Monads and their applications 4

Exercise 1.
Let $F\colon \mathscr{A} \to \mathscr{C}$, $K\colon \mathscr{A} \to \mathscr{B}$ be two functors where $\mathscr{A}$ is essentially small. Note that K is not assumed to be full and faithful.

(a) Show that $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](\widetilde{K}c, \widetilde{F}-)\colon \mathscr{D} \to \mathbf{Set}$ is representable if and only if the colimit of the diagram $K/c \to \mathscr{D}$ which sends $(a, \varphi\colon Ka \to c)$ to $Fa \in \mathscr{D}$ has a colimit.

(a) Assume that the colimit of part (a) always exist. By Yoneda, there exists a functor $L\colon \mathscr{C} \to \mathscr{D}$ with bijections

$$\mathscr{D}(Lc, d) \cong [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](\widetilde{K}c, \widetilde{F}d)$$

which are natural in c and d. Show that, in this case, there exists a natural transformation $\eta\colon F \Rightarrow LK$ which exhibits L as left Kan extension of F along K.

Exercise 2.

(a) Let $(\mathscr{C}_i)_{i\in I}$ be a family of locally finitely presentable categories. Show that the product $\prod_{i\in I} \mathscr{C}_i$ is locally finitely presentable.

(b) Let $\mathscr{A}$ be a small category and $\mathscr{C}$ locally finitely presentable. Show that $[\mathscr{A}, \mathscr{C}]$ is locally finitely presentable.

Exercise 3.
Let $\mathscr{C}$ be a cocomplete category. An object $a \in \mathscr{C}$ is called *small projective* if $\mathscr{C}(a, -)$ preserves *all* small colimits. Suppose there exists a small subcategory $\mathscr{A} \subseteq \mathscr{C}$ such that the closure of $\mathscr{A}$ under colimits is all of $\mathscr{C}$. Show that $\mathscr{C} \simeq [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$. (Hint: start by showing that the inclusion $K\colon \mathscr{A} \to \mathscr{C}$) is dense.

Exercise 4.
Let $\mathscr{C}$ be locally finitely presentable, $\mathscr{A}$ a small dense subcategory consisting of finitely presentable objects. Let $\mathscr{A}'$ be the closure of $\mathscr{A}$ under finite colimits. Let $\mathscr{C}_{\mathrm{fp}}$ denote the full subcategory consisting of finitely presentable objects. From the lecture, we know that $\mathscr{A}' \subseteq \mathscr{C}_{\mathrm{fp}}$. Show that this in fact an equality: every finitely presentable object lies in the closure of $\mathscr{A}$ under finite colimits.

Exercise 5. *(bonus)*
Show that the finitely presentable objects in the category of topological spaces are precisely the finite discrete spaces, and that they do not form a dense generator. Thus **Top** is not finitely presentable.