Übungen zu Quantenfeldtheorie

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Blatt 9

PCAC in the linear σ-model

Consider again the Lagrangian of the linear σ-model of the previous Problem set:

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}\left[\sum_{j=1}^{3}(\partial_\mu\pi_j)^2 + (\partial_\mu\sigma)^2\right] + \bar{N}i\gamma_\mu\partial_\mu N + g\bar{N}\left(\sigma + i\gamma_5\sum_{j=1}^{3}\pi_j\tau_j\right)N \\ &\quad + \tfrac{\mu^2}{2}\left(\sigma^2 + \sum_{j=1}^{3}\pi_j^2\right) - \tfrac{\lambda}{4}\left(\sigma^2 + \sum_{j=1}^{3}\pi_j^2\right)^2 \end{aligned} \tag{1}$$

Suppose we add a symmetry-breaking term to $\mathcal{L}$.

$$\mathcal{L}_{\text{SB}} = -c\,\sigma(x) \tag{2}$$

where c is a constant. In the following we use the notation $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$.

Aufgabe 1:

Find the new minimum for the effective potential

$$V = -\frac{\mu^2}{2}(\sigma^2 + \vec{\pi}^2) + \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2)^2 + c\sigma \tag{3}$$

[Find the corresponding algebraic equation. You do not need to solve it explicitly.]
Hint:
Minimize V as a function of σ and $|\vec{\pi}|$. Show that extremal conditions for $c \neq 0$ reduce to $|\vec{\pi}| = 0$ and

$$0 = \lambda\sigma^3 - \mu^2\sigma + c. \tag{4}$$

Aufgabe 2:

Show that in this case the pions are no longer massless and, on the tree level, their masses are proportional to the constant c. You should get

$$m_\sigma^2 = 2\mu^2 - \frac{3c}{v} + O(c^2)\,, \qquad m_\pi^2 = -\frac{c}{v} + O(c^2) \tag{5}$$

where v is the vacuum expectation value for σ.
Hint:
You need to find the minimum $\sigma = \tilde{v}$ of the potential. For this you can make an Ansatz $\tilde{v} = v + \delta v$, where $v \equiv \sqrt{\frac{\mu^2}{\lambda}}$ is the minimum of V for $c = 0$. Notice that you may ignore quadratic and higher terms in δv and solve for δv. Then make a field redefinition $\sigma \to \sigma + \tilde{v}$ and look for the quadratic terms in σ and $\vec{\pi}^2$. These are the mass terms.

Aufgabe 3:

Show that the axial-vector current A_μ derived in the Problem set 8 is no longer conserved. Calculate the divergence $\partial_\mu A^\mu$ and show that it is proportional to the pion field:

$$\partial_\mu \vec{A}^\mu = -c\vec{\pi} \tag{6}$$

To this end, notice that since A_μ by construction is a Noether current, its divergence is related to the variation of the Lagrangian with respect to the corresponding symmetry transformation: $\vec{\beta} \cdot \partial_\mu A^\mu = \delta_5 \mathcal{L}$. The right-hand-side would vanish were it not for the presence of the symmetry-breaking term that we introduce here.

Remark:

The constant c can be related to the pion mass and the pion decay constant, m_π and f_π. The definition reads

$$\langle 0|A_\mu^a(0)|\pi^b(p)\rangle = i\delta^{ab}\frac{1}{\sqrt{2}}f_\pi p_\mu \tag{7}$$

Thus the matrix element of the divergence is given by

$$\langle 0|\partial^\mu A_\mu^a(0)|\pi^b(p)\rangle = \delta^{ab}\frac{1}{\sqrt{2}}f_\pi m_\mu^2 = -c\langle 0|\pi^a(0)|\pi^b(p)\rangle \tag{8}$$

or

$$-c = \frac{1}{\sqrt{2}}f_\pi m_\mu^2 \tag{9}$$

Combining this with the result for the pion mass (Aufgabe 2) we see that the vacuum expectation value of the σ-field is simply the pion decay cosntant: $v = f_\pi/\sqrt{2}$.