Monads and their applications 3

Exercise 1.
Let $F\colon \mathscr{A} \to \mathscr{C}$, $K\colon \mathscr{A} \to \mathscr{B}$ and $L\colon \mathscr{B} \to \mathscr{C}$ be functors. A natural transformation $\eta\colon F \Rightarrow L \circ K$ is said to exhibit L as left Kan extension of F along K if the composite

$$[\mathscr{B},\mathscr{C}](L,G) \xrightarrow{-\circ K} [\mathscr{A},\mathscr{C}](LK,G) \xrightarrow{\eta^*} [\mathscr{A},\mathscr{C}](F,G)$$

is a bijection for all functors $G\colon \mathscr{B} \to \mathscr{C}$. If a left Kan extension of F along K exists, then it is unique up to unique natural isomorphism and it is denoted by $\operatorname{Lan}_K F$.

(a) Show that left adjoints preserve left Kan extensions in the following sense: if $\eta\colon F \Rightarrow LK$ exhibits L as left Kan extension of F along K and $H\colon \mathscr{C} \to \mathscr{D}$ is a left adjoint, then $H\eta$ exhibits HL as left Kan extension of HF along K.

(b) Show that left Kan extensions compose: if $\operatorname{Lan}_K F$ exists and

$$K'\colon \mathscr{B} \to \mathscr{B}'$$

is any functor, then $\operatorname{Lan}_{K'} \operatorname{Lan}_K F$ exists if and only if $\operatorname{Lan}_{K'K} F$ exists. Moreover, show that in this case there is a natural isomorphism $\operatorname{Lan}_{K'} \operatorname{Lan}_K F \cong \operatorname{Lan}_{K'K} F$.

Exercise 2.
The notion of *right* Kan extension is dual to left Kan extension: it is given by a universal natural transformation $\gamma\colon RK \Rightarrow F$ and denoted by $\operatorname{Ran}_K F$.

(a) Let $F\colon \mathscr{A} \to \mathscr{C}$ be a functor such that the right Kan extension

$$\operatorname{Ran}_F F\colon \mathscr{C} \to \mathscr{C}$$

of F along itself exists. Show that $\operatorname{Ran}_F F$ has the structure of a monad in a natural way. This monad is called the *codensity monad* of F.

(b) If $\mathscr{A} = *$ is the terminal category, then giving a functor $F\colon \mathscr{A} \to \mathscr{C}$ amounts to picking an object $c \in \mathscr{C}$, $c = F(*)$. Show that, in this case, $\operatorname{Ran}_F F$ exists if $\mathscr{C}$ has products. The resulting codensity monad is called the *endomorphism monad* of c and denoted by $\langle c, c\rangle$.

Exercise 3.
Let k be a field and $\mathbf{Vect}_k$ the category of k-vector spaces. Let $\mathscr{A} = \{k\}$ be the full subcategory on the one-dimensional vector space k. Note that every object of $\mathbf{Vect}_k$ is a colimit of some diagram that factors through $\mathscr{A}$ (since all vector spaces are free).

(a) Show that, nevertheless, the inclusion $\mathscr{A} \to \mathbf{Vect}_k$ is *not* dense.

(b) Let $\mathscr{B} = \{k \oplus k\}$ be the full subcategory one the two-dimensional vector space. Show that the inclusion $\mathscr{B} \to \mathbf{Vect}_k$ is dense.

Exercise 4.
Let $\mathscr{A}$ be a small category and let $Y \colon \mathscr{A} \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$ be the Yoneda embedding.

(a) Use the Yoneda lemma to show that the canonical cocone on Y/F exhibits F as colimit of the domain functor $\mathrm{dom} \colon Y/F \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$, $(\varphi \colon \mathscr{A}(-,a) \Rightarrow F) \mapsto \mathscr{A}(-,a)$.

(b) The category $\mathrm{el}(F)$ of elements of F has objects the pairs (a,x) where $a \in \mathscr{A}$ and $x \in Fa$ and morphisms $(a,x) \to (b,y)$ the morphisms $f \colon a \to b$ in $\mathscr{A}$ which satisfy $Ff(y) = x$. Show that there is an isomorphism $Y/F \cong \mathrm{el}(F)^{\mathrm{op}}$.

Exercise 5. *(bonus)*
An object $c \in \mathscr{C}$ is called *strongly finitely presentable* if the representable functor $\mathscr{C}(c,-) \colon \mathscr{C} \to \mathbf{Set}$ preserves sifted colimits. A cocomplete category $\mathscr{C}$ is called *locally strongly finitely presentable* if there exists a small dense subcategory $\mathscr{A}$ of $\mathscr{C}$ which consists of strongly finitely presentable objects.

(a) Show that finite coproducts of strongly finitely presentable objects are strongly finitely presentable.

(b) Let $U \colon \mathscr{D} \to \mathscr{C}$ have a left adjoint $F \colon \mathscr{C} \to \mathscr{D}$. Show the following claim: if U preserves sifted colimits, then F preserves strongly finitely presentable objects.

(c) Let $\mathscr{C}$ be a strongly finitely presentable category and let $T \colon \mathscr{C} \to \mathscr{C}$ be a monad which commutes with sifted colimits. Show that T-$\mathbf{Alg}$ is locally strongly finitely presentable. (Hint: let $\mathscr{A}$ be a dense subcategory of $\mathscr{C}$ consisting of strongly finitely presentable objects. Show that the objects (Ta, μ_a) form a dense subcategory of T-$\mathbf{Alg}$).