Monads and their applications II 1

Exercise 1.
Let (T, μ, η) be a 2-monad on a 2-category $\mathscr{K}$. Show that $T\text{-}\mathbf{Alg}_p$, $T\text{-}\mathbf{Alg}_\ell$, and $T\text{-}\mathbf{Alg}_c$ are 2-categories.

Exercise 2.
Let $\mathscr{M}$ be a strict monoidal 2-category (a monoid in 2-$\mathbf{CAT}$) and let

$$- \circ -\colon \mathscr{M} \times \mathscr{K} \to \mathscr{K}$$

be a strict action of $\mathscr{M}$ on $\mathscr{K}$. Assume that each 2-functor $- \circ A$ has a right 2-adjoint $\langle A, -\rangle$, so there are isomorphisms of categories

$$\mathscr{K}(M \circ A, B) \cong \mathscr{M}(M, \langle A, B\rangle)$$

2-natural in M, A, and B.

Show that $\langle A, A\rangle$ has a monoid structure in $\mathscr{M}$. Moreover, show that monoid morphisms $(M, \mu, \eta) \to \langle A, A\rangle$ are in bijection with M-actions on A, that is, 1-cells $a\colon M \circ A \to A$ subject to the two axioms $a.M \circ a = a.\mu \circ A$ and $a.\eta \circ A = \mathrm{id}_A$. Analogously to the case of monads we can define lax morphism of M-actions. Show that the above bijection extends to an isomorphism of categories between the category of monoid morphisms and 2-cells as defined in class on the one hand, and the category of M-actions on A and lax morphisms whose underlying 1-cell is the identity on A on the other.

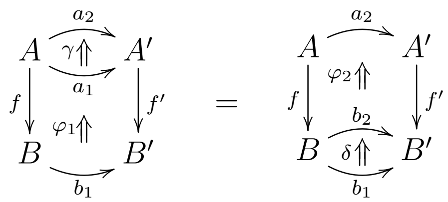
Exercise 3.
Let $\mathscr{K}$ be a complete 2-category. Apply Exercise 2 to the action

$$[\mathscr{K}, \mathscr{K}] \times \mathscr{K} \to \mathscr{K}, \quad (F, A) \mapsto FA$$

to show that the endo-2-functor $\langle A, A\rangle$ is a 2-monad and that 2-monad morphisms $T \to \langle A, A\rangle$ are in bijection with T-algebra structures on A. (Hint: why is $\langle A, -\rangle$ a right 2-adjoint?)

Exercise 4.
Let $\mathscr{K}$ be a complete 2-category. Let $\mathrm{Colax}[\mathbb{2}, \mathscr{K}]$ be the 2-category with objects the 1-cells $f\colon A \to B$ in $\mathscr{K}$, 1-cells the triples $(a, \varphi, b)\colon f \to f'$ where $a\colon A \to A'$ and $b\colon B \to B'$ are 1-cells in $\mathscr{K}$ and $\varphi\colon b.f \Rightarrow f'.a$ is a 2-cell in $\mathscr{K}$, and 2-cells $(a_1, \varphi_1, b_1) \Rightarrow (a_2, \varphi_2, b_2)$ the pairs (γ, δ) of 2-cells $\gamma\colon a_1 \Rightarrow a_2$

and $\delta\colon b_1 \to b_2$ such that the equation



holds. Show that composition with an endo-2-functor defines a strict action

$$[\mathscr{K},\mathscr{K}] \times \mathrm{Colax}[\mathbb{2},\mathscr{K}] \to \mathrm{Colax}[\mathbb{2},\mathscr{K}]$$

and show that $\{f,-\}_\ell\colon \mathrm{Colax}[\mathbb{2},\mathscr{K}] \to [\mathscr{K},\mathscr{K}]$ is a right 2-adjoint of $-\circ f$ ("acting on f"). Use this and the previous exercises to show that $\{f,f\}_\ell$ is a 2-monad, that the two 2-natural transformations $\partial_0\colon \{f,f\}_\ell \to \langle A,A\rangle$ and $\partial_1\colon \{f,f\}_\ell \to \langle B,B\rangle$ from the definition are 2-monad morphisms, and show that 2-monad morphisms $T \to \{f,f\}_\ell$ correspond to 2-cells which make f a lax T-morphism.

Exercise 5.
Recall that a comonad in a 2-category $\mathscr{K}$ is a 1-cell $c\colon C \to C$ with a counit $\varepsilon\colon c \to \mathrm{id}_C$ and a comultiplication $\gamma\colon c \to c.c$. An Eilenberg–Moore object for c is a universal c-coaction, that is, a 1-cell $w\colon A \to C$ together with a 2-cell $\rho\colon w \Rightarrow c.w$ satisfying the dual axioms of the Eilenberg–Moore object of a monad.

Let T be a 2-monad on $\mathscr{K}$. Show that the forgetful 2-functor

$$U_\ell\colon T\text{-}\mathbf{Alg}_\ell \to \mathscr{K}$$

creates Eilenberg–Moore objects for comonads. This gives an abstract explanation for the fact that comodules of a Hopf algebra inherit a monoidal structure which is strictly preserved by the forgetful functor.