Generalized Tannakian duality

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Classical Tannaka duality

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Reconstruction problem: Can a group-like object be reconstructed from its category of representations?

Recognition problem: Which categories are equivalent to categories of representations for some group-like object?
Classical Tannaka duality

Group-like objects

Categories equipped with suitable structures

Reconstruction problem

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Which categories are equivalent to categories of representations for some group-like object?
Tannaka duality for Hopf algebras over fields

Theorem
Every Hopf algebra can be reconstructed from the category of finite-dimensional comodules.

Theorem (Saavedra Rivano, Deligne)
Let $k$ be a field. If $A$ is an abelian autonomous symmetric monoidal $k$-linear category $w : A \to \text{Vect}_k$ is a faithful exact symmetric strong monoidal $k$-linear functor then there exists a Hopf algebra $H$ such that $A \cong \text{Rep}(H)$. 

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The classical proof

Deligne’s proof

A abelian, \( w: A \rightarrow \text{Vect}_k \) faithful & exact
\Rightarrow A \cong \text{Comod}(C)

Symmetric monoidal structure
\Rightarrow \text{bialgebra structure on C}

Autonomous
\Rightarrow \text{Hopf algebra structure on C}

Theorem (Street)
There is a biadjunction between \( k \)-linear categories over \( \text{Vect}_k \) and coalgebras.

Reconstruction problem: when is the counit an isomorphism?
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**Deligne’s proof**

- **Abelian**: $\mathcal{A}$ abelian, $w: \mathcal{A} \rightarrow \text{Vect}_k$ faithful & exact $\rightsquigarrow \mathcal{A} \simeq \text{Comod}(C)$
- **Monoidal Structure**: Symmetric monoidal structure $\rightsquigarrow$ bialgebra structure on $C$
- **Autonomous**: $\mathcal{A}$ autonomous $\rightsquigarrow$ Hopf algebra structure on $C$
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Definition
A cosmos is a complete and cocomplete symmetric monoidal closed category $V$.

Definition
A profunctor (also known as distributor or module) $A \to B$ is a cocontinuous functor $[A^{\text{op}}, V] \to [B^{\text{op}}, V]$. The category of profunctors is denoted by $\text{Prof}(V)$.

Observation
Coalgebras are precisely comonads $I \to I$ in $\text{Prof}(V)$.

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A left adjoint 1-cell in a bicategory is called a **map**.
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Lemma
Maps $\mathcal{A} \to \mathcal{B}$ in $\text{Prof}(\mathcal{V})$ are in bijection with $\mathcal{V}$-functors $\mathcal{A} \to \overline{\mathcal{B}}$. 

Proof.
Let $L: \mathcal{A}^{\text{op}}, \mathcal{V} \to \mathcal{B}^{\text{op}}, \mathcal{V}$ be cocontinuous. Then there exists $w: \mathcal{A} \to \mathcal{B}^{\text{op}}, \mathcal{V}$ such that $L = \text{Lan}_Y w$. 

Lan$_Y w$ has a right adjoint $X \mapsto \text{Hom}(w^\text{-}, X)$. The right adjoint is cocontinuous $\iff w(A) \in \mathcal{B}$ for all $A \in \mathcal{A}$.

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The Cauchy completion of $\mathcal{I}$ is the full subcategory of dualizable objects in $\mathcal{V}$. 

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Question
Can we characterize $\text{Comod}(C')$ in terms of a 2-categorical universal property in $\text{Prof}(\mathcal{V})$?
A coaction of a comonad \( c: B \to B \) is a morphism \( v: A \to B \), together with a 2-cell \( \rho: v \Rightarrow c.v \), compatible with the comonad structure.
Tannaka-Krein objects

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- A coaction \((v, \rho)\) is called a map coaction if \( v \) is a map (left adjoint).
Tannaka-Krein objects

**Definition**

- A *coaction* of a comonad $c : B \to B$ is a morphism $v : A \to B$, together with a 2-cell $\rho : v \Rightarrow c \cdot v$, compatible with the comonad structure.

