)\,,\\ A_c^\mu(x) &\to V_s(x_-)A_c^\mu(x)V_s^\dagger(x_-)\,.\end{aligned} \tag{10.187}$$

- There is no term with a "short" derivative of the gauge parameter: The sum $A_s^\mu(x)+A_p^\mu(x)$ must transform a full QCD field, and $V_s(x)\partial^\mu V_s^\dagger(x)$ is already included in (10.186).
- $V_s(x)\mapsto V_s(x_-)$. When transforming the collinear fields, one needs to expand the soft fields around x_- in order to avoid inducing higher order corrections:

$$V_s(x)=V_s(x_-)+\underbrace{x_\perp\cdot\partial_\perp V_s(x_-)}_{\mathcal{O}(\lambda)}+\mathcal{O}(\lambda^2)\,. \tag{10.188}$$

Note that with these definitions the "fat" covariant derivative $\mathbf{D}$ transforms as expected:

$$\begin{aligned}n\cdot A_c(x)+n\cdot A_s(x_-) &\to V_s(x_-)\,[n\cdot A_c(x)+n\cdot A_s(x_-)]\,V_s^\dagger(x_-)+\frac{i}{g}V_s(x_-)(n\cdot\partial)V_s^\dagger(x_-)\,,\\ in\cdot\mathbf{D} &\to V_s(x_-)\,in\cdot\mathbf{D}\,V_s^\dagger(x_-)\,.\end{aligned} \tag{10.189}$$

- **Collinear gauge trafos**

We cannot allow the soft fields to be transform under collinear trafos, as they would acquire large momenta. Thus define

$$\psi_s(x)\to\psi_s(x)\,,\qquad A_s^\mu(x)\to A_s^\mu(x)\,. \tag{10.190}$$

The collinear fields instead transform as follows

$$\begin{aligned}\xi(x) &\to V_c(x)\xi(x)\,,\\ A_c^\mu(x) &\to V_c(x)A_c^\mu(x)V_c^\dagger(x)+\frac{1}{g}V_c(x)\Big(i\partial^\mu+g\frac{\bar n^\mu}{2}n\cdot A_s(x_-)\Big)V_c^\dagger(x)\,,\end{aligned} \tag{10.191}$$

With these definitions the covariant derivatives transform properly under collinear trafos as well:

$$\mathbf{D}^\mu\to V_c(x)\,\mathbf{D}^\mu\,V_c^\dagger(x)\,,\qquad\qquad D_s^\mu\to D_s^\mu \tag{10.192}$$

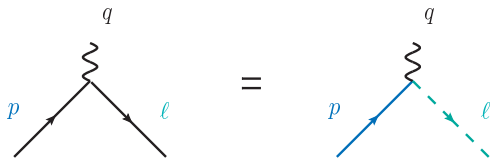
It is now straightforward to check that the Lagrangian in Eq. (10.181) is separately invariant under soft and collinear gauge transformations.

10.5.3 The vector current in SCET

In the Sudakov problem and many other applications we need the electromagnetic current operator connecting the collinear and anticollinear fields. The necessary hard momentum transfer is provided by the virtual photon on the external line. To leading order, obviously

$$J^\mu(x)=\bar\psi(x)\gamma^\mu\psi(x)\mapsto\bar\xi_c(x)\gamma^\mu\xi_{\bar c}(x)+\bar\xi_{\bar c}(x)\gamma^\mu\xi_c(x) \tag{10.193}$$

and we have to pick up one term depending on which particle (p-collinear or l-collinear) is in the initial/final state; e.g.



Since $\not{n}\xi_c = 0$ and $\not{\bar{n}}\xi_{\bar{c}} = 0$, the SCET operator can further be simplified as

$$\bar{\xi}_c\,\gamma^\mu\,\xi_{\bar{c}} = \bar{\xi}_c\left[n^\mu\frac{\not{\bar{n}}}{2} + \bar{n}^\mu\frac{\not{n}}{2} + \gamma_\perp^\mu\right]\xi_{\bar{c}} = \bar{\xi}_c\,\gamma_\perp^\mu\,\xi_{\bar{c}}\,. \tag{10.194}$$

Similar to scalar theory, including loop corections this expression will be modified including operators with derivatives in the directions of large momenta (and operators with more fields), so

$$J^\mu(x) \mapsto \int_{-\infty}^{+\infty}\!\!\!\!\!\!\int dsdt\, C_V(s,t,\mu)\,\bar{\xi}_c\,(x+s\bar{n})\gamma_\perp^\mu\xi_{\bar{c}}\,(x+tn) + \dots \tag{10.195}$$

with

$$C_V(s,t,\mu) = \delta(s)\delta(t) + \frac{\alpha_s}{4\pi}C_V^{(1)}(s,t,\mu) + \dots\,, \tag{10.196}$$

however, as written, this expression is not gauge-invariant. One should expect that gauge invariance will be restored adding three-particle contribuions with extra gluon fields, of the type $\xi_{\bar{c}}(x)\,\bar{n}\cdot A_c\,\xi_c(x)$ or $\xi_{\bar{c}}(x)\,n\cdot A_c\,\xi_c(x)$.

We had a very similar situation already with the definition of parton distributions (PDFs) and found the gauge invariance is restored by adding a Wilson line connecting the quark and the antiquark fields:

$$\langle N(p)|\bar{\psi}(x)\not{x}[x,0]\psi(0)|N(p)\rangle\Big|_{x^2=0} \;=\; 2(p\cdot x)\int_{-1}^{1}du\,e^{iupx}F(u) \tag{10.197}$$

with

$$[x,y] \;=\; \text{Pexp}\left[ig\int_0^1 du\,(x-y)_\mu A^\mu(ux+\bar{u}y)\right] \qquad\qquad \bar{u}=1-u \tag{10.198}$$

Since in this definition $x^2 = 0$, we can take e.g. $x^\mu = t\bar{n}^\mu$ and use that

$$\bar{\psi}(t\bar{n})\not{\bar{n}}[t\bar{n},0]\psi(0) = \bar{\xi}_c(t\bar{n})\not{\bar{n}}[t\bar{n},0]_c\xi_c(0) + \mathcal{O}(\lambda^2)$$

so that a PDF is nothing but a matrix element of an operator built of two collinear fields

$$\langle N(p)|\bar{\xi}_c(t\bar{n})\not{\bar{n}}[t\bar{n},0]_c\xi_c(0)|N(p)\rangle \tag{10.199}$$

connected by a path-ordered Wilson line with

$$[t\bar{n},0]_c \;=\; \text{Pexp}\left[ig\int_0^t du\,(\bar{n}\cdot A_c)(u\bar{n})\right]. \tag{10.200}$$

In the SCET context, it is convenient to take into account such factors by a suitable field redefinition. Wilson lines satisfy (by definition) a so-called transitivity property

$$[x,y] = [x,z][z,y] \equiv [x,z][y,z]^\dagger \tag{10.201}$$

so that introducing

$$W_c(x) = [x, x - \infty\, \bar{n}]_c\,, \tag{10.202}$$

we can write the finite segment as a product

$$[x + t\bar{n}, x]_c = W_c(x + t\bar{n}) W_c^\dagger(x)\,. \tag{10.203}$$

In other words, to move from x to $x + t\bar{n}$, we first move from x to infinity and then from infinity back to $x + t\bar{n}$.

We can use this trick to write everything in terms of

$$\chi_c(x) \stackrel{!}{=} W_c^\dagger(x)\xi_c(x)\,, \qquad\qquad \mathcal{A}_c^\mu \stackrel{!}{=} W_c^\dagger D_c^\mu W_c \tag{10.204}$$

which are invariant under collinear gauge transformations which vanish at the (properly chosen) infinity. With these building blocks, we can then easily build gauge invariant SCET operators. For example, for the PDF

$$\bar{\xi}_c(t\bar{n})\bar{\not{n}}[t\bar{n},0]_c\xi_c(0) = \bar{\xi}_c(t\bar{n})W_c(t\bar{n})\bar{\not{n}}W_c^\dagger(0)\xi_c(0) = \bar{\chi}_c(t\bar{n})\bar{\not{n}}\chi_c(0) \tag{10.205}$$

and for the matching of the vector current

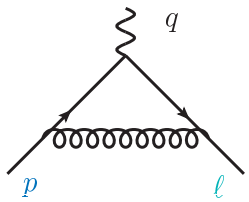
$$J^\mu(x) \mapsto \iint\limits_{-\infty}^{+\infty} dsdt\, C_V(s,t,\mu)\, \bar{\chi}_c\,(x + s\bar{n})\gamma_\perp^\mu \chi_{\bar{c}}\,(x + tn) + \ldots \tag{10.206}$$

where the dots correspond to gauge-invariant three-particle operators.

Similar to scalar theory, the simplest way to calculate

$$\tilde{C}_V^{(1)}(Q^2 = p_- l_+, \mu) = \iint dsdt\, e^{is\bar{n}\cdot p} e^{-itn\cdot l}\, C_V^{(1)}(s,t,\mu) \tag{10.207}$$

is to compute the vertex correction in QCD with on-shell momenta $p^2 = l^2 = 0$.



