Monads and their applications 11

Exercise 1.
Let $\mathscr{V}$ be the category $\mathrm{Ch}(\mathbf{Ab})$ of chain complexes over abelian groups ("differential graded $\mathbb{Z}$-modules"). The tensor product of $X_\bullet$ and $Y_\bullet$ is given in degree n by $\oplus_{i+j=n} X_i \otimes Y_j$, with differential $d(x \otimes y) = dx \otimes y + (-1)^i x \otimes dy$ for $x \otimes y \in X_i \otimes Y_j$. Show that this is a symmetric monoidal closed category and explicitly describe the self-enrichment of $\mathscr{V}$ (specifically, the composition morphism).

Exercise 2.
Let $\mathscr{C}$ be a $\mathscr{V}$-category with powers, $T \colon \mathscr{C} \to \mathscr{C}$ a $\mathscr{V}$-monad. Show that the forgetful $\mathscr{V}$-functor $T\text{-}\mathbf{Alg} \to \mathscr{C}$ creates powers (that is, for every T-algebra (A, a) and $V \in \mathscr{V}$ one can put a unique T-algebra structure on A^V making it a power of (A, a) in $T\text{-}\mathbf{Alg}$).

Exercise 3.
Let $t \colon C \to C$ be a monad in the 2-category $\mathscr{K}$. A t-action on a 1-cell $g \colon A \to C$ is a 2-cell $\alpha \colon tg \Rightarrow g$ satisfiying the laws for a t-algebra. This defines a 2-functor

$$t\text{-act}(-) \colon \mathscr{K}^{\mathrm{op}} \to \mathbf{Cat}$$

and we say that the *Eilenberg–Moore* object of t exists if this 2-functor is representable, that is, there exists a *universal* t-action $(u, \alpha) \colon C_t \to C$.

(a) Show that $U \colon T\text{-}\mathbf{Alg} \to \mathscr{C}$ is an Eilenberg–Moore object in $\mathscr{V}\text{-}\mathbf{Cat}$.

(b) Given any Eilenberg–More object $u \colon C_t \to C$, show that there exists a left adjoint f of u such that the monad uf is t. (Hint: define a suitable t-action on t itself).

Exercise 4.

(a) Let $\mathscr{V} = \mathbf{Set}$ and consider the discrete category $2 = \{0, 1\}$ on two objects. A weight W on this category amounts to a choice of two sets. What is the W-weighted colimit on a diagram $2 \to \mathscr{C}$?

(b) Let $\mathscr{V} = \mathbf{Ab}$ and let R be a ring, considered as a 1-object $\mathscr{V}$-category. Let $\mathscr{C} = \mathbf{Ab}$ as well. Show that weighted colimits over R correspond to the usual tensor product of a right and a left R-module.

(c) Let $\mathscr{V} = \mathbf{Cat}$ and consider the category $\mathscr{I} = 0 \longrightarrow 2 \longleftarrow 1$. Let W be the weight with $W(0) = *$, $W(1) = *$, and $W(2) = [1]$, with the morphisms $W(i) \to [1]$ picking out $i \in [1] = \{0 \to 1\}$. Show that W-weighted limits in a 2-category $\mathscr{K}$ are precisely comma-objects.

Exercise 5. *(bonus)*
Show that $\prod_{j \in J} \mathscr{V}$ is the free cocomplete $\mathscr{V}$-cateogry on the discrete category j. More precisely, given a cocomplete $\mathscr{V}$-categories $\mathscr{C}$, $\mathscr{D}$, write $\mathbf{Cocts}_0[\mathscr{C}, \mathscr{D}]$ for the category of cocontinuous $\mathscr{V}$-functors and $\mathscr{V}$-natural transformations. Show that the functor $\mathbf{Cocts}_0[\prod_{j \in J} \mathscr{V}, \mathscr{C}] \to \prod_{j \in J} \mathscr{C}_0$ given by $F \mapsto (FI_j)_{j \in J}$ is an equivalence of categories. Here I_j stands for the object which is given by I in degree j and by the initial object everywhere else. You need to be careful when showing that the functor is full!

If one checks all the requirements by only referring to copowers, $\mathscr{V}$-coproducts, and $\mathscr{V}$-coequalizers, (the latter is not even necessary), then one can use this to show that cocontinuous endofunctors of the product are equivalent, as a monoidal category, to $\mathscr{V}$-matrices on J. This can then be used to give a rigorous proof that any $\mathscr{V}$-category with object set J gives rise to the cocontinous $\mathscr{V}$-monad T defined in the lecture.