Monads and their applications II 7

Exercise 1.
Show that an equivalence in $T\text{-}\mathbf{Alg}_\ell$ is a pseudo T-morphism. In other words, any equivalence in $T\text{-}\mathbf{Alg}_\ell$ already lies in $T\text{-}\mathbf{Alg}_p$.

Exercise 2.
Let $\mathscr{K}$ be a complete and cocomplete 2-category, S, T accessible 2-monads on $\mathscr{K}$ and $\varphi\colon S \Rightarrow T$ a (strict) morphism of 2-monads. Then the induced 2-functor

$$\varphi^*\colon T\text{-}\mathbf{Alg}_s \to S\text{-}\mathbf{Alg}_s$$

has a left 2-adjoint φ_*. Show that φ_* preserves flexible algebras and use this fact to prove that the 2-functor

$$\varphi^*\colon T\text{-}\mathbf{Alg}_p \to S\text{-}\mathbf{Alg}_p$$

has a left biadjoint.

Exercise 3.
Let $W\colon \mathscr{A} \to \mathbf{Cat}$ be a 2-functor. Recall that W-weighted limits are called pseudo limits if W is isomorphic to QW_0 for some 2-functor $W_0\colon \mathscr{A} \to \mathbf{Cat}$. Show that equifiers are *not* pseudo limits. (Hint: Let W to be the weight for equifiers and derive a contradiction using the triangle identity $e_{W_0} n_{W_0} = \mathrm{id}_{W_0}$. For this, you first need to write down very explicitly what W is.)

Exercise 4.
Let $\mathscr{A}$ be the category with two objects u, v, morphisms $\iota\colon u \to v$ and $\rho\colon v \to u$ such that $\rho\iota = \mathrm{id}$. Let $I = \{0 \cong 1\}$ be the free isomorphism in $\mathbf{Cat}$ and let $F\colon \mathscr{A} \to \mathbf{Cat}$ be the 2-functor which sends u to I, v to $I + *$, ι to the coproduct inclusion, and ρ to the functor $I + * \to I$ which restricts to the identity on I and to the inclusion $0\colon * \to I$ on $*$.

Let $e_u = \mathrm{id}_{Fu}\colon Fu \to Fu$ and $e_v\colon Fv \to Fv$ the constant functor with value 0. Show that this extends (uniquely) to a pseudonatural transformation $e\colon F \to F$, that e is idempotent, and that e does *not* split in $[\mathscr{A}, \mathbf{Cat}]_p$.

Exercise 5.
Show that the free symmetric monoidal category on a commutative monoid is equivalent to the category of finite sets, with symmetric monoidal structure given by the coproduct, and with universal commutative monoid object given by the unique monoid structure on the terminal object.