The answer in $d = 4 - 2\varepsilon$ reads

$$\tilde{C}_V^{\text{bare}}(\varepsilon, Q^2) = 1 + \frac{\alpha_s(\mu)}{4\pi} C_F \left(-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} + \frac{\pi^2}{6} - 8 + \mathcal{O}(\varepsilon)\right)\left(\frac{Q^2}{\mu^2}\right)^{-\varepsilon} + \mathcal{O}\left(\alpha_s^2\right)\,, \tag{10.208}$$

Despite appearance, this expression does no depend on μ since, by definition, $\alpha_s^{\rm bare} = \alpha_s(\mu)\mu^{2\varepsilon}(1+\mathcal{O}(\alpha_s))$. Renormalization is done as usual, by absorbing all divergences in a multiplicative Z-factor

$$\tilde{C}_V(Q^2,\mu) = \lim_{\varepsilon\to 0} Z^{-1}(\varepsilon,Q^2,\mu)\tilde{C}_V^{\rm bare}(\varepsilon,Q^2)$$

$$Z(\varepsilon,Q^2,\mu) = 1 + \frac{\alpha_s(\mu)}{4\pi}C_F\left(-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} + \underbrace{\frac{3}{\varepsilon}\ln\frac{Q^2}{\mu^2}}_{!}\right) \tag{10.209}$$

so that

$$\tilde{C}_V(Q^2,\mu) = 1 + \frac{\alpha_s(\mu)}{4\pi}C_F\left(-\ln^2\frac{Q^2}{\mu^2} + 3\ln\frac{Q^2}{\mu^2} + \frac{\pi^2}{6} - 8\right) + \mathcal{O}(\alpha_s^2)\,. \tag{10.210}$$

The whole procedure is the same as renormalization in standard quantum field theory, apart from that we have to deal with $1/\epsilon^2$ divergences, which arise because we have both soft and collinear divergences. As a consequence, $\tilde{C}_V(Q^2,\mu)$ contains double logs. Due to the presence of the double logs, the anomalous dimension governing the RG equation for $\tilde{C}_V(Q^2,\mu)$ has a logarithmic piece,

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s}\right)\tilde{C}_V(Q^2,\mu) = \left[\gamma_{\rm cusp}(\alpha_s)\ln\frac{Q^2}{\mu^2} + \gamma_V(\alpha_s)\right]\tilde{C}_V(Q^2,\mu)\,, \tag{10.211}$$

where, at order α_s, the functions $\gamma_{\rm cusp}$ and γ_V are given by

$$\gamma_{\rm cusp}(\alpha_s) = 4C_F\frac{\alpha_s(\mu)}{4\pi}\,, \qquad \text{and} \qquad \gamma_V(\alpha_s) = -6C_F\frac{\alpha_s(\mu)}{4\pi}\,. \tag{10.212}$$

The solution of this renormalization group equation is

$$\tilde{C}_V(Q^2,\mu) = \exp\left\{\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\left[\gamma_{\rm cusp}(\alpha_s(\mu'))\ln\frac{Q^2}{\mu'^2} + \gamma_V(\alpha_s(\mu'))\right]\right\}\tilde{C}_V(Q^2,\mu_0) \tag{10.213}$$

10.5.4 The decoupling transformation

In addition to the collinear Wilson line W_c (and its anticollinear analogue $W_{\bar{c}}$) we will now introduce Wilson lines built from soft fields:

$$S_n(x) = \mathbf{P}\exp\left[ig\int_{-\infty}^{0} ds\, n\cdot A_s(x+sn)\right]\,. \tag{10.214}$$

We will find that by a suitable redefinition of field operators it is possible to remove soft-collinear interaction terms from the Lagrangian, so that soft and collinear sectors of our effective theory will be essentially decoupled.

The corresponding term in the interaction between collinear quarks and soft gluons is

$$\mathcal{L}_{c+s} = \bar{\xi}\frac{\bar{n}\!\!\!/}{2}in\cdot\mathbf{D}\xi\,. \tag{10.215}$$

where the "fat" derivative is defined in Eq. (10.182). We now redefine the fields ξ and $A_c^\mu(x)$ employing the soft Wilson line defined in Eq. (10.214)

$$\begin{aligned}\xi(x) &\to S_n(x_-)\xi^{(0)}(x)\,,\\ A_c^\mu(x) &\to S_n(x_-)A_c^{(0)\mu}(x)S_n^\dagger(x_-)\,.\end{aligned} \tag{10.216}$$

Then

$$
\begin{aligned}
in\cdot\mathbf{D}\xi(x) &\to in\cdot\mathbf{D}' S_n(x_-)\xi^{(0)}(x)\\
&= \Big(in\cdot\partial + gn\cdot S_n(x_-)A_c^{(0)}(x)S_n^\dagger(x_-) + gn\cdot A_s(x_-)\Big)\, S_n(x_-)\xi^{(0)}(x)\\
&= \Big(in\cdot\partial_- S_n(x_-) + S_n(x_-)in\cdot\partial + S_n(x_-)\, gn\cdot A_c^{(0)}(x) + gn\cdot A_s(x_-)S_n(x_-)\Big)\xi^{(0)}(x)\\
&= \Big[\underbrace{(in\cdot D_- S_n(x_-))}_{=0} + S_n(x_-)in\cdot\partial + S_n(x_-)gn\cdot A_c^{(0)}(x)\Big]\xi^{(0)}(x)\\
&= S_n(x_-)\Big(in\cdot\partial + gn\cdot A_c^{(0)}(x)\Big)\,\xi^{(0)}(x) \equiv S_n(x_-)in\cdot D_c^{(0)}\xi^{(0)}(x)\,,
\end{aligned}
\tag{10.217}
$$

where I used $\boxed{x_-^\mu = \bar n\cdot x\,\dfrac{n^\mu}{2}}$

$$
n^\alpha\frac{\partial}{\partial x^\alpha}S_n(x_-) = n^\alpha\frac{\partial x_-^\beta}{\partial x^\alpha}\frac{\partial}{\partial x_-^\beta}S_n(x_-) = \frac{n^\alpha\bar n_\alpha}{2}n^\beta\frac{\partial}{\partial x_-^\beta}S_n(x_-) \equiv n\cdot\partial_- S_n(x_-)\,, \tag{10.218}
$$

and the fact that the covariant derivative along the Wilson line is zero (can be shown):

$$
(x-y)\cdot D\,[x,y] = 0\,. \tag{10.219}
$$

Thus

$$
\mathcal{L}_{c+s} = \bar\xi\frac{\bar{\not n}}{2}in\cdot\mathbf{D}\xi \;=\; \bar\xi^{(0)}\frac{\bar{\not n}}{2}in\cdot D_c^{(0)}\xi^{(0)}(x)\,. \tag{10.220}
$$

The soft gluon field is eliminated from the covariant derivative and no longer appears explicitly — a *decoupling transformation*. It is possible to show that after a decoupling transformation also the interactions between soft and collinear gluons disappear from the leading power collinear Lagrangian.

$$
ig\mathbf{F}^c_{\mu\nu} = [i\mathbf{D}_\mu, i\mathbf{D}_\nu] \;=\; \left[iD_\mu^{c(0)}, iD_\nu^{c(0)}\right] = igF^{c(0)}_{\mu\nu} \qquad iD_\mu^{c(0)} \equiv i\partial_\mu + gA_\mu^{c(0)} \tag{10.221}
$$

Performing an analogous decoupling also for the anti-collinear fields, [‖] we get

$$
\mathcal{L}_{\rm SCET} = \mathcal{L}_c^{(0)} + \mathcal{L}_{\bar c}^{(0)} + \mathcal{L}_s\,. \tag{10.222}
$$

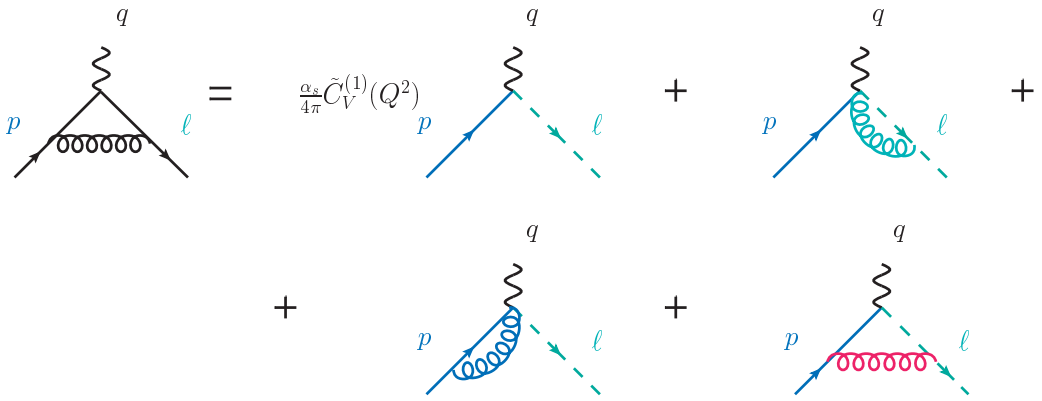
Since there are no longer any interactions, we are dealing with independent theories of soft and collinear particles and also the states can be separated as

$$
|X\rangle = |X_c\rangle\otimes|X_{\bar c}\rangle\otimes|X_s\rangle\,. \tag{10.223}
$$

[‖] For the decoupling for the anti-collinear fields one uses an outgoing soft Wilson line $S_{\bar n}$ along the $\bar n$ direction, while the collinear field we need the outgoing line in n direction, $\bar S_n$.

