Weakly Tannakian categories

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Outline

1 Introduction

- 2 A proof strategy
- **3** Weakly Tannakian categories
- 4 An application

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A quasi-coherent sheaf is a sheaf of \mathcal{O}_X -modules which locally admits a presentation

$$\bigoplus_{I} \mathcal{O}_U \longrightarrow \bigoplus_{J} \mathcal{O}_U \longrightarrow M \longrightarrow 0$$

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The category of quasi-coherent sheaves on X is denoted by QC(X).

Fact

The category QC(X) is a Grothendieck abelian k-linear symmetric monoidal closed category.

Basic question

For two varieties X, Y, can we describe the category $QC(X \times Y)$ in terms of the categories QC(X) and QC(Y)?

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 $\begin{array}{lcl} \operatorname{QC}_{\operatorname{fp}}(X) &= & \operatorname{full} \text{ subcategory of finitely presentable objects} \\ \boxtimes &= & \operatorname{Kelly's tensor product of finitely cocomplete} \\ & k-\text{linear categories} \end{array}$

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Definition

Let \mathcal{RM} denote the 2-category of \boxtimes -pseudomonoids:

- 0-cells = finitely cocomplete symmetric monoidal k-linear categories \mathscr{A} such that $A \otimes -$ preserves finite colimits for all $A \in \mathscr{A}$
- 1-cells = right exact symmetric monoidal functors
- 2-cells = symmetric monoidal natural transformations

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Theorem (Lurie '05, Brandenburg-Chirvasitu '12)

For reasonable varieties, the contravariant pseudofunctor $QC_{fp}(-)$: {varieties} $\rightarrow \mathscr{RM}$ is an equivalence on hom-categories.

Consequence

 ${\rm QC}_{\rm fp}(X\times Y)$ has the universal property of a bicategorical coproduct in the image of ${\rm QC}_{\rm fp}(-)$

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To prove the theorem, it suffices to show:

- (i) Bicategorical coproducts in \mathscr{RM} are given by \boxtimes
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- (ii) If \mathscr{A} and \mathscr{B} lie in the image of $QC_{fp}(-)$, then so does $\mathscr{A} \boxtimes \mathscr{B}$.

Indeed: both $QC_{fp}(X) \boxtimes QC_{fp}(Y)$ and $QC_{fp}(X \times Y)$ have the same universal property in the image.

Coproducts

The first requirement follows from:

Theorem (S.)

Let \mathscr{M} be a symmetric monoidal bicategory, and let (A, i, m) and (B, i, m) be two symmetric pseudomonoids in \mathscr{M} . Then the two morphisms

$$A \simeq A \otimes I \xrightarrow{A \otimes i} A \otimes B \quad \text{and} \quad B \simeq I \otimes B \xrightarrow{i \otimes B} A \otimes B$$

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Proof: M, N commutative monoids \Rightarrow coproduct is given by $M \otimes N$. Categorify!

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Theorem (Grothendieck, Saavedra-Rivano, Deligne-Milne)

If \mathscr{A} is a category satisfying (i)-(iii), then there exists an affine group scheme G and an equivalence $\mathscr{A} \simeq \operatorname{Rep}(G)$ of symmetric monoidal k-linear categories. Such categories are called Tannakian.

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Definition

 ${\bf Stacks}$ is the 2-category of pseudofunctors ${\bf CAlg} \to {\bf Gpd}$ which send certain colimits in ${\bf Aff} = {\bf CAlg}^{\rm op}$ to limits. The Yoneda embedding

$$\mathbf{Aff} = \mathbf{CAlg}^{\mathrm{op}} \to \mathbf{Stacks}$$

is therefore universal among pseudofunctors which preserve these colimits.

Coherent sheaves, revisited

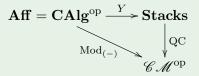
Let \mathscr{CM} be the 2-category of k-linear locally presentable symmetric monoidal closed categories, and strong symmetric monoidal left adjoints.

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Definition

Let QC be the essentially unique left biadjoint which makes the diagram



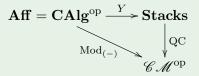
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Definition

A stack X is algebraic (in the sense of Goerss and Hopkins) if it is associated to a flat affine groupoid. It is called an Adams stack if, in addition, the duals form a strong generator in QC(X).

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Proof: full and faithful on 2-cells (Lurie), 1-cell is in the image if and only if it is tame (Lurie).

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Goal

Find a characterization of the image of $QC_{fp} \colon \mathscr{AS}^{op} \to \mathscr{RM}$.

Definition

Let \mathscr{A} be an abelian symmetric monoidal k-linear category. The category \mathscr{A} is called Tannakian if

 $({\sf i})$ There exists a strong symmetric monoidal functor

 $w\colon \mathscr{A} \to \operatorname{Vect}$

which is faithful and exact (called the fiber functor);

(ii) Every object of \mathscr{A} has a dual.

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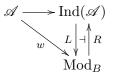
Theorem (S.)

 \mathscr{A} weakly Tannakian $\Leftrightarrow \mathscr{A} \simeq C(X)$ for some coherent Adams stack X.

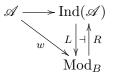
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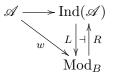


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Use duals to show: LR is a symmetric Hopf monoidal comonad Show it comes from a flat Hopf algebroid = affine groupoid

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Corollary

The functor C(-) gives a biequivalence between the 2-category of coherent Adams stacks and the 2-category of weakly Tannakian categories, right exact strong symmetric monoidal k-linear functors, and symmetric monoidal natural transformations.

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Problem

X, Y coherent Adams stacks $\Rightarrow X \times Y$ coherent Adams stack.

Definition

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(ii) Every object of $\mathscr A$ is a quotient of a dual.

The recognition theorem is still true (using essentially the same proof):

Theorem (S.)

 \mathscr{A} weakly Tannakian $\Leftrightarrow \mathscr{A} \simeq \mathrm{QC}_{\mathrm{fp}}(X)$ for some Adams stack X.

Theorem (S.)

Let X, Y be Adams stacks over k. Then the comparison functor

$$\operatorname{QC}_{\operatorname{fp}}(X) \boxtimes \operatorname{QC}_{\operatorname{fp}}(Y) \to \operatorname{QC}_{\operatorname{fp}}(X \times Y)$$

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- (iv) Check that \mathscr{A} , \mathscr{B} weakly Tannakian $\Rightarrow \mathscr{A} \boxtimes \mathscr{B}$ weakly Tannakian (use the fiber functor $w \boxtimes v$)