Weakly Tannakian categories

Daniel Schäppi

University of Chicago

July 12, 2013
CT 2013
Macquarie University

Outline

Categories arising in algebraic geometry

Categories arising in algebraic geometry

An algebraic variety is a space that is locally the zero-set of a set of polynomials with coefficients in k.

Categories arising in algebraic geometry

An algebraic variety is a space that is locally the zero-set of a set of polynomials with coefficients in k.
Closed subsets are sets of solutions to polynomial equations.

Categories arising in algebraic geometry

An algebraic variety is a space that is locally the zero-set of a set of polynomials with coefficients in k.
Closed subsets are sets of solutions to polynomial equations.
Sheaf of rings $\mathcal{O}_X$: rational functions whose denominator is non-zero

Categories arising in algebraic geometry

An algebraic variety is a space that is locally the zero-set of a set of polynomials with coefficients in k.
Closed subsets are sets of solutions to polynomial equations.
Sheaf of rings $\mathcal{O}_X$: rational functions whose denominator is non-zero

Definition

A quasi-coherent sheaf is a sheaf of $\mathcal{O}_X$-modules which locally admits a presentation

$$\bigoplus_I \mathcal{O}_U \longrightarrow \bigoplus_J \mathcal{O}_U \longrightarrow M \longrightarrow 0$$

Categories arising in algebraic geometry

An algebraic variety is a space that is locally the zero-set of a set of polynomials with coefficients in k.
Closed subsets are sets of solutions to polynomial equations.
Sheaf of rings $\mathcal{O}_X$: rational functions whose denominator is non-zero

Definition

A quasi-coherent sheaf is a sheaf of $\mathcal{O}_X$-modules which locally admits a presentation

$$\bigoplus_I \mathcal{O}_U \longrightarrow \bigoplus_J \mathcal{O}_U \longrightarrow M \longrightarrow 0$$

The category of quasi-coherent sheaves on X is denoted by $\mathrm{QC}(X)$.

Categories arising in algebraic geometry

An algebraic variety is a space that is locally the zero-set of a set of polynomials with coefficients in k.
Closed subsets are sets of solutions to polynomial equations.
Sheaf of rings $\mathcal{O}_X$: rational functions whose denominator is non-zero

Definition

A quasi-coherent sheaf is a sheaf of $\mathcal{O}_X$-modules which locally admits a presentation

$$\bigoplus_I \mathcal{O}_U \longrightarrow \bigoplus_J \mathcal{O}_U \longrightarrow M \longrightarrow 0$$

The category of quasi-coherent sheaves on X is denoted by $\mathrm{QC}(X)$.

Fact

The category $\mathrm{QC}(X)$ is a Grothendieck abelian k-linear symmetric monoidal closed category.

Categories arising in algebraic geometry

Basic question

For two varieties X, Y, can we describe the category $\mathrm{QC}(X \times Y)$ in terms of the categories $\mathrm{QC}(X)$ and $\mathrm{QC}(Y)$?

Categories arising in algebraic geometry

Basic question

For two varieties X, Y, can we describe the category $\mathrm{QC}(X \times Y)$ in terms of the categories $\mathrm{QC}(X)$ and $\mathrm{QC}(Y)$?

Answer

For reasonable varieties, there is an equivalence

$$\mathrm{QC}_{\mathrm{fp}}(X \times Y) \simeq \mathrm{QC}_{\mathrm{fp}}(X) \boxtimes \mathrm{QC}_{\mathrm{fp}}(Y)$$

of symmetric monoidal k-linear categories.

Categories arising in algebraic geometry

Basic question

For two varieties X, Y, can we describe the category $\mathrm{QC}(X \times Y)$ in terms of the categories $\mathrm{QC}(X)$ and $\mathrm{QC}(Y)$?

Answer

For reasonable varieties, there is an equivalence

$$\mathrm{QC}_{\mathrm{fp}}(X \times Y) \simeq \mathrm{QC}_{\mathrm{fp}}(X) \boxtimes \mathrm{QC}_{\mathrm{fp}}(Y)$$

of symmetric monoidal k-linear categories.

$\mathrm{QC}_{\mathrm{fp}}(X) \quad = \quad$ full subcategory of finitely presentable objects

Categories arising in algebraic geometry

Basic question

For two varieties X, Y, can we describe the category $\mathrm{QC}(X \times Y)$ in terms of the categories $\mathrm{QC}(X)$ and $\mathrm{QC}(Y)$?