10.5.5 Sudakov form factor in QCD

The one-loop contribution is



where the three-particle vertices come from the expansion of the collinear Wilson lines in the QCD-SCET matching of the vector current:

$$J^\mu(x) \mapsto \iint dsdt\, C_V(s,t,\mu)\, \bar\chi_c\,(x+s\bar n)\gamma_\perp^\mu \chi_{\bar c}\,(x+tn) + \dots$$

$$\equiv \iint dsdt\, C_V(s,t,\mu)\, \bar\xi_c(x+s\bar n)\, W_c(x+t\bar n)\gamma_\perp^\mu W^\dagger(x+tn)\chi_{\bar c}(x+tn) + \dots \qquad (10.224)$$

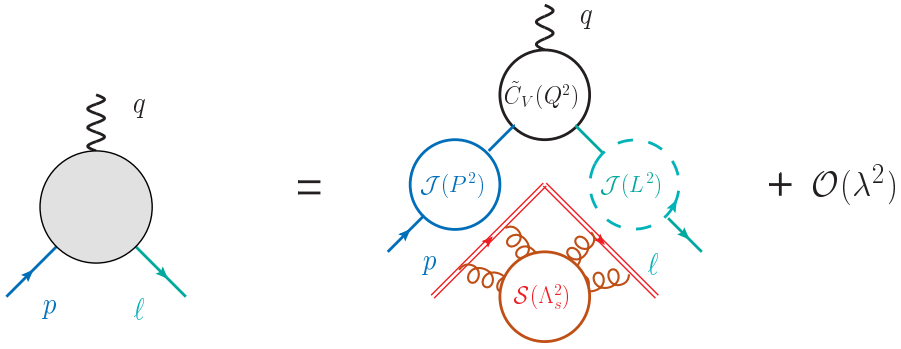
After the decoupling transformation, the soft-collinear interaction vertices as in the last diagram are no more present in the Lagrangian, but the current acquires an additional factor — a product of collinear and anticollinear Wilson lines

$$\bar\chi_c(x+t\bar n)\,\gamma_\perp^\mu\,\chi_{\bar c}(x+sn) = \bar\chi_c^{(0)}(x+t\bar n)\, S_n^\dagger(x_-)\,\gamma_\perp^\mu\, S_{\bar n}(x_+)\,\chi_{\bar c}^{(0)}(x+sn) \qquad (10.225)$$

Since in this formulation of the theory soft, collinear and anticollinear field do not interaract, the Sudakov form factor *to all orders of perturbation theory* will have the following factorized structure:

$$F(Q^2,P^2,L^2) = \widetilde{C}_V(Q^2,\mu)\,\mathcal{J}(P^2,\mu)\mathcal{J}(L^2,\mu)\,\mathcal{S}\left(\frac{P^2L^2}{Q^2},\mu\right) \qquad (10.226)$$

or



where soft Wilson lines are shown by the red double-lines and the soft scale is $\Lambda_s^2 = \frac{P^2L^2}{Q^2}$. The two jet functions are of course identical, just depend on different momentum.

If $Q^2, P^2, L^2, \Lambda_s^2 \gg \Lambda_{QCD}^2$, all these functions can be calculated in perturbation theory and will involve double logarithms, e.g.

$$\widetilde{C}_V(Q^2,\mu) = 1 + c_v\frac{\alpha_s}{4\pi}\ln^2\frac{Q^2}{\mu^2} + \dots$$

In order that this expansion remains under control, we must ensure that $\frac{\alpha_s}{4\pi}\ln^2\frac{Q^2}{\mu^2} \ll 1$, i.e. take $\mu^2 \sim Q^2$. Similarly, in order that the perturbative expansion for the jet functions is under control we have to take $\mu^2 \sim P^2, L^2$ and for the soft function $\mu^2 \sim \Lambda_s^2$. Clearly these conditions are not mutually consistent if, as we assume from the very beginning, $Q^2 \gg P^2, L^2 \gg \Lambda_s^2$. This means that with any choice of μ (e.g. we can take $\mu \sim Q$) one will have to to beyond the fixed-order calculation and resum the (double) logarithms in other entries. This can be accomplished using the **renormalization group** technique.

The final result must be independent on the choice of μ, thus

$$\begin{aligned} 0 &= \frac{d}{d\ln\mu}\ln\left[\tilde{C}_V\left(Q^2,\mu^2\right)\mathcal{J}\left(L^2,\mu^2\right)\mathcal{J}\left(P^2,\mu^2\right)\mathcal{S}\left(\Lambda_s^2,\mu^2\right)\right] \\ &= \frac{1}{\tilde{C}_V\left(Q^2,\mu^2\right)}\frac{d\tilde{C}_V\left(Q^2,\mu^2\right)}{d\ln\mu} + \frac{1}{\mathcal{J}\left(L^2,\mu^2\right)}\frac{d\mathcal{J}\left(L^2,\mu^2\right)}{d\ln\mu} \\ &\quad + \frac{1}{\mathcal{J}\left(P^2,\mu^2\right)}\frac{d\mathcal{J}\left(P^2,\mu^2\right)}{d\ln\mu} + \frac{1}{\mathcal{S}\left(\Lambda_s^2,\mu^2\right)}\frac{d\mathcal{S}\left(\Lambda_s^2,\mu^2\right)}{d\ln\mu}\,. \end{aligned} \tag{10.227}$$

This is a very nontrivial equation because the four terms are functions of different variables. The RG equation for the hard function is given in (10.228)

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s}\right)\tilde{C}_V(Q^2,\mu) = \left[\gamma_{\rm cusp}(\alpha_s)\ln\frac{Q^2}{\mu^2} + \gamma_V(\alpha_s)\right]\tilde{C}_V(Q^2,\mu)\,, \tag{10.228}$$

and the cancellation in (10.227) requires that the RG equations for the jet function and the soft function have a similar structure:

$$\begin{aligned} \left(\mu\frac{\partial}{\partial\mu} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s}\right)\mathcal{J}(P^2,\mu) &= \left[\gamma_{\rm cusp}(\alpha_s)\ln\frac{P^2}{\mu^2} + \gamma_J(\alpha_s)\right]\mathcal{J}(P^2,\mu)\,, \\ \left(\mu\frac{\partial}{\partial\mu} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s}\right)\mathcal{S}(\Lambda_s^2,\mu) &= \left[\gamma_{\rm cusp}(\alpha_s)\ln\frac{\Lambda_s^2}{\mu^2} + \gamma_S(\alpha_s)\right]\mathcal{S}(\Lambda_s^2,\mu)\,, \end{aligned} \tag{10.229}$$

with the same $\gamma_{\rm cusp}(\alpha_s)$ in front of the log terms. In this case, indeed, (10.227) can be satisfied if

$$\gamma_{\rm cusp}\ln\frac{Q^2}{\mu^2} + \gamma_V - \gamma_{\rm cusp}\left(\ln\frac{L^2}{\mu^2} + \ln\frac{P^2}{\mu^2}\right) - 2\gamma_J - \gamma_{\rm cusp}\ln\frac{\mu^2}{\Lambda_s^2} + \gamma_S = 0\,. \tag{10.230}$$

For this cancellation to work, it is crucial that the anomalous dimensions are (at most) linear in the logarithm. To prove linearity one can make a more general ansatz for the anomalous dimensions and then show that scale independence of the sum, together with the kinematic dependence of the individual pieces, imply that higher-log terms must be absent. A direct proof of this statement from Feynman diagrams exists but is much more complicated.

The solution of all these RG equations has the same form as for the hard function

$$\tilde{C}_V(Q^2,\mu) = \exp\left\{\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\left[\gamma_{\rm cusp}(\alpha_s(\mu'))\ln\frac{Q^2}{\mu'^2}+\gamma_V(\alpha_s(\mu'))\right]\right\}\tilde{C}_V(Q^2,\mu_0)$$
$$\mathcal{J}(P^2,\mu) = \exp\left\{\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\left[\gamma_{\rm cusp}(\alpha_s(\mu'))\ln\frac{P^2}{\mu'^2}+\gamma_J(\alpha_s(\mu'))\right]\right\}\mathcal{J}(P^2,\mu_0)$$
$$\mathcal{S}(\Lambda_s^2,\mu) = \exp\left\{\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\left[\gamma_{\rm cusp}(\alpha_s(\mu'))\ln\frac{\Lambda_s^2}{\mu'^2}+\gamma_S(\alpha_s(\mu'))\right]\right\}\mathcal{S}(\Lambda_s^2,\mu_0) \tag{10.231}$$

so that calculating the anomalous dimensions and the initial conditions for the evolution at their "natural" scales $\tilde{C}_V(\mu_0 = Q^2)$, $\mathcal{J}(\mu_0^2 = P^2)$, $\mathcal{S}(\mu_0^2 = \Lambda_s^2)$ one can can use these expressions to calculate all entries at one common scale (usually $\mu = Q$). This procedure is called **resummation**.

A disclaimer:
While all the steps we took to achieve factorization and to perform resummation would be perfectly appropriate for a physical observable, there is a problem for the Sudakov form factor: the off-shell form factor is unphysical and gauge dependent. We only consider it because it is the simplest example involving both soft and collinear physics. The decoupling trafo that we used to achieve factorization leave physical quantities invariant, but can and does lead to different off-shell Green's functions. In particular, one finds that the soft matrix elements change. After the field redefinition, they involve additional IR divergences and are given by scaleless integrals. As a consequence, in the calculation of the anomalous dimensions one needs to separate explicitly the UV from the IR divergences that requires additional regularization. As already said, these problems are not present for physical observables.

