Monads and their applications II 6

Exercise 1.
Show that the forgetful 2-functor $U_p \colon T\text{-}\mathbf{Alg}_p \to \mathscr{K}$ is "weakly conservative:" if $U_p(f)$ is an equivalence in $\mathscr{K}$, then f is an equivalence in $T\text{-}\mathbf{Alg}_p$. (Hint: use doctrinal adjunction.)

Exercise 2.
Let X be a set and consider the monoidal 2-category $[X \times X, \mathbf{Cat}]$ of $X \times X$-matrices in $\mathbf{Cat}$. Show that pseudomonoids for the matrix tensor product are precisely bicategories with object set X, and strong monoidal morphisms between such are the pseudofunctors which act as the identity on objects. Use Exercise 1 to show that a pseudofunctor which is bijective on objects is a biequivalence if and only if it acts via an equivalence on each hom-category.

Exercise 3.
Let $F \colon \mathscr{B} \to \mathscr{C}$ be a pseudofunctor which is locally an equivalence and essentially surjective up to equivalence. Show that F is a biequivalence, using the following steps.

(i) Show the claim in the case where F is the inclusion of a pseudo-skeleton: a full sub-bicategory consisting of one object of each equivalence class in $\mathscr{C}$;

(ii) Factor F as a bijective on objects pseudofunctor followed by one which acts as the identity on hom-categories;

(iii) Using Exercise 2, reduce to the case where F acts as the identity on hom-categories. To prove this case, consider the inclusion of a pseudo-skeleton $\mathscr{B}'$ of $\mathscr{B}$ and observe that the composite

$$\mathscr{B}' \longrightarrow \mathscr{B} \longrightarrow \mathscr{C}$$

is also a pseudo-skeleton.

Exercise 4.
Show that the terminal category $*$ in $\mathbf{Lex}$ is not flexible. (Hint: consider the strict and pseudo morphisms from $*$ to the category $\mathbf{Set}_{\text{fin}}$ with the standard assignment of limits.)

Exercise 5.
A T-algebra A is called *semi-flexible* if the counit $e_A \colon QA \to A$ is an equivalence in $T\text{-}\mathbf{Alg}_s$ (not necessarily surjective). Show that the following are equivalent:

(i) The algebra A is semi-flexible;

(ii) There exists an algebra B and an equivalence $A \simeq QB$ in $T\text{-}\mathbf{Alg}_p$;

(iv) For all B, each pseudo T-morphism $A \rightsquigarrow B$ is isomorphic to a strict T-morphism $A \to B$.