Übungen zu Quantenfeldtheorie

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Symmetries of the linear σ-model

The Lagrangian for the linear σ-model is given by (summation signs written out here for clarity; for the rest of the problem set: Einsteins sum convention)

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}\left[\sum_{j=1}^{3}(\partial_\mu \pi_j)^2 + (\partial_\mu \sigma)^2\right] + \bar{N} i \gamma_\mu \partial_\mu N + g\bar{N}\left(\sigma + i\gamma_5 \sum_{j=1}^{3}\pi_j\tau_j\right) N \\ &\quad + \tfrac{\mu^2}{2}\left(\sigma^2 + \sum_{j=1}^{3}\pi_j^2\right) - \tfrac{\lambda}{4}\left(\sigma^2 + \sum_{j=1}^{3}\pi_j^2\right)^2 \end{aligned} \tag{1}$$

where $N = \begin{pmatrix} p \\ n \end{pmatrix}$ is an isospin-1/2 nucleon field, τ_j are pauli-matrices, $\pi \equiv \sum_{j=1}^{3}\pi_j\tau_j$ an isospin one pion field (with real scalar field components π_j), and σ an isospin zero real scalar field. Do not confuse the different vector spaces involved. The γ-matrices and p, n fields live in the usual four-dimensional Dirac spin space, while the τ_j are proportional to the unit matrix in that space. This is not to be confused with the two-dimensional isospin space, where N is a vector, with components p and n, and the Pauli matrices τ_j act in this space (γ_μ is proportional to the unit matrix in isospin space).
It is convenient to use a 2×2 matrix to represent the spin zero fields collectively:

$$\Sigma = \sigma + i\pi \tag{2}$$

and it is also useful to define the left-handed and right-handed chiral nucleon fields as

$$N_L = \frac{1}{2}(1-\gamma_5)N\,, \qquad N_R = \frac{1}{2}(1+\gamma_5)N. \tag{3}$$

Aufgabe 1:

Show that the Lagrangian can be written in these notations as

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\,\mathrm{tr}\left(\partial_\mu\Sigma\,\partial^\mu\Sigma^\dagger\right) + \bar{N}_L i\gamma^\mu\partial_\mu N_L + \bar{N}_R i\gamma^\mu\partial_\mu N_R + g\left(\bar{N}_L\Sigma N_R + \bar{N}_R\Sigma^\dagger N_L\right) \\ &\quad + \frac{\mu^2}{4}\,\mathrm{tr}(\Sigma\Sigma^\dagger) - \frac{\lambda}{16}\left[\mathrm{tr}(\Sigma\Sigma^\dagger)\right]^2 \end{aligned} \tag{4}$$

Aufgabe 2:

Show that the Lagrangian is invariant under the isospin transformations:

$$N \to N' = U\,N\,, \qquad \Sigma \to \Sigma' = U\,\Sigma\,U^\dagger \tag{5}$$

where $U = \exp[\frac{i}{2}\alpha_j\tau_j]$ is an arbitrary 2×2 unitary matrix with $\alpha_j \in \mathbb{R}$. Use Noether approach to find the corresponding conserved isospin vector currents V_j^μ, $j = 1, 2, 3$. You should get

$$V_j^\mu = \bar{N}\gamma^\mu\frac{\tau_j}{2}N - \epsilon_{jkl}\pi_l\partial^\mu\pi_k \tag{6}$$

Aufgabe 3:

Show that the Lagrangian is invariant under the axial isospin transformations:

$$N \to N' = \exp\left(\frac{i}{2}\beta_j\tau_j\gamma_5\right) N\,, \qquad \Sigma \to \Sigma' = V^\dagger\,\Sigma\,V^\dagger \tag{7}$$

where $V = \exp[\frac{i}{2}\beta_j\tau_j]$ is an arbitrary 2×2 unitary matrix with $\beta_j \in \mathbb{R}$. Use Noether approach to find the corresponding conserved isospin axial-vector currents A_j^μ. You should find

$$A_j^\mu = \bar{N}\gamma^\mu\gamma_5\frac{\tau_j}{2}N - (\pi_j\partial^\mu\sigma - \sigma\partial^\mu\pi_i) \tag{8}$$

Aufgabe 4:

Thanks to the current conservation, the charges must be time-independent. Use this property and the canonical equal-time commutation relations to calculate the charge commutators

$$[Q_i, Q_j]\,, \quad [Q_i, Q_{5j}]\,, \quad [Q_{5i}, Q_{5j}]\,, \tag{9}$$

where

$$Q_i = \int d^3x\, V_i^0(x)\,, \qquad Q_{5i} = \int d^3x\, A_i^0(x) \tag{10}$$

For fermion fields, you may find the following identity useful:

$$[AB, CD] = -AC\{B,D\} + A\{B,C\}D - C\{A,D\}B + \{C,A\}DB \tag{11}$$

where $\{\ldots,\ldots\}$ stands for an anticommutator. You should find

$$[Q_i, Q_j] = i\epsilon_{ijk}Q_k\,, \quad [Q_i, Q_{5j}] = i\epsilon_{ijk}Q_{5k}\,, \quad [Q_{5i}, Q_{5j}] = i\epsilon_{ijk}Q_k \tag{12}$$

Remark:

One can combine vector and axial-vector transformations as follows:

$$N_R \to N_R' = RN_R\,, \quad N_L \to N_L' = LN_L\,, \quad \Sigma \to \Sigma' = L\Sigma R^\dagger \tag{13}$$

where we have introduced the right-handed and left-handed transformations:

$$R = \exp\left(\frac{i}{2}\gamma_j\tau_j\right)\,, L = \exp\left(\frac{i}{2}\delta_j\tau_j\right)\,, \tag{14}$$

with $\gamma_j = \delta_j = \alpha_j$ for the vector transformation and $\gamma_j = -\delta_j = \beta_j$ for the axial transformation. Under the infinitisemal transformation $R \simeq 1 + \frac{i}{2}\gamma_j\tau_j$ one obtains, for example $\delta_R N_R = \frac{i}{2}\gamma_j\tau_j N_R$ and $\delta_R N_L = 0$, etc. From the corresponding field variations one can immediately work out the corresponding conserved currents R^μ (right-handed) and L^μ (left-handed) and it is easy to see that the vector and axial currents are given in terms of them as

$$V_i^\mu = R_i^\mu + L_i^\mu\,, \qquad A_i^\mu = R_i^\mu - L_i^\mu \tag{15}$$

The same relation obviously holds for the conserved charges. If we define

$$Q_{Ri} = Q_i + Q_{5i}\,, \qquad Q_{Ri} = Q_i - Q_{5i} \tag{16}$$

the algebra becomes

$$[Q_{Li}, Q_{Lj}] = i\epsilon_{ijk}Q_{Lk}\,, \quad [Q_{Li}, Q_{5Rj}] = 0\,, \quad [Q_{R5i}, Q_{R5j}] = i\epsilon_{ijk}Q_{Rk}\,, \tag{17}$$

Namely, each set $\{Q_{Ri}\}$ and $\{Q_{Li}\}$ separately form an $SU(2)$ algebra. This is why it is referred as the $SU(2)_L \times SU(2)_R$ algebra.