- A coaction $(v, \rho)$ is called a *map coaction* if $v$ is a map (left adjoint).

- A morphism of (map) coactions $(v, \rho) \to (w, \sigma)$ is a 2-cell $\alpha : v \Rightarrow w$ compatible with $\rho$ and $\sigma$.

- A Tannaka-Krein object is a *universal* map coaction, i.e., a map coaction $(v, \rho)$ such that every map coaction is isomorphic to $v \cdot f$ for some map $f$. For all maps $f$ and all 1-cells $g$, whiskering with $v$ induces a bijection between 2-cells $g \Rightarrow f$ and morphisms of coactions $v \cdot g \to v \cdot f$. 

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Tannaka-Krein objects in $\text{Prof}(\mathcal{V})$

**Definition**

Let $C$ be a cocontinuous comonad on $[\mathcal{B}^{\text{op}}, \mathcal{V}]$. A **Cauchy comodule of $C$** is a comodule whose underlying object lies in $\overline{\mathcal{B}}$.
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**Definition**

Let $C$ be a cocontinuous comonad on $[\mathcal{B}^{\text{op}}, \mathcal{V}]$. A Cauchy comodule of $C$ is a comodule whose underlying object lies in $\mathcal{B}$. The $\mathcal{V}$-category of Cauchy comodules of $C$ is denoted by $\text{Rep}(C)$. 
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Theorem (S.)

The forgetful functor $\text{Rep}(C) \to \mathcal{B}$ is a Tannaka-Krein object in $\text{Prof}(\mathcal{V})$. 
Theorem (S.)

If $M$ is a 2-category with Tannaka-Krein objects, then the functor

\[ L : \text{Map}(M) / B \to \text{Comon}(B) \]

given by $w \mapsto \Delta(w)$ has a right biadjoint $\text{Rep}(-)$ (which sends a comonad $c$ to the Tannaka-Krein object of $c$).

The category $\text{Map}(M) / B$ has morphisms the triangles that commute up to invertible 2-cell.

This theorem does not require the full strength of the definition of Tannaka-Krein objects.
Theorem (S.)

If $\mathcal{M}$ is a 2-category with Tannaka-Krein objects, then the functor

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given by $w \mapsto w \cdot w$ has a right biadjoint $\text{Rep}(\cdot)$ (which sends a comonad $c$ to the Tannaka-Krein object of $c$).
Theorem (S.)

If $\mathcal{M}$ is a 2-category with Tannaka-Krein objects, then the functor

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given by $w \mapsto w.\overline{w}$ has a right biadjoint $\text{Rep}(-)$ (which sends a comonad $c$ to the Tannaka-Krein object of $c$).

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- The category $\text{Map}(\mathcal{M})/B$ has morphisms the triangles that commute up to invertible 2-cell.
- This theorem does not require the full strength of the definition of Tannaka-Krein objects.
Let $\mathcal{M}$ be a monoidal 2-category, and $(B, m, u) \in \mathcal{M}$ a map pseudomonoid.
Monoidal structure on the slice category

Let $\mathcal{M}$ be a monoidal 2-category, and $(B, m, u) \in \mathcal{M}$ a map pseudomonoid. Given $w$ and $w'$ in $\text{Map}(\mathcal{M})/B$, let $w \bullet w'$ be the composite

$$A \otimes A' \xrightarrow{w \otimes w'} B \otimes B \xrightarrow{m} B$$
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**Proposition**

The above assignment endows $\text{Map}(\mathcal{M})/B$ with the structure of a monoidal 2-category.
Let $\mathcal{M}$ be a monoidal 2-category, let $(A, d, e)$ be a pseudocomonoid in $\mathcal{M}$, and let $(B, m, u)$ be pseudomonoid in $\mathcal{M}$.