Answer

For reasonable varieties, there is an equivalence

$$\mathrm{QC}_{\mathrm{fp}}(X \times Y) \simeq \mathrm{QC}_{\mathrm{fp}}(X) \boxtimes \mathrm{QC}_{\mathrm{fp}}(Y)$$

of symmetric monoidal k-linear categories.

$\mathrm{QC}_{\mathrm{fp}}(X)$ = full subcategory of finitely presentable objects

$\boxtimes$ = Kelly's tensor product of finitely cocomplete k-linear categories

A proof strategy

Problem

Need to compare colimit-like universal property of $\boxtimes$ to the limit-like property of $\mathrm{QC}_{\mathrm{fp}}(-)$ (glueing).

A proof strategy

Problem

Need to compare colimit-like universal property of $\boxtimes$ to the limit-like property of $\mathrm{QC}_{\mathrm{fp}}(-)$ (glueing).

Definition

Let $\mathscr{RM}$ denote the 2-category of $\boxtimes$-pseudomonoids:

0-cells	$=$	finitely cocomplete symmetric monoidal k-linear categories $\mathscr{A}$ such that $A \otimes -$ preserves finite colimits for all $A \in \mathscr{A}$
1-cells	$=$	right exact symmetric monoidal functors
2-cells	$=$	symmetric monoidal natural transformations

A proof strategy

Problem

Need to compare colimit-like universal property of $\boxtimes$ to the limit-like property of $\mathrm{QC}_{\mathrm{fp}}(-)$ (glueing).

Definition

Let $\mathscr{RM}$ denote the 2-category of $\boxtimes$-pseudomonoids:

0-cells	$=$	finitely cocomplete symmetric monoidal k-linear categories $\mathscr{A}$ such that $A \otimes -$ preserves finite colimits for all $A \in \mathscr{A}$
1-cells	$=$	right exact symmetric monoidal functors
2-cells	$=$	symmetric monoidal natural transformations

Theorem (Lurie '05, Brandenburg-Chirvasitu '12)

For reasonable varieties, the contravariant pseudofunctor $\mathrm{QC}_{\mathrm{fp}}(-)\colon \{\text{varieties}\} \to \mathscr{RM}$ is an equivalence on hom-categories.

A proof strategy

Consequence

$\mathrm{QC}_{\mathrm{fp}}(X \times Y)$ has the universal property of a bicategorical coproduct in the image of $\mathrm{QC}_{\mathrm{fp}}(-)$

A proof strategy

Consequence

$\mathrm{QC}_{\mathrm{fp}}(X \times Y)$ has the universal property of a bicategorical coproduct in the image of $\mathrm{QC}_{\mathrm{fp}}(-)$

Strategy

To prove the theorem, it suffices to show:

A proof strategy

Consequence

$\mathrm{QC}_{\mathrm{fp}}(X \times Y)$ has the universal property of a bicategorical coproduct in the image of $\mathrm{QC}_{\mathrm{fp}}(-)$

Strategy

To prove the theorem, it suffices to show:

(i) Bicategorical coproducts in $\mathscr{R}\mathscr{M}$ are given by $\boxtimes$

A proof strategy

Consequence

$\mathrm{QC}_{\mathrm{fp}}(X \times Y)$ has the universal property of a bicategorical coproduct in the image of $\mathrm{QC}_{\mathrm{fp}}(-)$

Strategy

To prove the theorem, it suffices to show:

(i) Bicategorical coproducts in $\mathscr{RM}$ are given by $\boxtimes$

(ii) If $\mathscr{A}$ and $\mathscr{B}$ lie in the image of $\mathrm{QC}_{\mathrm{fp}}(-)$, then so does $\mathscr{A} \boxtimes \mathscr{B}$.

A proof strategy

Consequence

$\mathrm{QC}_{\mathrm{fp}}(X \times Y)$ has the universal property of a bicategorical coproduct in the image of $\mathrm{QC}_{\mathrm{fp}}(-)$

Strategy

To prove the theorem, it suffices to show:

(i) Bicategorical coproducts in $\mathscr{RM}$ are given by $\boxtimes$

(ii) If $\mathscr{A}$ and $\mathscr{B}$ lie in the image of $\mathrm{QC}_{\mathrm{fp}}(-)$, then so does $\mathscr{A} \boxtimes \mathscr{B}$.