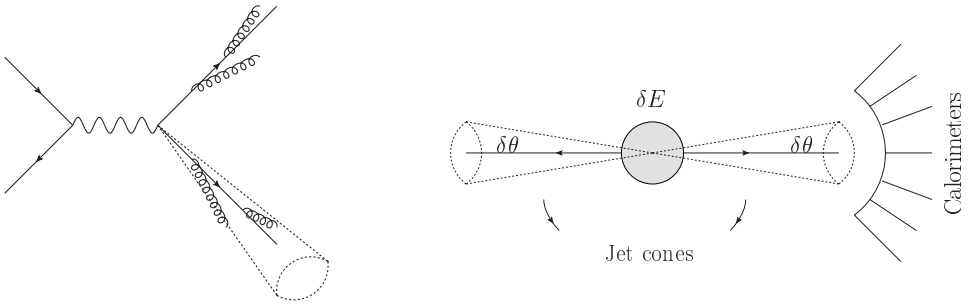
10.6 Application to jet physics: The thrust distribution

My presentation in this section follows: T. Becher, *Les Houches Lectures on Soft-Collinear Effective Theory,* [arXiv:1803.04310].

In this section we return to the process

$$e^+e^- \to X \tag{10.232}$$

where at large energy the final state has a multi-jet structure, e.g. a two jet event:



The classification as a two-jet or three-jet event is rather crude and aiming at a more detailed understanding of the shape of the particle flow in the final state one introduces a large variety of the so-called **event shape** observables. Their definition follows the general principle that "good" event shapes have to be **infrared safe**, if we want to be able to predict them in QCD. One standard observable of this kind is the **thrust distribution** which we will consider now.

The definition of thrust (Farhi, 1977) is

$$T = \frac{1}{Q} \max_{\vec{n}_T} \sum_i |\vec{n}_T \cdot \vec{p}_i| \,, \qquad \tau = 1 - T \tag{10.233}$$

It involves a sum over all particles in the event and one sums the projections of their momenta along the thrust axis $\vec{n}_T$ which must be chosen to maximize the sum. The momentum flow along $\vec{n}_T$ is then normalized to the center-of-mass energy, which for massless particles is given by

$$Q = \sum_i |\vec{p}_i| \,. \tag{10.234}$$

The thrust T thus measures the fraction of momentum flowing along the thrust axis. For $T = 1$ all momentum must flow along $\vec{n}_T$, i.e. all particles are parallel or anti-parallel to $\vec{n}_T$. Events with large thrust T are thus pencil-like and involve two narrow jets. The maximum value of τ (minimum value of T) is obtained for a completely spherical event. It is easy to verify that such a configuration has $\tau = 1/2$. Thrust is infrared-safe, i.e. its value does not change under collinear or

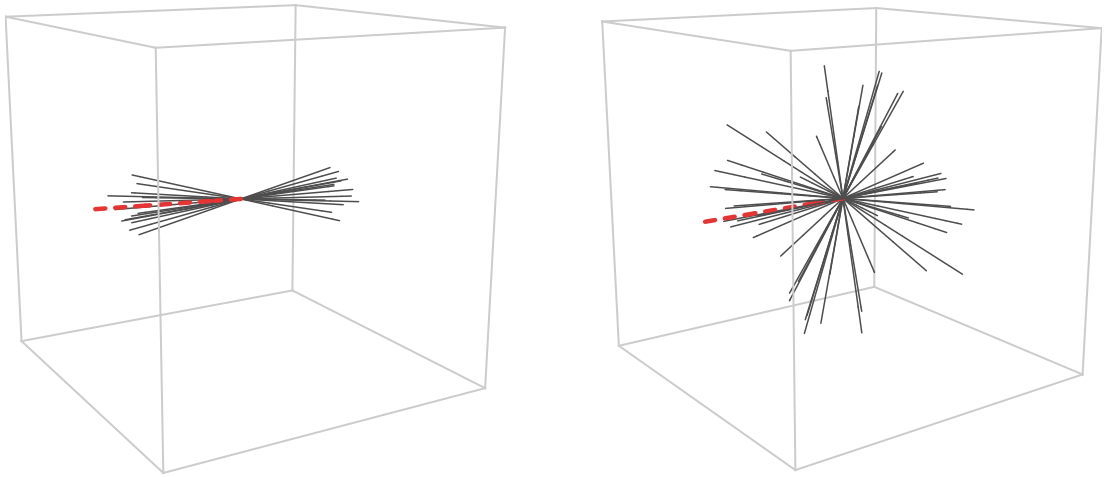


Figure 1: Two sample collider events. On the left side a pencil-like two-jet event with small $\tau = 1 - T \approx 0.002$, on the right side an almost spherical event with large $\tau \approx 0.35$. The thrust axis is shown as a red dashed line. Taken from [arXiv:1803.04310].

soft gluon emissions.

The leading contribution to thrust comes from gluon emission and the result is (De Rujula *et al.*,, NPB **138** (1978), 387)

$$\frac{1}{\sigma_{tot}}\frac{d\sigma_{e^+e^-\to q\bar{q}g)}}{d\tau} = \frac{4\alpha_s}{3\pi}\frac{1}{\tau}\left\{\left(3\tau+\frac{2}{1-\tau}\right)\ln\frac{1-2\tau}{\tau} - \frac{3}{2}(1-3\tau)(1+\tau)\right\} \tag{10.235}$$

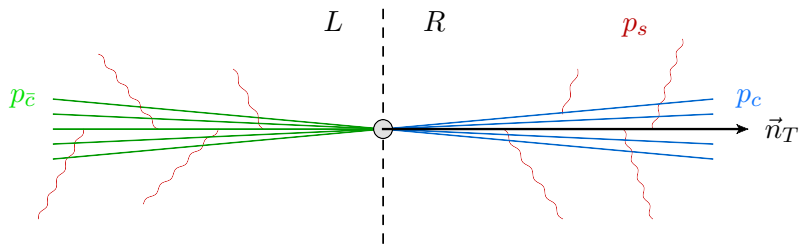
The $\mathcal{O}(\alpha_s^2)$ and $\mathcal{O}(\alpha_s^3)$ corrections are known, but there are no analytic expressions.

The SCET machinery is useful if τ is small. It defines the expansion parameter which was denoted by λ in the SCET discussion of the Sudakov problem. As we have seen already in Sec. 5.2.3, the cross section for production of narrow jets contains large logs, and utility of the SCET approach is that it allows one to resum all log-enhanced contributions to all orders.

To start the construction, assume that we are working in the C.M. frame $q^\mu = (Q, \vec{0})$ and define the two light-like vectors n^μ, $\bar{n}^\mu$ as

$$n^\mu = (1, \vec{n}_T)\,, \qquad \bar{n}^\mu = (1, -\vec{n}_T) \qquad q^\mu = \frac{n^\mu + \bar{n}^\mu}{2}Q\,. \tag{10.236}$$

Here we sketch an event with small τ:



It involves energetic particles collinear to n^μ and $\bar{n}^\mu$, together with soft large-angle radiation. Performing a region analysis, one finds that the same regions contribute as in the Sudakov form factor, but with expansion parameter $\lambda = \tau$:

$$p_c \sim (\tau^2, 1, \tau)\,, \qquad p_{\bar{c}} \sim (1, \tau^2, \tau)\,, \qquad p_s \sim (\tau^2, \tau^2, \tau^2) \tag{10.237}$$

We can thus separate the sum over particles in the definition of thrust into individual sums in the soft and collinear sectors and write

$$\begin{aligned}
\tau Q &= \sum_i |\vec{p}_i| - |\vec{n}_T \cdot \vec{p}_i| \\
&= \sum_i n\cdot p_{ci} + \sum_i \bar{n}\cdot p_{\bar{c}i} + \sum_i n\cdot p_{si}^R + \sum_i \bar{n}\cdot p_{si}^L \\
&\equiv \underbrace{n\cdot p_{X_c} + n\cdot p_{X_s}^R}_{\text{right movers}} + \underbrace{\bar{n}\cdot p_{X_{\bar{c}}} + \bar{n}\cdot p_{X_s}^L}_{\text{left movers}}\,,
\end{aligned} \tag{10.238}$$

where we have split (also) the soft particles into left- and right-moving ones.

Due to the definition of thrust the total transverse momentum of final state particles with respect to the thrust axis is zero in each hemisphere. Thus the *total* collinear transverse momentum $p_{X_c}^\perp$

is also zero, up to terms $\mathcal{O}(\tau^2)$, the soft momentum. As a consequence, the invariant mass of all particles in the right hemisphere, up to power corrections, is given by

$$\begin{aligned} M_R^2 &\stackrel{!}{=} (p_{X_c} + p_{X_s}^R)^2 \\ &= p_{X_c}^2 + (\bar{n}\cdot p_{X_c})(n\cdot p_{X_s}^R) + \cancel{(p_{X_s}^R)^2} \\ &= (\bar{n}\cdot p_{X_c})(n\cdot p_{X_c}) + \cancel{(p_{X_c}^\perp)^2} + (\bar{n}\cdot p_{X_c})(n\cdot p_{X_s}^R) \qquad \boxed{(\bar{n}\cdot p_{X_c}) = Q} \\ &= Q(n\cdot p_{X_c} + p_{X_s}^R)\,. \end{aligned} \tag{10.239}$$

The expression in the last line is exactly the contribution of the right movers to (10.238). Up to power corrections

$$\tau Q^2 = M_L^2 + M_R^2 = p_{X_c}^2 + p_{X_{\bar{c}}}^2 + Q\big(n\cdot p_{X_s}^R + \bar{n}\cdot p_{X_s}^L\big)\,. \tag{10.240}$$

For $\tau \ll 1$, thrust is equal to the sum of the invariant masses in the two hemispheres, normalized to Q^2. The fact that thrust is additive in the soft and collinear contributions, will be important to establish factorization.