**Definition**

The convolution product $f \star g$ of two 1-cells $f, g \in \mathcal{M}(A, B)$ is given by:

$A \xrightarrow{d} A \otimes A \xrightarrow{f \otimes g} B \otimes B \xrightarrow{m} B$

**Proposition**

Let $(B, m, u)$ be a map pseudomonoid in $\mathcal{M}$. Then $(B, m, u)$ is a pseudocomonoid, and the convolution product on $\mathcal{M}(B, B)$ lifts to the category $\text{Comon}(B)$ of comonads on $B$.

A monoid in $\text{Comon}(B)$ is precisely a monoidal comonad.
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The Tannakian biadjunction is monoidal

**Theorem (S.)**

If $\mathcal{M}$ is a monoidal 2-category and $(B, m, u)$ is a map pseudomonoid in $\mathcal{M}$, then the left adjoint of the Tannakian biadjunction is strong monoidal.
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**Proof.** Let $w: A \to B$, $w': A' \to B$ be two objects in the domain of $L$. 

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**Proof.** Let $w: A \to B$, $w': A' \to B$ be two objects in the domain of $L$. Since $\otimes$ is a pseudofunctor, we have

$$L(w \bullet w') = B \xrightarrow{m} B \otimes B \xrightarrow{w \otimes w'} A \otimes A' \xrightarrow{w \otimes w'} B \otimes B \xrightarrow{m} B$$
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By definition, $L(w) \star L(w')$ is given by

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Thus $L(w) \star L(w') \cong L(w \bullet w')$. 
Braiding, syllepsis and symmetry

If $\mathcal{M}$ is braided and $B$ is a braided map pseudomonoid, then

$\text{Map}(\mathcal{M})/B$ is a braided 2-category.

$\text{Comon}(B)$ is a braided category.

The left adjoint of the Tannakian biadjunction is a braided strong monoidal 2-functor.

Analogous facts hold for sylleptic and symmetric monoidal 2-categories.

Corollary
If $\mathcal{M}$ is (braided, sylleptic) monoidal, then the Tannakian biadjunction lifts to (braided, symmetric) pseudomonoids.

Theorem (S.)
If $A$ and $B$ are autonomous map pseudomonoids, and $w: A \to B$ is a strong monoidal map, then $L(w) = w$. $w$ is a Hopf monoidal comonad.
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If $A$ and $B$ are autonomous map pseudomonoids, and $w: A \to B$ is a strong monoidal map, then $L(w) = w$. $w$ is a Hopf monoidal comonad.
Braiding, syllepsis and symmetry

If $\mathcal{M}$ is braided and $B$ is a braided map pseudomonoid, then

- $\text{Map}(\mathcal{M})/B$ is a braided 2-category.
- $\text{Comon}(B)$ is a braided category.
- The left adjoint of the Tannakian biadjunction is a \textit{braided} strong monoidal 2-functor.

Analogous facts hold for sylleptic and symmetric monoidal 2-categories.
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**Theorem (S.)**

If $A$ and $B$ are autonomous map pseudomonoids, and $w: A \to B$ is a strong monoidal map, then $L(w) = w \cdot \overline{w}$ is a Hopf monoidal comonad.
Theorem (S.)
Let $B$ be a commutative $R$-algebra, and let $A$ be an additive autonomous symmetric monoidal $R$-linear category. Let $w : A \to \text{Mod}_B$ be a symmetric strong monoidal $R$-linear functor. Suppose that:

1. $w$ is faithful and reflects isomorphisms;
2. $w$ is flat;
3. whenever the cokernel of $w(f)$ is finitely generated projective, then the cokernel of $f$ exists and is preserved by $w$.

Then there exists a Hopf algebroid $(H, B)$ and an equivalence $A \cong \text{Rep}(H, B)$.
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The category of Cauchy comodules has the universal property of a TK-object in $\text{Prof}(V)$. The existence of TK-objects in $M$ implies that the Tannakian biadjunction exists. If $M$ is monoidal, then the Tannakian biadjunction is monoidal. The same is true for braided, sylleptic and symmetric $M$. This explains why the Tannakian biadjunction lifts to (braided or symmetric) pseudomonoids.

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