Indeed: both $\mathrm{QC}_{\mathrm{fp}}(X) \boxtimes \mathrm{QC}_{\mathrm{fp}}(Y)$ and $\mathrm{QC}_{\mathrm{fp}}(X \times Y)$ have the same universal property in the image.

Coproducts

The first requirement follows from:

Theorem (S.)

Let $\mathscr{M}$ be a symmetric monoidal bicategory, and let (A, i, m) and (B, i, m) be two symmetric pseudomonoids in $\mathscr{M}$. Then the two morphisms

$$A \simeq A \otimes I \xrightarrow{A \otimes i} A \otimes B \quad \text{and} \quad B \simeq I \otimes B \xrightarrow{i \otimes B} A \otimes B$$

exhibit $A \otimes B$ as bicategorical coproduct in the bicategory of symmetric pseudomonoids.

Coproducts

The first requirement follows from:

Theorem (S.)

Let $\mathscr{M}$ be a symmetric monoidal bicategory, and let (A, i, m) and (B, i, m) be two symmetric pseudomonoids in $\mathscr{M}$. Then the two morphisms

$$A \simeq A \otimes I \xrightarrow{A \otimes i} A \otimes B \quad \text{and} \quad B \simeq I \otimes B \xrightarrow{i \otimes B} A \otimes B$$

exhibit $A \otimes B$ as bicategorical coproduct in the bicategory of symmetric pseudomonoids.

Proof: M, N commutative monoids
$\Rightarrow$ coproduct is given by $M \otimes N$.

Coproducts

The first requirement follows from:

Theorem (S.)

Let $\mathscr{M}$ be a symmetric monoidal bicategory, and let (A, i, m) and (B, i, m) be two symmetric pseudomonoids in $\mathscr{M}$. Then the two morphisms

$$A \simeq A \otimes I \xrightarrow{A \otimes i} A \otimes B \quad \text{and} \quad B \simeq I \otimes B \xrightarrow{i \otimes B} A \otimes B$$

exhibit $A \otimes B$ as bicategorical coproduct in the bicategory of symmetric pseudomonoids.

Proof: M, N commutative monoids
$\Rightarrow$ coproduct is given by $M \otimes N$.
Categorify!

Recognizing categories in the image

We need a recognition theorem.

Recognizing categories in the image

We need a recognition theorem. Such a theorem exists for categories of representations.

Recognizing categories in the image

We need a recognition theorem. Such a theorem exists for categories of representations.

Properties of $\mathrm{Rep}(G)$

(i) $\mathrm{Rep}(G)$ is an abelian symmetric monoidal k-linear category

Recognizing categories in the image

We need a recognition theorem. Such a theorem exists for categories of representations.

Properties of $\mathrm{Rep}(G)$

(i) $\mathrm{Rep}(G)$ is an abelian symmetric monoidal k-linear category

(ii) there exists a functor $\mathrm{Rep}(G) \to \mathrm{Vect}_k$ which is strong symmetric monoidal, faithful, and exact (the forgetful functor).

Recognizing categories in the image

We need a recognition theorem. Such a theorem exists for categories of representations.

Properties of $\mathrm{Rep}(G)$

(i) $\mathrm{Rep}(G)$ is an abelian symmetric monoidal k-linear category

(ii) there exists a functor $\mathrm{Rep}(G) \to \mathrm{Vect}_k$ which is strong symmetric monoidal, faithful, and exact (the forgetful functor).

(iii) it is rigid (or autonomous), every object has a dual.

Recognizing categories in the image

We need a recognition theorem. Such a theorem exists for categories of representations.

Properties of $\mathrm{Rep}(G)$

(i) $\mathrm{Rep}(G)$ is an abelian symmetric monoidal k-linear category

(ii) there exists a functor $\mathrm{Rep}(G) \to \mathrm{Vect}_k$ which is strong symmetric monoidal, faithful, and exact (the forgetful functor).

(iii) it is rigid (or autonomous), every object has a dual.

Theorem (Grothendieck, Saavedra-Rivano, Deligne-Milne)

If $\mathscr{A}$ is a category satisfying (i)-(iii), then there exists an affine group scheme G and an equivalence $\mathscr{A} \simeq \mathrm{Rep}(G)$ of symmetric monoidal k-linear categories. Such categories are called Tannakian.

Recognizing categories in the image

Goal: generalize this theorem to a context which encompasses varieties and schemes.

Recognizing categories in the image

Goal: generalize this theorem to a context which encompasses varieties and schemes.

