Monads and their applications 10

Exercise 1.
Let $\mathscr{V}$ be a braided monoidal category. Show that the monoidal structure of $\mathscr{V}$ lifts to a monoidal structure of $\mathbf{Mon}(\mathscr{V})$ (that is, put a monoid structure on $M \otimes M'$ so that α, ρ, λ become monoid morphisms). If $\mathscr{V}$ is symmetric, show that γ lifts to a symmetry on $\mathbf{Mon}(\mathscr{V})$.

Exercise 2.
Show that a monoid in $\mathbf{Mon}(\mathscr{V})$ is a commutative monoid (this is known as the "Eckmann-Hilton argument"). More precisely, if $\big((M, \mu, \eta), \mu', \eta'\big)$ is a monoid in $\mathbf{Mon}(\mathscr{V})$, show that $\eta' = \eta$, $\mu' = \mu$ and $\mu\gamma_{M,M} = \mu$. (Hint: it is best to first prove this for $\mathscr{V} = \mathbf{Set}$ and then translate to general monoidal categories. First show that the two units $\eta = \eta'$ are the same and then turn the following pictorial "proof" into a precise argument, where we write one multiplication vertically, one horizontally:

$$\begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} a & \eta \\ \eta & b \end{bmatrix} = \begin{bmatrix} \eta & a \\ b & \eta \end{bmatrix} = \begin{bmatrix} b & a \end{bmatrix}$$

for all $a, b \in M$.

Exercise 3.
Let $\mathscr{V}$ be a symmetric monoidal closed category, $\mathscr{C}$ a $\mathscr{V}$-category, and $C \in \mathscr{C}$. Show that the functor $\mathscr{C}(C, -)\colon \mathscr{C} \to \mathscr{V}$ defined in class is indeed a $\mathscr{V}$-functor.

Exercise 4.
Let $\mathscr{V}$ be a symmetric monoidal category. If $A \in \mathscr{V}$ is a monoid, then $A \otimes -$ is a monad. The $(A \otimes -)$-aglebras are called *A-modules* and we write $\mathscr{V}_A$ for $(A \otimes -)$-$\mathbf{Alg}$. Now suppose that A is commutative, that $\mathscr{V}$ has reflexive coequalizers, and that $V \otimes -$ preserves these for all $V \in \mathscr{V}$. Given A-modules M and N, define $M \otimes_A N$ by the reflexive coequalizer

$$M \otimes A \otimes N \rightleftarrows M \otimes N \longrightarrow M \otimes_A N$$

in $\mathscr{V}$. Show that this lifts to a functor $- \otimes_A -\colon \mathscr{V}_A \otimes \mathscr{V}_A \to \mathscr{V}_A$ making $\mathscr{V}_A$ into a symmetric monoidal category with unit A (Hint: use the fact that reflexive coequalizers are sifted and split coequalizers are absolute).

Exercise 5. *(bonus)*
Give an example of a braided monoidal category $\mathscr{V}$ such that the braiding does not lift to $\mathbf{Mon}(\mathscr{V})$.