ONE PLUS ONE EQUALS ONE TIMES ONE

DANIEL SCHÄPPI

1. Autonomous categories and semiadditivity

Let $\mathscr{C}$ be an autonomous symmetric monoidal category with a zero object 0. Assume that the coproduct $I+I$ of the unit object I with itself exists. Write $\iota_i \colon I \to I+I$ for the structure maps of this coproduct. For all objects $C \in \mathscr{C}$, the functors $C \otimes - \colon \mathscr{C} \to \mathscr{C}$ are left adjoint to $C^\vee \otimes - \colon \mathscr{C} \to \mathscr{C}$, so they preserve all colimits. In particular, the maps $C \otimes \iota_i \colon C \otimes I \to C \otimes (I+I)$ exhibit $C \otimes (I+I)$ as the coproduct of two copies of $C \otimes I \cong C$. Furthermore, we have

$$\mathscr{C}\big(C,(I+I)^\vee\big) \cong \mathscr{C}\big(C \otimes (I+I), I\big) \cong \mathscr{C}(C,I) \times \mathscr{C}(C,I)$$

which shows that $(I+I)^\vee$ is a product of I with itself. This shows in particular that $\mathscr{C}^{\mathrm{op}}$ is again a category satisfying all our assumptions. We fix a product $I \times I$ of I with itself, and we write $\pi_i \colon I \times I \to I$ for the projection maps. Let $p_i \colon I+I \to I$ and $q_i \colon I \to I \times I$ be the unique maps making the diagrams

$$\begin{tikzcd}
I \arrow[d, "\iota_1"'] \arrow[dr, "\mathrm{id}"] & \\
I+I \arrow[r, "p_1"] & I \\
I \arrow[u, "\iota_2"] \arrow[ur, "0"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
I \arrow[d, "\iota_1"'] \arrow[dr, "0"] & \\
I+I \arrow[r, "p_2"] & I \\
I \arrow[u, "\iota_2"] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
& I \\
I \arrow[ur, "\mathrm{id}"] \arrow[r, "q_1"] \arrow[dr, "0"'] & I \times I \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \\
& I
\end{tikzcd}
\quad
\begin{tikzcd}
& I \\
I \arrow[ur, "0"] \arrow[r, "q_2"] \arrow[dr, "\mathrm{id}"'] & I \times I \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \\
& I
\end{tikzcd}$$

commutative.

Proposition 1.1. *The object $(I+I, p_1, p_2)$ is a weak product of I with itself, i.e., given an object $C \in \mathscr{C}$ and maps $f_i \colon C \to I$, there exists a map $f \colon C \to I+I$ with $p_i f = f_i$ for $i=1,2$. Dually, the object $(I \times I, q_1, q_2)$ is a weak coproduct of I with itself.*

Before proving this, we show how to use this to prove that the canonical map $\varphi \colon I+I \to I \times I$ is an isomorphism. First note φ is the unique map with $\pi_i \circ \varphi = p_i$, and it is also the unique map with $\varphi \circ \iota_i = q_i$. From Proposition 1.1 it follows that the dotted morphisms l and r in the commutative diagrams

$$\begin{tikzcd}
& I & \\
I \times I \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] \arrow[r, dashed, "r"] & I+I \arrow[u, "p_1"'] \arrow[d, "p_2"] \arrow[r, "\varphi"] & I \times I \arrow[ul, "\pi_1"'] \arrow[dl, "\pi_2"] \\
& I &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& I \arrow[dl, "\iota_1"'] \arrow[d, "q_1"] \arrow[dr, "\iota_1"] & \\
I+I \arrow[r, "\varphi"] & I \times I \arrow[r, dashed, "l"] & I+I \\
& I \arrow[ul, "\iota_2"] \arrow[u, "q_2"'] \arrow[ur, "\iota_2"'] &
\end{tikzcd}$$

exist. Commutativity immediately implies that $\varphi r = \mathrm{id}$ and $l\varphi = \mathrm{id}$, which shows that the canonical morphism $I + I \to I \times I$ is indeed an isomorphism.

Corollary 1.2. *Let $\mathscr{C}$ be an autonomous symmetric monoidal category with a zero object 0 such that the coproduct $I + I$ of the unit object I with itself exists. Then $\mathscr{C}$ is enriched in commutative monoids. If $\mathscr{C}$ has all finite coproducts, then $\mathscr{C}$ is semiadditive.*

Proof. The functors $C \otimes -$ are both left and right adjoint, hence the coproduct $C + C$ and the product $C \times C$ exist (they are the images of $I + I$ and $I \times I$ under $C \otimes -$). Moreover, it follows from the above considerations that $C + C \cong C \times C$, i.e., that the biproduct of C with itself exists. Using the diagonal and codiagonal maps one obtains the desired monoid structure on hom-sets. □

Corollary 1.3. *Let $\mathscr{V}$ be a cosmos (a complete and cocomplete symmetric monoidal closed category) with a zero object such that the object $I + I$ is dualizable. Then $\mathscr{V}$ is semiadditive.*

