Monads and their applications II 4

Exercise 1.
Write down a presentation for the 2-monad whose algebras are ($\mathscr{V}$-)categories with chosen (conical) *limits* of a given shape. What are the lax morphisms in this case?

Exercise 2.
A 2-monad is called *lax idempotent* if every 1-cell in the underlying 2-category $\mathscr{K}$ has a *unique* lax T-morphism structure (for example, the 2-monad for finitely cocomplete categories is lax idempotent). Show that the following are equivalent:

(i) The 2-monad T is lax idempotent;

(ii) For each algebra (A, a), there exists a 2-cell $\theta_{(A,a)}\colon \mathrm{id}_{TA} \Rightarrow \eta_A.a$ such that $(a, \eta_A, \theta_{(A,a)}, 1_{\mathrm{id}_A})$ is an adjunction in $\mathscr{K}$;

(iii) In $[\mathscr{K}, \mathscr{K}]$, there exists a modification $\lambda\colon \mathrm{id} \Rightarrow \eta T.\mu\colon T^2 \to T^2$ giving an adjunction $(\mu, \eta T, \lambda, 1)$ with identity counit in $[\mathscr{K}, \mathscr{K}]$.

Exercise 3.
Let B be a small category. Show that the 2-functor $\mathbf{Cat}/B \to \mathbf{Cat}/B$ which sends $p\colon E \to B$ to $p \downarrow B \to B$ is a lax idempotent 2-monad whose algebras are the Grothendieck opfibrations over B.

Exercise 4.
Let $T = \mathrm{id}_{\mathscr{K}}\colon \mathscr{K} \to \mathscr{K}$ be the identity 2-monad on $\mathscr{K}$. What are the lax T-algebras?

Exercise 5.
Let $\mathscr{K}$ be a cocomplete and let $\mathscr{A}$ be a small 2-category. Let T be the 2-monad whose 2-category of algebras is given by the 2-functors $\mathscr{A} \to \mathscr{K}$, (strict/pseudo/lax) 2-natural transformations, and modifications. Show that the pseudo T-algebras are precisely the pseudofunctors and the lax T-algebras are the lax functors.