A common generalization

X variety or scheme $\quad \rightsquigarrow \quad$ functor of points: $X(-) \colon \mathbf{CAlg} \to \mathbf{Set}$

Recognizing categories in the image

Goal: generalize this theorem to a context which encompasses varieties and schemes.

A common generalization

X variety or scheme	$\rightsquigarrow$	functor of points: $X(-)\colon \mathbf{CAlg} \to \mathbf{Set}$
G affine group scheme	$\rightsquigarrow$	functor $G(-)\colon \mathbf{CAlg} \to \mathbf{Grp}$

Recognizing categories in the image

Goal: generalize this theorem to a context which encompasses varieties and schemes.

A common generalization

X variety or scheme $\rightsquigarrow$ functor of points: $X(-)\colon \mathbf{CAlg} \to \mathbf{Set}$

G affine group scheme $\rightsquigarrow$ functor $G(-)\colon \mathbf{CAlg} \to \mathbf{Grp}$

Both of these are groupoid valued presheaves.

Recognizing categories in the image

Goal: generalize this theorem to a context which encompasses varieties and schemes.

A common generalization

X variety or scheme	$\rightsquigarrow$	functor of points: $X(-)\colon \mathbf{CAlg} \to \mathbf{Set}$
G affine group scheme	$\rightsquigarrow$	functor $G(-)\colon \mathbf{CAlg} \to \mathbf{Grp}$

Both of these are groupoid valued presheaves.

To get the desired embedding theorem, we need to pass to associated stacks (Lurie '05). (Recall that we need recognition and embedding.)

Recognizing categories in the image

Goal: generalize this theorem to a context which encompasses varieties and schemes.

A common generalization

X variety or scheme $\rightsquigarrow$ functor of points: $X(-)\colon \mathbf{CAlg} \to \mathbf{Set}$
G affine group scheme $\rightsquigarrow$ functor $G(-)\colon \mathbf{CAlg} \to \mathbf{Grp}$

Both of these are groupoid valued presheaves.

To get the desired embedding theorem, we need to pass to associated stacks (Lurie '05). (Recall that we need recognition and embedding.)

Definition

$\mathbf{Stacks}$ is the 2-category of pseudofunctors $\mathbf{CAlg} \to \mathbf{Gpd}$ which send certain colimits in $\mathbf{Aff} = \mathbf{CAlg}^{\mathrm{op}}$ to limits. The Yoneda embedding

$$\mathbf{Aff} = \mathbf{CAlg}^{\mathrm{op}} \to \mathbf{Stacks}$$

is therefore universal among pseudofunctors which preserve these colimits.

Coherent sheaves, revisited

Let $\mathscr{C}\mathscr{M}$ be the 2-category of k-linear locally presentable symmetric monoidal closed categories, and strong symmetric monoidal left adjoints.

Coherent sheaves, revisited

Let $\mathscr{C}\mathscr{M}$ be the 2-category of k-linear locally presentable symmetric monoidal closed categories, and strong symmetric monoidal left adjoints.

Definition

Let QC be the essentially unique left biadjoint which makes the diagram

$$\begin{tikzcd}
\mathbf{Aff} = \mathbf{CAlg}^{\mathrm{op}} \arrow[r, "Y"] \arrow[dr, "\mathrm{Mod}_{(-)}"'] & \mathbf{Stacks} \arrow[d, "\mathrm{QC}"] \\
& \mathscr{C}\mathscr{M}^{\mathrm{op}}
\end{tikzcd}$$

commute up to isomorphism. (This is a left Kan extension.)

Coherent sheaves, revisited

Let $\mathscr{C}\mathscr{M}$ be the 2-category of k-linear locally presentable symmetric monoidal closed categories, and strong symmetric monoidal left adjoints.

Definition

Let QC be the essentially unique left biadjoint which makes the diagram

$$\begin{tikzcd} \mathbf{Aff} = \mathbf{CAlg}^{\mathrm{op}} \arrow[r, "Y"] \arrow[dr, "\mathrm{Mod}_{(-)}"'] & \mathbf{Stacks} \arrow[d, "\mathrm{QC}"] \\ & \mathscr{C}\mathscr{M}^{\mathrm{op}} \end{tikzcd}$$

commute up to isomorphism. (This is a left Kan extension.)

Definition

A stack X is algebraic (in the sense of Goerss and Hopkins) if it is associated to a flat affine groupoid. It is called an Adams stack if, in addition, the duals form a strong generator in $\mathrm{QC}(X)$.