- Our aim is to calculate the cross section

$$\frac{d\sigma}{d\tau} = \frac{1}{2Q^2}\sum\!\!\!\!\!\!\int_X \big|\mathcal{M}(e^+(q_2)e^-(q_1)\to\gamma^*\to X)\big|^2\,(2\pi)^4\delta^{(4)}(q-p_X)\,\delta(\tau-\tau(X))\,, \tag{10.241}$$

where
— τ is the thrust value we prescribe
— $\tau(X)$ is the thrust value for the final state X, as given in (10.240) (at leading power)
— $q = q_1 + q_2$ is the total lepton momentum with $q^2 = Q^2$.
— $\sum\!\!\!\!\!\!\int_X$ stands for the sum over all final states X and the phase space integrals $\int \frac{d^3p_i}{2|p_i|}$

The amplitude can be written as a product of a leptonic amplitude times a hadronic one (which is a "trivial" example of factorization). We get

$$|\mathcal{M}|^2 = L^{\mu\nu}H_{\mu\nu}, \tag{10.242}$$

with

$$H_{\mu\nu}(q,\tau) = \sum\!\!\!\!\!\!\int_X \langle 0|\,J_\nu^\dagger(0)\,|X\rangle\langle X|\,J_\mu(0)\,|0\rangle (2\pi)^4\delta^{(4)}(q-p_X)\,\delta(\tau-\tau(X))\,, \tag{10.243}$$

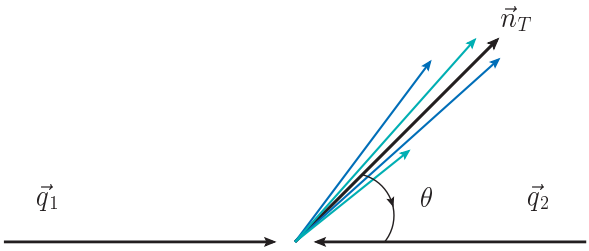
where J^μ is the electromagnetic current, and

$$L^{\mu\nu}(q_1,q_2) = \frac{e^4Q_q^2}{Q^4}\,\bar{v}(q_2)\,\gamma^\mu\,u(q_1)\,\bar{u}(q_1)\,\gamma^\nu\,v(q_2) \tag{10.244}$$

Averaging over the spins of the incoming leptons produces a trace of Dirac matrices and the lepton tensor becomes

$$L^{\mu\nu}(q_1,q_2) = \frac{e^4Q_q^2}{Q^4}\,\left(q_1^\mu q_2^\nu + q_2^\mu q_1^\nu - q_1\cdot q_2\,g^{\mu\nu}\right)\,. \tag{10.245}$$

• In the cross section (10.241) the value of thrust is fixed, but the direction of the thrust axis $\vec{n}_T$ can be arbitrary. Sine we define space components of our light-like vectors n, $\bar{n}$ as $\pm\vec{n}_T$, the $d\sigma/d\tau$ involves an integral over all directions which is inconvenient. We can separate this integration and fix the direction of $\vec{n} = \vec{n}(\theta, \phi)$ by defining the angles w.r.t. the the (initial) electron momentum, i.e. $\vec{q_1} \cdot \vec{n} = \cos\theta$.



This can be achieved, formally, by inserting in (10.241) a dummy integration

$$1 = \int d\phi \int d\cos\theta\, \delta^{(2)}(\vec{n} - \vec{n}_T(X))\,, \tag{10.246}$$

where $\vec{n}_T(X)$ is the thrust direction for the particular final state X. Up to power corrections $\vec{n}_T = \vec{p}_{X_c}/|\vec{p}_{X_c}| = (2/Q)\vec{p}_{X_c}$ since $|\vec{p}_{X_c}| = Q/2$ due to energy-momentum conservation. Thus

$$1 = \int d\phi \int d\cos\theta \left(\frac{Q}{2}\right)^2 \delta^{(2)}(p^\perp_{X_c})\,, \tag{10.247}$$

where $p^\perp_{X_c}$ is the total collinear momentum component in the direction orthogonal to $\vec{n}(\theta, \phi)$. Suppressing the integral $\int d\cos\theta$ will give us $\frac{d\sigma}{d\tau d\cos\theta}$ and $\int d\phi$ just produces a factor 2π. What remains is the $Q^2/4$ factor and a δ-function which fixes the total transverse momentum of the collinear radiation to be zero, as required by the thrust definition. Combining it with the momentum conservation δ-functions and expanding away small momentum components, we get

$$\delta^{(4)}(q - p_{X_c} - p_{X_{\bar{c}}} - p_{X_s})\,\delta^{(2)}(p^\perp_{X_c}) = 2\,\delta(\bar{n}\cdot p_{X_c} - Q)\,\delta(n\cdot p_{X_{\bar{c}}} - Q)\delta^{(2)}(p^\perp_{X_c})\,\delta^{(2)}(p^\perp_{X_{\bar{c}}})\,. \tag{10.248}$$

The factor of 2 is the Jacobian for converting to light-cone components.

• We now use the expression for the em. current in terms of SCET fields:**

$$J^\mu(0) \mapsto \iint dsdt\, C_V(s,t,\mu)\, \bar{\chi}^{(0)}_c(t\bar{n})\, S^\dagger_n(0)\, \gamma^\mu_\perp\, S_{\bar{n}}(0)\, \chi^{(0)}_{\bar{c}}(sn) \tag{10.249}$$

and that after decoupling transformation there are no interactions left between collinear, anti-collinear and soft sectors, so that the projectors on the particular final state also factorize as

$$|X\rangle\langle X| = |X_c\rangle\langle X_c| \cdot |X_{\bar{c}}\rangle\langle X_{\bar{c}}| \cdot |X_s\rangle\langle X_s| \tag{10.250}$$

**In the Sudakov form factor, we had an incoming and an outgoing particle, so that the current (10.225) had a soft Wilson line incoming from minus infinity to the current position, and an an outgoing Wilson line to plus infinity, after decoupling. In the present case the particle and anti-particle are both outgoing so that the appropriate soft Wilson-lines structure in the current involves both Wilson line going to plus infinity (in time).

Thus the hadronic tensor contains a collinear, an anti-collinear and a soft matrix elements, tied together by the thrust constraint (10.240). To separate the individual contributions, we insert three more unity integrals

$$1 = \int dM_c^2\, \delta(M_c^2 - p_{X_c}^2) \int dM_{\bar c}^2\, \delta(M_{\bar c}^2 - p_{X_{\bar c}}^2) \int d\omega\, \delta(\omega - n\cdot p_{X_s}^R - \bar n \cdot p_{X_s}^L) \tag{10.251}$$

Putting all this together, one obtains the cross section in the factorized form

$$\begin{aligned}\frac{d\sigma}{d\tau d\cos\theta} = \frac{\pi}{2}\, L_{\mu\nu}\, |\tilde C_V(-Q^2 - i0,\mu)|^2 \int dM_c^2 \int dM_{\bar c}^2 \int d\omega\, \delta(\tau - \frac{M_c^2 + M_{\bar c}^2 + Q\omega}{Q^2}) \\ \times \sum\!\!\!\!\!\!\int_{X_c} \langle 0|\, \chi_{c,\delta}^a(0) |X_c\rangle\langle X_c|\, \bar\chi_{c,\alpha}^b\, |0\rangle\, \delta(M_c^2 - p_{X_c}^2)\, \delta^{(2)}\big(p_{X_c}^\perp\big) \delta(\bar n\cdot p_{X_c} - Q) \\ \times \sum\!\!\!\!\!\!\int_{X_{\bar c}} \langle 0|\, \bar\chi_{\bar c,\gamma}^d(0) |X_{\bar c}\rangle\langle X_{\bar c}|\, \chi_{\bar c,\beta}^e\, |0\rangle\, \delta(M_{\bar c}^2 - p_{X_{\bar c}}^2)\, \delta^{(2)}\big(p_{X_{\bar c}}^\perp\big) \delta(n\cdot p_{X_{\bar c}} - Q) \\ \times \sum\!\!\!\!\!\!\int_{X_s} \langle 0|\, \big[S_n^\dagger S_{\bar n}\big]_{da} |X_s\rangle\langle X_s|\, \big[S_{\bar n}^\dagger S_n\big]_{be}\, |0\rangle\, \delta(\omega - n\cdot p_{X_s}^R - \bar n\cdot p_{X_s}^L) \\ \times (2\pi)^4\, (\gamma_\perp^\mu)_{\alpha\beta}\, (\gamma_\perp^\nu)_{\gamma\delta}\,. \end{aligned} \tag{10.252}$$

In this expression, the Latin letters a, b, d, e denote the quark colors, and the Greek letters stand for the Dirac indices.

