Übungen zu Quantenfeldtheorie

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SS 2023
Blatt 3

Problem 1: Determinant formula

Let K be a positive definite (i.e. symmetric and all eigenvalues are positive) $n \times n$ matrix and $J \in \mathbb{R}^n$. Show that

$$\int d^n x \; e^{-\frac{1}{2}x^T K x + J^T x} = \frac{(2\pi)^{n/2}}{\sqrt{\det(M)}} e^{\frac{1}{2}J^T K^{-1} J}. \tag{1}$$

Remark: Consider an operator K, which can be viewed as the limit of a positive definite matrix acting on functions as

$$(Kx)(t) = \int_{t_i}^{t_f} dt' \; K(t,t')x(t'),$$

where the integration kernel $K(t,t')$ may be distribution-valued (i.e. may depend on delta functions and its derivatives). The continuum version of eq. (1) reads

$$\int_{q(t_i)=q_i}^{q(t_f)=q_f} \mathcal{D}q \; e^{-\frac{1}{2}\int_{t_i}^{t_f} dt\, q(t)(Kq)(t) + \int_{t_i}^{t_f} dt\, J(t)q(t)} = \lim_{n\to\infty} \frac{(2\pi)^{n/2}}{\sqrt{\det(K)}} e^{\frac{1}{2}\int_{t_i}^{t_f} dt \;\; J(t)(K^{-1}J)(t)}, \tag{2}$$

where the inverse operator K^{-1} is defined by

$$\delta(t-t') = \int_{t_i}^{t_f} dt''\, K(t,t'')K^{-1}(t',t'').$$

The determinant of K is the product of eigenvalues of the differential equation

$$(K-\lambda)q = 0$$

subject to the boundary conditions $q(t_i) = q_i$, $q(t_f) = q_f$.

Problem 2: Propagator of the harmonic oscillator from path integral

We calculate the path integral representation of the propagator

$$\langle q_f, t_f | q_i, t_i \rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f} \mathcal{D}q \, e^{iS[q]}$$

for the harmonic oscillator

$$S[q] = \frac{1}{2}\int_{t_i}^{t_f} dt \, \left(\dot{q}^2 - \omega^2 q^2\right).$$

a) Determine the classical path $q_{\rm cl}$, the solution of the equation $\delta S = 0$ subject to the boundary conditions $q(t_i) = q_i$, $q(t_f) = q_f$.

b) Show that

$$\langle q_f, t_f | q_i, t_i \rangle = e^{iS[q_{\rm cl}]} \int_{q(t_i)=0}^{q(t_f)=0} \mathcal{D}q \, e^{iS[q]}.$$

c) Evaluate the remaining path integral by using eq. (2) and compute the determinant to get the final result

$$\langle q_f, t_f | q_i, t_i \rangle = \sqrt{\frac{\omega}{2\pi i \sin(\omega(t_f - t_i))}} \exp\left(\frac{i\omega}{2} \frac{(q_i^2 + q_f^2)\cos(\omega(t_f - t_i)) - 2q_f q_i}{\sin(\omega(t_f - t_i))} \right).$$

Hints:

- $\frac{\sin x}{x} = \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 k^2}\right)$
- You may ignore all (possibly infinite) constant factors and at the end, determine the overall constant factor by comparing your result with the free propagator for $\omega \to 0$, which was calculated in Blatt 2, Problem 1.

Problem 3: Path integral for an infinite potential barrier

Consider the Hamiltonian

$$H = \frac{p^2}{2m} + V(q),$$

where

$$V(q) = \begin{cases} 0 & , q \geq 0 \\ \infty & , q < 0 \end{cases}.$$

a) Show that

$$\langle q_f, t_f | q_i, t_i \rangle = \Theta(q_f)\Theta(q_i)\Big(\langle q_f, t_f | q_i, t_i \rangle_{\rm free} - \langle -q_f, t_f | q_i, t_i \rangle_{\rm free} \Big), \tag{3}$$

where $\Theta(x)$ is the Heaviside function and $\langle q_f, t_f | q_i, t_i \rangle_{\rm free}$ denotes the propagator for a free particle ($V = 0$), which was computed in Blatt 2, Problem 1.

Hint: Find continuous energy eigenstates $\psi_E(x) = \langle x | E \rangle$ such that

$$\int_0^\infty dE \, \psi_E^*(x) \psi_E(y) = \Theta(x)\delta(x - y)$$

and use this completeness relation.

b) Give an intuitive argument, based on the path integral representation of $\langle q_f, t_f | q_i, t_i \rangle$, for the correctness of eq. (3).