The Embedding Theorem

Remark

If X is algebraic, then $\mathrm{QC}(X)$ is Grothendieck abelian.

The Embedding Theorem

Remark

If X is algebraic, then $\mathrm{QC}(X)$ is Grothendieck abelian.

Let $\mathscr{A}\mathscr{S}$ denote the 2-category of Adams stacks.

The Embedding Theorem

Remark

If X is algebraic, then $\mathrm{QC}(X)$ is Grothendieck abelian.

Let $\mathscr{A}\mathscr{S}$ denote the 2-category of Adams stacks.

Theorem (S.)

The restriction $\mathrm{QC}_{\mathrm{fp}} \colon \mathscr{A}\mathscr{S}^{\mathrm{op}} \to \mathscr{R}\mathscr{M}$ of $\mathrm{QC}_{\mathrm{fp}}(-)$ to Adams stacks is an equivalence on hom-categories.

The Embedding Theorem

Remark

If X is algebraic, then $\mathrm{QC}(X)$ is Grothendieck abelian.

Let $\mathscr{A}\mathscr{S}$ denote the 2-category of Adams stacks.

Theorem (S.)

The restriction $\mathrm{QC}_{\mathrm{fp}} \colon \mathscr{A}\mathscr{S}^{\mathrm{op}} \to \mathscr{R}\mathscr{M}$ of $\mathrm{QC}_{\mathrm{fp}}(-)$ to Adams stacks is an equivalence on hom-categories.

Proof: full and faithful on 2-cells (Lurie), 1-cell is in the image if and only if it is tame (Lurie).

The Embedding Theorem

Remark

If X is algebraic, then $\mathrm{QC}(X)$ is Grothendieck abelian.

Let $\mathscr{A}\mathscr{S}$ denote the 2-category of Adams stacks.

Theorem (S.)

The restriction $\mathrm{QC}_{\mathrm{fp}} \colon \mathscr{A}\mathscr{S}^{\mathrm{op}} \to \mathscr{R}\mathscr{M}$ of $\mathrm{QC}_{\mathrm{fp}}(-)$ to Adams stacks is an equivalence on hom-categories.

Proof: full and faithful on 2-cells (Lurie), 1-cell is in the image if and only if it is tame (Lurie).
Need to show that tameness is automatic in case of Adams stacks.

The Embedding Theorem

Remark

If X is algebraic, then $\mathrm{QC}(X)$ is Grothendieck abelian.

Let $\mathscr{A}\mathscr{S}$ denote the 2-category of Adams stacks.

Theorem (S.)

The restriction $\mathrm{QC}_{\mathrm{fp}} \colon \mathscr{A}\mathscr{S}^{\mathrm{op}} \to \mathscr{R}\mathscr{M}$ of $\mathrm{QC}_{\mathrm{fp}}(-)$ to Adams stacks is an equivalence on hom-categories.

Proof: full and faithful on 2-cells (Lurie), 1-cell is in the image if and only if it is tame (Lurie).
Need to show that tameness is automatic in case of Adams stacks.

Goal

Find a characterization of the image of $\mathrm{QC}_{\mathrm{fp}} \colon \mathscr{A}\mathscr{S}^{\mathrm{op}} \to \mathscr{R}\mathscr{M}$.

Weakly Tannakian categories

Definition

Let $\mathscr{A}$ be an abelian symmetric monoidal k-linear category. The category $\mathscr{A}$ is called Tannakian if

(i) There exists a strong symmetric monoidal functor

$$w \colon \mathscr{A} \to \mathrm{Vect}$$

which is faithful and exact (called the fiber functor);

(ii) Every object of $\mathscr{A}$ has a dual.

Weakly Tannakian categories

Definition

Let $\mathscr{A}$ be an abelian symmetric monoidal k-linear category. The category $\mathscr{A}$ is called weakly Tannakian if

(i) There exists a strong symmetric monoidal functor

$$w \colon \mathscr{A} \to \mathrm{Vect}$$

which is faithful and exact (called the fiber functor);

(ii) Every object of $\mathscr{A}$ has a dual.

Weakly Tannakian categories

Definition

Let $\mathscr{A}$ be an abelian symmetric monoidal k-linear category. The category $\mathscr{A}$ is called weakly Tannakian if

(i) There exists a strong symmetric monoidal functor

$$w \colon \mathscr{A} \to \mathrm{Mod}_B$$

which is faithful and exact (called the fiber functor) for some commutative algebra B;

(ii) Every object of $\mathscr{A}$ has a dual.