We have now managed to separate the contributions to the different sectors: colinear in line two, anticollinear in line three and soft in line four. As they stand, these functions are matrices in color and Dirac space, but the expressions can still be massaged and simplified further, see [arXiv:1803.04310]. After these manipulations and a little algebra which I will skip, one arrives at the final result

$$\begin{aligned}\frac{d\sigma}{d\tau d\cos\theta} = \frac{\pi N_c Q_f^2 \alpha^2}{2Q^2} (1+\cos^2\theta) |\tilde C_V(-Q^2 - i0,\mu)|^2 \int dM_c^2 \int dM_{\bar c}^2 \int d\omega \\ \times \delta\Big(\tau - \frac{M_c^2 + M_{\bar c}^2 + Q\omega}{Q^2}\Big)\, J(M_c^2,\mu)\, J(M_{\bar c}^2,\mu)\, S(\omega,\mu)\,, \end{aligned} \tag{10.253}$$

where the soft function is defined as[††]

$$S(\omega) = \frac{1}{N_c} \sum\!\!\!\!\!\!\int_{X_s} \langle 0|\, \big[S_n^\dagger S_{\bar n}\big]_{ab} |X_s\rangle\langle X_s|\, \big[S_{\bar n}^\dagger S_n\big]_{ba} |0\rangle\, \delta(\omega - n\cdot p_{X_s}^R - \bar n\cdot p_{X_s}^L)\,. \tag{10.254}$$

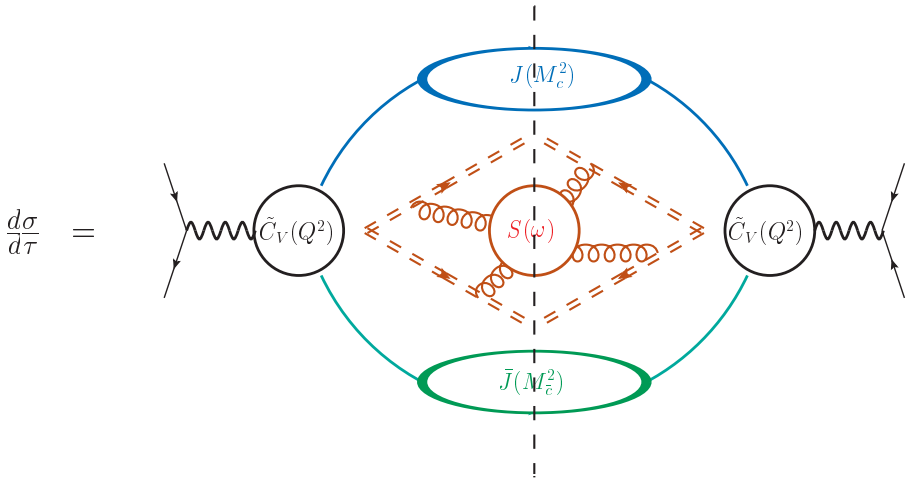
and the jet function can be represented as the imaginary part of the collinear propagator

$$\begin{aligned} J(p^2) &= \frac{1}{\pi}\, \mathrm{Im}\,\big[i\mathcal{J}(p^2)\big]\,, \\ \frac{\not n}{2}\, \bar n\cdot p\, \mathcal{J}(p^2) &= \int d^4x\, e^{-ip\cdot x}\, \langle 0\,|\,\mathrm{T}\,\{\chi_c(0)\,\overline{\chi}_c(x)\}\,|\,0\rangle\,. \end{aligned} \tag{10.255}$$

[††]The prefactor $1/N_c$ has been added so that $S(\omega) = \delta(\omega)$ at lowest order.

Recall that the collinear Lagrangian is equivalent to the QCD Lagrangian and the field $\chi_c = W_c^\dagger P_+ \psi_c$ can be obtained in QCD using the same projection operator P_+ and the Wilson line. In the light-cone gauge $\bar{n} \cdot A = 0$ the Wilson line is absent and we can thus compute the jet function as the imaginary part of the quark propagator in this particular gauge.

Diagrammatically:



The hard, jet and soft functions depend on μ. To resum large logarithms, one has to write and solve the RG equations and evolve μ to a common reference scale just as for the Sudakov form factor. A technical complication is that the factorization formula contains a convolution of the jet and soft functions, and not a product. This can be handled by a Laplace transformation:

$$\begin{aligned}\int d\tau\, e^{-\nu\tau} \frac{d\sigma}{d\tau} &\sim \int d\tau\, e^{-\nu\tau} \int dM_c^2 \int dM_{\bar{c}}^2 \int d\omega\, \delta\Big(\tau - \frac{M_c^2 + M_{\bar{c}}^2 + Q\omega}{Q^2}\Big) J(M_c^2,\mu)\, J(M_{\bar{c}}^2,\mu)\, S(\omega,\mu) \\ &= \int dM_c^2 e^{-\nu M_c^2/Q^2} J(M_c^2,\mu) \int dM_{\bar{c}}^2\, e^{-\nu M_{\bar{c}}^2/Q^2} J(M_{\bar{c}}^2,\mu) \int d\omega\, e^{-\nu\omega/Q} S(\omega,\mu)\end{aligned} \tag{10.256}$$

The state-of-the art calculation is currently done to the so-called N^{3}LL accuracy, corresponding to taking into account two-loop results for the hard, jet and soft functions, together with three-loop anomalous dimensions. A comparison with the experimental data is shown in the Figure. The plots show that fixed-order perturbation theory becomes bad at small τ, and demonstrate that resummation dramatically improves the convergence of the expansion. One-loop calculations for the hard, jet and soft functions can be found in the textbook M. D. Schwartz, *"Quantum Field Theory and the Standard Model,"* Cambridge University Press, 2014,

In summary, Soft-Collinear Effective Theory (SCET) is the effective field theory for processes with energetic particles such as jet production at high-energy colliders. The result of a SCET analysis is often a factorization theorem of the schematic form

$$\sigma = H \cdot J \otimes \bar{J} \otimes S\,. \tag{10.257}$$

Its main virtue is scale separation — the property that each of the functions in this formula is only sensitive to a single scale. The individual functions furthermore fulfill renormalization group (RG)

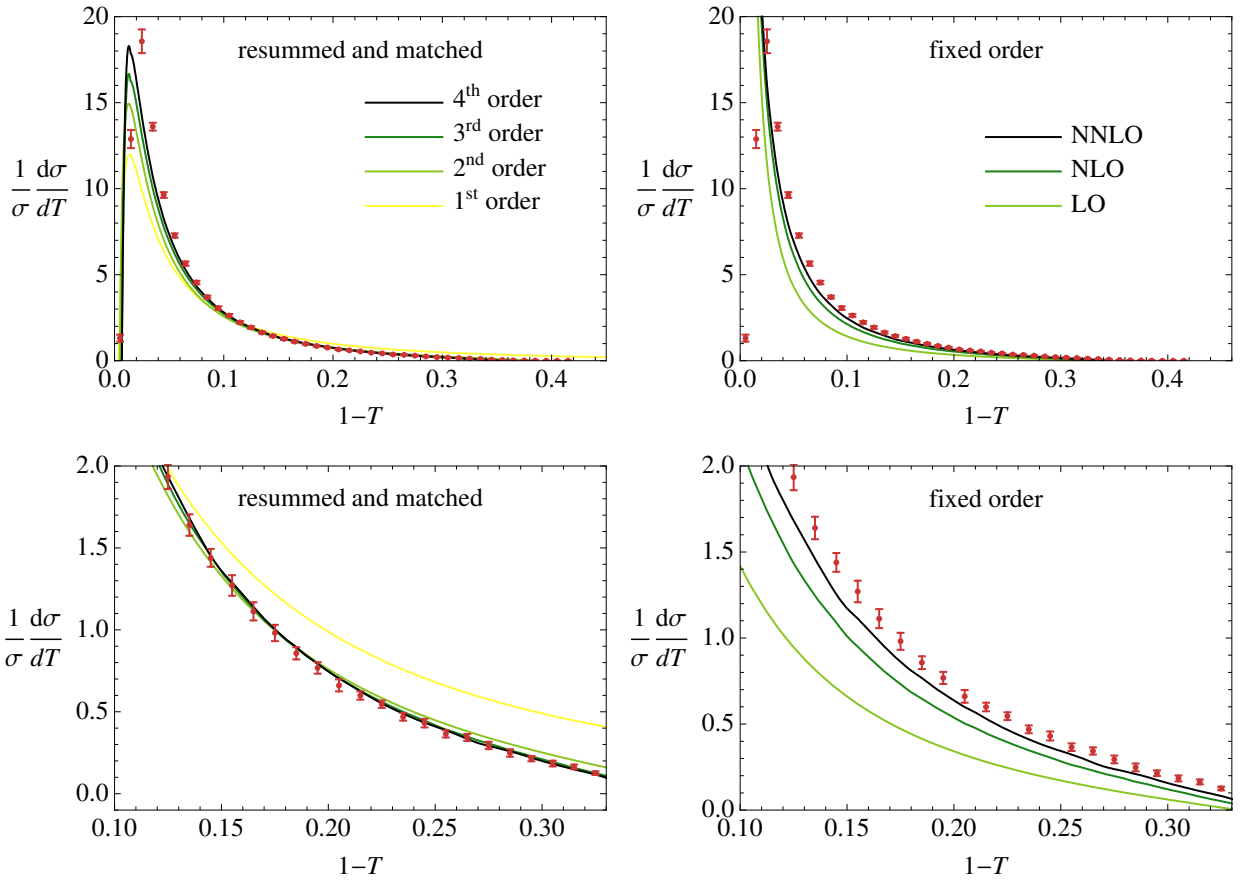


Figure 2: Convergence of resummed and fixed-order distributions. ALEPH data (red) at 91.2 GeV are included for reference. This figure is taken from [Becher:2008cf].

equations and by solving them one can resum a class of the large perturbative corrections to all orders.

