Übungen zu Quantenfeldtheorie

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The nonlinear sigma model discussed in the lectures can be thought as a quantum theory of fields that are coordinates of the unit sphere. A slightly more complicated space of high symmetry is complex projective space, called CP^N. This space can be defined as the space of (N+1)-dimensional complex vectors $(z_1, \ldots, z_{N+1})$ subject to the condition

$$\sum_j |z_j|^2 = 1, \tag{1}$$

with points related by an overall phase rotation identified, that is

$$(e^{i\alpha} z_1, \ldots, e^{i\alpha} z_{N+1}) \;\text{ identified with }\; (z_1, \ldots, z_{N+1})\,. \tag{2}$$

In this problem (Peskin, Schröder, pp 466-467) we study the two-dimensional quantum field theory whose fields are coordinates on this space - the so-called CP^N-model.

Aufgabe 1:

One way to represent a theory of coordinates of CP^N is to write a Lagrangian depending on the fields $z_j(x)$, where x is a two-dimensional coordinate, subject to the constraint in Eq. (1), which also has the local symmetry

$$z_j(x) \to e^{i\alpha(x)} z_j(x) \tag{3}$$

independently at each point x. Show that the following Lagrangian has this symmetry:

$$\mathcal{L} = \frac{1}{g^2}\Big[\sum_j |\partial_\mu z_j|^2 - \Big|\sum_j z_j^* \partial_\mu z_j\Big|^2\Big]. \tag{4}$$

To prove the invariance, you will need to use the constraint on the z_j and its consquence:

$$\sum_j z_j^* \partial_\mu z_j = -\sum_j (\partial_\mu z_j^*) z_j$$

Show that the nonlinear sigma model for the case N=3 can be converted to the CP^N model for the case $N = 1$ by the substitution

$$n^i = z^* \sigma^i z$$

where σ^i are Pauli sigma matrices. You may find the following identity useful:

$$\sigma^a_{sr}\sigma^a_{s'r'} = 2\delta_{sr'}\delta_{s'r} - \delta_{sr}\delta_{s'r'} \tag{5}$$

where $r, s, \ldots$ are two-dimensional indices and the sum over a is implied.

Aufgabe 2:

Show that the above Lagrangian of the CP^N model can be obtained from the Lagrangian

$$\mathcal{L} = \frac{1}{g^2}\Big[\sum_j |D_\mu z_j(x)|^2 - \lambda(x)\Big(\sum_j |z_j|^2 - 1\Big)\Big]. \tag{6}$$

where $D_\mu = \partial_\mu + iA_\mu$, by taking the functional integral over the (auxilary) fields $A_\mu(x)$ and $\lambda(x)$.
Hint: Use eq. (3.93) in the script.

Aufgabe 3:

One can solve the CP^N model in the limit $N \to \infty$ by using the Lagrangian in Eq. (6) and integrating over the fields z_j. Show that this integral leads to the expression

$$Z = \int \mathcal{D}A\,\mathcal{D}\lambda\, \exp\left[-N\mathrm{Tr}\ln(-D^2 - \lambda) + \frac{i}{g^2}\int d^2x\lambda\right] \tag{7}$$

where we kept only the leading terms for $N \to \infty$, $g \to 0$ with g^2N fixed. Using the methods similar to those we used for the sigma model, examine the conditions for minimizing the exponent (i.e. the action) with respect to $\lambda(x)$ and $A_\mu(x)$. Show that these conditions have a solution at $A_\mu = 0$ and constant $\lambda = m^2 > 0$. Show that, if g^2 is defined at the scale M, m can be written as

$$m \approx M \exp\left[-\frac{2\pi}{g^2N}\right] \tag{8}$$

Hints:

- For a complex scalar field ϕ we have $\int \mathcal{D}\phi\ \exp\Big(-\int d^dx\ \phi^* K\phi\Big) = \mathcal{N}(\det K)^{-1}$
- Assuming λ to be constant will produce a "volume" factor $\int d^2x = \delta^2(0)$, which you can just treat as a finite number.
- As usual, you may treat $\int d^dp$ as being over $\mathbb{R}^d$ and introduce the cutoff M only in the last step after you performed the angular integration $\int_0^\infty dp \to \int_0^M dp$.
- Alternatively, you can also use dimensional regularization (with $\overline{\mathrm{MS}}$ and $\epsilon = 2 - d/2$) $\int \frac{d^2p}{(2\pi)^2} \to \Big(\frac{\mu^2 e^{\gamma_E}}{4\pi}\Big)^\epsilon \int \frac{d^dp}{(2\pi)^d}$. You will encounter a $\frac{1}{\epsilon}$ pole term, which you can drop (it is taken into account by renormalization). You should get $\lambda = \mu^2\exp\Big(-\frac{4\pi}{g^2N}\Big)$.

Aufgabe 4: (optional)

Now expand the exponent around $A_\mu = 0$. Show that the first nontrivial term in this expansion is proportional to the vacuum polarization of massive scalar fields. Evaluate this expression using dimensional regularization, and show that it yields a standard kinetic energy term for A_μ. Thus the strange nonlinear field theory that we started with is finally transformed into a theory of (N+1) massive scalar fields interacting with a massless photon!
Hints:

- For this exercise you may treat $\lambda = m^2 =$ const..
- There are two ways to do this. Option 1 is to directly expand the $\ln(...)$ in (7) in terms of A_μ up to the quadratic terms and take the trace. Option 2 is two rewrite the $\exp \mathrm{Tr} \ln(...)$ in terms of scalar fields and calculate the two one-loop diagrams with two external A legs.
- It is convenient to work with Fourier transformed A fields, i.e. $A^\mu(x) = \int \frac{d^d k}{(2\pi)^d} e^{-ikx} \widetilde{A}^\mu(k)$.
- In both cases you will encounter the loop integral $\int \frac{d^d p}{(2\pi)^d} \frac{p_\mu p_\nu}{[(p+k/2)^2 - m^2][(p-k/2)^2 - m^2]}$. Combine the denominators using Feynman parameters and evaluate the p-integral. In the resulting expression expand in $\epsilon = 2 - d/2$. To evaluate the integral over the Feynman parameter you should expand the integrand in k/m and drop terms $O((k/m)^4)$.
- In the end you want to get something proportional to $\int \frac{d^2 k}{(2\pi)^2} \widetilde{A}^\mu(k) \widetilde{A}^\nu(-k) \Big(-g_{\mu\nu} k^2 + k_\mu k_\nu \Big) = -\frac{1}{2} \int d^2 x \; F^{\mu\nu} F_{\mu\nu}$.