Weakly Tannakian categories

Definition

Let $\mathscr{A}$ be an abelian symmetric monoidal k-linear category. The category $\mathscr{A}$ is called weakly Tannakian if

(i) There exists a strong symmetric monoidal functor

$$w \colon \mathscr{A} \to \mathrm{Mod}_B$$

which is faithful and exact (called the fiber functor) for some commutative algebra B;

(ii) Every object of $\mathscr{A}$ is a quotient of a dual.

Weakly Tannakian categories

Definition

Let $\mathscr{A}$ be an abelian symmetric monoidal k-linear category. The category $\mathscr{A}$ is called weakly Tannakian if

(i) There exists a strong symmetric monoidal functor

$$w \colon \mathscr{A} \to \mathrm{Mod}_B$$

which is faithful and exact (called the fiber functor) for some commutative algebra B;

(ii) Every object of $\mathscr{A}$ is a quotient of a dual.

An algebraic stack X is called coherent if $\mathrm{QC}_{\mathrm{fp}}(X)$ is an abelian subcategory of $\mathrm{QC}(X)$. Then we write $\mathrm{C}(X) := \mathrm{QC}_{\mathrm{fp}}(X)$.

Weakly Tannakian categories

Definition

Let $\mathscr{A}$ be an abelian symmetric monoidal k-linear category. The category $\mathscr{A}$ is called weakly Tannakian if

(i) There exists a strong symmetric monoidal functor

$$w \colon \mathscr{A} \to \mathrm{Mod}_B$$

which is faithful and exact (called the fiber functor) for some commutative algebra B;

(ii) Every object of $\mathscr{A}$ is a quotient of a dual.

An algebraic stack X is called coherent if $\mathrm{QC}_{\mathrm{fp}}(X)$ is an abelian subcategory of $\mathrm{QC}(X)$. Then we write $\mathrm{C}(X) := \mathrm{QC}_{\mathrm{fp}}(X)$.

Theorem (S.)

$\mathscr{A}$ weakly Tannakian $\Leftrightarrow \mathscr{A} \simeq \mathrm{C}(X)$ for some coherent Adams stack X.

Why is this true?

Classical Tannakian category $\mathscr{A} \rightsquigarrow$ group $\mathrm{Aut}_{\otimes}(w)$

Why is this true?

Classical Tannakian category $\mathscr{A} \rightsquigarrow$ group $\mathrm{Aut}_{\otimes}(w)$
This does not help to get a stack from a weakly Tannakian category.

Why is this true?

Classical Tannakian category $\mathscr{A} \rightsquigarrow$ group $\mathrm{Aut}_{\otimes}(w)$
This does not help to get a stack from a weakly Tannakian category.
Instead we use the symmetric monoidal adjunction induced by w:

$$\begin{tikzcd} \mathscr{A} \arrow[r] \arrow[dr, "w"'] & \mathrm{Ind}(\mathscr{A}) \arrow[d, shift right, "L"'] \arrow[d, phantom, "\dashv"] \\ & \mathrm{Mod}_B \arrow[u, shift right, "R"'] \end{tikzcd}$$

Why is this true?

Classical Tannakian category $\mathscr{A} \rightsquigarrow$ group $\mathrm{Aut}_{\otimes}(w)$
This does not help to get a stack from a weakly Tannakian category.
Instead we use the symmetric monoidal adjunction induced by w:

$$\begin{tikzcd} \mathscr{A} \arrow[r] \arrow[dr, "w"'] & \mathrm{Ind}(\mathscr{A}) \arrow[d, shift right, "L"'] \arrow[d, phantom, "\dashv"] \\ & \mathrm{Mod}_B \arrow[u, shift right, "R"'] \end{tikzcd}$$

Use duals to show: LR is a symmetric Hopf monoidal comonad

Why is this true?

Classical Tannakian category $\mathscr{A} \rightsquigarrow$ group $\mathrm{Aut}_{\otimes}(w)$
This does not help to get a stack from a weakly Tannakian category.
Instead we use the symmetric monoidal adjunction induced by w:

$$\begin{tikzcd} \mathscr{A} \arrow[r] \arrow[dr, "w"'] & \mathrm{Ind}(\mathscr{A}) \arrow[d, shift right=2, "L"'] \arrow[d, phantom, "\dashv"] \\ & \mathrm{Mod}_B \arrow[u, shift right=2, "R"'] \end{tikzcd}$$

Use duals to show: LR is a symmetric Hopf monoidal comonad
Show it comes from a flat Hopf algebroid = affine groupoid

What about the original question?