This is not he only approach possible and many such factorization theorems can be derived purely diagrammatically. An advantage of SCET is seen in that it provides an operator formulation of the low-energy physics, which simplifies and systematizes the analysis. This is especially important for complicated problems such as the factorization of power suppressed contributions, which is a very hot topic.

The End

11 Epilogue

Eternal QCD problems:

- confinemement
- chiral symmetry breaking
- (strong CP problem)

Modern QCD physics issues:

- Heavy-quark physics
- Large-energy physics, BFKL and beyond
- Multi-jet production (Higgs searches)
- Hard exclusive processes, generalized parton distributions, nucleon tomography
- Heavy-ion physics, large parton densities

Modern QCD methods (theory):

- Lattice QCD
- Event generators
- Feynman diagrams in high orders
- Techniques based on dispersion relations and duality
- Learning from SUSY models and AdS/CFT correspondence, conformal symmetry, integrability
- Effective theories: SCET, HQET, NRQCD, low energy (chiral) ET

QCD in Regensburg:

- Lattice QCD
- High-order calculations using computer algebra
- Light Cone Sum Rules
- Conformal symmetry based methods
- Resummation of large logarithms in semiinclusive reactions
- Entropy generation in heavy-ion collisions
- Higher-twist effects

Appendices

A Collection of formulas

Dirac algebra in 4 Dimensions

Traces with even number of γ-matrices

$$\mathrm{Tr}\{\mathbf{1}\} = 4 \tag{A.1}$$

$$\mathrm{Tr}\{\gamma_\mu\gamma_\nu\} = 4g_{\mu\nu} \tag{A.2}$$

$$\mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta\} = 4[g_{\mu\nu}g_{\alpha\beta} + g_{\mu\beta}g_{\nu\alpha} - g_{\mu\alpha}g_{\nu\beta}] \tag{A.3}$$

Traces with odd number of γ-matrices

$$\mathrm{Tr}\{\gamma_{\mu_1}\dots\gamma_{\mu_{2k+1}}\} = 0\,, \qquad k = 0, 1, 2, \dots \tag{A.4}$$

Traces including a γ_5-matrix

$$\mathrm{Tr}\{\gamma_5\} = 0 \tag{A.5}$$

$$\mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_5\} = 0 \tag{A.6}$$

$$\mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta\gamma_5\} = 4i\epsilon_{\mu\nu\alpha\beta} \tag{A.7}$$

$$\mathrm{Tr}\{\gamma_{\mu_1}\dots\gamma_{\mu_{2k+1}}\gamma_5\} = 0\,, \qquad k = 0, 1, 2, \dots \tag{A.8}$$

Useful identities for products of γ-matrices

$$\gamma_\mu\gamma^\mu = 4 \tag{A.9}$$

$$\gamma_\mu\gamma_\alpha\gamma^\mu = -2\gamma_\alpha \tag{A.10}$$

$$\gamma_\mu\gamma_\alpha\gamma_\beta\gamma^\mu = 4g_{\alpha\beta} \tag{A.11}$$

$$\gamma_\mu\gamma_\alpha\gamma_\beta\gamma_\rho\gamma^\mu = -2\gamma_\rho\gamma_\beta\gamma_\alpha \tag{A.12}$$

$$\gamma_\mu\gamma_\alpha\gamma_\nu = g_{\alpha\mu}\gamma_\nu + g_{\alpha\nu}\gamma_\mu - g_{\mu\nu}\gamma_\alpha + i\epsilon_{\mu\alpha\nu\beta}\gamma_5\gamma_\beta \tag{A.13}$$

Useful identities for products of ϵ-tensors

$$\epsilon_{\alpha\beta\mu\nu}\epsilon^{\alpha\beta\mu\nu} = -24 \tag{A.14}$$

$$\epsilon_{\alpha\beta\mu\nu}\epsilon^{\rho\beta\mu\nu} = -6g^\rho_\alpha \tag{A.15}$$

$$\epsilon_{\alpha\beta\mu\nu}\epsilon^{\rho\sigma\mu\nu} = -2[g^\rho_\alpha g^\sigma_\beta - g^\sigma_\alpha g^\rho_\beta] \tag{A.16}$$

$$\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}\epsilon^{\beta_1\beta_2\beta_3\beta_4} = -\det\left(g^{\beta_k}_{\alpha_i}\right) \tag{A.17}$$

$$\frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\sigma^{\mu\nu} = i\sigma_{\alpha\beta}\gamma_5 \tag{A.18}$$

!!! We use definitions from Bjorken and Drell:

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3\,, \qquad \epsilon_{0123} = +1 \tag{A.19}$$

Be careful, some other (equally famous) books use different definitions:

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3\,, \qquad \epsilon^{0123} = -\epsilon_{0123} = +1 \qquad \text{Itzykson, Zuber} \tag{A.20}$$

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = -i\gamma^0\gamma^1\gamma^2\gamma^3\,, \qquad \epsilon^{0123} = -\epsilon_{0123} = +1 \qquad \text{Okun} \tag{A.21}$$

This ambiguity is a standard source of sign errors!

Identities involving Dirac spinors

$$\begin{aligned}
\bar{u}^\lambda(p)\, u^{\lambda'}(p) &= 2m\delta_{\lambda\lambda'} \\
\bar{v}^\lambda(p)\, v^{\lambda'}(p) &= -2m\delta_{\lambda\lambda'} \\
\bar{u}^\lambda(p)\, v^{\lambda'}(p) &= \bar{v}^\lambda(p)\, u^{\lambda'}(p) \;=\; 0
\end{aligned} \tag{A.22}$$

$$\bar{u}^\lambda(p)\gamma_\mu u^{\lambda'}(p) = \bar{v}^\lambda(p)\gamma_\mu v^{\lambda'}(p) \;=\; 2p_\mu\delta_{\lambda\lambda'} \tag{A.23}$$

$$\sum_{\lambda=\pm 1/2}\Big[u^\lambda_\alpha(p)\bar{u}^\lambda_\beta(p) - v^\lambda_\alpha(p)\bar{v}^\lambda_\beta(p)\Big] = 2m\delta_{\alpha\beta} = 2m(\mathbb{I})_{\alpha\beta} \tag{A.24}$$

$$\begin{aligned}
\sum_{\lambda=\pm 1/2} u^\lambda_\alpha(p)\bar{u}^\lambda_\beta(p) &= (\not{p} + m)_{\alpha\beta} \\
\sum_{\lambda=\pm 1/2} v^\lambda_\alpha(p)\bar{v}^\lambda_\beta(p) &= (\not{p} - m)_{\alpha\beta}
\end{aligned} \tag{A.25}$$

Hermitian and Charge conjugation

$$\gamma^0\gamma^\dagger_\nu\gamma^0 = \gamma_\nu \tag{A.26}$$

$$C = i\gamma^2\gamma^0\,, \qquad C^{-1}\gamma_\mu C = -\gamma^T_\mu\,, \qquad C = -C^{-1} = -C^\dagger = -C^T \tag{A.27}$$

Integration in the 4 dimensional Euclidean space

Definitions:

$$k_o \to ik_4 \tag{A.28}$$

$$d^4k \;=\; dk_o d^3\vec{k} = id^4k_E \tag{A.29}$$

$$k^2 \;=\; k_0^2 - \vec{k}^2 = -(k_1^2 + k_2^2 + k_3^2 + k_4^2) = -k_E^2 \tag{A.30}$$

Integration:

$$\int d^D k_E f(k_E^2) = \int d\Omega_D \int_0^\infty dk_E k_E^{D-1} f(k_E^2) \tag{A.31}$$

$$= \frac{\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \int_0^\infty dk_E^2 \left(k_E^2\right)^{\frac{D}{2}-1} f(k_E^2) \tag{A.32}$$

Dimensional Regularization ($D = 4 - 2\epsilon$)

Definitions:

$$\int d^4 k \to \int d^D k \tag{A.33}$$

$$e_0 \to e_0 \mu^{2-\frac{D}{2}} \tag{A.34}$$

Dirac algebra in D Dimensions

Defining the ϵ-tensor and γ_5 in D dimensions involves subtleties that would require a detailed explanation; we will leave out the corresponding formulas.

$$\gamma_\mu \gamma^\mu = D \tag{A.35}$$

$$\gamma_\mu \gamma_\alpha \gamma^\mu = (2 - D)\gamma_\alpha \tag{A.36}$$

$$\gamma_\mu \gamma_\alpha \gamma_\beta \gamma^\mu = 4 g_{\alpha\beta} + (D - 4)\gamma_\beta \gamma_\alpha \tag{A.37}$$

$$\gamma_\mu \gamma_\alpha \gamma_\beta \gamma_\rho \gamma^\mu = -2\gamma_\rho \gamma_\beta \gamma_\alpha + (4 - D)\gamma_\alpha \gamma_\beta \gamma_\rho \tag{A.38}$$

