Übungen zu Quantenfeldtheorie

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Blatt 2

Problem 1: Transition amplitude for a free particle

Show that the transition amplitude for a free particle with mass m moving in one dimension has the expression

$$\langle q', t'|q, t\rangle = \langle q'|e^{-i\widehat{H}(t'-t)}|q\rangle = \left[\frac{m}{2\pi i(t'-t)}\right]^{1/2} \exp\left[\frac{im}{2}\frac{(q'-q)^2}{t'-t}\right] \tag{1}$$

To do the calculation, give t' a small imaginary part, $t' \to t' - i\epsilon$, and take $\epsilon \to 0^+$ at the end of the calculation. Show that this is justified, given that we interpret the position space propagator $\langle q', t'|q, t\rangle$ as a distribution, i.e. it is supposed to be integrated with an integrable wave function ψ.

$$\psi(q', t') = \int_{-\infty}^{\infty} dq\ \langle q', t'|q, t\rangle\, \psi(q, t). \tag{2}$$

Hint: Use the dominated convergence theorem.

Problem 2: Spreading of a wave packet

For the free particle, suppose $\psi(q, t = 0)$ is a Gaussian wave packet

$$\psi(q, t = 0) = \left(\frac{1}{2\pi\sigma_0^2}\right)^{1/4} \exp\left[-\frac{(q-a)^2}{4\sigma_0^2}\right].$$

Show that it will spread as time evolves:

$$|\psi(q, t)|^2 = \left(\frac{1}{2\pi\sigma(t)^2}\right)^{1/2} \exp\left[-\frac{(q-a)^2}{2\sigma(t)^2}\right],$$

where

$$\sigma(t) = \sqrt{\sigma_0^2 + \frac{t^2}{4m^2\sigma_0^2}}.$$

Hint: Use eqs. (1) and (2).

Problem 3: Non-standard path-integral representation

Consider the Lagrangian with a position-dependent “mass”

$$L = \frac{1}{2}m(q)\dot{q}^2\,.$$

Thus the canonical momentum is $p = m(q)\dot{q}$ and the Hamiltonian $H = \frac{1}{2m(q)}p^2$. Show that the path-integral representation of the transition amplitude has the form

$$\langle q'|e^{-i\widehat{H}(t'-t)}|q\rangle = \mathcal{N}\int \mathcal{D}q \, \exp\left[i\int_t^{t'} ds\, \left(L(q(s),\dot{q}(s)) - \frac{i}{2}\widetilde{\mathcal{N}}\log m(q(s))\right)\right],$$

where $\widetilde{\mathcal{N}} \equiv$ "$\lim_{\Delta t\to 0}\frac{1}{\Delta t}$" is an infinite constant, similar to $\mathcal{N}$.
Remark 1: Recall that the path integral is usually ill-defined in the continuum limit, where the discretization spacing Δt goes to zero. The continuum limit can only be taken for observable quantities.
Remark 2: This is a counterexample, given by Lee and Yang (1962), showing that the path-integral representation is not always of the "standard" form.

Problem 4: Poles in correlation functions

Consider a Fourier transform of the vacuum expectation value of a time-ordered product

$$T(q^2) = i\int d^4x e^{iqx}\langle\Omega|\mathrm{T}\{\widehat{A}(x)\widehat{A}^\dagger(0)\}|\Omega\rangle,$$

where the local operator $\widehat{A}(x)$ can be either elementary scalar field operator like $\widehat{\phi}(x)$ or composite operator like $\widehat{\phi}^2(x)$ etc. The main restriction is that $\widehat{A}$ is composed of fields at the same space-time point, which makes it "local".
Suppose $\widehat{A}$ has a non-zero matrix element between the vacuum and the one-particle state $|\vec{p}\rangle = \hat{a}^\dagger_{\vec{p}}|\Omega\rangle$. Here $\vec{p}$ is the three-momentum and $p = (E(\vec{p}),\vec{p})$, where $E(\vec{p}) = \sqrt{m^2+\vec{p}^2}$. Recall that in the relativistic normalization $\langle\vec{p}|\vec{q}\rangle = (2\pi)^3 2E(\vec{p})\delta^{(3)}(\vec{p}-\vec{q})$.
Show that the function $T(q^2)$ considered as a function of the complex variable $q^2 = q_0^2 - \vec{q}^2$ has a pole in the vicinity of the point $q^2 = m^2$ and we can write

$$T(q^2) = \frac{|\langle\Omega|\widehat{A}(0)|\vec{q}\rangle|^2}{m^2 - q^2 - i\epsilon}.$$

Check that this agrees with the expected result of the propagator in the non-interacting scalar field theory if $\widehat{A} = \widehat{\phi}$.
Hints:

- Write out the T product explicitly (in terms of Heaviside functions Θ) and insert

$$1_{\text{one-particle subspace}} = \int \frac{d^3p}{(2\pi)^3 2E(\vec{p})}|\vec{p}\rangle\langle\vec{p}|$$

  in between the operators. Explain when inserting this "unity" is correct and when not. How does this completeness relation have to be modified to be valid more generally?

- Prove and use the relation $\langle\Omega|\widehat{A}(x)|\vec{p}\rangle = \langle\Omega|\widehat{A}(0)|\vec{p}\rangle e^{-ipx}$ (Hint: How does $\widehat{A}$ transform under translations?)

- Use the identity

$$\Theta(t) = \lim_{\epsilon\to 0^+}\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\tau\, \frac{e^{it\tau}}{\tau - i\epsilon}.$$