# Problem set 1 for Quantum field theory

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**Problem 1: Correlation function of $n$ fields**

Show that

$$\langle\Omega|\mathrm{T}\{\widehat{\phi}(t_1)...\widehat{\phi}(t_n)\}|\Omega\rangle = \frac{\langle 0|\mathrm{T}\{\widehat{\phi}_I(t_1)...\widehat{\phi}_I(t_n)\exp\left[-i\int_{-\infty}^{\infty}dt'\widehat{H}_I(t')\right]\}|0\rangle}{\langle 0|\mathrm{T}\{\exp\left[-i\int_{-\infty}^{\infty}dt'\widehat{H}_I(t')\right]\}|0\rangle},$$

where the notation is that from chapter 1 of the lecture notes.

**Problem 2: Feynman propagator of the harmonic oscillator**

Consider the 0+1-dimensional QFT of a single free mode, i.e. the harmonic oscillator, where

$$\widehat{H} = \frac{1}{2}(\partial_t\widehat{\phi})^2 + \frac{1}{2}\omega^2\widehat{\phi}^2, \qquad \widehat{\phi}(t) = \frac{1}{\sqrt{2\omega}}\Big(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\Big), \qquad [\hat{a},\hat{a}^\dagger] = 1. \tag{1}$$

a) Show that

$$\widehat{H} = \omega\Big(\hat{a}^\dagger\hat{a} + \frac{1}{2}\Big).$$

b) The Feynman propagator is defined by

$$D(t-t') = \langle 0|T\{\widehat{\phi}(t)\widehat{\phi}(t')\}|0\rangle.$$

Show that

$$D(t-t') = \frac{1}{2\omega}e^{-i\omega|t-t'|} = \lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\,e^{-iE(t-t')}\frac{i}{E^2-\omega^2+i\epsilon}. \tag{2}$$

**Aufgabe 3: Ground state energy**

a) As in the script, consider

$$\widehat{H} = \widehat{H}_0 + \widehat{H}_1,$$

where $\widehat{H}_1$ is viewed as a perturbation. In the interaction picture

$$\widehat{H}_I(t) = e^{i\widehat{H}_0(t-t_0)}\widehat{H}_1(t_0)e^{-i\widehat{H}_0(t-t_0)}.$$

Show that

$$\log\Big[\langle 0|\mathrm{T}\{\exp\Big(-i\int_{-T}^{T}dt\,\widehat{H}_I(t)\Big)\}|0\rangle\Big] \overset{T\to\infty}{\sim} -2iT(E_{\mathrm{vac}}-E_0), \tag{3}$$

where $\widehat{H}|\Omega\rangle = E_{\mathrm{vac}}|\Omega\rangle$ and $\widehat{H}_0|0\rangle = E_0|0\rangle$.

Hint: Use the relation obtained from $\langle\Omega|\Omega\rangle = 1$ derived in the script and take the logarithm of this equation.

b) Assume that the $\widehat{H}_I(t)$ is small and expand the left-hand side in eq. (3) up to leading order in $\widehat{H}_I(t)$. Show that

$$E_{\mathrm{vac}} - E_0 \approx \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}dt\,\langle 0|\widehat{H}_I(t)|0\rangle = \langle 0|\widehat{H}_1|0\rangle \tag{4}$$

for large $T$. Show that you get the same result using the usual Rayleigh-Schrödinger perturbation theory.

c) Consider now as the unperturbed Hamiltonian $\widehat{H}_0$ the harmonic oscillator in eq. (1). We add the anharmonic quartic perturbation

$$\widehat{H}_1 = \frac{\lambda}{4!}\widehat{\phi}^4.$$

Calculate $E_{\mathrm{vac}} - E_0$ to first order in $\lambda$ from eq. (4) using Wicks theorem and the propagator in eq. (2). Identify the corresponding Feynman diagram. Check your result by calculating $\langle 0|\widehat{H}_1|0\rangle$ by brute force, using $\hat{a}|0\rangle = 0$ and $[\hat{a},\hat{a}^\dagger] = 1$.

Remark: This pertubative series

$$E_{\mathrm{vac}}(\lambda) = E_0 + \sum_{j=1}^{\infty}\lambda^j\,E_0^{(j)}$$

is actually divergent for any non-zero value of $\lambda$ (has zero radius of convergence), see Bender and Wu (1969). This does not mean that the series is not useful. It is a so-called asymptotic expansion. This means that for a given small value of $\lambda$ the magnitude of successive terms generally decrease until, at some point, they start growing again. A good approximation to the desired answer is obtained by including only the part of the sum where the terms are decreasing. This situation is very common for perturbative expansion in physics. For observables of QFT it is practically always the case.