Monads and their applications 7

Exercise 1.
Complete the proof that $(-)\text{-}\,\mathbf{Alg}\colon \mathbf{Mnd}(\mathscr{C})^{\mathrm{op}} \to \mathbf{CAT}/\mathscr{C}$ is a functor and that it is full and faithful.

Exercise 2.
Recall that the *codensity monad* $\mathrm{Ran}_F F$ of $F\colon \mathscr{A} \to \mathscr{C}$ is the right Kan extension of F along itself.

(a) Show that the right Kan extension of any functor G along a right adjoint U always exists and is given by GL ("by adjunction").

(b) Show that the codensity monad of a right adjoint is precisely the monad $(UL, \eta, U\varepsilon L)$ associated to the adjunction.

Exercise 3.
We call a functor $F\colon \mathscr{A} \to \mathscr{C}$ *admissible* if $\mathrm{Ran}_F F$ exists and we write $\mathbf{CAT}'/\mathscr{C}$ for the full subcategory of admissible functors. From Exercise 2 it follows that $(-)\text{-}\,\mathbf{Alg}$ factors through the admissible functors. We denote the codensity monad of F by $S(F)$.

(a) Show that

$$(-)\text{-}\,\mathbf{Alg}\colon \mathbf{Mnd}(\mathscr{C})^{\mathrm{op}} \to \mathbf{CAT}'/\mathscr{C}$$

is right adjoint to

$$S\colon\ \mathbf{CAT}'/\mathscr{C} \to \mathbf{Mnd}(\mathscr{C})^{\mathrm{op}}$$

(this is called the *semantics-structure adjunction*).

(b) Use this adjunction to give a rigorous argument that the algebra functor

$$(-)\text{-}\,\mathbf{Alg}\colon \mathbf{Mnd}(\mathscr{C})^{\mathrm{op}} \to \mathbf{CAT}/\mathscr{C}$$

with target the full slice category sends colimits to limits if $\mathscr{C}$ is complete.

(c) Use the adjunction to give an alternative proof that $(-)\text{-}\,\mathbf{Alg}$ is full and faithful.

Exercise 4.

Directed graphs are presheaves on $G = \{ 0 \rightrightarrows 1 \}$, that is, pairs of sets E and V with source and target maps $s\colon E \to V$ and $t\colon E \to V$. Use finitary endofunctors of the presheaf category such as $(E, V, s, t) \mapsto (V, V, \mathrm{id}, \mathrm{id})$ or the functor which sends a graph to the graph consisting of paths of length two in the original (that is, the edges in the new graph are given by the pullback of s along t) to construct a finitary monad on $[G^{\mathrm{op}}, \mathbf{Set}]$ whose category of algebras is isomorphic to **Cat**.

Exercise 5. *(bonus)*

Let $\mathscr{C}$ be the category whose objects are small categories with a choice of colimit for each finite diagram and whose morphisms are the functors which preserve these colimits strictly. Use free finitary monads and colimits of such to show that $\mathscr{C}$ is finitarily monadic over **Cat**.