Monads and their applications 2

Exercise 1.
Let $\mathscr{C}$ be a category with finite coproducts. For an object $c \in \mathscr{C}$, let $c/\mathscr{C}$ denote the slice category, whose objects are morphisms with domain c and whose morphisms are commutative triangles. Show that the forgetful functor $c/\mathscr{C} \to \mathscr{C}$ is monadic (using Beck's theorem) and describe the monad in question.

Exercise 2.
Let $\mathscr{C}$ be the category of torsion free abelian groups.

(a) Show that the inclusion $\mathscr{C} \to \mathbf{Ab}$ is monadic (using the monadicity theorem).

(b) Show that the forgetful functor $\mathbf{Ab} \to \mathbf{Set}$ is monadic.

(c) Show that the composite $\mathscr{C} \to \mathbf{Set}$ of the above two functors is *not* monadic. (Hint: what happens to the canonical presentation of a finite abelian group?)

Exercise 3.
An object $g \in \mathscr{C}$ is called a *strong generator* if $\mathscr{C}(g, -)\colon \mathscr{C} \to \mathbf{Set}$ is conservative.

(a) Assume that $\mathscr{C}$ has small colimits and that g is a strong generator such that $\mathscr{C}(g, -)$ preserves small colimits (an object satisfying the latter condition is sometimes called *small projective*). Show that there exists a monoid M such that $\mathscr{C}$ is equivalent to the category of M-sets. (Hint: is $\mathscr{C}(g, -)$ monadic?)

(b) Show that the monoid M above is isomorphic to the endomorphism monoid $\mathscr{C}(g, g)$ of g.

Exercise 4.
Let $F\colon \mathscr{C} \to \mathscr{C}$ be an endofunctor. An F-*algebra* is a pair (c, γ) of an object $c \in \mathscr{C}$ and a morphism $\gamma\colon Fc \to c$ (not subject to any axioms). A morphism of algebras $(c, \gamma) \to (d, \delta)$ is a morphism $f\colon c \to d$ making the evident square commutative. We denote the category of F-algebras by $F\text{-}\mathbf{Alg}$. Show that the forgetful functor $F\text{-}\mathbf{Alg} \to \mathscr{C}$ is monadic if and only if it has a left adjoint.

Exercise 5.

Complete the proof of the monadicity theorem by showing that the natural transformations $\overline{\eta}$ and $\overline{\varepsilon}$ described in the lecture satisfy the triangle identities.