Proof. Let $\mathscr{C} \subseteq \mathscr{V}$ be the full subcategory of dualizable objects. First note that the existence of internal homs implies that $V \otimes -$ will preserve colimits, so we have

$$\mathscr{V}\big(V, (I+I)^\vee\big) \cong \mathscr{V}\big(V \otimes (I+I), I\big) \cong \mathscr{V}(V,I) \times \mathscr{V}(V,I)$$

for any object $V \in \mathscr{V}$. This shows that $(I+I)^\vee$ is isomorphic to $I \times I$, i.e., that $I \times I$ lies in $\mathscr{C}$. Thus we can apply Corollary 1.2 to find that the canonical map $I + I \to I \times I$ is an isomorphism. Moreover, for any $M, N \in \mathscr{V}$ we have isomorphisms

$$\mathscr{V}\big(M, N \otimes (I \times I)\big) \cong \mathscr{V}\big(M \otimes (I+I), N\big) \cong \mathscr{V}(M,N) \times \mathscr{V}(M,N)$$

which are natural in M. This shows that $N \otimes (I \times I) \cong N \times N$, which implies that the canonical map $N + N \to N \times N$ is an isomorphism. It follows that $\mathscr{V}$ is enriched in commutative monoids. □

In order to prove Proposition 1.1 we need to introduce some auxiliary maps. We first fix an object $C \in \mathscr{C}$, together with two morphisms $f_i \colon C \to I$. Let

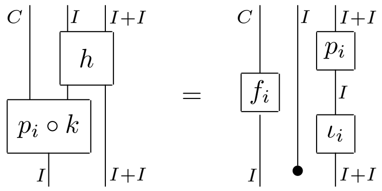
$$h \colon I \otimes (I+I) \to (I+I) \otimes (I+I) \quad \text{and} \quad k \colon C \otimes (I+I) \to I + I$$

be the unique maps which make the diagrams

$$\begin{tikzcd}
I \otimes I \arrow[d, "I \otimes \iota_1"'] \arrow[dr, "\iota_1 \otimes \iota_1"] & \\
I \otimes (I+I) \arrow[r, "h"] & (I+I) \otimes (I+I) \\
I \otimes I \arrow[u, "I \otimes \iota_2"] \arrow[ur, "\iota_2 \otimes \iota_2"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
C \otimes I \arrow[r, "f_1 \otimes \iota_1"] \arrow[d, "C \otimes \iota_1"'] & I \otimes (I+I) \arrow[d, "\cong"] \\
C \otimes (I+I) \arrow[r, "k"] & I + I \\
C \otimes I \arrow[u, "C \otimes \iota_2"] \arrow[r, "f_2 \otimes \iota_2"'] & I \otimes (I+I) \arrow[u, "\cong"']
\end{tikzcd}$$

commutative.

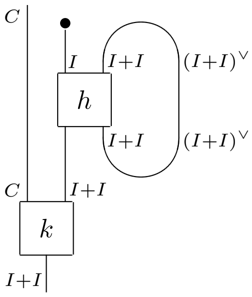
Lemma 1.4. *The equality*



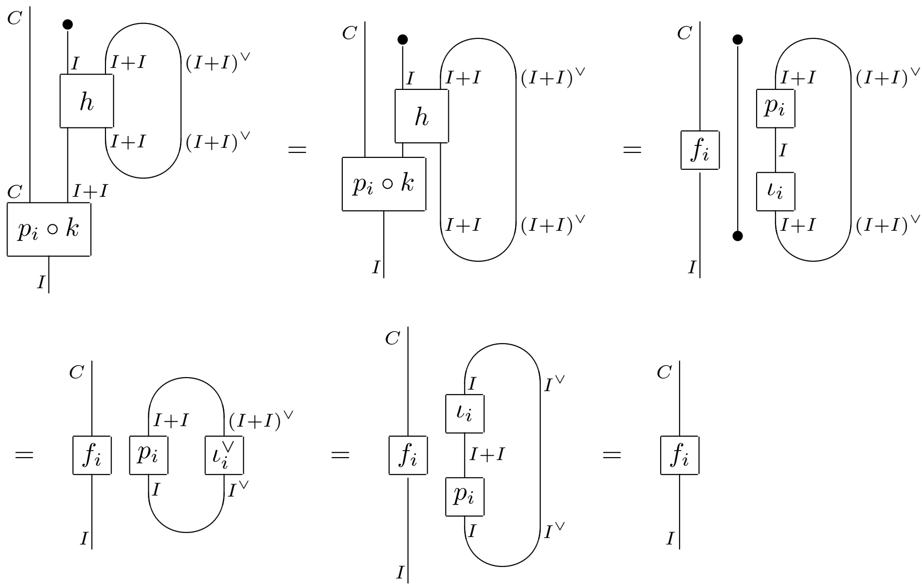
holds.

Proof. Since $C \otimes I \otimes -$ preserves coproducts, it suffices to check that both sides of the above equation coincide when precomposed with $C \otimes I \otimes \iota_i$ and $C \otimes I \otimes \iota_{3-i}$. It follows easily from the definitions that these composites are $(l_I \circ f_i \otimes I) \otimes \iota_i$ and 0 respectively, where $l_I \colon I \otimes I \to I$ is the canonical isomorphism. $\square$

Proof of Prop 1.1. We define a map $f \colon C \to I + I$ by the string diagram

$f \quad =$ 

Lemma 1.4 gives one of the equalities in the sequence



which shows that $p_i \circ f = f_i$, as desired. This proves that $(I + I, p_1, p_2)$ is a weak product of I with itself. Moreover, we already showed that the product of I with itself exists (it is isomorphic to the dual of $I + I$), so we can apply our reasoning to $\mathscr{C}^{\mathrm{op}}$ to find that $(I \times I, q_1, q_2)$ is a weak coproduct of I with itself. $\square$

University of Chicago, Department of Mathematics, 5734 S University Avenue, 60637 Chicago