Monads and their applications 8

Exercise 1.
Let $\mathscr{V}$, $\mathscr{W}$, $\mathscr{U}$ be monoidal categories and let $(F, \varphi_0, \varphi)\colon \mathscr{V} \to \mathscr{W}$ and $(G, \psi_0, \psi)\colon \mathscr{W} \to \mathscr{U}$ is lax monoidal with structure morphisms $\gamma_0 = G(\varphi_0)\cdot\psi_0$ and $\gamma_{X,Y} = G(\varphi_{X,Y}) \cdot \psi_{FX,FY}$. Show that monoidal natural transformations can be whiskered on either side with lax monoidal functors.

Exercise 2.
Let $(F, \varphi_0, \varphi)\colon \mathscr{V} \to \mathscr{W}$ be strong monoidal and suppose that the underlying functor F has a right adjoint U. Show that the composites

$$I_{\mathscr{V}} \xrightarrow{\eta_{I_{\mathscr{V}}}} UF(I) \xrightarrow{U(\varphi_0^{-1})} U(I_{\mathscr{W}})$$

and

$$\begin{array}{ccc} UX \otimes_{\mathscr{V}} UY & \xrightarrow{\eta_{UX \otimes_{\mathscr{V}} UY}} & UF(UX \otimes_{\mathscr{V}} UY) \\ \Big\downarrow & & \Big\downarrow U\varphi^{-1}_{UX,UY} \\ U(X \otimes_{\mathscr{W}} Y) & \xleftarrow[U(\varepsilon_X \otimes_{\mathscr{W}} \varepsilon_Y)]{} & U(FUX \otimes_{\mathscr{W}} FUY) \end{array}$$

endow U with the structure of a lax monoidal functor, and that η, ε are monoidal natural transformations for this structure if the composites UF and FU are given the lax monoidal structure of Exercise 1.

Exercise 3.
Let $\mathscr{V}$, $\mathscr{W}$, be monoidal categories, $(F, \varphi_0, \varphi)\colon \mathscr{V} \to \mathscr{W}$ a strong monoidal left adjoint, and $f\colon S \to T$ a function of sets.

(a) Show that f induces a strong monoidal $f_*\colon \mathbf{Mat}(\mathscr{V}, S) \to \mathbf{Mat}(\mathscr{W}, T)$.

(b) Show that F induces a strong monoidal $F\colon \mathbf{Mat}(\mathscr{V}, S) \to \mathbf{Mat}(\mathscr{V}, S)$.

(c) Use Exercise 2 to show that these are both monoidal adjunctions.

Exercise 4.
Let $\mathscr{C}$ be a complete category, a, b objects of $\mathscr{C}$. Recall that $\langle a, b\rangle$ is defined to be the right Kan extension of $b\colon * \to \mathscr{C}$ along $a\colon * \to \mathscr{C}$. Show that

$$\big(\mathrm{Ob}(\mathscr{C}), (\langle a, b\rangle)_{(a,b)\in \mathrm{Ob}\,\mathscr{C} \times \mathrm{Ob}(\mathscr{C})}\big)$$

defines a category enriched in the monoidal category $[\mathscr{C}, \mathscr{C}]$ of endofunctors.

Exercise 5. *(bonus)*
There are two natural monoidal functors $\mathscr{V} \to [\mathscr{V}, \mathscr{V}]$ given by tensoring in either side. Under what conditions do these have a right adjoint? (For example, is locally presentable enough?) Applying Exercise 3(b) to the enriched category of Exercise 4, we get two $\mathscr{V}$-category structures on $\mathscr{V}$. Describe them explicitly. (Hint: you only need to know what the right adjoint does to functors of the form $\langle V, W\rangle$).