Combining the recognition theorem with the embedding theorem, we get:

What about the original question?

Combining the recognition theorem with the embedding theorem, we get:

Corollary

The functor $\mathrm{C}(-)$ gives a biequivalence between the 2-category of coherent Adams stacks and the 2-category of weakly Tannakian categories, right exact strong symmetric monoidal k-linear functors, and symmetric monoidal natural transformations.

What about the original question?

Combining the recognition theorem with the embedding theorem, we get:

Corollary

The functor $\mathrm{C}(-)$ gives a biequivalence between the 2-category of coherent Adams stacks and the 2-category of weakly Tannakian categories, right exact strong symmetric monoidal k-linear functors, and symmetric monoidal natural transformations.

Can we use this to prove the result mentioned at the beginning?

What about the original question?

Combining the recognition theorem with the embedding theorem, we get:

Corollary

The functor $\mathrm{C}(-)$ gives a biequivalence between the 2-category of coherent Adams stacks and the 2-category of weakly Tannakian categories, right exact strong symmetric monoidal k-linear functors, and symmetric monoidal natural transformations.

Can we use this to prove the result mentioned at the beginning? No!

What about the original question?

Combining the recognition theorem with the embedding theorem, we get:

Corollary

The functor $\mathrm{C}(-)$ gives a biequivalence between the 2-category of coherent Adams stacks and the 2-category of weakly Tannakian categories, right exact strong symmetric monoidal k-linear functors, and symmetric monoidal natural transformations.

Can we use this to prove the result mentioned at the beginning? No!

Problem

X, Y coherent Adams stacks $\not\Rightarrow$ $X \times Y$ coherent Adams stack.

Back to the drawing board!

Definition

Let $\mathscr{A}$ be an abelian symmetric monoidal k-linear category. The category $\mathscr{A}$ is called weakly Tannakian if

(i) There exists a strong symmetric monoidal functor

$$w\colon \mathscr{A} \to \mathrm{Mod}_B$$

which is faithful and exact (called the fiber functor) for some commutative algebra B;

(ii) Every object of $\mathscr{A}$ is a quotient of a dual.

Back to the drawing board!

Definition

Let $\mathscr{A}$ be an ind-abelian symmetric monoidal k-linear category. The category $\mathscr{A}$ is called weakly Tannakian if

(i) There exists a strong symmetric monoidal functor

$$w \colon \mathscr{A} \to \mathrm{Mod}_B$$

which is faithful and exact (called the fiber functor) for some commutative algebra B;

(ii) Every object of $\mathscr{A}$ is a quotient of a dual.

Back to the drawing board!

Definition

Let $\mathscr{A}$ be an ind-abelian symmetric monoidal k-linear category. The category $\mathscr{A}$ is called weakly Tannakian if

(i) There exists a strong symmetric monoidal functor

$$w \colon \mathscr{A} \to \mathrm{Mod}_B$$

which is faithful, right exact, and flat (called the fiber functor) for some commutative algebra B;

(ii) Every object of $\mathscr{A}$ is a quotient of a dual.

Back to the drawing board!

Definition

Let $\mathscr{A}$ be an ind-abelian symmetric monoidal k-linear category. The category $\mathscr{A}$ is called weakly Tannakian if

(i) There exists a strong symmetric monoidal functor

$$w \colon \mathscr{A} \to \mathrm{Mod}_B$$

which is faithful, right exact, and flat (called the fiber functor) for some commutative algebra B;

(ii) Every object of $\mathscr{A}$ is a quotient of a dual.

The recognition theorem is still true (using essentially the same proof):

Theorem (S.)

$\mathscr{A}$ weakly Tannakian $\Leftrightarrow$ $\mathscr{A} \simeq \mathrm{QC}_{\mathrm{fp}}(X)$ for some Adams stack X.

$\mathrm{QC}_{\mathrm{fp}}(-)$ preserves products

Theorem (S.)

Let X, Y be Adams stacks over k. Then the comparison functor

$$\mathrm{QC}_{\mathrm{fp}}(X) \boxtimes \mathrm{QC}_{\mathrm{fp}}(Y) \to \mathrm{QC}_{\mathrm{fp}}(X \times Y)$$

is an equivalence of symmetric monoidal k-linear categories.

