Monads and their applications 5

Exercise 1.
Let (S, σ) be a well-pointed endofunctor of $\mathscr{D}$. Let $\mathscr{C}$ be a finitely cocomplete category and let $F\colon \mathscr{D} \to \mathscr{C}$ be left adjoint to $U\colon \mathscr{C} \to \mathscr{D}$. Show that the endofunctor (S', σ') of $\mathscr{C}$ defined by the pushout diagram

$$\begin{array}{ccc} FU & \xrightarrow{F\sigma U} & FSU \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow \\ \mathrm{id} & \xrightarrow[\sigma']{} & S' \end{array}$$

in $[\mathscr{C}, \mathscr{C}]$ is a well-pointed endofunctor and that there is an induced diagram

$$\begin{array}{ccc} (S', \sigma')\text{-}\,\mathbf{Alg} & \xrightarrow{\overline{U}} & (S, \sigma)\text{-}\,\mathbf{Alg} \\ \downarrow & & \downarrow \\ \mathscr{C} & \xrightarrow[U]{} & \mathscr{D} \end{array}$$

which is a (strict) pullback diagram of categories.

Exercise 2.
Let $\big((S_i, \sigma_i)\big)_{i \in I}$ be a family of accessible well-pointed endofunctors of a locally presentable category $\mathscr{C}$. Let S be the colimit in $[\mathscr{C}, \mathscr{C}]$ of the "star-shaped" diagram $\sigma_i\colon \mathrm{id}_{\mathscr{C}} \Rightarrow S_i$ (that is, a star with center $\mathrm{id}_{\mathscr{C}}$). Write σ for the induced composite $\mathrm{id} \Rightarrow S_i \Rightarrow S$ (which is by construction independent of i). Show that (S, σ) is an accessible well-pointed endofunctor and that (S, σ)- **Alg** is the intersection $\bigcap_{i \in I} (S_i, \sigma_i)$- **Alg**.

Exercise 3.
Let $k\colon a \to b$ be a morphism of a category $\mathscr{C}$. An object $c \in \mathscr{C}$ is called *orthogonal* to k if for any $f\colon a \to c$, there exists a unique dashed arrow making the triangle

$$\begin{array}{ccc} a & \xrightarrow{f} & c \\ {\scriptstyle k}\downarrow & \nearrow & \\ b & & \end{array}$$

commutative. The full subcategory of objects orthogonal to k is denoted by $\{k\}^{\perp} \subseteq \mathscr{C}$. Similarly, given a set Σ of morphisms in $\mathscr{C}$, we write $\Sigma^{\perp} \subseteq \mathscr{C}$ for the class of objects which are orthogonal to all the morphisms in Σ.

(a) Let $\mathscr{C}$ be locally presentable and let $k\colon a \to b$ be a morphism in $\mathscr{C}$. This defines a functor $k\colon [1] \to \mathscr{C}$, where $[1] = \{0 \to 1\}$ denotes the category consisting of a single non-trivial morphism. Show that the right adjoint of the induced adjunction $\mathrm{Lan}_Y k\colon \mathbf{Set}^{[1]^{\mathrm{op}}} \rightleftarrows \mathscr{C} : \widetilde{k}$ is accessible.

(b) Show that there exists an accessible well-pointed endofunctor (S_k, σ_k) of $\mathscr{C}$ such that (S_k, σ_k)- $\mathbf{Alg}$ is equal to the full subcategory $\{k\}^{\perp} \subseteq \mathscr{C}$ of objects orthogonal to k. (Hint: apply Exercise 1 to the adjunction of (a).)

(c) Let Σ be a set of morphisms in $\mathscr{C}$. Show that there exists an accessible well-pointed endofunctor of $\mathscr{C}$ whose category of algebras is $\Sigma^{\perp} \subseteq \mathscr{C}$.

Exercise 4.
Let $\mathscr{A}$ be a small category and consider a set $\big(D^k\colon \mathscr{I}_k \to \mathscr{A}\big)_{k\in K}$ of diagrams in $\mathscr{A}$. For each diagram D^k, fix a cocone $\kappa_i\colon D^k_i \to a_k$ in $\mathscr{A}$. Let $\mathscr{C} \subseteq [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$ be the full subcategory of presheaves $F\colon \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$ with the property that $F\kappa_i\colon Fa_k \to FD^k_i$ is a limit cone for each $k \in K$. Show that there exists a set Σ of morphisms in $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$ such that $\mathscr{C} = \Sigma^{\perp}$.

Exercise 5. *(bonus)*
From the lecture, we know that the category $\mathscr{C}$ in Exercise 4 is reflective. Show that the composite

$$\mathscr{A} \xrightarrow{Y} [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}] \longrightarrow \mathscr{C}$$

of the Yoneda embedding and the left adjoint of the inclusion is the universal functor to a cocomplete category which sends the given cones to colimit cones.