Feynman parameter integrals for products of propagators:

$$\frac{1}{A \cdot B} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2} \tag{A.39}$$

$$\frac{\Gamma(a)\Gamma(b)}{A^a \cdot B^b} = \int_0^1 dx\, dy\, x^{a-1} y^{b-1}\, \delta(1 - x - y) \frac{\Gamma(a+b)}{[xA + yB]^{a+b}}$$

$$= \int_0^1 dx\, x^{a-1} (1-x)^{b-1} \frac{\Gamma(a+b)}{[xA + (1-x)B]^{a+b}} \tag{A.40}$$

This representation can be generalized to an arbitrary number of the denominators, e.g.,

$$\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{A^a \cdot B^b \cdot C^c} = \int_0^1 dx\, dy\, dz\, x^{a-1} y^{b-1} z^{c-1}\, \delta(1 - x - y - z) \frac{\Gamma(a+b+c)}{[xA + yB + zC]^{a+b+c}} \tag{A.41}$$

Loop integrals in D Dimensions

$$\int d^D k \, \frac{\Gamma(a)}{[-k^2 - A - i\epsilon]^a} = i\pi^{\frac{D}{2}} \frac{\Gamma\left(a - \frac{D}{2}\right)}{[-A - i\epsilon]^{a - \frac{D}{2}}} \tag{A.42}$$

$$\int d^D k \, \frac{\Gamma(a)}{[-k^2 - A - i\epsilon]^a} \, k_\mu k_\nu = i\pi^{\frac{D}{2}} \left(-\frac{g_{\mu\nu}}{2}\right) \frac{\Gamma\left(a - 1 - \frac{D}{2}\right)}{[-A - i\epsilon]^{a - 1 - \frac{D}{2}}} \tag{A.43}$$

$$\int d^D x \, \frac{\Gamma(\alpha)}{(-x^2 - a^2 + i\epsilon)^\alpha} = -i\pi^{D/2} \frac{\Gamma(\alpha - D/2)}{[-a^2 + i\epsilon]^{\alpha - D/2}}$$

$$\int d^D x \, \frac{\Gamma(\alpha)}{(-x^2 - a^2 + i\epsilon)^\alpha} x_\mu x_\nu = -i\pi^{D/2} \left(-\frac{g_{\mu\nu}}{2}\right) \frac{\Gamma(\alpha - D/2 - 1)}{[-a^2 + i\epsilon]^{\alpha - D/2 - 1}} \tag{A.44}$$

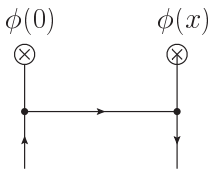
Fourier integrals in D Dimensions

$$\int d^D x \, e^{iqx} \frac{\Gamma(\alpha)}{[-x^2 + i\epsilon]^\alpha} = -i\pi^{D/2} 2^{D-2\alpha} \frac{\Gamma(D/2 - \alpha)}{[-q^2 - i\epsilon]^{D/2 - \alpha}}$$

$$\int d^D q \, e^{-iqx} \frac{\Gamma(\alpha)}{[-q^2 - i\epsilon]^\alpha} = +i\pi^{D/2} 2^{D-2\alpha} \frac{\Gamma(D/2 - \alpha)}{[-x^2 + i\epsilon]^{D/2 - \alpha}} \tag{A.45}$$

B OPE in $g\phi^3$ theory

In first order in g^2 we have the following diagram:



$+ \quad crossed$ (B.1)

Let us calculate it: [cf. *Balitsky, Braun, NPB361 (1991) pp. 103-105*]

$$D_1 = (ig)^2 \int \frac{d^D k}{(2\pi)^D} \, e^{-ikx} \frac{i}{(p+k)^2 - m^2 + i\epsilon} \left(\frac{i}{k^2 - m^2 + i\epsilon}\right)^2 \tag{B.2}$$

Use:

$$\frac{i}{(p+k)^2 - m^2 + i\epsilon} = \int_0^\infty ds_1 \, e^{is_1[(p+k)^2 - m^2]} \,,$$

$$\left(\frac{i}{k^2 - m^2 + i\epsilon}\right)^2 = \int_0^\infty ds_2 \, s_2 \, e^{is_2[k^2 - m^2]} \tag{B.3}$$

Thus

$$D_1 = (ig)^2 \int \frac{d^D k}{(2\pi)^D} \, e^{-ikx} \int_0^\infty ds_1 \int_0^\infty ds_2 \, s_2 \, e^{is_1[(p+k)^2 - m^2] + is_2[k^2 - m^2]} \tag{B.4}$$

Change variables $s_1, s_2 \to s, u$:

$$s_1 = us\,, \qquad s_2 = \bar{u}s\,, \qquad \boxed{\bar{u} = 1 - u} \tag{B.5}$$

Jacobian

$$\frac{\partial(s_1, s_2)}{\partial(s, u)} = \begin{vmatrix} u & \bar{u} \\ s & -s \end{vmatrix} = -s\,, \qquad \Longrightarrow \qquad \int_0^\infty ds_1 \int_0^\infty ds_2 \; = \int_0^1 du \int_0^\infty s\, ds \tag{B.6}$$

Obtain

$$D_1 = (ig)^2 \int \frac{d^D k}{(2\pi)^D}\, e^{-ikx} \int_0^1 du\, \bar{u} \int_0^\infty s^2\, ds\, \exp\Big\{ is\big[u(p+k)^2 - um^2 + \bar{u}k^2 - \bar{u}m^2\big]\Big\} \tag{B.7}$$

Write

$$\big[\,**\,\big] = (k+up)^2 - m^2(1 - u\bar{u})\,, \qquad \boxed{m_u^2 = m^2(1-u\bar{u})} \tag{B.8}$$

and shift integration variable $k \to k - up$:

$$D_1 = (ig)^2 \int_0^1 du\, \bar{u}\, e^{-uipx} \int_0^\infty s^2\, ds \int \frac{d^D k}{(2\pi)^D}\, e^{-ikx + is[k^2 - m_u^2]} \tag{B.9}$$

Use the same trick again:

$$-ikx + is[k^2 - m_u^2] = is\left[\left(k - \frac{1}{2s}x\right)^2 - \frac{x^2}{4s^2} - m_u^2\right] \tag{B.10}$$

and shift $k \to k + x/(2s)$:

$$D_1 = (ig)^2 \int_0^1 du\, \bar{u}\, e^{-uipx} \int_0^\infty s^2\, ds \int \frac{d^D k}{(2\pi)^D}\, e^{is\left[k^2 - x^2/(4s^2) - m_u^2\right]} \tag{B.11}$$

Useful formula:

$$\int \frac{d^D k}{(2\pi)^D}\, e^{isk^2} = \frac{ie^{-i\pi D/4}}{(4\pi s)^{D/2}} \qquad \stackrel{D=6}{\longrightarrow} \qquad -\frac{1}{(4\pi s)^3} \tag{B.12}$$

Obtain

$$D_1 = \frac{g^2}{(4\pi)^3} \int_0^1 du\, \bar{u}\, e^{-uipx} \int_0^\infty \frac{ds}{s}\; \exp\left[-\frac{ix^2}{4s} - ism_u^2\right] \tag{B.13}$$

The remaining integral is a Bessel function. Adding the contribution of the crossed diagram obtain the answer:

$$\boxed{D_1 = 2\frac{g^2}{(4\pi)^3} \int_0^1 du\, \bar{u} \left[e^{-uipx} + e^{uipx}\right] K_0(\sqrt{m_u^2(-x^2 + i\epsilon)})} \tag{B.14}$$

We are interesting in the expansion at $x^2 \to 0$:

$$e^{-uipx} + e^{uipx} = 2 - u^2(px)^2 + \mathcal{O}((px)^4)$$

$$K_0(z) = (-\ln z - \gamma_E + \ln 2) + \frac{1}{4}z^2(-\ln z - \gamma_E + 1 + \ln 2) + \mathcal{O}(z^4) \tag{B.15}$$

For the remaining integrals I used MATHEMATICA:

$$\begin{aligned}\int_0^1 du\,\bar{u}\,\ln(1-u\bar{u}) &= \frac{\pi}{2\sqrt{3}} - 1\\ \int_0^1 du\,\bar{u}\,(1-u\bar{u})\ln(1-u\bar{u}) &= \frac{\pi}{2\sqrt{3}} - \frac{19}{36}\\ \int_0^1 du\,\bar{u}\,u^2\ln(1-u\bar{u}) &= \frac{\pi}{2\sqrt{3}} - \frac{17}{36}\end{aligned} \tag{B.16}$$

and obtain

$$\begin{aligned}D_1 =& 2\frac{g^2}{(4\pi)^3}\Big\{2\Big[\frac{1}{2}\Big(-\frac{1}{2}\ln(-x^2m^2) - \gamma_E + \ln 2\Big) - \frac{1}{2}\left(\frac{\pi}{2\sqrt{3}} - 1\right)\Big]\\ &+ 2\frac{1}{4}(-x^2m^2)\Big[\frac{5}{12}\Big[-\frac{1}{2}\ln(-x^2m^2) + 1 - \gamma_E + \ln 2\Big] - \frac{1}{2}\left(\frac{\pi}{2\sqrt{3}} - \frac{19}{36}\right)\Big]\\ &- (px)^2\Big[\frac{1}{6}\Big[-\frac{1}{2}\ln(-x^2m^2) - \gamma_E + \ln 2\Big] - \frac{1}{2}\left(\frac{\pi}{2\sqrt{3}} - \frac{17}{36}\right)\Big]\Big\} + \dots\end{aligned} \tag{B.17}$$

Note that the answer is finite (no divergences) but the limit $x^2 \to$ cannot be taken