$\mathrm{QC}_{\mathrm{fp}}(-)$ preserves products

Theorem (S.)

Let X, Y be Adams stacks over k. Then the comparison functor

$$\mathrm{QC}_{\mathrm{fp}}(X) \boxtimes \mathrm{QC}_{\mathrm{fp}}(Y) \to \mathrm{QC}_{\mathrm{fp}}(X \times Y)$$

is an equivalence of symmetric monoidal k-linear categories.

Proof: (sketch)

$\mathrm{QC}_{\mathrm{fp}}(-)$ preserves products

Theorem (S.)

Let X, Y be Adams stacks over k. Then the comparison functor

$$\mathrm{QC}_{\mathrm{fp}}(X) \boxtimes \mathrm{QC}_{\mathrm{fp}}(Y) \to \mathrm{QC}_{\mathrm{fp}}(X \times Y)$$

is an equivalence of symmetric monoidal k-linear categories.

Proof: (sketch)

(i) X, Y Adams stacks $\Rightarrow X \times Y$ Adams stack

$\mathrm{QC}_{\mathrm{fp}}(-)$ preserves products

Theorem (S.)

Let X, Y be Adams stacks over k. Then the comparison functor

$$\mathrm{QC}_{\mathrm{fp}}(X) \boxtimes \mathrm{QC}_{\mathrm{fp}}(Y) \to \mathrm{QC}_{\mathrm{fp}}(X \times Y)$$

is an equivalence of symmetric monoidal k-linear categories.

Proof: (sketch)

(i) X, Y Adams stacks $\Rightarrow X \times Y$ Adams stack

(ii) RHS is bicategorical coproduct in $\mathscr{RM}$

$\mathrm{QC}_{\mathrm{fp}}(-)$ preserves products

Theorem (S.)

Let X, Y be Adams stacks over k. Then the comparison functor

$$\mathrm{QC}_{\mathrm{fp}}(X) \boxtimes \mathrm{QC}_{\mathrm{fp}}(Y) \to \mathrm{QC}_{\mathrm{fp}}(X \times Y)$$

is an equivalence of symmetric monoidal k-linear categories.

Proof: (sketch)

(i) X, Y Adams stacks $\Rightarrow$ $X \times Y$ Adams stack

(ii) RHS is bicategorical coproduct in $\mathscr{RM}$

(iii) LHS is bicategorical coproduct in the image of $\mathrm{QC}_{\mathrm{fp}}(-)$ by the embedding theorem.

$\mathrm{QC}_{\mathrm{fp}}(-)$ preserves products

Theorem (S.)

Let X, Y be Adams stacks over k. Then the comparison functor

$$\mathrm{QC}_{\mathrm{fp}}(X) \boxtimes \mathrm{QC}_{\mathrm{fp}}(Y) \to \mathrm{QC}_{\mathrm{fp}}(X \times Y)$$

is an equivalence of symmetric monoidal k-linear categories.

Proof: (sketch)

(i) X, Y Adams stacks $\Rightarrow$ $X \times Y$ Adams stack

(ii) RHS is bicategorical coproduct in $\mathscr{RM}$

(iii) LHS is bicategorical coproduct in the image of $\mathrm{QC}_{\mathrm{fp}}(-)$ by the embedding theorem.

(iv) Check that $\mathscr{A}$, $\mathscr{B}$ weakly Tannakian $\Rightarrow$ $\mathscr{A} \boxtimes \mathscr{B}$ weakly Tannakian

$\mathrm{QC}_{\mathrm{fp}}(-)$ preserves products

Theorem (S.)

Let X, Y be Adams stacks over k. Then the comparison functor

$$\mathrm{QC}_{\mathrm{fp}}(X) \boxtimes \mathrm{QC}_{\mathrm{fp}}(Y) \to \mathrm{QC}_{\mathrm{fp}}(X \times Y)$$

is an equivalence of symmetric monoidal k-linear categories.

Proof: (sketch)

(i) X, Y Adams stacks $\Rightarrow$ $X \times Y$ Adams stack

(ii) RHS is bicategorical coproduct in $\mathscr{RM}$

(iii) LHS is bicategorical coproduct in the image of $\mathrm{QC}_{\mathrm{fp}}(-)$ by the embedding theorem.

(iv) Check that $\mathscr{A}$, $\mathscr{B}$ weakly Tannakian $\Rightarrow$ $\mathscr{A} \boxtimes \mathscr{B}$ weakly Tannakian (use the fiber functor $w \boxtimes v$)